Notes on fundamental groups and covering spaces

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These are the notes for the course M1-Topologie Algébrique, Master en Mathématiques at the Sorbonne University (2022-2023).

The lectures will follow closely the notes by Ilia Itenberg (in french).

The present notes are intended as complementary study material, written in english. If you are an ERASMUS student, feel free to ask me for other studying materials in english.

If you have any comments, suggestions, questions, corrections, please write to me: marco.robalo at sorbonne-universite.fr

All the exercises (TD) are available in the links below (in french) but the most important are also incorporated in the text, in english.

- Feuille 1
- Feuille 2
- Feuille 3
- Feuille 4
- Feuille 5

There is no claim of original content in this notes (except the mistakes!). These notes are essentially a compilation of materials from the following references:

- **Main reference**: Le Polycopie d'Ilia Itenberg (in french)
- Michèle Audin - Notes revêtements et groupe fondamental.
- Notes by Pierre Schapira, General Topology and Algebra and Topology, Chapter 7
- Munkres, Topology (2nd edition)
- Emily Riehl, Categories in Context - Chapters 1 to 4 and these notes.
- Hatcher, Algebraic Topology, Chapters 0 and 1.
- P. May, A concise course in algebraic topology - Chapters 1 to 3
- Analysis Situs
- R. Brown, Topology and Groupoids, Chapter 10
- T. Szamuely Galois groups and fundamental groups - Chapter 2

Contents

CHAPTER I

Introduction

In today's lecture we will make some experiments.

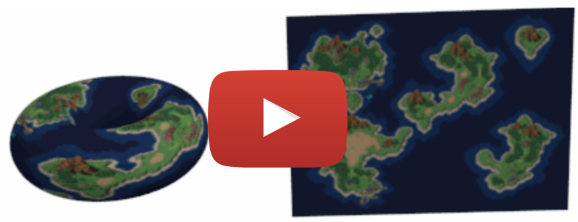
I.1. World's with different shapes

Exercice I.1.1. Click on the picture and play this snake game trying to cross the walls

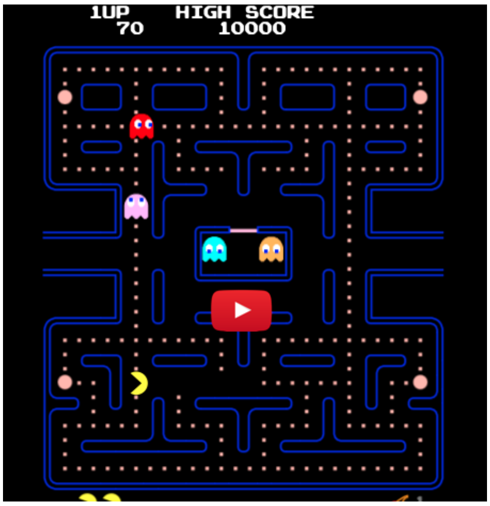
Is the planet where the snake game takes place, a sphere like our planet?

Solution I.1.2. Check the solution here.

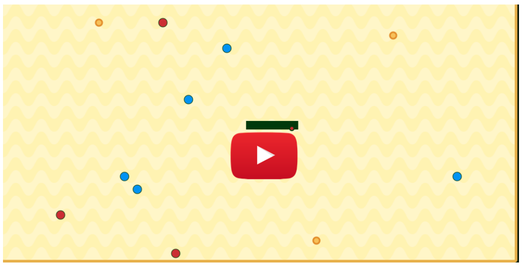
Remark I.1.3. Many video games take place in a doughnut-like planet but you will miss this if you are not topology-aware. Here is another example from Chrono Trigger.



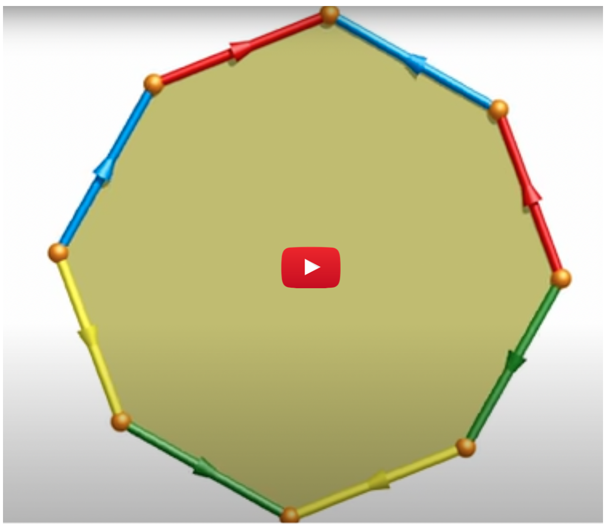
Exercice I.1.4. What is the shape of the planet where the first level of Pacman takes place?



Exercice I.1.5. Try to describe the shape of the planet where the next snake game takes place:

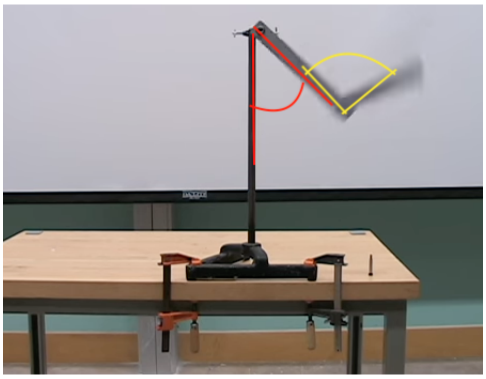


Exercice I.1.6. Imagine now that the snake game takes place in the region delimited in the picture below where if you cross a border of a given color, you re-appear in the other border of the same color, exactly as in the world of the original snake game. What is the shape of the snake planet this time?



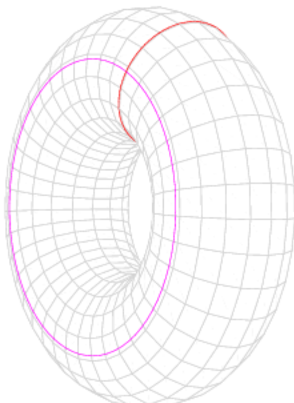
Question I.1.7. In both cases of Exercice **I.1.1**, Exercice **I.1.5** and Exercice **I.1.6**, how can the snake itself detect/measure the shape of the world it lives in without any references to the outside world? In this course we will learn some algebraic tools to make this possible.

Exercice I.1.8. Consider the double-pendulum as in the following video:



It has two degrees of freedom: the angle α around the vertical axis of the first arm and the angle β of the second arm around the main arm. The position of the pendulum is therefore given by two angles (α, β). What is the shape of the space of all positions?

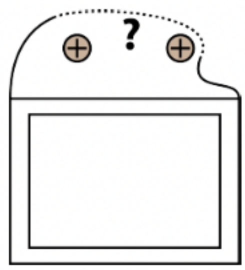
Solution I.1.9. This is yet another presentation of the torus. Indeed, there is a whole circle of possibilities for the first angle α. In topology we usually denote the circle of radius 1 by S^1. At each choice of angle α, we then have a second whole circle of possibilities, corresponding to the angle β. It follows that the collection of all pairs (α, β) is an element in the *product space* $S^1 \times S^1$. We will define the product topology in the next lecture. Here's a picture of the situation: the larger circle represents the choice of the angle α and for every such choice, we have a smaller circle, corresponding to the choice of β:



At the end of the second week you should be able to show that the presentation of the torus as a result of gluing the sides of a square as in the Exercice **I.1.1** is *equivalent* to the product topological space $\mathsf{S}^1 \times \mathsf{S}^1$.

I.2. Strings,Words and Labyrinths

Exercice I.2.1 (This puzzle and the pictures are taken from this book which is highly recommended!)**.** Imagine you have picture frame hung by one string. Nailed to the wall are two pins. How can you wrap the string around the two nails (figure below) such that the picture does not fall, but as soon as either of the nails are removed the picture will fall? (as an extension, what about a situation with N = 100 pins, where the frame is hung but will again fall as soon as you remove any one of the nails?)



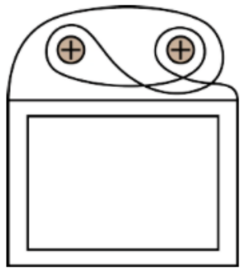
Solution I.2.2. The solution to this puzzle is somehow the universal example of how we will use algebra to solve problems in topology. Here's what we can do. Consider the first nail. If we wrap the string around it clockwise, we denote it with the letter a. If we wrap it counter-clockwise, we denote it by a^{-1}. For the second nail we use the letter b and b^{-1}. Now, the different ways to wrap the string around the two nails will correspond to words written in the alphabet a, a^{-1}, b and b^{-1}. But there is more to it: if we wrap the string around the first nail clockwise (ie, if we write a) and then wrap it again counter-clockwise, (ie, a^{-1}), the net result is that we didn't wrap anything at all! Ie, the word $a^{-1}a$ is identified with the empty word which we will simply denote as $\emptyset$. The same goes for the words aa^{-1}, bb^{-1} and $b^{-1}b$.

Notice that the operation of making consecutive wraps around the nails corresponds to the operation of concatenating words in the alphabet $\{a, b, b^{-1}, a^{-1}\}$.

Let us write $\pi :=< a, b >$ for the *free group* generated by the symbols a and b. By definition this is exactly the set of all words written using the alphabet $\{a, b, b^{-1}, a^{-1}\}$ with the relations $a^{-1}a = aa^{-1} = bb^{-1} = b^{-1}b = \emptyset$. The group operation is given by concatenation of words. For instance, the word

$$a^{-1}b^{-1}ab \tag{1}$$

is an element of π and corresponds to start by wrapping the string clockwise around the second nail, then clockwise around the first nail, then counter-clockwise around the second, and finally counter-clockwise around the first. By convention we read the word from the left to the right. Here's a picture of this situation:



Let us now suppose that we take away the second nail. In this case all instances of b and b^{-1} will disappear. For instance, the previous word becomes

$$a^{-1}a$$

But this is the empty word and therefore, corresponds to wrapping and un-wrapping and therefore to doing nothing. In this case, this means that if we take away the second nail, the picture frame will fall.

Here's another example. Consider the word

$$aaba^{-1}$$

and again take away the second nail. This eliminates all the b's from the words, and we get simply the word

$$aaa^{-1} = a$$

This means that the picture frame will still hold since there is a wrapping that survives. It is easy now to understand the general mechanism: the frame will hold every time there is at least one wrapping around the first nail that survives.

Remark I.2.3. Notice that the group π introduced in the Solution **I.2.2** is not *abelian*. Indeed, in a abelian group, we should always have the relation

$$ab = ba$$

Or equivalently,

$$\emptyset = b^{-1}a^{-1}ba$$

The fact that the word (1) in the Solution **I.2.2** is not the empty word means that the group π is definitively non-abelian.

Definition I.2.4. Let $(G, \circ, e)$ be a group with operation $\circ$ and unit element e. For elements $g, h \in G$, the element $[g, h] := ghg^{-1}h^{-1}$ is called the commutator of g and h. An element of G of the form $[g, h]$ is called a commutator. We denote by $[G, G] \subseteq G$ the smallest subgroup of G containing all commutators.

Remark I.2.5. $[G, G] \subseteq G$ is a normal subgroup. Indeed, assume that $c \in [G, G]$ and let $g \in G$. We want to show that gcg^{-1} is again in $[G, G]$. But $gcg^{-1}c^{-1}$ is a commutator, so it must belong to $[G, G]$. Since c is a commutator, and $[G, G]$ is a subgroup, it is close for the group operation, and therefore $(gcg^{-1}c^{-1}).c$ must also belong to $[G, G]$. But $(gcg^{-1}c^{-1}).c = g.c.g^{-1}$.

It follows that the quotient $G/[G, G]$ is a group.

Proposition I.2.6. *The quotient group $G^{ab} := G/[G, G]$ is abelian. Moreover, it is the universal abelian group built out of G, in the sense that if $f : G \to H$ is a group homomorphism, with H abelian, then f factors canonically in a unique way through G^{ab}:*

$$\begin{array}{ccc} G & \xrightarrow{f} & H \\ \downarrow & \nearrow & \\ G^{ab} & & \end{array}$$

Proof. We start by showing that the quotient $G^{ab} := G/[G, G]$ is an abelian group. Let us denote by $[g]$ the image of $g \in G$ in the quotient. We must show that $[g].[h] = [h].[g]$ for every g, h in G. In other words, that $[ghg^{-1}h^{-1}] = 0$ in the quotient. But this is precisely the quotient relation.

Now suppose that $f : G \to H$ is a group homomorphism with H abelian. It follows the commutator group of H, $[H, H] = \{0\}$ is the trivial subgroup and therefore $H \simeq H^{ab}$. By definition of group homomorphism, f must send commutator subgroups to commutator subgroups and therefore descend to the quotients

$$\begin{array}{ccc} G & \xrightarrow{f} & H \\ \downarrow & & \downarrow \wr \\ G^{ab} & \dashrightarrow & H^{ab} \end{array}$$

∎

Definition I.2.7. The group G^{ab} of the Proposition **I.2.6** is called the *abelianization* of G

Example I.2.8. The abelianization of the group π of the Solution **I.2.2** is the free *abelian* group with two generators a and b, ie

$$\pi^{ab} = a.\mathbb{Z} \oplus b.\mathbb{Z}$$

For instance, the wrapping corresponding to a word such as $ababbaa^{-1} \in \pi$ is sent to the element $2a + 3b$, obtained simply by counting the occurrences of each symbol and canceling everytime we have inverses. In conclusion, the frame will hold if after eliminating the second nail, the coefficient attached to a in π^{ab} is non-zero.

Exercice I.2.9. Considering wrapping the frame following the prescription

$$ababa^{-1}ba^{-1}a^{-1}$$

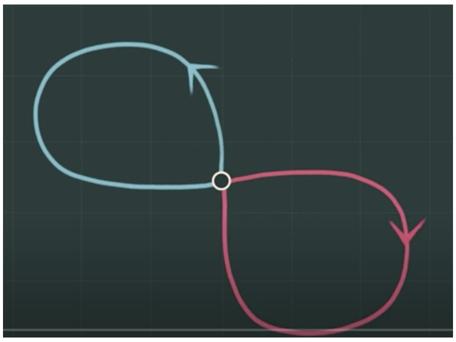
What happens when we take away the second pin?

Here is another example of how to turn a topological problem into an algebraic one:

Exercise I.2.10. [The Infinite Labyrinth]
Watch this video (in french).

and draw the shape of the labyrinth corresponding to the subgroup $< rr, bb, rb >$ (with the the notations used in the video).

Remark I.2.11. Notice that in the Exercise **I.2.10**, the group associated to the simplest labyrinth (the double loop)



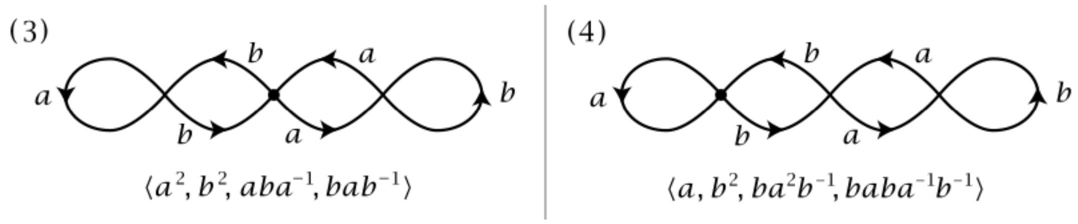
is the free group generated by the two symbols r and b. It is exactly the same group (up to isomorphism) as the group π from the Solution **I.2.2**. In this case what the video suggests is some form of correspondence

$$\{\text{subgroups of } \pi\} \leftrightarrow \{\text{ shapes of the labyrinth "covering" the double loop}\}$$

The video illustrates an example of how this correspondence works in the example

$< r, bb, brb^{-1} >$ $\longleftrightarrow$ 

However, this example has a particularly nice feature that hides the real form of the correspondence: in this example we don't see how the subgroup we get depends on the choice of the room we use as reference. This is because this is a normal subgroup! Here's an example where this dependency shows up:



(Picture taken from Hatcher's book)

These two examples differ by the choice of the reference room. The two subgroups we get differ by conjugation with the path b connecting the two different rooms. Indeed, write

$$H_1 :=< a^2, b^2, aba^{-1}, bab^{-1} > \qquad H_2 :=< a, b^2, ba^2b^{-1}, baba^{-1}b^{-1} >$$

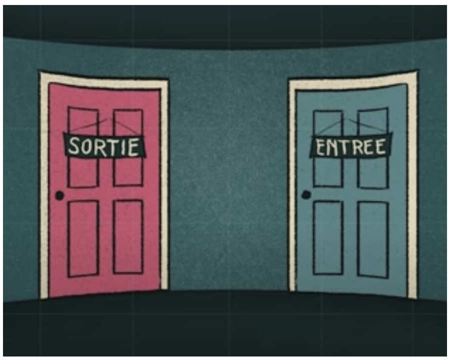
and notice that $b.H_1.b^{-1} = H_2$ (check this on the generators!).

In fact, the real content of the correspondence is of the form

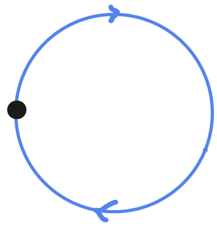
$$\{\text{Conjugacy classes of subgroups of } \pi\} \leftrightarrow \{\text{ Labyrinths "covering" the double loop}\}$$

The last main theorem in this course (Theorem **VI.4.14**) is a generalization of this example, replacing the simplest double loop labyrinth by any space X, the group π by the so-called *fundamental group of the space* X and the different possible shapes of the labyrinth covering the double loop by the different covering spaces of X.

Example I.2.12. Let us try to repeat the Exercise **I.2.10** this time with a simpler labyrinth, namely, a room where there is simply an entrance door and an exit door.



The simplest possibility is that everytime we go out through the red door with come in through the blue door. In this case, the labyrinth is a single circle



The first main theorem in these lectures (Theorem **V.2.2**) is that the group π associated to this labyrinth is the free group with one generator a corresponding to going around the loop one time clockwise, ie

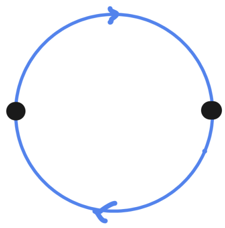
$$\pi \simeq (\mathbb{Z}, +)$$

determined by

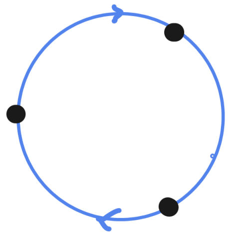
$$a^n \mapsto n$$

In this case we can start imagining the other possible shapes of the labyrinth and match them with subgroups of $\pi \simeq \mathbb{Z}$ as we did in Exercise **I.2.10**:

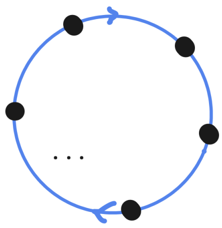
- There are two different rooms and taking the red door takes us to the blue door of the second room and taking the red door in the second room takes us back to the blue door in the first:



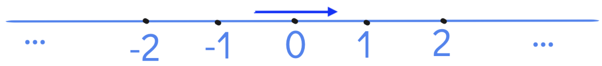
- There are three different rooms and the last one brings us back to the first



- ...
- There are n different rooms and the last one brings us back to the first.



- Every room is a different room. We never come back to the starting one.



In terms of subgroups, these correspond exactly to the different subgroups given by multiples of n, for each n

$$n\mathbb{Z} := \{a \in \mathbb{Z} : n|a\} \subseteq \mathbb{Z}$$

The case with infinitely many rooms corresponds to the trivial subgroup $\{0\}$

CHAPTER II

Quick review of general topology

II.1. Topological Spaces

Goal II.1.1. Our goal in this chapter is to define a useful notion of topological space accommodating the questions and examples of the first lecture. By the end of this chapter we will have precise definitions for the whole zoo: the circle, sphere, torus, square, gluings, quotient, product space, loops, paths, topological equivalence, etc.

Warning II.1.2. This first chapter covers more materials than the lectures. You should use it as a complement for both the main course and the exercise sessions.

In lack of a better strategy, a topological space will be *defined* as a set with extra structure. The purpose of this extra structure is to give us a way of saying that two points/elements of the set are close to each other. We will do this in a somehow minimalist way, avoiding to talk about distances. In this sense, we will be authorized to talk about open subsets - which we think of as regions of the set. We control how the points are close or far to each other, by looking at the open subsets they share in common.

Notation II.1.3. Let X be a set. We denote by $P(X)$ the set of all subsets of X.

Definition II.1.4. Let X be a set. A *topology* on X is a collection τ of subsets of X, ie, $\tau \subseteq P(X)$, satisfying the following properties:

- Both the empty subset $\emptyset$ and X belong to τ;
- For any family of subsets belonging to τ, their union belongs to τ;
- For any finite family of subsets belonging to τ, their intersection belongs to τ

In this case we call the pair (X, τ) a topological space.
As an abuse of notation, and whenever it is clear, we will simply write X to denote a topological space (X, τ)

Terminology II.1.5. The elements of τ are called open subsets and their complements *closed subsets*. The elements of X are called points. If $x \in X$ and U is an open subset containing x we call U an open neighborhood of x.

Exercice II.1.6. Translate the three properties of open sets into properties of closed subsets: closure under finite unions and arbitrary intersections.

Example II.1.7.

(i) Let $X = \{0, 1\}$ be a set with two elements. Then the collection of subsets $\tau := \{\{0\}, X, \emptyset\}$ forms a topology on X. Indeed,

$$\emptyset \bigcap \{0\} = \emptyset \in \tau, \quad \emptyset \bigcup \{0\} = \{0\} \in \tau, \quad X \bigcap \{0\} = \{0\} \in \tau$$

$$X \bigcup \{0\} = X \in \tau$$

In this topology we cannot isolate the element 0 from the element 1 by open subsets: every open subset that contains 1 also contains 0.

(ii) The set of real numbers $\mathbb{R}$ with topology where the open sets are unions of open intervals, ie, intervals of the form $]a, b[$ with $a < b$, $a, b \in \mathbb{R}$. Let us check the axioms: first $\mathbb{R}$ and $\emptyset$ are both open by definition. Secondly, since open subsets are by definition arbitrary unions of open intervals, they are stable under unions. Let us now observe that finite intersections of open intervals are open intervals. We can see this in the simplest case of a double intersection:

$$]a, b[\cap]c, d[=]\mathsf{sup}(a, c), \mathsf{inf}(b, d)[$$

and proceed by induction. Assume now $U = \bigcup_i]a_i, b_i[$ and $V = \bigcup_j]c_j, b_j[$ are arbitrary unions of open intervals. Then their intersection is

$$U \bigcap V = (\bigcup_i]a_i, b_i[) \bigcap (\bigcup_j]c_j, b_j[)$$

which by the distributive property of unions over intersections (*), gives

$$= \bigcup_{i.j} \left( ]a_i, b_i[\bigcap]c_j, b_j[\right) = \bigcup_{i.j}]\mathsf{sup}(a_i, c_j), \mathsf{inf}(b_i, d_j)[$$

which is again open, since it is an arbitrary union of open intervals.

We denote this topological space simply by $\mathbb{R}$.

(iii) The set of real numbers $\mathbb{R}$ with the discrete topology, where we declare every subset to be open. In this case, for instance, the singleton $\{0\}$ is open, something which brings us away from our standard intuition. We denote this topological space by $\mathbb{R}_{disc}$

(*) $A \bigcap (B \bigcup C) = (A \bigcap B) \bigcup (A \bigcap C)$. More general, this also works with infinite unions. Prove it!

(iv) The set of real numbers $\mathbb{R}$ with the trivial topology, where we declare only two open subsets: $\mathbb{R}$ and $\emptyset$. This indeed satisfies the axioms of Definition **II.1.4** but it is so stupid that for instance we cannot isolate the points 0 and 0.001. We denote this topological space by $\mathbb{R}_{triv}$.

(v) The set of real numbers $\mathbb{R}$ with the *Zariski topology*, where we declare a subset to be open if its complement is finite. TD Exo 1, Feuille 1

(vi) The set of real numbers $\mathbb{R}$ with the *Arrow topology*, where τ consists of $\mathbb{R}$, $\emptyset$ and all subsets of the form $]a, +\infty[$ with $a \in \mathbb{R}$. TD Exo 1, Feuille 1

Exercice II.1.8. As in previous example, define the discrete and trivial topologies on any set X.

Exercise II.1.9. List all the possible topologies on a set with two elements. Now with three elements.

Exercise II.1.10. Show that the intersection of topologies is a topology.

Proposition II.1.11. *Let X be a topological space and $A \subseteq X$ a subset. Then A is open if and only if for every point $x \in A$ there exists an open subset U with $x \in U \subseteq A$.*

Proof. If A is open we can take $U = A$. Assume now the condition. Then for every $x \in A$ we can find U_x open subset with $x \in U_x \subseteq A$. It follows that $\bigcup_{x \in A} U_x = A$ and A is open as a consequence of the axioms. □

Definition II.1.12. Let (X, τ) be a topological space. We say that a subcollection $\beta \subseteq \tau$ forms a *basis* of the topology τ if every element of τ can be written as a union of elements in β.

Example II.1.13. Open intervals of $\mathbb{R}$ form a basis for the topology of $\mathbb{R}$.

Proposition II.1.14. *Let X be a set and let β be a collection of subsets of X satisfying the two conditions:*

(i) For each $x \in X$, there exists at least one element B of β containing x;

(ii) If x belongs to the intersection of two elements B_1 and B_2 in β, then there exists a third basis element B_3 containing x such that $B_3 \subseteq B_1 \bigcap B_2$.

Then, the collection of subsets τ_β defined by

(*) *$U \in \tau_\beta$ if and only if for every $x \in U$, there exists $B \in \beta$ such that $x \in B \subseteq U$.*

forms a topology with β as a basis. We call it the topology generated by β.

Proof. Let us start by showing that τ_β forms a topology if the two conditions (i) and (ii) are verified.

- The empty set $\emptyset$ is in τ_β because it satisfies the condition in $(*)$ vacuously.
- X satisfies the condition in $(*)$ because of (i) ;
- Let U_i be a family of subsets of X satisfying $(*)$. It is automatic that their union satisfies X.
- Let U_1 and U_2 be subsets of X satisfying the condition in $(*)$. Let $x \in U_1 \bigcap U_2$. Since U_1 satisfies $(*)$, there exists $B_1 \in \beta$ with $x \in B_1 \subseteq U_1$. Similarly, since U_2 satisfies $(*)$, there exists $B_2 \in \beta$ with $x \in B_2 \subseteq U_2$. Therefore $x \in B_1 \bigcap B_2$. By (ii), there exists B_3 with $x \in B_3 \subseteq B_1 \bigcap B_2 \subseteq U_1 \bigcap U_2$. So $U_1 \bigcap U_2 \in \tau_\beta$. We can proceed by induction for finitely many intersections.

To conclude let us show that β is a basis for the topology τ_β. Notice that since elements of β satisfy the condition $(*)$, we have $\beta \subseteq \tau_\beta$. Since τ_β is a topology, the union of elements of β is in τ_β. Finally, given $U \in \tau_\beta$, we can choose for each $x \in U$ an element $B_x \in \beta$ with $x \in B_x \subseteq U$. Therefore, $U = \bigcup_{x\in U} B_x$. □

Construction II.1.15. Let X be a set and β a collection of subsets of X satisfying only the condition (i) in Proposition **II.1.14**. Let us form $I(\beta)$ the collection of all finite intersections of elements in β. Then $I(\beta)$ satisfies conditions (i) and (ii) in Proposition **II.1.14**. Indeed, condition (i) is a consequence of the same assumption for β and condition (ii) is because we have saturated β to include all its finite intersections: if $S_1 := B_1 \bigcap \cdots \bigcap B_n$ and $S_2 := C_1 \bigcap \cdots \bigcap C_m$ with $B_1, \cdots B_n, C_1, \cdots C_m \in \beta$, then

$$S_1 \bigcap S_2 = \big(B_1 \bigcap \cdots \bigcap B_n()\bigcap\big(C_1 \bigcap \cdots \bigcap C_m\big)$$

being a finite intersection of elements of β, is also in $I(\beta)$ so we can take $S_3 := S_1 \bigcap S_2$. Therefore the collection of subsets $\tau_{I(\beta)}$ forms a topology. In this case we say that β is a *subbasis* for the topology and $\tau_{I(\beta)}$ is the topology generated by β.

Exercise II.1.16. Let X be a topological space and β a collection of subsets of X. Then we can form the intersection of all topologies that contain β (see the Exercise **II.1.10** above)

$$\tau_\beta := \bigcap_{\beta \subset \tau, \tau \text{ topology}} \tau$$

This forms a topology without any conditions on β. The problem of this topology is that we do not have a useful criterium to understand what are the open subsets. Show that when β satisfies the conditions of a basis, then this definition of τ_β coincides with the one of the Proposition **II.1.14**.

Example II.1.17. The collection of open intervals of the form $]a, +\infty[$ and $]-\infty, b[$ gives a subbasis for the standard topology in $\mathbb{R}$.

Proposition II.1.18. *Let X be a set and τ_1 and τ_2 topologies on X. Assume τ_1 is generated by a basis β_1 and τ_2 is generated by a basis β_2. Assume that for every $x \in X$ and for every $B_2 \in \beta_2$ containing x, there exists $B_1 \in \beta_1$ with $x \in B_1 \subseteq B_2$. Then $\tau_2 \subseteq \tau_1$.*

PROOF. By definition, elements of τ_2 are obtained as unions of finite intersections of elements of β_2. Since τ_1 is a topology, to show that $\tau_2 \subseteq \tau_1$, by the axioms, it suffices to show that $\beta_2 \subseteq \tau_1$.
Let $B_2 \in \beta_2$. Then using the assumption, we can pick for every $x \in B_2$ an element $B_x \in \beta_1$ with $x \in B_x \subseteq B_2$. Therefore, $B_2 = \bigcup_x B_x$, so $B_2 \in \tau_1$.

□

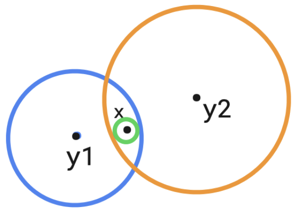
Definition II.1.19. Let (X, d) be a metric space. Then the collection β_{balls} of *open balls*, ie, all subsets of the form

$$B(x, r) := \{y \in X : d(x, y) < r\}$$

with x running through the all points of X and $r \in \mathbb{R}$, with $r > 0$, satisfy the conditions (i) and (ii) of Proposition **II.1.14**. First, notice that for any $x \in X$, $x \in B(x, \epsilon)$ for any $\epsilon > 0$. Secondly, notice that if $y \in B(x, \epsilon)$, then we can set $\delta := \epsilon - d(x, y)$ and have $y \in B(y, \delta) \subseteq B(x, \epsilon)$.

Finally, given $x \in B(y_1, r_1) \bigcap B(y_2, r_2)$, then by the argument before, we $\delta_1 > 0$ with $x \in B(x, \delta_1) \subseteq B(y_1, r_1)$ and $\delta_2 > 0$, with $x \in B(x, \delta_2) \subseteq B(y_2, r_2)$. Choosing δ be the smallest of δ_1 and δ, we conclude

$$B(x, \delta) \subseteq B(y_1, r_1) \bigcap B(y_2, r_2)$$



We call the topology $\tau_{\mathcal{B}_{balls}}$ the standard topology on (X, d).

Example II.1.20. Let $n \geq 1$. The set $\mathbb{R}^n := \underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{n}$ with the standard euclidean distance

$$d(x, y) := \sqrt{(x_1 - y_1)^2 + \cdots (x_n - y_n)^2}$$

is a metric space. The induced topology with a basis given by open balls, makes it a topological space. We denote it simply by $\mathbb{R}^n$.

Definition II.1.21. Let X be a set and let τ_1 and τ_2 be topologies on X. We say that τ_1 is finer than τ_2 if $\tau_2 \subseteq \tau_1$, ie, τ_1 is obtained from τ_2 by adding extra open subsets. In this case we also say that τ_2 is coarser than τ_1.

Proposition II.1.22. *Let (X, τ) be a topological space and $S \subseteq X$ a subset. Then the collection of subsets of S defined by*

$$\tau_{|S} := \{S \cap U : U \in \tau\}$$

defines a topology on S.

Proof. Let us check the axioms:

- $S = S \cap X$ with $X \in \tau$, so $S \in \tau_{|S}$;
- $\emptyset = S \cap \emptyset$ with $\emptyset \in \tau$, so $\emptyset \in \tau_{|S}$;
- Given a family $\{S \cap U_i\}_{i \in I}$ with $U_i \in \tau$ for each $i \in I$, then

$$\left(\bigcup_i S \cap U_i\right) = S \cap \left(\bigcup_i U_i\right)$$

  so that $(\bigcup_i S \cap U_i) \in \tau_{|S}$ since $(\bigcup_i U_i) \in \tau$.
- Given a finite family $\{S \cap U_i\}_{i \in I}$ with $U_i \in \tau$ for each $i \in I$, then

$$\left(\bigcap_i S \cap U_i\right) = S \cap \left(\bigcap_i U_i\right)$$

  so that $(\bigcup_i S \cap U_i) \in \tau_{|S}$ since $(\bigcap_i U_i) \in \tau$ because the family is finite.

□

Definition II.1.23. The topology of the Proposition **II.1.22** is called the induced or subspace topology. If $S \subseteq X$ is a subset equipped with a subspace topology, we call it a subspace.

Exercise II.1.24. Let X be a topological space and U an open subset. Show that the induced topology on U corresponds exactly to the collection of open subsets of X contained in U.

Example II.1.25. We consider $\mathbb{R}$ and the closed subset $[0,1]$. We endow $[0,1]$ with the induced topology. In this case the subset $[0,\frac{1}{2}[$ is an open subset for the induced topology, since it is obtained as an intersection $[0,1]\cap]-1,\frac{1}{2}[$. However, $[0,\frac{1}{2}[$ is not an open subset of $\mathbb{R}$.

Example II.1.26 (The circle and the spheres)**.** The n-dimensional sphere S^n is the subset

$$S^n := \{(x_1,\cdots,x_n,x_{n+1}) : \sqrt{x_1^2+\cdots+x_n^2+x_{n+1}^2}=1)\} \subseteq \mathbb{R}^{n+1}$$

endowed with the subspace topology. The circle is S^1.

Example II.1.27. The closed disk D^n is the subspace of $\mathbb{R}^n$ defined by

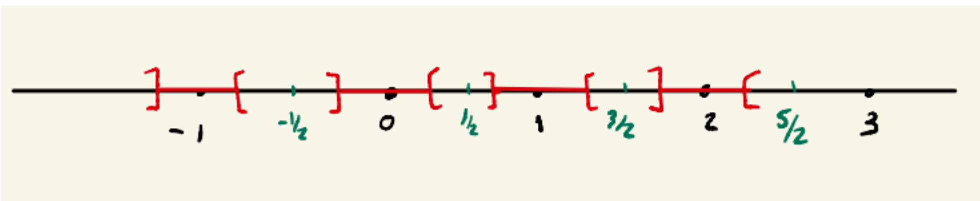
$$D^n := \{x \in \mathbb{R}^n : d(x,0) \leq 1\}$$

with the subspace topology.

Example II.1.28. Let us describe $\mathbb{Z}$ as a subset of $\mathbb{R}$ endowed with the induced topology. Open subsets for the induced topology are of the form $\mathbb{Z}\cap U$ where U is an open in $\mathbb{R}$. It is therefore enough to check what happens when U is an open interval $]a,b[$. In particular, given an interger n, we can consider the open interval $]n-\frac{1}{4},n+\frac{1}{4}[$ to get

$$\mathbb{Z}\bigcap]n-\frac{1}{4},n+\frac{1}{4}[=\{n\}$$

an open subset of $\mathbb{Z}$ for the induced topology. This is true for every n.



In conclusion, the induced topology on $\mathbb{Z}$ coincides with the discrete topology of the Example **II.1.7**:

Definition II.1.29. Let X be a topological space and $A\subset X$ a subset. We define:

- The *closure* of A, denoted $\overline{A}$, the smallest closed subset that contains A, ie

$$\overline{A} := \bigcap_{A\subseteq F : F \text{ is closed}} F$$

- The *interior* of A, denoted $Int(A)$ to be the biggest open subset contained in A, ie
$$Int(A) := \bigcup_{U \subseteq A : U \text{ is open}} U$$
- The *boundary* of A, $\partial A := \overline{A} \setminus Int(A)$.
- A is *dense* if $\overline{A} = X$

Exercise II.1.30. Let X be a topological space and A, B subsets. Show that:

(i) $Int(A) \subseteq A \subseteq \overline{A}$
(ii) A is open if and only if $Int(A) = A$.
(iii) A is closed if and only if $A = \overline{A}$.
(iv) $Int(A) = X \setminus \overline{X \setminus A}$.
(v) $\overline{A} = X \setminus Int(X \setminus A)$
(vi) $x \in \overline{A}$ if and only if every neighborhood of x intersects A
(vii) $\overline{A \bigcup B} = \overline{A} \bigcup \overline{B}$
(viii) $Int(A \bigcap B) = Int(A) \bigcap Int(B)$

Exercise II.1.31. Describe an open subset A of $\mathbb{R}^2$, different from $\mathbb{R}^2$ but with $Int(\overline{A}) = \mathbb{R}^2$.

II.2. Continuous Maps

Definition II.2.1. cl Let (X, τ_X) and (Y, τ_Y) be topological spaces. We say that a map of sets $f : X \to Y$ is continuous if for every open subset $U \in \tau_Y$, the pre-image $f^{-1}(U)$ is an open in X, ie, τ_X.

Remark II.2.2. Let (X, τ_X) and (Y, τ_Y) be topological spaces. A map $f : X \to Y$ is continuous if and only if the pre-image of closed subsets are closed. Indeed, assume f is continuous and V is closed in Y. Then by definition $Y \setminus V$ is open. Therefore $f^{-1}(Y \setminus V) = X \setminus f^{-1}(V)$ is open, so $f^{-1}(V)$ is closed. Vice-versa, assume that f^{-1} sends closed subsets to closed subsets. The symmetry of the argument by taking complements concludes the proof.

Proposition II.2.3. *The following conditions are equivalent for a set-theoretic map $f : X \to Y$ between two topological spaces*

(i) f is continuous;

(ii) for each x and for each open neigh. V of $f(x)$ there exists an open neigh. U of x with $U \subseteq f^{-1}(V)$ ($\epsilon - \delta$ characterization)

Proof. This is a consequence of the characterization of open subsets in Proposition **II.1.11**. ☐

Proposition II.2.4. *The composition of continuous maps is again continuous.*

PROOF. Indeed, if $f : X \to Y$ and $g : Y \to Z$ are continuous, given $U \in \tau_Z$ an open in Z, $g^{-1}(U)$ is open in Y and therefore, by definition of continuity in Definition **II.2.1**, $f^{-1}(g^{-1}(U))$ is an open in X. To conclude we only need to observe that $f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$. □

Remark II.2.5. Let X and Y be topological spaces. Assume the topology in Y is of the form τ_β for β as in the Proposition **II.1.14**. Then a map $f : X \to Y$ is continuous if and only if the $f^{-1}(B)$ is open in X for every $B \in \beta$. This follows because every open subset in Y is obtained as a union of elements of β, and taking set-theoretic pre-images commutes with unions.

In the same way, if the topology in Y is of the form $\tau_{I(\beta)}$ for β as in the Construction **II.1.15**, then a map $f : X \to Y$ is continuous if and only if the $f^{-1}(B)$ is open in X for every $B \in \beta$. This follows because every open subset in Y is obtained as a union of finite intersections of elements of β, and taking set-theoretic pre-images commutes with unions and finite intersections.

Exercise II.2.6. A continuous map $f : X \to Y$ is said to be open if for every open subset U of X, its *image* $f(U)$ is open in Y. Show that if β is a basis for the topology of X, then f is open if and only if $f(B)$ is open for every $B \in \beta$.

Example II.2.7. The complex exponential function $\exp : \mathbb{C} \to \mathbb{C}^*$ sending $z \mapsto e^z$ is continuous with respect to the subspace topology on $\mathbb{C}^*$. Thanks to Remark **II.2.5** this can be proved using the familiar $\epsilon - \delta$ definition of continuity. We can even prove something better - we can show it is holomorphic. Any holomorphic function is continuous. We refer to your complex analysis course.

Example II.2.8. Let (X, d) be a metric space. Then the topology of the Definition **II.1.19** is the coarsest topology on X such that the maps $d(x, -) : X \to \mathbb{R}_{\geq 0}$ are continuous for every $x \in X$, where $\mathbb{R}_{\geq 0}$ is endowed with the subspace topology as open balls are by definition, inverse images of open intervals under this map. We conclude with the Remark **II.2.5**.

We now explain some rules to construct continuous maps.

Proposition II.2.9 (Constant maps)**.** *Let X and Y be topological spaces and $f : X \to Y$ a set map. Then if f is constant with values y, f is continuous.*

PROOF. Let V be an open in Y. Then $f^{-1}(V)$ is either empty of X depending on V containing y or not. □

Proposition II.2.10. *[Restricting the domain] Let $f : X \to Y$ be a continuous map and $A \subseteq X$ a subspace. Then the restriction $f_{|_A} : A \to Y$ where A is equipped with the subspace topology, is continuous.*

Proof. Let V be an open in Y. Then $f_{|A}^{-1}(V) = f^{-1}(V) \bigcap A$ is open in the subspace topology. □

Proposition II.2.11. *[Restriction the range] Let $f : X \to Y$ be a continuous map and $Z \subseteq Y$ a subspace containing the image $f(X)$. Then the restriction $f : X \to Z$ is continuous.*

Proof. This follows because for any open subset U of Z we have $f^{-1}(U \bigcap f(X)) = f^{-1}(U)$ and by assumption f is continuous. □

Remark II.2.12. The inclusion of a subspace $A \subseteq X$ is continuous.

Proposition II.2.13. *[gluings: local formulation of continuity] Let X be a topological space written as a union of open subsets U_α. Show that a map $X \to Y$ is continuous if and only if each restriction $f_{|U_\alpha}$ is continuous.*

Proof. Let V be an open subset of Y. Then, since $X = \bigcup U_\alpha$ we have

$$f^{-1}(V) = \bigcup_\alpha f^{-1}(V) \cap U_\alpha$$

It is therefore enough to show that each $f^{-1}(V) \cap U_\alpha$ is open in U_α. But by definition, $f^{-1}(V) \cap U_\alpha = f_{|U_\alpha}^{-1}(V)$ and by assumption, $f_{|U_\alpha}$ is continuous. □

TD, Exo 5, Feuille 1

Exercise II.2.14. [gluing continuous maps along closed subsets] Let X be a topological space and $F_1, F_2 \subseteq X$ two closed subsets such that $X = F_1 \cup F_2$. Let $f_1 : F_1 \to Z$ and $f_2 : F_2 \to Z$ be continuous maps such that f_1 and f_2 agree on $F_1 \cap F_2$. Show that f_1 and f_2 glue to a well-defined unique continuous map $X \to Z$.

Definition II.2.15. We say that a continuous map $f : X \to Y$ is a *homeomorphism*, if there exists a continuous map $g : Y \to X$ such that $f \circ g = Id_Y$ and $g \circ f = Id_X$. In other words, if the map f admits an inverse and the inverse is also continuous. Two topological spaces X and Y are said to be *homeomorphic* if there exists a homeomorphism between them.

Example II.2.16.

- The set of complex numbers $\mathbb{C}$ together with the distance function $|z - z'|$ is a metric space. The natural identification of $\mathbb{C}$ with $\mathbb{R}^2$ via

  $$z = a + ib \mapsto (a, b)$$

  defines a homeomorphism, with $\mathbb{R}^2$ equipped with the euclidean distance.

- The tangent map $]-\frac{\pi}{2}, \frac{\pi}{2}[\to \mathbb{R}$ is a homeomorphism with inverse given by the function arctan.

- The real exponential map $\mathbb{R} \to \mathbb{R}_{>0}$ is an homemorphism with inverse given by the real logarithm.

- Let $a \leq b$. Then the interval $[a, b]$ is homeomorphic to $[0, 1]$. Indeed, the affine function $[0, 1] \to [a, b]$ sending $x \mapsto (b - a)x + a$ has inverse $y \mapsto \frac{1}{b-a}y - \frac{a}{b-a}$.

Exercise II.2.17. Let $]a, b[$ be an open interval in $\mathbb{R}$. Show that it is homeomorphic to $\mathbb{R}$.

Example II.2.18. The homeomorphism of the Example **II.2.16**, restricts to a homeomorphism between $\mathsf{S}^1 \subseteq \mathbb{R}^2$ and the subspace of complex numbers z with $|z| = 1$. For the 1-dimensional circle, it will be convenient to take this as a definition instead.

Example II.2.19. The space of real numbers can be seen as a subspace of the complex numbers via the continuous inclusion $x \mapsto 2\pi i x$. Using the Proposition **II.2.11** and Proposition **II.2.10**, the complex exponential of the Example **II.2.7** restricts to a continuous map $\mathbb{R} \to \mathsf{S}^1$ via $x \mapsto e^{2\pi i x} = cos(2\pi x) + i \sin(2\pi x)$.

Example II.2.20. Here is an example of a continuous map that admits a set-theoretic inverse but the inverse is not continuous. Take the topological spaces $\mathbb{R}_{disc}$ and $\mathbb{R}_{triv}$ as in the Example **II.1.7**. Notice that the set-theoretical identity $Id : \mathbb{R} \to \mathbb{R}$ defines a continuous map $\mathbb{R}_{disc} \to \mathbb{R}_{triv}$. Indeed, the pre-image of any open set for the trivial topology (either $\mathbb{R}$ or $\emptyset$) are open for the discrete topology (where all subsets are open). Therefore this map is a continuous bijection. However, its inverse - the identity map - does not define a continuous map $\mathbb{R}_{triv} \to \mathbb{R}_{disc}$ since for instance, the inverse image of the singleton $\{0\}$ is not empty for the trivial topology.

Exercise II.2.21. Show that the open ball $B(0, 1)$ in $\mathbb{R}^2$ is topologically isomorphic to $\mathbb{R}^2$. Hint: use the map $(x, y) \mapsto (\frac{x}{\sqrt[2]{1-||(x,y)||^2}}, \frac{y}{\sqrt[2]{1-||(x,y)||^2}})$ with inverse given by $(x, y) \mapsto (\frac{x}{\sqrt[2]{1+||(x,y)||^2}}, \frac{y}{\sqrt[2]{1+||(x,y)||^2}})$.

Exercise II.2.22. Let $N = (0, 0, 1)$ denote the North pole in the sphere S^2. The stereographic projection from the north pole is described in the video below

Show that it is given by the formula

$$S^2 \setminus \{N\} \subseteq \mathbb{R}^3 \to \mathbb{R}^2$$

$$(x, y, z) \mapsto (\frac{x}{1-z}, \frac{y}{1-z})$$

and its inverse by

$$\vec{v} = (a, b) \in \mathbb{R}^2 \mapsto (\frac{2a}{1+||\vec{v}||^2}, \frac{2b}{1+||\vec{v}||^2}, \frac{1-||\vec{v}||^2}{1+||\vec{v}||^2})$$

Show these formulas define a homeomorphism from $S^2 \setminus \{N\}$ to $\mathbb{R}^2$.

Here's another video illustrating this:

II.3. Compact Spaces

Definition II.3.1. A topological space X is said to be *Hausdorff* (séparé in french) if for every pair of points x, y in X, there exists open neighborhoods U and V with $x \in U$ and $y \in V$ and such that $U \cap V = \emptyset$.

Example II.3.2. The topological space from the Example **II.1.7**-(i) is not Hausdorff since we can not isolate 0 and 1 by disjoint open subsets.

Remark II.3.3. Let X be a topological space and $A \subseteq X$ a subspace. If X is Hausdorff then so is A

Exercise II.3.4. Show that a topological space X is Hausdorff if and only if the diagonal of $\Delta := \{(x,x) : x \in X\} \subseteq X \times X$ is a closed subset for the product topology. TD, Exo 3, Feuille 1

Exercise II.3.5. Show that in a Hausdorff space, the singletons $\{x\}$ are closed subsets. TD, Exo 6, Feuille 1

Proposition II.3.6. *Let (X, d) be a metric space. Then the induced topology is Hausdorff.*

PROOF. Let x and $y \in X$ and let $\epsilon := \frac{1}{2}d(x,y)$. We want to show that $B(x,\epsilon)$ and $B(y,\epsilon)$ are disjoint. Suppose z exists in their intersection. Then $d(x,z) < \epsilon$ and $d(y,z) < \epsilon$. By the triangle inequality for d, we have

$$d(x,y) \leq d(x,z) + d(z,y) < \epsilon + \epsilon = d(x,y)$$

which is a contradiction.

□

Exercise II.3.7. (line with double origin) We consider the set of all real numbers different from zero $\mathbb{R} \setminus \{0\}$ and formally add to it two elements 0_A and 0_B,ie, we define

$$X := \mathbb{R} \setminus \{0\} \bigcup \{0_A, 0_B\}$$

.

We consider the collections of subsets

$$\beta_1 := \{ \text{ all open intervals of } \mathbb{R} \text{ which do not contain zero } \}$$

$$\beta_2 := \{ \text{ all subsets of the form }]-a,0[\cup\{0_A\}\cup]0,a[, a > 0 \ \}$$

$$\beta_3 := \{ \text{ all subsets of the form }]-a,0[\cup\{0_B\}\cup]0,a[, a > 0 \ \}$$

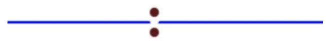
and defined

$$\beta = \beta_1 \bigcup \beta_2 \bigcup \beta_3$$

(i) Check β satisfies the conditions of the Proposition **II.1.14** and therefore generates a topology τ_β

(ii) Show that each of the spaces $X \setminus \{0_A\}$ and $X \setminus \{0_B\}$ is topologically isomorphic to $\mathbb{R}$

(iii) Show that τ_β does not separate the points 0_A and 0_B, namely, that it is impossible to find an open subset containing 0_A that does not intersect an open subset containing 0_B.



In the next example we finally define the circle and spheres as topological spaces:

Example II.3.8. The n-dimensional sphere S^n of the Example **II.1.26** are closed subsets. Indeed, let d denote the standard euclidean metric in $\mathbb{R}^n$ and let $f_n : \mathbb{R}^n \to \mathbb{R}$ be the function sending $x \mapsto d(x, 0) = ||x||$. By Example **II.2.8** this function is continuous with $\mathbb{R}$ also endowed with the standard topology. By the Proposition **II.3.6**, singletons are closed in $\mathbb{R}$. In particular, $\{1\} \subseteq \mathbb{R}$ is a closed subset. By Remark **II.2.2**, $S^{n-1} = f_n^{-1}\{\{1\}\}$ is a closed subset of $\mathbb{R}^n$. In particular, S^{n-1} is Hausdorff.

TD, Exo 4, Feuille 1

Exercise II.3.9. Is the property of being Hausdorff stable under:

- Unions?
- Intersections?
- closure?
- interiors?
- boundary?
- passing a closed subspace?

Exercise II.3.10. Let X be a set. When is the trivial topology on X, Hausdorff?

Definition II.3.11. Let X be a topological space. An open covering of X is a family (possibly infinite) of open subsets $\mathscr{U} := \{U_\alpha\}_{\alpha\in A}$ whose union is X, ie, $\cup_\alpha U_\alpha = X$. A subcover of $\mathscr{U}$ is a open covering of X - $\mathscr{U}'$ - with $\mathscr{U}' \subseteq \mathscr{U}$.

Definition II.3.12. A topological space X is said to be *quasi-compact* if for every open cover $\mathscr{U} := \{U_\alpha\}_{\alpha\in A}$ of X we can always extract a subcover $\mathscr{U}'$ which is finite.

Definition II.3.13. Let X be a topological space. We say that a subset $A \subseteq X$ is quasi-compact if it is quasi-compact for the subspace topology.

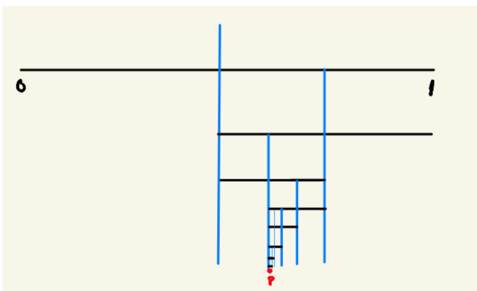
Remark II.3.14. Following the definition Definition **II.3.13**, an open covering for a subspace $A \subseteq X$ consists of a collection of subsets of X, $\{U_i\}_{i\in I}$, where each U_i is open in X and such that $A = \bigcup_i (A \cap U_i)$, ie, $A \subseteq \bigcup_i U_i$. Then A is quasi-compact if for any collection $\{U_i\}_{i\in I}$ of opens in X with $A \subseteq \bigcup_i U_i$ we can find a subcollection $\{U_{i_1}, \cdots U_{i_n}\}$ with $A \subseteq \bigcup_{k=1}^n U_{i_k}$.

Exercise II.3.15. Is the property of being quasi-compact stable under: TD, Exo 4, Feuille 1

- Unions?
- Intersections?
- closure?
- interiors?
- boundary?
- passing a closed subspace?

Proposition II.3.16. *Consider the closed interval* $[0,1]$ *as a subspace of* $\mathbb{R}$. *Then* $[0,1]$ *is quasi-compact.*

PROOF. Let $\mathscr{U} := \{U_i\}_{i\in I}$ be a cover of $[0,1]$. Assume that $[0,1]$ is not quasi-compact. Then at least one of the intervals $[0,\frac{1}{2}]$ or $[\frac{1}{2},1]$ is not contained in a finite subcover of $\mathscr{U}$ since if they were both, we would have a contradiction with $[0,1]$ not being quasi-compact. Lets pick one of these halves (one that is contained in a finite subcover), and write it as $[a_1,b_1]$. Re-applying the same argument, we can cut this half again in two and at least one of the parts will not be contained in a finite subcover of $\mathscr{U}$ since if they were both, we would again find a contradiction. Call $[a_2,b_2]$ one of the sides which is not contained in a finite subcover of $\mathscr{U}$. By induction we can build this way a sequence of nested intervals $[a_n,b_n]$ of length $\frac{1}{2^n}$, none of which is contained in a finite subcover of $\mathscr{U}$. We remark that the infinite intersection $\bigcap_n [a_n,b_n]$ is non-empty and consists of a single point: indeed, the sequence $\{a_n\}$ is increasing and bounded by 0 on the left and 1 on the right. Therefore by the axioms of the real numbers, it admits a supremum sup. Similarly, the sequence $\{b_n\}$ is decreasing and bounded by 0. It admits an infimum inf. By construction, $[\mathsf{sup},\mathsf{inf}] \subseteq \bigcap_n [a_n,b_n]$ and since the length of the intervals goes to zero with $\frac{1}{2^n} \to 0$, we must have $\mathsf{sup} = \mathsf{inf}$. Call this point p.



Since $p \in [0,1]$ and $\mathscr{U}$ is a covering family, there must exist i_0, such that $p \in U_{i_0}$. By definition of open sets for the standard topology, there must exist $\epsilon > 0$ such that $p \in B(p,\epsilon) \subseteq U_{i_0}$. Using the Archimedean property, choose $N > 0$ such that $\epsilon > \frac{1}{2^N}$. Since $p \in [a_N,b_N]$, it follows that

$$p \in [a_N,b_N] \subseteq B(p,\epsilon) \subseteq U_{i_0}$$

But this contradicts the fact that $[a_N,b_N]$ by construction is not contained in a finite subcover of $\mathscr{U}$. □

Example II.3.17. The interval $[0,1[$ is not quasi-compact. Indeed, take the open cover $\{U_n\}_{n\in\mathbb{N}}$ where $U_n := [0, 1-\frac{1}{n}[$. The union $\bigcup_n U_n = [0,1[$ but the cover does not admit a finite subcover. Indeed, for any finitely many elements in the cover $U_{n_1}, \cdots U_{n_k}$, take $n = max(n_1, \cdots n_k)$ and take $N = n+1$. Then $1 - \frac{1}{N}$ is not contained in any of the $U_1, \cdots, U_k$.

Definition II.3.18. A topological space X is said to be *compact* if it is quasi-compact and Hausdorff.

TD, Exo 6, Feuille 1 (a,b)

Exercise II.3.19.

(i) Let X be quasi-compact. Show that that if $F \subseteq X$ is a closed subset then F is quasi-compact.
(ii) Let X be a Hausdorff space. Suppose $F \subseteq X$ is quasi-compact and suppose x is a point of $X \setminus F$. Show that there exists disjoint open sets U and V of X with $F \subseteq U$ and $x \in V$.
(iii) Let X be a Hausdorff space. Suppose $F \subseteq X$ is quasi-compact. Then F is closed. (Hint: use (ii)).

TD, Exo 6, Feuille 7 (c)

Exercise II.3.20. Let $f : X \to Y$ be a continuous surjection. Show that if X quasi-compact then so is Y.

TD, Exo 6, Feuille 7 (c)

Exercise II.3.21. Let $f : X \to Y$ be a continuous bijection. Show that if Y Hausdorff then so is X.

TD, Exo 6, Feuille 7 (c)

Proposition II.3.22. *Let $f : X \to Y$ be a continuous bijection. If X is quasi-compact and Y is Hausdorff then f is a homeomorphism.*

Exercise II.3.23. A topological group is a group G endowed with a topology that renders the maps $G \times G \to G$ given by $(x,y) \mapsto x.y$ and $G \to G$ given by $x \mapsto x^{-1}$, continuous.

(i) Let $H \subseteq G$ be a subgroup. Then if H is open, it is closed.
(ii) G is Hausdorff if and only if the singleton $\{e\}$ is closed.

Example II.3.24. The unit square $[0,1] \times [0,1]$ and the disk D^2 are homeomorphic with respect to the subspace topologies induced from $\mathbb{R}^2$.

II.4. Sequences and Continuity

Definition II.4.1. Let (X, τ) be a topological space. We say that a sequence $(a_n)_{n\in\mathbb{N}}$ *converges* to an element $x \in X$ if for every open subset U of X containing x there exists an integer $N > 0$ such that all the terms of the sequence of order $n \geq N$ are inside U.

Exercise II.4.2. Show that if X is Hausdorff, if a sequence has a limit then this limit is unique.

Exercise II.4.3. Let X be a topological space. Let $A \subseteq X$.

(i) Show that if there is a sequence of points in A converging to x then $x \in \overline{A}$.

(ii) Show that the converse holds if X is metrizable, ie, the topology of X is induced by a distance function.

Exercise II.4.4. Let $f : X \to Y$ be a map of sets between topological spaces.

(i) Show that if f is continuous then for every sequence of points x_n in X converging to x the sequence $f(x_n$ converges to $f(x) in Y$.

(ii) Show that the converse holds if X is metrizable.

II.5. Constructions: Products, Quotients, Gluings, Function Spaces

Products.

Construction II.5.1. Let (X, τ_X) and (Y, τ_Y) be topological spaces. Consider the set theoretic product $X \times Y$. We define $\beta \subset P(X \times Y)$ the collection of *open boxes*

$$\beta_{box} := \{U \times V : U \in \tau_X, V \in \tau_Y\}$$

Lemma II.5.2. *The collection β_{box} of the Construction **II.5.1** satisfies the conditions (i) and (ii) of the Proposition **II.1.14**.*

Proof. (i) Let $(x, y) \in X \times Y$. Since τ_X is a topology, there exists $U \in \tau_X$ with $x \in U$. Same argument gives us $V \in \tau_Y$ with $y \in V$. Therefore $U \times V \in \beta_{box}$ contains (x, y).

(ii) Let $U_1, U_2 \in \tau_X$ and $V_1, V_2 \in \tau_Y$. Notice that set-theoretically

$$(U_1 \times V_1) \bigcap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2)$$

Since U_1, U_2, V_1, V_2 are opens in a topology, by the axioms we have $U_1 \cap U_2 \in \tau_X$ and $V_1 \cap V_2 \in \tau_Y$ so that $(U_1 \cap U_2) \times (V_1 \cap V_2) \in \beta_{box}$

□

Definition II.5.3. Let (X, τ_X) and (Y, τ_Y) be topological spaces. The topological space obtained by endowing the set $X \times Y$ with the topology generated by β_{box} is called the *product space*. This topology is called the product topology.

Proposition II.5.4. *Let (X, τ_X) and (Y, τ_Y) be topological spaces. Then the product topology coincides with the coarsest topology rendering the two canonical projections $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$, continuous.*

Proof. Let us denote by $\pi \subseteq P(X \times Y)$ the collection

$$\pi := \{\pi_X^{-1}(U) : U \in \tau_X\} \bigcup \{\pi_Y^{-1}(V) : V \in \tau_V\}$$

Then β_{box} coincides with the collection $I(\pi)$ obtained by adjoining to π all finite intersections of elements of π as in the Construction **II.1.15**. This follows from the formula

$$U \times V = \pi_X^{-1}(U) \bigcap \pi_Y^{-1}(V)$$

for $U \in \tau_X$ and $V \in \tau_Y$. In particular π is a subbasis for the product topology and it follows from the Remark **II.2.5** that the two projections are continuous. □

TD. Ex. 2 Feuille 1

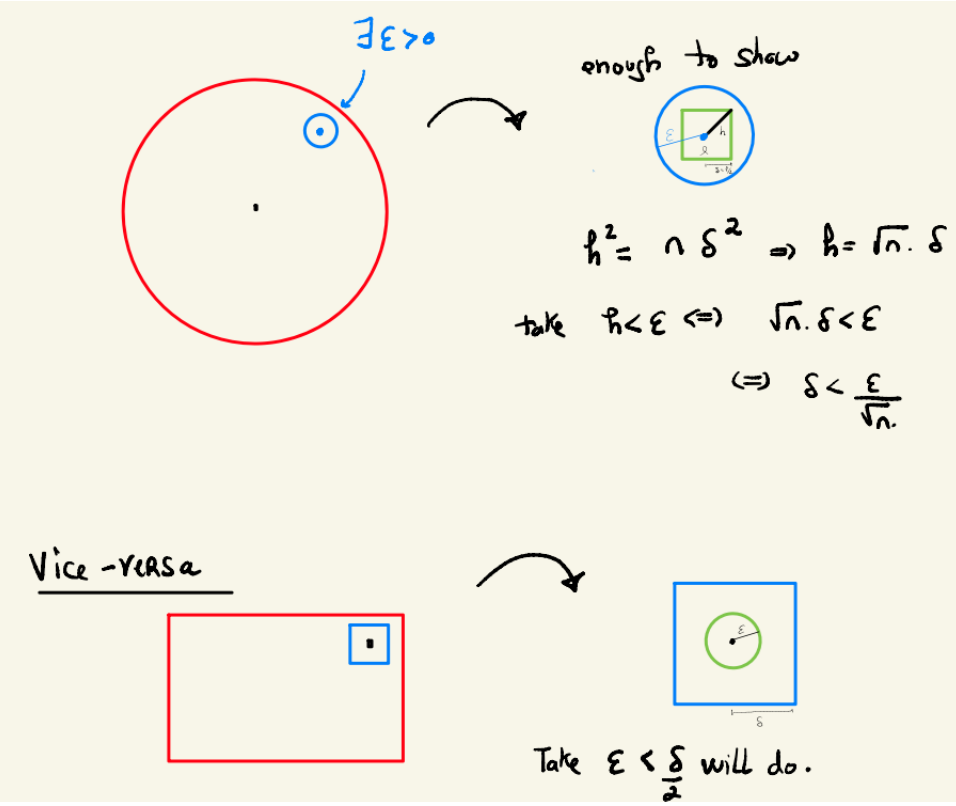
Exercise II.5.5. A map of topological spaces $Z \to X \times Y$ is continuous if and only if the composition with the two projections π_X and π_Y are continuous.

Exercise II.5.6 (Tube Lemma)**.** Let X and Y be topological spaces with Y quasi-compact. Let N be an open subset of $X \times Y$ containing a subset of the form $\{x\} \times Y$. Then there exists an open neighborhood W of x in X such that $W \times Y$ is still contained in N

Exercise II.5.7. Let X and Y be quasi-compact topological spaces. Then $X \times Y$ is quasi-compact. (Hint: Use the Exercise **II.5.6**).

Exercise II.5.8. Let X and Y be quasi-compact topological spaces. Then $X \times Y$ is Hausdorff if and only if both X and Y are Hausdorff.

Example II.5.9. Let $n \geq 2$. The standard euclidean topology on $\mathbb{R}^n := \underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{n}$ of the Example **II.1.20** coincides with the product topology. We use the criterium of the Proposition **II.1.18** to show that the two topologies coincide. The standard euclidean topology has a basis β_{balls} given by the collection of open balls. The product topology has a basis β_{box} given by the collection of open boxes. Let $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$. Given $\epsilon > 0$ and taking δ such that $\delta\sqrt{n} < \epsilon$ the open box $]-x_1-\delta, x_1+\delta[\times \cdots \times]x_n-\delta, x_n+\delta[$ is contained in the open ball $B(x, \epsilon)$. Conversely, if $x = (x_1, \cdots, x_n)$ belongs to an open box $U_1 \times U_2 \times \cdots \times U_n$, we can always find $\delta > 0$ such that $x \in] - x_1 - \delta, x_1 + \delta[\times \cdots \times]x_n - \delta, x_n + \delta[\subseteq U_1 \times \cdots \times U_n$. In this case, by taking an open ball $B(x, \epsilon)$ of radius ϵ smaller than $\delta.\sqrt{n}$, we will have $x \in B(x, \epsilon) \subseteq U_1 \times \cdots \times U_n$.



Example II.5.10. The subspace topology on $\{(x_1, \cdots x_n) \in \mathbb{R}^n : 0 \leq x_i \leq n, \forall i \in \{1, \cdots, n\}\} \subseteq \mathbb{R}^n$ coincides with the product topology $[0,1]^n$. By induction from the Proposition **II.3.16**, it is quasi-compact.

Example II.5.11. The spheres S^{n-1} can be described as closed subspaces of the product interval $[-1,1]^n$ which is quasi-compact as a consequence of the Example **II.5.10** (since $[0,1]$ and $[-1,1]$ are homeomorphic). It follows from the Exercise **II.3.19** that the spheres are also quasi-compact, and therefore compact (since we already knew they were Hausdorff).

Definition II.5.12. The n-dimensional torus $\mathbb{T}^n$ is the product space $\underbrace{\mathsf{S}^1 \times \cdots \times \mathsf{S}^1}_{n}$.

In particular, it is Hausdorff.

Disjoint unions.

Reminder II.5.13. If X and Y are sets, we consider their disjoint union

$$X \coprod Y := (X \times \{0\}) \bigcup (Y \times \{1\})$$

We denote by $i_X : X \to X \coprod Y$ and $i_Y : Y \to X \coprod Y$ the two inclusions. A pair of maps of sets $f : X \to Z$ and $g : Y \to Z$, uniquely defines a map $X \coprod Y \to Z$ sending an element $a \in X \coprod Y$ to $f(a)$ if a belongs to X and a to $g(a)$ if a belongs to Y

Construction II.5.14. If (X, τ_X) and (Y, τ_Y) are topological spaces, we can endow the disjoint union of the Reminder **II.5.13** with a topology. Namely, we consider

$$\tau := \{A \in P(X \coprod Y) : i_X^{-1}(A) \in \tau_X, \quad \text{and} \quad i_Y^{-1}(A) \in \tau_Y\}$$

The compatibility of taking inverse images, with unions and intersections, guarantees that this forms a topology which renders the two inclusions $i_X : X \to X \coprod Y$ and $i_Y : Y \to X \coprod Y$ continuous. We call $X \coprod Y$ the disjoint union space.

Example II.5.15. Let X and Y be topological spaces. Then $i_X(X) \subseteq X \coprod Y$ is both open and closed in $X \coprod Y$. Indeed, set-theoretically we have $i_X^{-1}(i_X)(X)) = X$ which is open in X and $i_Y^{-1}(i_X)(X)) = \emptyset$ which is open in Y.

Exercise II.5.16. Let X, Y, Z be topological spaces. Show that that a map $\Psi : (X \coprod Y, \tau) \to (Z, \tau_Z)$ is continuous if and only if the two compositions $\Psi \circ i_X$ and $\Psi \circ i_Y$ are continuous.

Exercise II.5.17. Let $X, \{Y_i\}_{i \in I}$ be topological spaces. Show that the canonical map induced by the inclusions $Y_i \to \coprod_I Y_i$

$$X \times (\coprod_i Y_i) \to \coprod_i (X \times Y_i)$$

is a homeomorphism.

Quotients.

Reminder II.5.18. Let X be a set. An equivalence relation on X is a subset $\mathscr{R} \subseteq X \times X$ satisfying the following three properties:

- Identities: $\mathscr{R}$ contains the diagonal subset Δ;
- Symmetry: if $(x, y) \in \mathscr{R}$, then $(y, x) \in \mathscr{R}$;
- Transitivity: if $(x, y) \in \mathscr{R}$ and $(y, z) \in \mathscr{R}$, then $(y, x) \in \mathscr{R}$;

Given an equivalence relation $\mathscr{R}$ on X we can form the *quotient set* $X/\mathscr{R}$ of all equivalence classes $[x]$ where $[x] = [y]$ if and only if $(x, y) \in \mathscr{R}$. In denote by

$$\pi : X \to X/\mathscr{R}$$

the map of sets sending $x \in X$ to its class $[x]$.

Example II.5.19. Let $f : X \to Y$ be a map of sets. Then the subset $\mathscr{R} \subseteq X \times X$ defined by

$$\mathscr{R} = \{(x, y) : f(x) = f(y)\}$$

defines an equivalence relation on X. By the nature of this relation, the map $f : X \to Y$ factors through the quotient set

$$\begin{array}{ccc} X & & \\ {\scriptstyle\pi}\downarrow & \searrow^{f} & \\ X/\mathscr{R} & \underset{\tilde{f}}{\dashrightarrow} & Y \end{array}$$

by $\tilde{f}([x]) = f(x)$.

The map $\tilde{f}$ is injective: if $\tilde{f}([x]) = \tilde{f}([y])$, then $f(x) = f(y)$ so by definition $[x] = [y]$.

The map $\tilde{f}$ is surjective if and only if f is surjective. Indeed, for any composition of maps of sets $u = v \circ h$, with h surjective, then u is surjective if and only if v is surjective.

In particular, all surjective maps of sets are quotient maps.

Exercise II.5.20. Let X be a set.

(i) Let $\{\mathscr{R}_i\}_{i\in I}$ be a family of equivalence relations on X. Show that the intersection $\bigcap_i \mathscr{R}_i \subseteq X \times X$ defines an equivalence relation on X.

(ii) Let $S \subseteq X \times X$ be a subset. Define $< S > \subseteq X \times X$ as follows: $(x, y) \in < S >$ if and only if there exists $x_0, x_1, \cdots x_n, x_{n+1} \in X$, with $x_0 = x,\ x_{n+1} = y$ such that, either
- $x_i = x_{i+1}$
- $(x_i, x_{i+1}) \in S$
- $(x_{i+1}, x_i) \in S$

Show that $< S >$ defines an equivalence relation on X.

(iii) Show that if $\mathscr{R}$ is any equivalence relation with $S \in \mathscr{R}$ then $< S > \subseteq \mathscr{R}$.

(iv) Conclude that

$$< S > = \bigcap_{S\subseteq\mathscr{R}:\mathscr{R} \text{ is an equivalence relation}} \mathscr{R}$$

We call $< S >$ the equivalence relation *generated by* S. It is the finest equivalence relation containing S

Construction II.5.21. (Quotient topology) Let (X, τ) be a topological space and let R be an equivalence relation on the set X. The topology of X produces a topology in X/R where declare that a subset $V \subseteq X/R$ is open if and only if $\pi^{-1}(V)$ is open in X. The fact that this defines a topology is immediate from the properties of the operation π^{-1} and from the properties of opens in X. Moreover, this topology makes π continuous by definition.

Definition II.5.22. Let X be a topological space, $\mathscr{R}$ an equivalence relation. We say that an open subset U of X is *saturated* if and only if $U = p^{-1}(p(U))$.

TD, Exo 8-(ii), Feuille 1

Remark II.5.23. Let X be a topological space, $\mathscr{R}$ an equivalence relation. Then the map π is open (image of open subsets is open) if and only if every open subset is saturated.

TD, Exo 8-(iii), Feuille 1

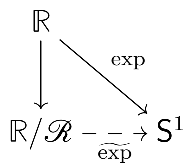
Proposition II.5.24. *Let X and Z be a topological spaces and let R be an equivalence relation on the set X. Then a map $u : X/R \to Z$ is continuous if and only if the composition $u \circ \pi : X \to Z$ is continuous. Moreover, continuous maps $\tilde{f} : X/R \to Z$ are in bijection with continuous maps $f : X \to Z$ which identify the equivalence classes, ie, $f(x) = f(y)$ if and only if $x \sim y$.*

TD, Exo 13, Feuille 1

Example II.5.25. Consider the exponential map $\exp : \mathbb{R} \to \mathsf{S}^1$ of the Example **II.2.19**. For every $x \in \mathbb{R}$ and for every $n \in \mathbb{Z}$, we have

$$e^{2\pi(x+n)} = e^{2\pi x}$$

Let us define an equivalence relation $\mathscr{R}$ on the real line by declaring $x \sim y$ if and only if $y - x \in \mathbb{Z}$. This satisfies the symmetry and transitivity relations. Therefore, we can form the quotient space $\mathbb{R}/\mathscr{R}$ of the Construction **II.5.21** and by the Proposition **II.5.24**, exp factors through a continuous map



The map $\widetilde{\exp}$ is a continuous bijection. Let us prove that it is injective. Suppose

$$\widetilde{\exp}([x]) = \widetilde{\exp}([y])$$

ie,

$$\exp(2\pi i\, x) = \exp(2\pi i y)$$

Since the exponential is a map of groups, this is equivalent to

$$e^{2\pi i(x-y)} = 1$$

therefore $x - y \in \mathbb{Z}$ so that $[x] = [y]$.

For surjectivity, it is the fact that every point of the circle can be written in exponential form via Euler's formula.

TD, Exo 10, Feuille 1

Exercise II.5.26. Let X be a topological space and $\mathscr{R}$ an equivalence relation. Then:

(i) Show that if $\mathscr{R} \subseteq X \times X$ is closed and $\pi : X \to X/\mathscr{R}$ is open, then the quotient is Hausdorff.

(ii) Show that if $\mathscr{R} \subseteq X \times X$ is closed and X is compact (quasi-compact Hausdorff), then $X/\mathscr{R}$ is compact.

Exercise II.5.27. Show that the group $(\mathbb{R}^n, +, 0)$ is a topological group.

Exercise II.5.28. Consider the additive group $(\mathbb{C}, +, 0)$ and the multiplicative group $\mathbb{C}^*$ with the subspace topology from $\mathbb{C}$. Show that they are both topological groups. Check that the exponential map is a map of topological groups.

Exercise II.5.29. Let G be a topological group and H a subgroup. Then the relation $g_1 \sim g_2$ if and only if $g_2^{-1} g_1 \in H$, defines an equivalence relation $\mathscr{R}_H$ in G (check this!). Show that the quotient G/H is Hausdorff if and only if the subgroup H is closed in G. TD, Exo 12, Feuille 1

Exercise II.5.30. Let G be a topological group and H a subgroup. Then the relation $g_1 \sim g_2$ if and only if $g_2^{-1} g_1 \in H$, defines an equivalence relation $\mathscr{R}_H$ in G (check this!). Show that the quotient G/H is Hausdorff if and only if the subgroup H is closed in G. TD, Exo 12, Feuille 1

Exercise II.5.31. Let G be a topological group and H is a subgroup which is closed and normal. Show that G/H is a topological group and the quotient map is a map of topological groups.

Exercise II.5.32. The quotient $\mathbb{R}/\mathscr{R}$ of the Example **II.5.25** is Hausdorff. Indeed, this quotient can also be identified, by construction, with the quotient $\mathbb{R}/\mathbb{Z}$ for the closed additive subgroup $\mathbb{Z} \subseteq \mathbb{R}$.

Example II.5.33. Consider $\mathbb{R}$ with the equivalence relation $\mathscr{R}$ of the Example **II.5.25** TD, Exo 13, Feuille 1
Let us consider the closed interval $[0,1] \subset \mathbb{R}$ as a closed subspace. The equivalence relation $\mathscr{R}$ on $\mathbb{R}$ restricts to an equivalence relation $\mathscr{R}'$ on $[0,1]$, consisting of

$$\mathscr{R}' := \{(0,1),(1,0)\} \bigcup \Delta$$

identifying the extremities 0 and 1 of the interval to the same point, ie, $[0] = [1]$ in the quotient. The inclusion $[0,1] \subseteq \mathbb{R}$ being compatible with the equivalences relations, passes to the quotient

$$\begin{array}{ccc} [0,1] & \longrightarrow & \mathbb{R} \\ \downarrow & & \downarrow \\ [0,1]/\mathscr{R}' & \overset{\phi}{\dashrightarrow} & \mathbb{R}/\mathscr{R} \end{array}$$

ϕ is a continuous bijection. Continuity is a consequence of the universal property. For injectivity, notice that if $\phi([x]) = [x] = \phi([y]) = [y]$ in $\mathbb{R}/\mathscr{R}$ with $x, y \in [0,1]$, then there exists n such that $x = y + n$. Since by assumption both x and y are in $[0,1]$, this can only mean $x = 0$ and $y = 1$ or $x = 1$ and $y = 0$. In both cases $[x] = [y]$ in $[0,1]/\mathscr{R}'$. Surjectivity is a consequence of the fact that any $x \in \mathbb{R}$ can be written as $x = y + n$ for some $y \in [0,1]$.

In particular, since $[0,1]$ is quasi-compact (Proposition **II.3.16**), by the Exercise **II.3.20**, $[0,1]/\mathscr{R}'$ is quasi-compact.

By the Exercise **II.5.32**, $\mathbb{R}/\mathscr{R}$ is Hausdorff. Therefore, by the Proposition **II.3.22**, the dotted map is a homeomorphism.

Returning to the exponential map, we obtain

$$\begin{array}{ccccc} [0,1] & \longrightarrow & \mathbb{R} & & \\ \downarrow & & \downarrow & \overset{\exp}{\searrow} & \\ [0,1]/\mathscr{R}' & \xrightarrow{\sim} & \mathbb{R}/\mathscr{R} & \dashrightarrow & \mathsf{S}^1 \end{array}$$

We claim that the continuous composition

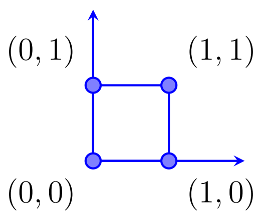
$$[0,1]/\mathscr{R}' \xrightarrow{\sim} \mathbb{R}/\mathscr{R} \dashrightarrow \mathsf{S}^1$$

is an homemorphism. Indeed, since S^1 is Hausdorff, by the same arguments as above, it is enough to remark that this composition is a bijection because both maps are bijections.

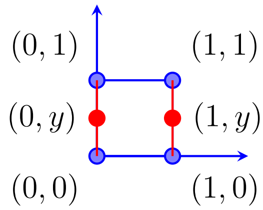
This example gives another a characterization of the circle, this time as a quotient space.

Exercise II.5.34. This exercise constructs the torus as in the snake game Exercice **I.1.1** and explains why it is homemorphic to the torus as a product of circles, as seen in the Exercice **I.1.8**.

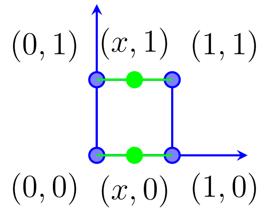
Consider the square $X := [0,1] \times [0,1]$ as a subspace of $\mathbb{R}^2$ with the standard topology, pictured as



We consider the equivalence relation $\mathscr{R}$ forcing the identifications of the snake game Exercice **I.1.1**. First, a point of the form $(0, y)$ should be identified with the point $(1, y)$



Secondly, a point of the form $(x, 0)$ should be identified with the point $(x, 1)$



More precisely, $\mathscr{R} \subseteq X \times X$ is defined by the union of

$$\{((x, y), (z, w)) : x = 0, y = w, z = 1\}$$

$$\{((x, y), (z, w)) : x = z, y = 0, w = 1\}$$

and the diagonal Δ. This defines an equivalence relation on X. Denote by $X/\mathscr{R}$ the quotient space. Show that

(i) $X/\mathscr{R}$ is quasi-compact.
(ii) The inclusion $X \subseteq \mathbb{R}^2$ renders the equivalence relation $\mathscr{R}$ compatible with the equivalence relation associated to the additive subgroup $\mathbb{Z}^2 \subseteq \mathbb{R}^2$.
(iii) The induced map quotient $X/\mathscr{R} \to \mathbb{R}^2/\mathbb{Z}^2$ is a continuous bijection.
(iv) $\mathbb{R}^2/\mathbb{Z}^2$ is Hausdorff
(v) The quotient map $X/\mathscr{R} \to \mathbb{R}^2/\mathbb{Z}^2$ is an homemorphism.
(vi) The map $(\exp, \exp) : \mathbb{R}^2 \to \mathsf{S}^1 \times \mathsf{S}^1$ descends to the quotient $\mathbb{R}^2/\mathbb{Z}^2$ and induces a homeomorphism.

Warning II.5.35. The product of quotient maps is not a quotient map in general. See the Example 7, page 143 in Munkres Topology book. See also the discussion below in the Exercise **II.5.65**.

Exercise II.5.36. Use n copies of the exponential function to show that the quotient $\mathbb{R}^n/\mathbb{Z}^n$ is homeomorphic to the product $(\mathsf{S}^1)^{\times_n}$. Suggestion: use the cubes $[0, 1]^n$ as above.

Exercise II.5.37. Consider the disk $X = D^2$ with the equivalence relation identifying its boundary $\partial X = \mathsf{S}^1$ to a single point. Show that the quotient space is homeomorphic to the 2-sphere S^2.

Exercise II.5.38. The real projective espace of dimension n, $\mathbb{R}P^n$ is the quotient of $\mathbb{R}^{n+1} \setminus \{0\}$ by the equivalence relation $x \sim y$ iff there exists $\lambda \in \mathbb{R}^*$ with $x = \lambda y$. TD, Exo 8, Feuille 3

(i) Show that $\mathbb{R}P^n$ is Hausdorff.

(ii) Show that $\mathbb{R}P^n$ is homeomorphic to the quotient of the sphere S^n by the antipodal relation , ie, $x \sim -x$. Conclude that $\mathbb{R}P^n$ is compact.
(iii) Show that $\mathbb{R}P^1$ is homeomorphic to S^1.

Here's a video illustrating the real projective plane (n=2):

Exercise II.5.39. The Mobius band M is the quotient of $\mathbb{R}^2$ by the relation generated by $(x, y) \sim (x + 1, -y)$.

(i) Show that M is Hausdorff.
(ii) Show that M is homeomorphic to the quotient of $[0, 1] \times [0, 1]$ by the relation $(0, y) \sim (1, 1 - y)$. Conclude that M is compact.

TD, Exo 8, Feuille 4

Exercise II.5.40. The Klein bottle K is the quotient of $\mathbb{R}^2$ by the equivalence relation given by the identifications $(x, y) \sim (x + 1, y)$ and $(x, y) \sim (-x, y + 1)$.

(i) Show that K is Hausdorff.
(ii) Show that K is homeomorphic to the quotient of $[0, 1] \times [0, 1]$ by the equivalence relation $(0, y) \sim (1, y)$ for all $y \in [0, 1]$ and $(x, 0) \sim (1 - x, 1)$ for all $x \in [0, 1]$.
(iii) Conclude that K is compact.

Exercise II.5.41. Let X be a topological space and let G be a group acting on X. We define a relation on X by declaring that $x \sim y$ if there exists $g \in G$ such that $y = g(x)$.

(i) Show that this defines an equivalence relation on X;
(ii) Let $p : X \to X/G$ denote the quotient map. Show that if U is an open subset in X, then $p^{-1}p(U) = \bigcup_{g \in G} g(U)$.
(iii) Conclude that the quotient map is an open map.

Gluings. Now that we know about the existence of quotient spaces and disjoint unions, we can define gluings in general. Given two topological spaces X and Y we would like to *glue them* along a common subspace Z. We can do something slightly more general:

Construction II.5.42 (Gluing)**.** Let $f : Z \to X$ and $g : Z \to Y$ be continuous maps. We consider the disjoint union $X \coprod Y$ endowed with the finest equivalence relation $\mathscr{R}$ such that $f(z) \sim g(z)$ for all $z \in Z$. In the notations of the Exercise **II.5.20**, we set $S := \{(f(z), g(z)) : z \in Z\}$ and $\mathscr{R} :=< S >$. We denote the quotient space by

$$X \coprod_Z Y := (X \coprod Y)/\mathscr{R}$$

by $i_X : X \to X \coprod Y \to X \coprod_Z Y$ and $i_Y : X \to X \coprod Y \to X \coprod_Z Y$ the maps induced by the two canonical inclusions. It follows from the quotient relations that the diagram commutes

$$\begin{array}{ccc} Z & \xrightarrow{g} & Y \\ \downarrow f & & \downarrow i_Y \\ X & \xrightarrow{i_X} & X \coprod_Z Y \end{array}$$

Proposition II.5.43. *[Universal property of the gluing] Let T be a topological space with two maps $u : X \to T$ and $v : Y \to T$ such that $u \circ f = v \circ g$. Then by the universal property of the quotient topology there exists a unique continuous map $\Psi : X \coprod_Z Y \to T$ such that $u = \Psi \circ i_X$ and $v = \Psi \circ i_Y$. Moreover, a map Ψ with this property is continuous if and only if both u and v are continuous.*

Proof. This is a combination of the universal property of disjoint unions with the universal property of quotient spaces. □

Definition II.5.44. Consider $Y = *$ and $f : Z \hookrightarrow X$ the inclusion of a subspace. Then we write $X/Z := X \coprod_Z Y$ for the result of the gluing. We call it the collapsed space, since Z becomes a single point in X/Z.

Example II.5.45. Let X be a topological space. The *cone* of X is the collapsed space $C(X) := [0, 1] \times X/\{1\} \times X$.

Example II.5.46. Let X be a topological space. The *suspension* of X is the collapsed space $S(X) := [0, 1] \times X/(\{1\} \times X \cup \{0\} \times X)$.

Example II.5.47. Consider the case where $Z = \emptyset$. Then the gluing $X \coprod_Z Y$ is the disjoint union space $X \coprod Y$ since no relations are forced.

Example II.5.48. Let X be a topological space with U and V open subsets such that $U \bigcup V = X$. Then the diagram of inclusions

$$\begin{array}{ccc} U \bigcap V & \longrightarrow & V \\ \downarrow & & \downarrow \\ U & \longrightarrow & X \end{array}$$

exhibits X as the gluing of U and V along the intersection. This is a direct consequence of Proposition **II.2.13**.

Example II.5.49. In this example we explain how the 2-sphere can be obtained as a gluing of two disks. Before addressing the proof, there's a video to explain what we are trying to do.

Let us formalize this. Consider the 2-sphere $S^2 \subseteq \mathbb{R}^3$. Write D_N and D_S for the northern and southern hemispheres.

$$D_N := \{(x, y, z) \in S^2 : z \geq 1\}$$

$$D_S := \{(x, y, z) \in S^2 : z \leq 1\}$$

Their intersection is the equator circle

$$D_N \cap D_S = \{(x, y, z) \in S^2 : z = 0\} = \{(x, y, z) : x^2 + y^2 = 1, z = 0\}$$

As in the Exercise **II.2.22** D_N is homeomorphic to the closed 2-disk D^2 via the. In the same way, D_S is also homeomorphic to the 2-disk via the stereographic projection from the north pole. The intersection of the two hemispheres is homeomorphic to the circle S^1. Using these identifications, we can fit S^2 in a commutative square

$$\begin{array}{ccc} S^1 & \longrightarrow & D_N \\ \downarrow & & \downarrow \\ D_S & \longrightarrow & S^2 \end{array}$$

where the maps correspond to the inclusions of the subspaces. This diagram exhibits S^2 as the gluing of the two hemispheres, ie, the induced map

$$\Psi : D_N \coprod_{S^1} D_S \to S^2$$

is a homeomorphism. To see this we start by constructing a set-theoretic map $\phi : S^2 \to D_N \coprod_{S^1} D_S$ by

$$\phi(x, y, z) = \begin{cases} i_N(x, y, z) & \text{if } (x, y, z) \in D_N \\ i_S(x, y, z) & \text{if } (x, y, z) \in D_S \end{cases}$$

where $i_N : D_N \to D_N \coprod_{S^1} D_S$ and $i_S : D_S \to D_N \coprod_{S^1} D_S$ are the canonical maps. Clearly this agrees on the intersection of the two hemispheres because of the definition of the equivalence relation. Notice that the restriction of ϕ to D_N is the inclusion i_N and ϕ restricted to D_S is i_S. Therefore, since D_N and D_S are closed subsets, ϕ is continuous because of the Exercise **II.2.14**. Notice that

$$\Psi \circ \phi(x, y, z) = \begin{cases} \Psi(i_N(x, y, z)) = (x, y, z) & \text{if } (x, y, z) \in D_N \\ \Psi(i_S(x, y, z)) = (x, y, z) & \text{if } (x, y, z) \in D_S \end{cases}$$

since the compositions $\Psi \circ i_S$ and $\Psi \circ i_N$ are the inclusions of the two hemispheres.

Finally, we also remark that the composition

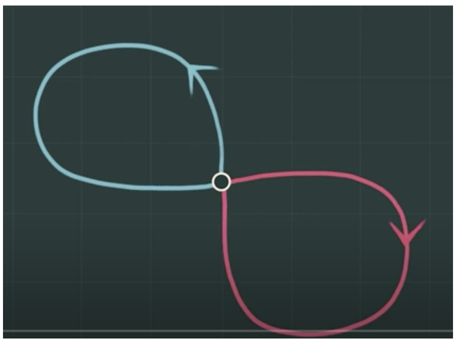
$$\phi \circ \Psi : D_N \coprod_{S^1} D_S \to S^2 \to D_N \coprod_{S^1} D_S$$

is the identity. Indeed, by the universal property, it suffices to check that $\phi \circ \Psi \circ i_N = i_N$ and $\phi \circ \Psi \circ i_S = i_S$. But $\Psi \circ i_S$ and $\Psi \circ i_N$ are the canonical inclusions of the northern and southern hemispheres, so the definition of ϕ by cases guarantees $\phi \circ \Psi \circ i_N = i_N$ and $\phi \circ \Psi \circ i_S = i_S$.

In conclusion, $\phi \circ \Psi$ and $\Psi \circ \phi$ are the identity maps.

Definition II.5.50. Let X and Y be topological spaces and $Z = *$. The maps $f : Z \to X$ and $g : Z \to Y$ correspond to a point $x \in X$ and a point y in Y. In this case the gluing is called the *wedge sum* and denoted by $X \vee Y$.

Example II.5.51. The wedge of two circles $\mathsf{S}^1 \vee \mathsf{S}^1$ recovers the space



of the Exercise **I.2.10**.

Exercise II.5.52. Let X be a topological space. Show that the suspension $S(X)$ (see Example **II.5.46**) can be obtained as a gluing of two copies of the cone of $C(X)$ along X

TD, Exo 11, Feuille 1

Exercise II.5.53. Consider X and Y topological spaces with $A \subseteq X$ a subspace and $f : A \to Y$ a continuous map. Show that if A is closed in X and both X and Y are Hausdorff, then A identifies with a subspace of $X \coprod_A Y$

Function spaces (optional, not part of the program).

Reminder II.5.54. Let X, Y be sets. Then the collection of maps of sets $X \to Y$ forms itself a set, which we will denote as $\mathrm{Hom}(X, Y)$. This set has a particular nice feature: to give a map of sets $\Psi : Z \to \mathrm{Hom}(X, Y)$ is the same thing as giving for every element $z \in Z$, a map $\Psi_z : X \to Y$,ie, for every element $x \in X$, an element $\Psi_z(x) \in Y$. We can arrange this as a function on pairs $(z, x) \mapsto \Psi_z(x)$, or simply, a map of sets $Z \times X \to Y$. Inversely, every map of sets $\phi : Z \times X \to Y$ determines for each $z \in Z$ a map $\phi(z, -) : X \to Y$, ie, a map $Z \to \mathrm{Hom}(X, Y)$

We would like to have a similar mechanism for topological spaces, namely, given two topological spaces, X and Y, construct a new topological space, denoted $\mathrm{Map}(X, Y)$ whose points are continuous maps $X \to Y$ and such that continuous maps $Z \to \mathrm{Map}(X, Y)$ are in bijection with continuous maps $Z \times X \to Y$ by the same principle as above. This is not always possible, but in some cases, it is:

Notation II.5.55. Let X and Y be topological spaces. We denote by $C(X, Y)$ the set of all continous maps from X to Y.

Construction II.5.56 (Compact-open topology)**.** Let X and Y be topological spaces. Given a compact subset K of X and an open subset U of Y, we denote by

$$W(K, U) := \{f \in C(X, Y) : f(K) \subseteq U\}$$

and consider the collection of subsets of $C(X, Y)$ given by

$$\beta := \{W(K; U) : K \text{ compact in } X, U \text{ open in } Y\}$$

The collection β verifies the condition for a subbasis of a topology as in Construction **II.1.15**:

(i) Indeed, for any continuous map $f : X \to Y$, choose $K = \emptyset$ and $U = Y$. Then $f \in W(K, U) = C(X, Y)$

We denote by $\mathrm{Map}(X, Y)$ the topological space obtained by endowing the set $C(X, Y)$ with the topology generated by the subbasis β as in Construction **II.1.15**. We call it the compact-open topology

Definition II.5.57. A space X is said to be locally compact if it is Hausdorff and every point has a compact neighborhood, ie, for every point $x \in X$ there exists an open neighborhood U and a compact subset K such that $x \in U \subseteq K$.

Remark II.5.58. Every compact space is also locally compact. Indeed, take $U = X$ for every point.

Example II.5.59. The spaces $\mathbb{R}^n$ are locally compact. Indeed, any point lies in a basis element $]a_1, b_1[\times \cdots \times]a_n, b_n[$ which is inside the compact subspace $[a_1, b_1] \times \cdots \times [a_n, b_n]$

Exercise II.5.60 (One point compactification)**.** Let X be a locally compact space. TD1, Exo 16, Feuille 1

(i) Let $\widehat{X}$ denote the set obtained by adjoining to X a point ∞. Define $\tau \subseteq P(\widehat{X})$ by

$$\tau := \{U : U \text{ open in } X\} \cup \{(X \setminus K) \cup \{\infty\} : K \text{ is compact in X }\}$$

  (a) Show that τ forms a topology on $\widehat{X}$.
  (b) Show that $\widehat{X}$ is compact.

(ii) Show that if Y is any other compact space containing X as a subspace and such that $Y \setminus X$ consists of a single point, then Y is canonically homeomorphic to $\widehat{X}$ through an isomorphism that preserves X. (We call $\widehat{X}$ the *one-point* compactification of X)

(iii) Use the stereographic projection to show that $\widehat{\mathbb{R}^n}$ is homeomorphic to S^n.

(iv) Show that the one-point compactification of the Mobius band is the real projective plane.

Exercise II.5.61. We define the complex projective space of dimension n - $\mathbb{C}P^n$- to be the quotient of $\mathbb{C}^n \setminus \{0\}$ by the equivalence relation given by $x \sim y$ if and only if there exists $\lambda \in \mathbb{C}^*$ such that $x = \lambda.y$. Show that $\mathbb{C}P^1$ is homeomorphic to the 2-sphere S^2.

Lemma II.5.62. *Let X be a locally compact space. Then for every point $x \in X$ and every open neighborhood U of x there exists an open neighborhood V such that $x \in V \subseteq \overline{V} \subseteq U$ with $\overline{V}$ compact.*

Proof. Let $x \in X$ and U be an open neighborhood of x. Let $\widehat{X}$ denote the one-point compactification of X (Exercise **II.5.60**), which exists since X is locally compact. Let $C = \widehat{X} \setminus U$. Since U is also open in $\widehat{X}$ by definition of the one-point compactification, C is closed in $\widehat{X}$. Since $\widehat{X}$ is compact, C is quasi-compact subspace (by the Exercise **II.3.19**). By the Exercise **II.3.19**-(ii), we can find open disjoint subsets V and W, with $x \in V$, and $C \subseteq W$. Then, $\overline{V}$ is compact in $\widehat{X}$ and disjoint from C. Therefore $\overline{V} \subseteq U$. □

Lemma II.5.63. *Let X be a locally compact space, then the evaluation map E : $\mathrm{Map}(X,Y) \times X \to Y$ defined set-theoretically by $(f,x) \mapsto f(x)$, is continuous with respect to the compact-open topology.*

PROOF. Let T be an open in Y. By the Proposition **II.1.11**. to show that $E^{-1}(T)$ is open it is enough to show that for $(f,x) \in E^{-1}(T)$ there exists an open subset S of $\mathrm{Map}(X,Y) \times X$ such that $(f,x) \in S \subseteq E^{-1}(T)$. In other words, $E(S) \subseteq T$. Since X is locally compact Hausdorff, by the Lemma **II.5.62**, given $U = f^{-1}(T)$ there exists an open V with $x \in V \subseteq \overline{V} \subseteq U$ with $\overline{V}$ compact. Form $S := U \times W(\overline{U}, T)$ which is open in the box topology. We notice that $(f,x) \in S$ since $x \in U$ and $\overline{U} \subseteq f^{-1}(T)$, ie, $f(\overline{U}) \subseteq T$. Also, notice that by definition of the pre-image we have $S \subseteq E^{-1}(T)$.

□

TD, Exo 6, Feuille 2

Proposition II.5.64. *Assume X is locally compact. Then a map $Z \to \mathrm{Map}(X,Y)$ is continuous if and only the the composition with the evaluation map $Z \times X \to \mathrm{Map}(X,Y) \times X \to Y$ is continuous.*

PROOF. As a consequence of the Lemma **II.5.63**, if X is locally compact, the evaluation map is continuous. Therefore, if $Z \to \mathrm{Map}(X,Y)$ is a continuous map, the composition

$$Z \times X \to \mathrm{Map}(X,Y) \times X \to Y$$

is continuous.
Conversely, assume that $\Psi : Z \times X \to Y$ is a continuous map. We want to show that the induced map $\phi : Z \to \mathrm{Map}(X,Y)$ is continuous. By the Remark **II.2.5**, it is enough to show that the pre-image of an open subset $W(K,V)$ in $\mathrm{Map}(X,Y)$ is open in Z. By definition

$$\phi^{-1}(W(K,V)) = \{z \in Z : \phi_z(K) \subseteq V\} = \{z \in Z : \{z\} \times K \subseteq \Psi^{-1}(V)\}$$

Since Ψ is continuous by assumption, $\Psi^{-1}(V)$ is open in $Z \times X$. In particular $\Psi^{-1}(V) \cap Z \times K$
is an open subset of $Z \times K$ (with the subspace topology). The tube lemma (Exercise **II.5.6**) applied to $\{z\} \times K \subseteq Z \times K$ implies that there exists an open neighborhood A of z in Z such that the whole $A \times K$ is still inside $\Psi^{-1}(V)$. It follows that $A \subseteq \phi^{-1}(W(K,V))$.

□

Exercise II.5.65. Let Z be a locally compact space and let X be a topological space with an equivalence relation $R \subseteq X \times X$. Endow $X \times Z$ with the equivalence relation $R \times \Delta_Z \subseteq X \times X \times Z \times Z$. Use the universal property of mapping spaces to show that the canonical map

$$(X/R) \times Z \to (X \times Z)/(R \times \Delta_Z)$$

is a homeomorphism.

Remark II.5.66. The Warning **II.5.35** shows that in general the product of quotient maps is not a quotient map. However, when one of the spaces is locally compact, the result holds as shown by the Exercise **II.5.65**. In fact, to make this result hold in full generality one needs to work with a smaller class of topological spaces, namely, those called compactly generated. Locally compact spaces belong to this class. We will not show it in this course, but for the purpose of algebraic topology any space can be replaced by one that is compactly generated without losing information. For more details see P. May's book (chapter 5) referenced in the bibliography.

II.6. Connected spaces

Proposition II.6.1. *Let Z be a topological space. The following conditions are equivalent:*

(i) There exists a non-empty subset $U \subseteq Z$ which is both open and closed in Z;

(ii) There exists U and V open subsets of Z such that $U \cap V = \emptyset$, $U \cup V = Z$;

(iii) The same as in (ii) but U and V closed.

(iv) There exists topological spaces U and V and a homeomorphism $Z \simeq U \coprod V$.

(v) There exists a non-constant continuous map $Z \to \{0, 1\}$ where $\{0, 1\}$ is equipped with the discrete topology.

Proof.

- $(i) \Rightarrow (ii)$: Let U be as in (i). Take $V = Z \setminus U$. Both U and V are open because U is both open and closed. Finally, $U \bigcap (Z \setminus U) = \emptyset$ and $U \bigcup (Z \setminus U) = Z$.

- $(ii) \Leftrightarrow (iii)$: Take complementary subsets.

- $(ii) \Leftrightarrow (iv)$: By Exercise **II.5.16** the two inclusions $U \hookrightarrow Z$ and $V \hookrightarrow Z$ induce a continuous map

$$U \coprod V \to Z$$

This is a continuous bijection. It remains to compare the topology. Let W be open in Z. Then the two intersections $U \bigcap W$ and $V \bigcap W$ are open by the axioms. Conversely, assume that both $U \bigcap W$ and $V \bigcap W$ are open. Then we have

$$(W \bigcap U) \bigcup (W \bigcup V) = W \bigcap (U \bigcup V) = W \bigcap Z = W$$

which by the axioms for a topology, is also open.

- $(ii) \Rightarrow (v)$. Given U and V as in (ii), define $f : Z \to \{0, 1\}$ by

$$f(z) = \begin{cases} 1 & z \in U \\ 0 & z \in V \end{cases}$$

This function is continuous since $f^{-1}(\{1\}) = U$ and $f^{-1}(\{0\}) = V$ are open.

- $(v) \Rightarrow (ii)$. Given such a continuous function, set $U := f^{-1}(\{1\})$ and $V := f^{-1}(\{0\})$. It follows that U and V are both non-empty (because the function is non-constant), open and disjoint and their union is Z.

□

Definition II.6.2. We say that a topological space is *disconnected* if it satisfies one of the equivalent conditions of the Proposition **II.6.1**. We say that a space is connected if it is not disconnected. We says that a subspace is connected if it is connected for the induced topology.

Remark II.6.3. In particular, a space X is connected if and only if every continuous function $X \to \{0, 1\}$ is constant. In particular, if X is connected, every function to a discrete space must be constant.

Proposition II.6.4. *The space $\mathbb{R}$ is connected.*

Proof. Suppose $\mathbb{R}$ is not connected. Then there exists a subset A which is simultaneously open, closed and non-empty and different from $\mathbb{R}$. Let $x \in \mathbb{R} \setminus A$. Then since $\mathbb{R} =]-\infty, x] \cup [x, +\infty]$, one of the intersections $[x, +\infty[\cap A$ or $]-\infty, x] \cap A$ has to be non-empty. Without loss of generality, assume it is $[x, +\infty[\cap A$ that is non-empty. Notice that $[x, +\infty[$ is the complement of the open subset $]-\infty, x[$ so it is closed. Since A is closed, $[x, +\infty[\cap A$ is also closed. However, since x does not belong to A, $[x, +\infty[\cap A =]x, +\infty[\cap A$ which is also open since A is open. It follows that $[x, +\infty[\cap A$ is both open and closed. Now, $[x, +\infty[\cap A$ is non-empty and admits a lower bound by x. Therefore, it has a greatest lower bound $s = \inf([x, +\infty[\cap A)$. Since $[x, +\infty[\cap A$ is closed, $s \in [x, +\infty[\cap A$. But at the same time since $[x, +\infty[\cap A$ is open, there exists $\epsilon > 0$ such that $]s - \epsilon, s + \epsilon[\subseteq [x, +\infty[\cap A$. Take z such that $s - \epsilon < z < s$. But then $z \in [x, +\infty[\cap A$ and $z < s$ contradicting the fact that s was the greatest lower bound.

□

Example II.6.5. The space $\mathbb{R} \setminus \{0\}$ is not connected. Indeed, we can write it as a disjoint union

$$\mathbb{R} \setminus \{0\} =]-\infty, 0[\bigcup]0, +\infty[$$

In particular, this example shows that the closure of a subspace A in X might be connected without A being connected.

Proposition II.6.6. *Let $A \subseteq B \subseteq \overline{A} \subseteq X$ and assume that A is connected for the subspace topology. Then B is connected for the subspace topology*

Proof. Assume there exists open subsets U and V in X with $B \subseteq U \cup V$ and $U \cap V = \emptyset$. We shall show that either $B \cap U$ or $B \cap V$ is empty. By definition of the subspace topology, both $U \cap A$ and $V \cap A$ are open in A. But then since $A \subseteq B$, we find

$$A = (A \cap U) \cup (A \cap V)$$

and

$$(A \cap U) \cap (A \cap V) = \emptyset$$

Since A is assumed to be connected, it follows that either $A \cap U$ is empty or $A \cap V$ is empty. Assume it is $A \cap U$ that is empty. It follows that $A \subseteq X \setminus U$. Since $X \setminus U$ is closed, by definition of the closure as the intersection of all closed subsets that contain A, we have $\overline{A} \subseteq X \setminus U$. Therefore $B \subseteq X \setminus U$ so that $B \cap U$ is empty. □

Remark II.6.7. The proof of Proposition **II.6.4** also works to show that any open interval of $\mathbb{R}$ is connected. The Proposition **II.6.6** now shows that any interval is connected.

Exercise II.6.8. Let A be a connected subspace of $\mathbb{R}$. Show that A is an interval

Remark II.6.9. The Proposition **II.6.6** shows that the closure of a connected subset is connected.

Proposition II.6.10. *Let $f : X \to Y$ be a continuous map and assume that X is connected. Then $f(X)$ is connected.*

Proof. We prove by contradiction. Suppose $f(X)$ is not connected for the subspace topology, ie, there exists non-empty open subsets V_1, V_2 of Y with $V_1 \cap V_2 = \emptyset$ and such that $f(X) \subseteq V_1 \cup V_2$. Then $f^{-1}(V_1)$, $f^{-1}(V_2)$ are such that

$$f^{-1}(V_1) \bigcap f^{-1}(V_2) = f^{-1}(V_1 \cap V_2) = f^{-1}(\emptyset) = \emptyset$$

$$f^{-1}(V_1) \bigcup f^{-1}(V_2) = f^{-1}(V_1 \cup V_2) = f^{-1}(f(X)) = X$$

showing that X is not connected. □

Proposition II.6.11 (Intermediate Value theorem)**.** *Let $f : X \to \mathbb{R}$ be a continuous map with X connected. Let $x, y \in X$ and let $r \in \mathbb{R}$ such that $f(a) \leq r \leq b$. Then there exists $z \in X$ with $r = f(z)$.*

PROOF. Since X is connected, by Proposition **II.6.10**, $f(X) \subseteq \mathbb{R}$ is a connected subspace. Let

$$A = f(X) \cap]-\infty, r[\quad , \quad B = f(X) \cap]r, +\infty[$$

Then A and B are disjoint by design. They are also non-empty since $f(a) \in A$ and $f(b) \in B$. Each is open in $f(X)$ by definition of the subspace topology.
Notice that by construction also

$$f(X) = A \cup B \cup (f(X) \cap \{r\})$$

Suppose there does not exist z with $f(z) = r$. Then the intersection $f(X) \cap \{r\}$ is empty and we have

$$f(X) = A \cup B$$

with A and B disjoint. This would imply that $f(X)$ is disconnected, which is a contradiction.

□

Proposition II.6.12. *Let X be a topological space and $\mathcal{R}$ an equivalence relation. Then the quotient space $X/\mathcal{R}$ is connected.*

PROOF. Indeed, apply Proposition **II.6.10** to the quotient map π and use that π is surjective. □

Exercise II.6.13. Let $X \to Y$ be a homeomorphism. Then X is connected if and only if Y is connected.

Proposition II.6.14. *The union of a collection of connected subspaces of X that have a point in common, is connected.*

PROOF. Let $\{A_i\}$ be a family of connected subsets of X and denote by $Y = \bigcup_I A_i$ their union. Let $p \in \bigcap_i A_i$.

Assume by contradiction that Y is not connected, ie, there exists non-empty opens U and V with $Y = U \bigcup V$ and $U \bigcap V = \emptyset$. The point p is in one of the subsets U or V. Assume it is in U. Since A_i is connected, it must lie entirely in either in U or in V. Since $p \in U$, A_i must lie in U, ie $A_i \subseteq U$. This argument applies to each i, so that $Y = \bigcup_i A_i \subseteq U$, implying that V is empty, which is a contradiction.

□

Corollary II.6.15. *A finite product of connected spaces is connected*

PROOF. Let us do the proof for the product $X \times Y$. Fix a point (a, b) in the product and consider the subset of the product given by

$$T_x := X \times \{b\} \bigcup \{x\} \times Y$$

This is a subset of $X \times Y$ which is the union of two connected subsets: $X \times \{y\}$ is connected because it is homeomorphic to X and $\{x\} \times Y$ is connected because it is homeomorphic to Y. Their intersection is non-empty, consisting of the single point

$$X \times \{y\} \bigcap \{x\} \times Y = \{(x, y)\}$$

Therefore, by Proposition **II.6.14** this union is connected.

Now form the union $\bigcup_{x \in X} T_x$. Each T_x is connected and the intersection $\bigcap_{x \in X} T_x$ is given by the horizontal line $X \times \{b\}$. In particular, the intersection is non-empty, say $(a, b) \in \bigcap_{x \in X} T_x$. Again by the Proposition **II.6.14**, the union $\bigcup_{x \in X} T_x$ is connected. But this union is precisely $X \times Y$ $\square$

Example II.6.16. The spaces $\mathbb{R}^n$ are all connected for $n \geq 0$.

Definition II.6.17. Let X be a topological space and $x \in X$. The union of all connected subsets of X which contain x is called the *connected component* of x.

Remark II.6.18. It follows from the Proposition **II.6.14** that a connected component of a point x is connected.

Remark II.6.19. Any connected component of X is closed in X. Indeed, if $C \subseteq X$ is the connected component of x, it is connected (by the Remark **II.6.18**), so $\overline{C}$ is also connected by Remark **II.6.9** and contains x. But C is the union of all connected subsets that contain x so $\overline{C} \subseteq C$. Therefore $C = \overline{C}$.

Remark II.6.20. A connected component is not necessarily open. Here's an example. Take $X = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}^*\} \subseteq \mathbb{R}$ as a subspace of the real numbers for the induced topology. Notice that each of the $\frac{1}{n}$ are isolated points, in the sense that $\{\frac{1}{n}\}$ is both open and closed for the induced topology. Let A be the connected component of 0. If some of the $\frac{1}{n}$ belongs to A. Then $A = A \setminus \{\frac{1}{n}\} \cup \{\frac{1}{n}\}$ is a disjoint union of open subsets contradicting the fact that A is connected. Therefore $A = \{0\}$ is the connected component of 0. But this set is not open for the induced topology since any neighbourhood of 0 for the induced topology intersects points of the form $\frac{1}{n}$. See Exercise **II.1.30**-(vi).

Definition II.6.21. Let X and Y be topological spaces. We say that a function $X \to Y$ is locally constant if for every point $x \in X$ there exists an open neighbourhood U such that $f_{|_U} : U \to Y$ is constant.

Proposition II.6.22. *If X is connected then every locally constant map is constant.*

PROOF. Assume X is connected. Fix $x_0 \in X$ and consider

$$U = \{x \in X; f(x) = f(x_0)\} = f^{-1}(\{f(x_0)\})$$

$$V = \{x \in X; f(x) \neq f(x_0)\} = X \setminus f^{-1}(f(x_0))$$

Clearly $U \cap V = \emptyset$. We claim that both U and V are open. Let $x \in U$. Since f is locally constant, there exists an open neighbourhood W of x such that f is constant in U, with value y. But since $x \in U$, we must have $y = f(x_0)$ and in this case $W \subseteq U$. The same argument applies for $x \in V$.

Since X is connected, this is a contradiction, so V must be empty and therefore f is constant.

Conversely, assume that every locally constant map is constant. Let □

TD, Exo 12-(4), Feuille 1

Exercise II.6.23. Let G be a topological group and $e \in G$ the unit. Show that the connected component G_0 of e in G is

- a closed subgroup
- a normal subgroup

TD, Exo 4, Feuille 1

Exercise II.6.24. Is the property of being connected stable under:

- Unions?
- Intersections?
- closure?
- interiors?
- boundary?
- passing a closed subspace?

Exercise II.6.25.
Show that the Klein bottle is connected (see Exercise **II.5.40**).

CHAPTER III

Paths and Homotopies

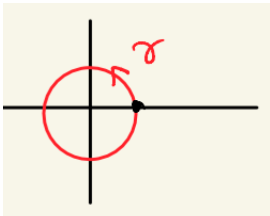
III.1. Operations on Paths

Notation III.1.1. Throughout these notes we denote by I the closed interval $[0, 1]$.

Definition III.1.2. Let X be a topological space. A *path* on X is a continuous map $\gamma : I \to X$. We say that $\gamma(0) = x$ is the starting point of γ and $y = \gamma(1)$ is the ending point. We say that γ is a path from x to y.

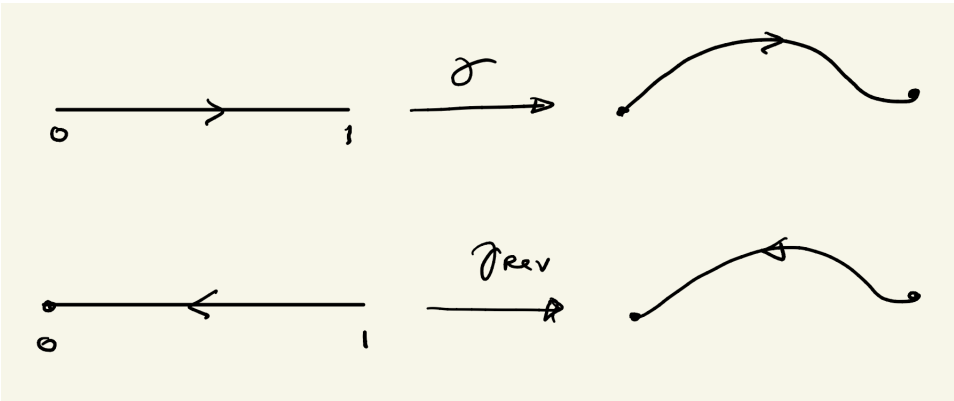
Example III.1.3. Let $X = \mathbb{R}^2$ and consider the path $\gamma : [0, 1] \to X$ given by

$$\gamma(t) = (\cos(2\pi t), \sin(2\pi t))$$

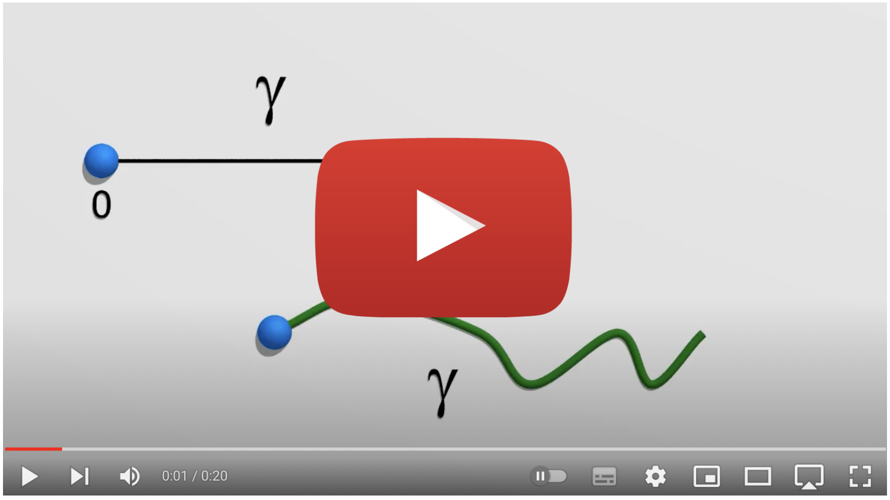


Remark III.1.4. Since the interval I is connected (Remark **II.6.7**), the Proposition **II.6.10** guarantees that $\gamma(I) \subseteq$ is connected.

Construction III.1.5. Let $\gamma : I \to X$ be a path from x to y. Then the map $I \to X$ given by the formula $\gamma_{rev}(t) := \gamma(1 - t)$ is a path from y to x. Indeed, it is continuous since it is obtained as a composition of γ with the map $I \to I$ given by $t \mapsto 1 - t$. We call γ_{rev} the reverse path of γ.

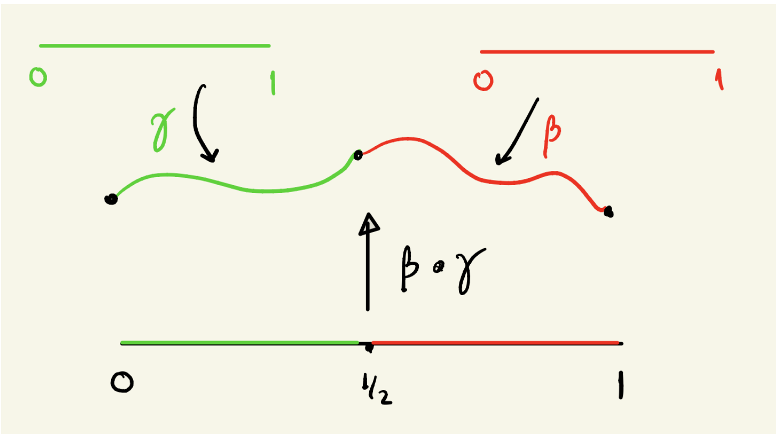


Here's a video illustrating this:



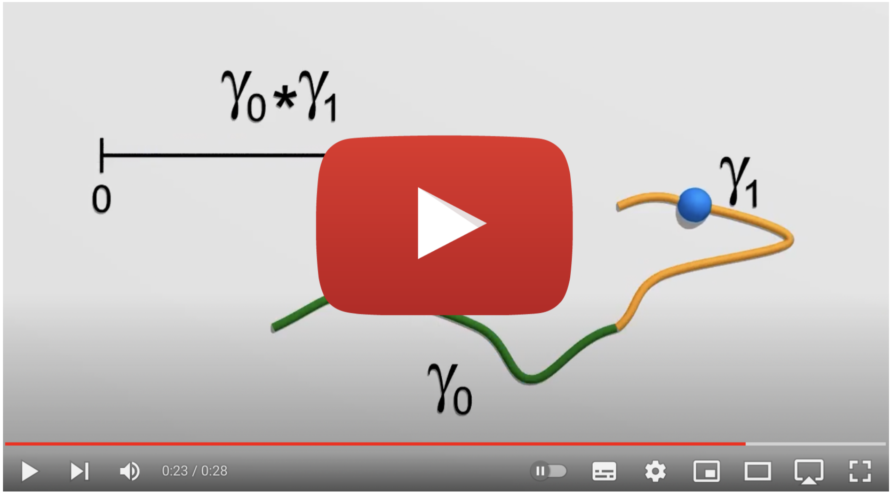
Construction III.1.6.
Given paths $\gamma : I \to X$ and $\beta : \gamma \to X$ where the end point of γ is the starting point of β, ie, $\beta(0) = \gamma(1)$, we can form a new path obtained by following γ twice as fast and then β also twice the speed. Namely, we define $\beta * \gamma : I \to X$ by the formula

$$(\beta \circ \gamma)(t) := \begin{cases} \gamma(2t) & 0 \leq t \leq \frac{1}{2} \\ \beta(2t-1) & \frac{1}{2} \leq t \leq 1 \end{cases}$$



We observe that $\beta * \gamma$ is continuous as a consequence of the Exercise **II.2.14**. We call $\beta * \gamma$ the concatenation of β with γ.

Here's a video with an animation:



III.2. Path-Connected Spaces

Definition III.2.1. Let X be a topological space. One says that X is *path-connected* if every pair of points x, y can be connected by a continuous path, ie, there exists $\gamma : I \to X$ with $\gamma(0) = x, \gamma(1) = y$.

Exercise III.2.2. Let X be a topological space and $\mathscr{R}$ an equivalence relation. Show that if X is path-connected then so is $X/\mathscr{R}$.

We turn to the relation between the notion of being connected and path-connected:

Proposition III.2.3. *If X is path-connected then X is connected.*

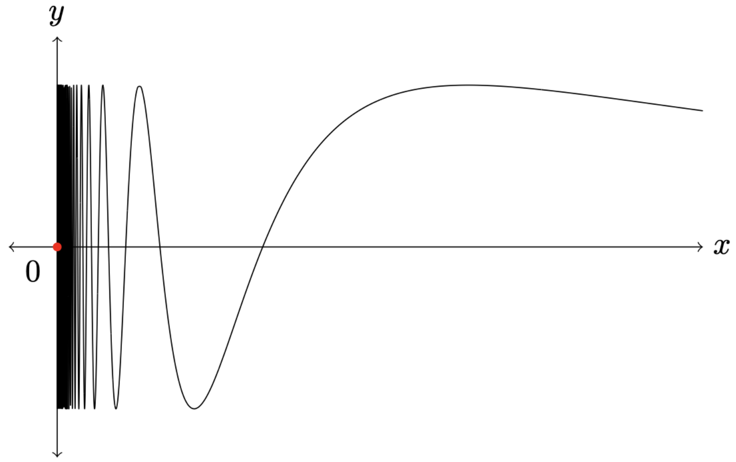
Proof. Let $x \in X$. For any other point $y \in X$ there exists a path γ_y connecting x to y. Therefore, we can write $X = \bigcup_{y \in X} \gamma_y(I)$. Each $\gamma_y(I)$ is connected because of the Proposition **II.6.10**. Therefore, we managed to write X as a union of connected subsets, with the point x in common. The conclusion follows from Proposition **II.6.14**. □

The converse fails:

Example III.2.4. Consider the union

$$X = \{(0,0)\} \bigcup \{(x,y) \in \mathbb{R}^2 : y = \sin(\frac{1}{x}), x > 0\}$$

endowed with the subspace topology.



(Pictures taken from here)

- X is connected: Indeed the space $A = \{(x, y) \in \mathbb{R}^2 : y = \sin(\frac{1}{x}), x > 0\}$ is the image of the continuous map $\sin(1/t) : \mathbb{R}_{>0} \to \mathbb{R}$. By Proposition **II.6.4** we know that $\mathbb{R}$ is connected, so using the homeomorphism given by the real logarithm $\log : \mathbb{R}_{>0} \simeq \mathbb{R}$, we establish that $\mathbb{R}_{>0}$ is connected. By Proposition **II.6.10** is connected. The set X is such that $A \subseteq X \subseteq \overline{A}$ with A connected. By the Proposition **II.6.6**, X is connected.

- X is not path-connected: Assume there exists a continuous path γ with $\gamma(0) = (0, 0)$ and $\gamma(1)$ lying over the graph. Let $\pi_1 : \mathbb{R}^2 \to \mathbb{R}$ denote the projection onto the x-coordinate. At some time t_0, the path γ must jump from having 0 as x-coordinate, to having a strictly positive x-coordinate.

$$t_0 := \inf\{t \in [0, 1] : \pi_1(\gamma(t)) > 0\}$$

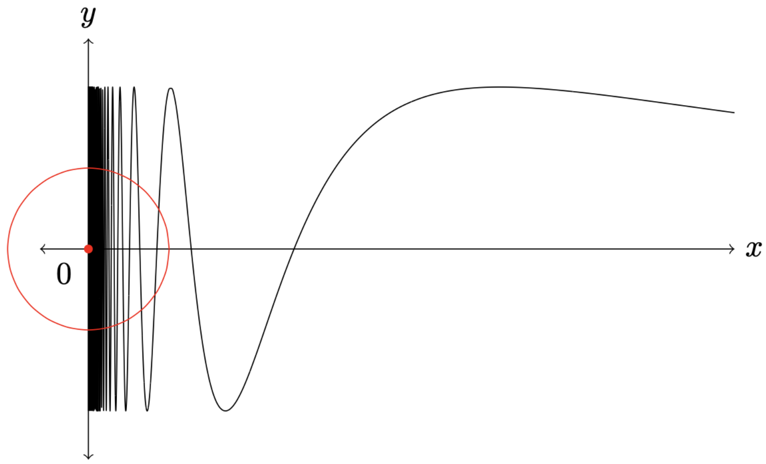
Notice that the inf exists since this set by assumption is non-empty and by design is bounded below by $t = 0$.

For $t < t_0$, $\pi_1(\gamma(t)) = 0$. By limit-point left-continuity of the path, it follows that $\pi_1(\gamma(t_0)) = 0$ so $\gamma(t_0) \in \{0\} \times [-1, 1]$.

At the same time, since γ is continuous at t_0, using the $\epsilon - \delta$-definition of continuity we see that for $\epsilon = \frac{1}{2}$, there exists $\delta > 0$ such that

$$\forall t : t_0 \leq t < t_0 + \delta \Rightarrow d(\gamma(t), (0, 0)) < \frac{1}{2}$$

ie, for all t slightly bigger than t_0, $\gamma(t)$ should be in a small ball of radius $\frac{1}{2}$:



By definition of t_0 as the infimum, there exists $t_0 + \delta > t_1 > t_0$ with $\pi_1(\gamma(t_1)) > 0$. By continuity, the image $\pi_1(\gamma([t_0, t_1]))$ is a connected interval in $\mathbb{R}$, $[0, a]$, with $0 = \pi_1(\gamma(t_0))$ and $a := \pi_1(\gamma(t_1))$. But as we can see from the picture, in order to progress to the right of the graph, the path will need to escape from the ball infinitely many times reaching $\sin = 1$ and $\sin = -1$. This occurs for values of x arbitrary small inside the ball, infinitely many times. In particular, we see that for values $\pi_1(\gamma(t))$ (such as the ones between t_0 and t_1), we can never get the whole interval $[0, a]$ - In order to get it we would need to reach values of the sinus functions outside the ball.

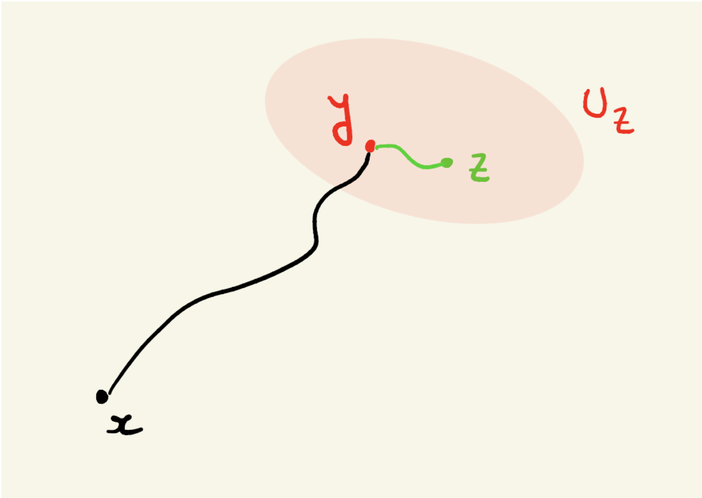
This has a fix:

Definition III.2.5. Let X be a topological space. One says that X is *locally path connected* if the topology admits a basis by path-connected open neighborhoods.

Proposition III.2.6. *If X is locally path-connected and connected, then X is path-connected.*

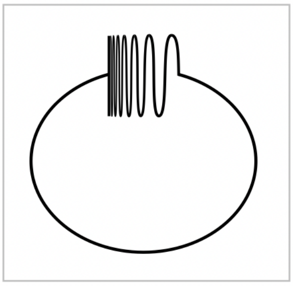
Proof. Assume X is non-empty. Let $x \in X$. Let A denote the subset of X of points y such that there exists a path from x to y. First of all A is non-empty because X is locally path-connected.
We show that A is open: since X is locally path-connected, for every $y \in A$ there exists an open neighborhood U_y of y which is path-connected. Therefore, any point in $z \in U_y$ can be connected to x by concatenation of a path from x to y and a path from y to z. It follows that all points in U_y are by definition of A, still inside A. This shows that A is open.
We now show that A is closed. Let $z \in \overline{A}$ and let U_z be a path-connected open neighborhood of z. By the Exercise **II.1.30**-(vi) the intersection $A \cap U_z$ is non-empty. Let $y \in A \cap U_z$. Therefore, z can be connected to x by a path that passes trough the intermediate point y. It follows that $z \in A$ by the definition of A.

Since X is connected, it cannot have A both non-empty. open and closed unless $A = X$

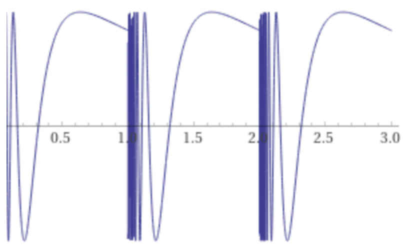


□

Warning III.2.7. A path-connected space is not necessarily locally path-connected. The standard example is the "topologist's sinus curve" (Circle de Varsovie)



Here's an explicit construction of this space: take in $\mathbb{R}^2$ the graph Γ of the function $\mathbb{R} \setminus \mathbb{Z} \to \mathbb{R}$ sending $x \mapsto \sin(\frac{1}{x-[x]})$ where $[x]$ means the integer part of x:



Finally, take $E := \overline{\Gamma}$ the closure of Γ. This is equal to

$$E = \Gamma \cup \left(\bigcup_{n \in bbZ} \{\{n\} \times [-1, 1]\} \right)$$

Now we consider the quotient space obtained by identifying $(x, y) \sim (x', y')$ if and only if $x' - x \in \mathbb{Z}$ and $y' = y$. The quotient $E/\sim$ is the topologists circle represented in the picture above.

Proposition III.2.8. *Let X be a topological space. Let $x, y \in X$. The relation $x \sim y$ if and only if there exists a path from x to y, is an equivalence relation on the set of points of X.*

PROOF. Obviously $x \sim x$, since we can always take the constant path $c_x : I \to X$ defined by $c_x(t) = x$ for all t. Symmetry is a consequence of the Construction **III.1.5**. Trransitivity is a consequence of the Construction **III.1.6**. □

Construction III.2.9. Let X be a locally path connected topological space. Then the two notions of connectedness coincide (Proposition **III.2.6** and Proposition **III.2.3**). We denote by $\pi_0(X)$ the quotient of the set of points of X by the path-connected relation of the Proposition **III.2.8**. If $f : X \to Y$, we have a well-defined map of sets

$$\pi_0(f) : \pi_0(X) \to \pi_0(y)$$

sending the $[x] \mapsto [f(x)]$. This is indeed well-defined on equivalence classes of points: if γ is a path from x to y, then $f \circ \gamma$ is a path from $f(x)$ to $f(y)$.
Moreover, if $g : Y \to Z$ is another continuous map, we have $\pi_0(g \circ f) = \pi_0(g) \circ \pi_0(f)$.

Exercise III.2.10. Is the property of being path-connected stable under: TD, Exo 4, Feuille 1

- Unions?
- Intersections?
- closure?
- interiors?
- boundary?
- passing a closed subspace?

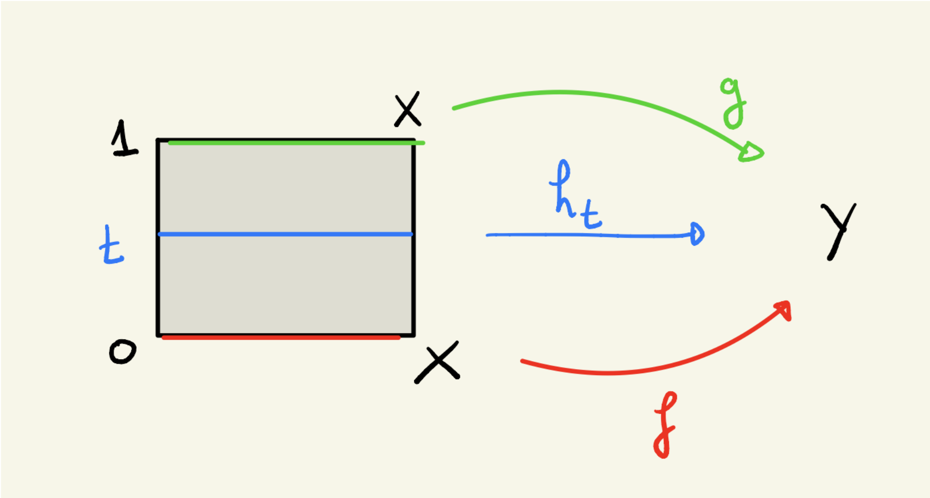
Exercise III.2.11. Consider X and Y topological spaces with $A \subseteq X$ a subspace and $f : A \to Y$ a continuous map. TD, Exo 11, Feuille 1

(i) Show that if X and Y are connected and A is non-empty, then $X \coprod_A Y$ is connected.
(ii) Show that the same holds replacing connected by path-connected.

III.3. Homotopies

Definition III.3.1. Let $f, g : X \to Y$ be continuous maps. We say that f and g are homotopic if there exists a continuous map $H : I \times X \to Y$, such that $H(0, -) = f$ and $H(1, -) = g$.

Remark III.3.2. We can think of a homotopy as a continuous 1 -parameter family of maps $h_t : H(t, -) : X \to Y$ with $h_0 = f$ and $h_1 = g$.



Remark III.3.3. If X is locally compact, Proposition **II.5.64** tells us that a homotopy $I \times X \to Y$ from f to g is the same thing as a continuous map $I \to \mathrm{Map}(X, Y)$ sending 0 to f and 1 to g, ie a path in $\mathrm{Map}(X, Y)$ from f to g.

Definition III.3.4. We say that a continuous map $f : X \to Y$ is null-homotopic if it is homotopic to a constant map.

Example III.3.5. Any two continuous maps $f, g : X \to \mathbb{R}^n$ are homotopic via

$$H(t, x) := (1 - t)f(x) + t.g(x)$$

In particular, any continuous map $X \to \mathbb{R}^n$ is null-homotopic.

Construction III.3.6. [Reverse homotopy] If f is homotopic to g then g is homotopic to f. Indeed, let $H : I \times X \to Y$ be a homotopy from f to g. We consider the *reverse* homotopy $H_{rev} : I \times X \to Y$ defined by the formula

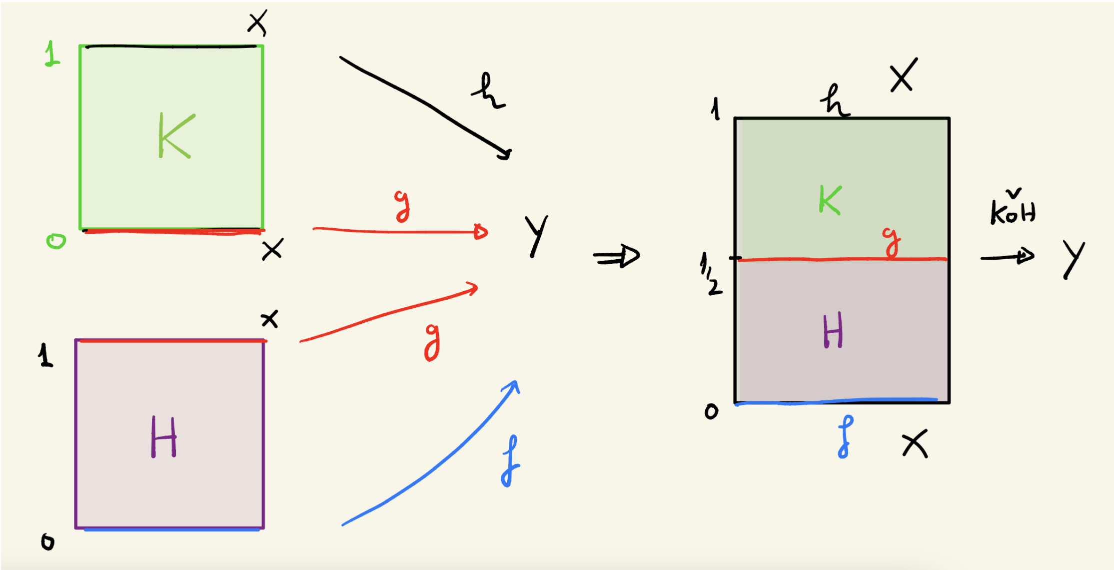
$$H_{rev}(t, x) := H(1 - t, x)$$

which is obtained by composing H with the continuous map I$\to I$ sending $t \mapsto 1-t$.

Construction III.3.7. [Vertical Concatenation of Homotopies] If f is homotopic to g and g is homotopic to h then f is homotopic to h. Let H be a homotopy from f to g and K from g to h. We set

$$(K \circ^v H)(t, x) = \begin{cases} H(2t, x) & 0 \leq t \leq \frac{1}{2} \\ K(2t - 1, x) & \frac{1}{2} \leq t \leq 1 \end{cases}$$

This is again a continuous function.



Proposition III.3.8. *The relation "f is homotopic to g" is an equivalence relation on the set of continuous maps $X \to Y$.*

Proof. We check the three axioms for an equivalence relation:

- f is homotopic to f via the homotopy obtained by composing f with the canonical projection $\pi_X : I \times X \to X$.
- Symmetry is a consequence of the Construction **III.3.6**.
- Transitivity is a consequence of the Construction **III.3.7**.

□

Homotopies behave well under composition of continuous maps:

Proposition III.3.9. *Let $f, g : X \to Y$, $h : Z \to X$ and $k : Y \to W$ be continuous maps. Assume f and g are homotopic. Then:*

- *$k \circ f$ and $k \circ g$ are homotopic*
- *$f \circ h$ and $g \circ h$ are homotopic.*

Proof. Let $H : I \times X \to Y$ be a homotopy from f to g:

- We obtain a homotopy from $k \circ f$ to $k \circ g$ by composing $k \circ H : I \times X \to Y \to W$.
- We obtain a homotopy from $f \circ h$ to $g \circ f$ by composing $H : I \times X \to Y$ with the continuous map $I \times Z \to I \times X$ given by $\mathsf{id}_I \times h$.

□

Proposition III.3.10. *Let $f, g : Z \to X \times Y$ be continuous maps. Then f and g are homotopic if and only if both projections $\pi_X \circ f, \pi_X \circ g$ (respectively $\pi_Y \circ f$ and $\pi_Y \circ g$) are homotopic.*

Proof. Suppose f and g are homotopic via $H : I \times Z \to X \times Y$, then the two projections $\pi_X \circ H$ and $\pi_Y \circ H$ provides the required homotopies between the projections. Conversely, assume that we have $H_1 : I \times Z \to X$ a homotopy between $\pi_X \circ f$ and $\pi_X \circ g$ and $H_2 : I \times Z \to Y$ a homotopy between $\pi_Y \circ f$ and $\pi_Y \circ g$. By the universal property of product space, the map $(H_1, H_2) : I \times Z \to X \times Y$ is continuous and provides the required homotopy between f and g □

Definition III.3.11. Let X be a topological space and A a subspace. Let $f, g : X \to Y$ be continuous maps with $f(a) = g(a) \forall a \in A$. We say that f and g are homotopic *relative* to A if there exists a homotopy $H : I \times X \to Y$ from f to g, that fixes A for all $t \in I$, ie

$$H(t, a) = H(0, a)$$

for all $t \in I$ and $a \in A$.

Exercise III.3.12. Let X and A be as in Definition **III.3.11**. Show that the construction of reverse homotopies Construction **III.3.6** and vertical compositions Construction **III.3.7** make, as in the Proposition **III.3.8**, the notion of homotopy relative to A an equivalence relation.

III.4. Homotopy Types and Homotopy equivalences

Definition III.4.1. Let $f : X \to Y$ be a continuous map. We say that f is an *homotopy equivalence* if there exists $g : Y \to X$ and homotopies H_1 and H_2 rendering, respectively $g \circ f$ homotopic to id_X and $f \circ g$ homotopic to id_Y. When two spaces are homotopy equivalence we say they have the same *homotopy type*.

Remark III.4.2. Every homeomorphism is a homotopy equivalence. Indeed, if $f : X \to Y$ is a homeomorphism with inverse $g : Y \to X$, then we can pick the homotopies $H : I \times X \to X$ and $H' : I \times Y \to Y$ given by the projection maps.

Remark III.4.3. The composition of homotopy equivalences is a homotopy equivalence. This is a direct consequence of Proposition **III.3.9**.

A particular type of homotopy equivalence is when we can collapse a space onto a smaller subspace:

Definition III.4.4. Let X be a topological space and A a subspace with inclusion $i : A \hookrightarrow X$. A retraction of X onto A is a continuous map $r : X \to A$ such that $r \circ i = \mathsf{id}_A$. A deformation retraction of X onto A is a homotopy $H : I \times X \to X$ such that $H(0, -) = \mathsf{id}_X : X \to X$, $H(1, -)(X) = A$, and for every t, $H(t, -) : X \to X$ satisfies $H(t, -)_{|A} = \mathsf{id}_A$. In other words, H is a homotopy relativity to A from the identity of X to a retraction.

Remark III.4.5. If H is a deformation retraction of X onto A then the inclusion $A \hookrightarrow X$ is an homotopy equivalence. Indeed, let $f_1 := H(1, 0) : X \to X$. By definition of deformation retract, f_1 factors as a map $f_1 : X \to A$ and we have $f_1 \circ i = \mathsf{id}_A$. At the same time, the inclusion $i \circ f_1 : X \to A \to X$ give by f_1 is homotopic to the identity of X via H

Exercise III.4.6. Show that $\mathbb{C} \setminus \{0, 1\}$ deformation retracts onto the space X given by the union of the circles centered at 0 and 1 of radius $\frac{1}{2}$. TD, Exo 5.1, Feuille 2

Exercise III.4.7. Let X be a topological space and A and B two subspaces. Show that we can have A and B homotopy equivalent without $X \setminus A$ and $X \setminus B$ being homotopy equivalent. TD, Exo 5.2, Feuille 2

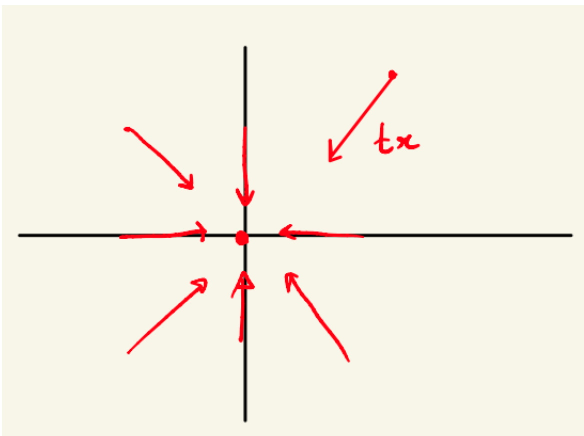
Exercise III.4.8. Show that a punctured torus is homotopy equivalent to a wedge of two circles. TD, Exo 5.4, Feuille 2

Exercise III.4.9. Let E be a linear subspace of $\mathbb{R}^n$ of dimension $k < n$. Show that $\mathbb{R}^n \setminus E$ is homotopy equivalent to S^{n-k-1} TD, Exo 3.1, Feuille 2

Definition III.4.10. One says that a topological space X is *contractible* if it is homotopy equivalent to a point, ie, there exists a homotopy equivalence $f : \{x_0\} \to X$ and $g : X \to \{x_0\}$. In this case the composition $g \circ f$ is automatically equal to the identity $\mathsf{id}_{\{x_0\}}$.

Exercise III.4.11. Show that a space X is contractible if and only if its identity map id_X is null-homotopic

Example III.4.12. The Example **III.3.5** shows that $\mathbb{R}^n$ is contractible for every $n \geq 0$. Indeed, the homotopy $H : I \times \mathbb{R}^n \to \mathbb{R}^n$ given by $H(t, x) = t.x$ gives a homotopy between the identity at $t = 1$ and the constant map $0 \in \mathbb{R}^n$, at $t = 0$. In fact, since the point $0 \in \mathbb{R}^n$ is fixed by the homotopy, this is deformation retract of the inclusion $\{0\} \subseteq \mathbb{R}^n$.



Example III.4.13. The homotopy H of the Example **III.3.5** restricts to the open interval $]-1, 1[$ and to the closed $[-1, 1]$ and shows that both are contractible.

In particular, since they are both homotopy equivalent to a point, they are homotopy equivalent. As a consequence, for any $a < b$, the intervals $]a, b[$ and $[a, b]$ are homotopy equivalent (use the homeomorphisms of the Example **II.2.16**).

Example III.4.14. The circle S^1 is not contractible (proving this is somehow the goal of the next chapter). The inclusion $\{1\} \hookrightarrow \mathsf{S}^1$ admits a canonical retraction sending every point of the circle to 1. However, since the circle is not contractible, this cannot be a deformation retraction.

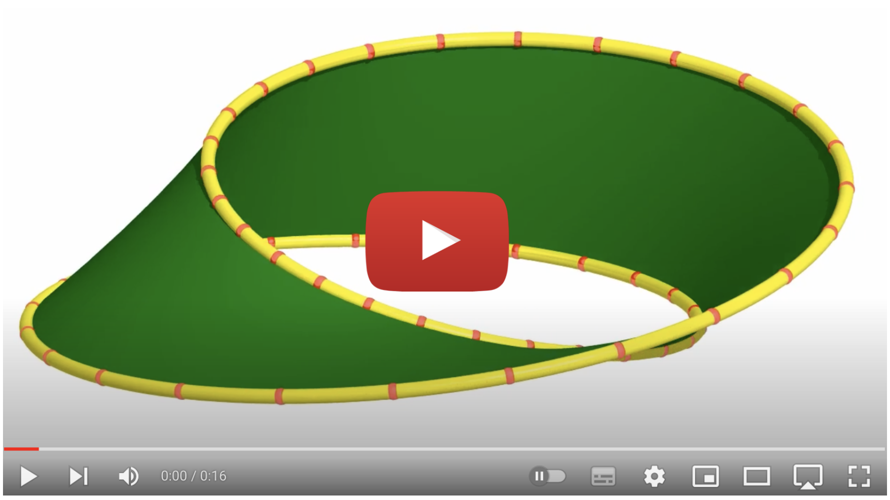
Example III.4.15. The sphere S^2 is not contractible. Proving this is beyond the scope of this course. See the Remark **V.3.23**.

Example III.4.16. The homotopy H of the Example **III.3.5** restricts to the n-dimensional disk $H : I \times D^n \to D^n$ showing that D^n is contractible.

TD, Exo 1.2, Feuille 2 **Exercise III.4.17.** Let Y be a contractible space. Show that any two continuous maps $X \to Y$ are homotopic.

TD, Exo 1.3, Feuille 2 **Exercise III.4.18.** Show that a space X is contractible if and only if for any topological space Y, any continuous map $X \to Y$ is null-homotopic. Show that X is contractible if and only if for every topological space Z, any continuous map $Z \to X$ is null-homotopic.

TD, Exo 2.1, Feuille 2 **Exercise III.4.19.** The Mobius band has a deformation retraction to its equator circle:



Write this deformation retraction explicitly.

TD, Exo 2.2, Feuille 2 **Exercise III.4.20.** Show that if X_1 and X_2 are homotopy equivalent, and Y_1 and Y_2 are homotopy equivalent, then $X_1 \times Y_1$ and $X_2 \times Y_2$ are homotopy equivalent.

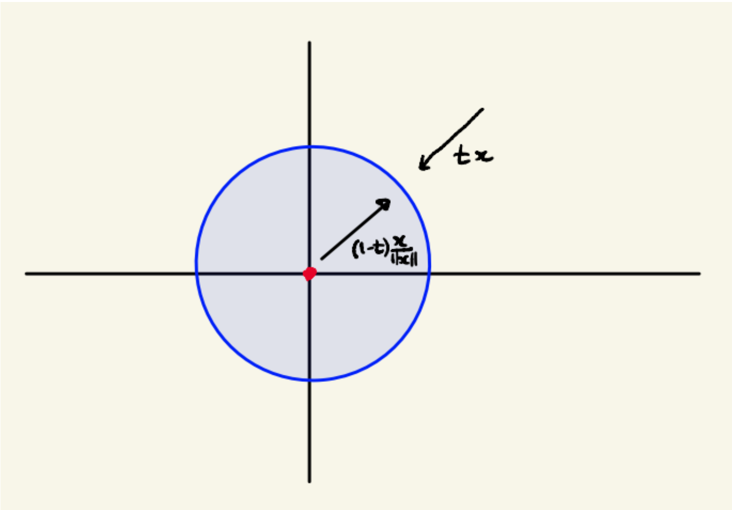
Exercise III.4.21. If C is contractible then $X \times C$ has the homotopy type of X. TD, Exo 2.3, Feuille 2

Exercise III.4.22. Let $f : X \to Y$ and assume f is homotopic to a map $g : X \to Y$ which is a homotopy equivalence. Show that f is a homotopy equivalence. TD, Exo 2.4, Feuille 2

Example III.4.23. The inclusion $S^{n-1} \subseteq \mathbb{R}^n \setminus \{0\}$ admits a deformation retraction. Indeed, the homotopy

$$H(t, x) := tx + (1-t)\frac{x}{||x||}$$

is continuous away from $x = 0$, fixes the sphere for all t (since if $x \in S^{n-1}$, $||x|| = 1$ we have $H(t, x) = x$) and for $t = 1$, $H(1, x) = \frac{x}{||x||} \in S^{n-1}$.



Exercise III.4.24. Let X be a topological space and $x \in X$. Let P_xX denote the set of all paths in X starting at x. Endow P_xX with the subspace topology from the space $\mathrm{Map}([0, 1], X)$ of the *Proposition* **II.5.64**. TD, Exo6, Feuille 2

(i) Show that the map $P_xX \to X$ sending a path γ to its end point $\gamma(1)$, is continuous.
(ii) Show that P_xX is contractible.

Exercise III.4.25. Show that $\mathbb{R}P^2$ minus a point, deformation retracts into the image of ∂D^2 in the quotient. Check carefully the continuity of the retract. TD, Exo 8, Feuille 3

Exercise III.4.26. Let X be a topological space. Show that the cone $C(X)$ is contractible.

Exercise III.4.27. Show that a retraction of a contractible space is contraction.

CHAPTER IV

Category Theory

IV.1. Categories and Functors

Goal IV.1.1. In this chapter we will axiomatize some of the features observed in the previous for topological spaces. For our purposes, category theory is a convenient language and an organizational principle. The abstraction of this chapter will become more meaningful in the next chapter when we introduce the fundamental group and prove the Van Kampen theorem.

We avoid set-theoretical issues:

Definition IV.1.2. A category C consists of

- a collection of objects, denoted $Obj(\mathsf{C})$;
- for every pair of objects, X, Y, a set of morphisms $\mathrm{Hom}_{\mathsf{C}}(X, Y)$. We will write elements $f \in \mathrm{Hom}_{\mathsf{C}}(X, Y)$ as arrows $f : X \to Y$;
- For every object X, a distinguished morphism $Id_X \in \mathrm{Hom}_{\mathsf{C}}(X, X)$ called the identity of X
- For every triple of objects, X, Y, Z, a *composition law*

$$\circ : \mathrm{Hom}_{\mathsf{C}}(X, Y) \times \mathrm{Hom}_{\mathsf{C}}(Y, Z) \to \mathrm{Hom}_{\mathsf{C}}(X, Z)$$

satisfying the following properties:

- The composition law is associative;
- The elements Id_X are unit elements for the composition.

Example IV.1.3. The category where objects are sets, morphisms are set-theoretic maps, identities $Id_X : X \to X$ are given by the identity maps of sets. and the composition law is the standard composition of maps. We will denote it by SETS.

Example IV.1.4. The category where objects are topological spaces, morphisms are continuous maps, the identity $Id_X : X \to X$ is given by the identity map and the composition law is the standard composition of maps (since the composition of continous maps is continuous). We will denote it by TOP.

Example IV.1.5. For any category C with an object $X \in \mathsf{C}$, the collection of morphisms $X \to Y$ forms a category, with morphisms $u : (f_1 : X \to Y_1) \to (f_2 : X \to Y_2)$ given by morphisms $u : Y_1 \to Y_2$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{f_1} & Y_1 \\ & \searrow^{f_2} & \downarrow u \\ & & Y_2 \end{array}$$

commutes, forms a category. We denote it by $\mathsf{C}_{X/.}$ and call it the category of objects under X.

Example IV.1.6. The category of pointed spaces TOP_* where objects are pairs (X, x) with x is a point in X, and morphisms $f : (X, x) \to (Y, y)$ are continuous maps $f : X \to Y$ satisfying the condition $f(x) = y$. This coincides with the category $\mathsf{TOP}_{*/}$ of the Example **IV.1.5**.

Example IV.1.7. The category with one object 0 and a single morphisms given by the identity. We denote it by $[0]$.

Example IV.1.8. The category with two objects 0 and 1, the identity of 0, the identity of 1 and and one single arrow from 0 to 1 is called the *interval category*. We denote it by $[1]$.

Exercice IV.1.9. Show that collection of all integers $\mathbb{Z}$ forms a category with objects given by the integers and $\mathrm{Hom}(n, m) = \{*\}$ if $a \leq b$ and $\emptyset$ otherwise.

Exercice IV.1.10. Let k be a field. Show that the collection of vector spaces over k together with linear mappings, forms a category, denoted VECT_k.

Exercice IV.1.11. Show that the collection of all associative monoids together with monoid homomorphisms forms a category, denoted $\mathsf{MONOIDS}$.

Exercice IV.1.12. Show that the collection of all groups together with group homomorphisms forms a category, denoted GROUPS.

Exercice IV.1.13. Show that the collection of all commutative rings together with ring homomorphisms forms a category, denoted CRINGS.

Example IV.1.14. The homotopy category, $\mathsf{HO}(\mathsf{TOP})$, whose objects are topological spaces, and whose morphisms are homotopy classes of continuous maps. Composition is well-defined as a consequence of the Proposition **III.3.9**.

The notion of *functor* allows us to navigate between different categories:

Definition IV.1.15. Let C and D be categories. A functor $F : \mathsf{C} \to \mathsf{D}$ consists of the following data:

- A map of sets $F : Obj(\mathsf{C}) \to Obj(\mathsf{D})$, denoted $X \mapsto F(X)$;
- For every pair of objects X, Y in C, a map of sets $F : \mathrm{Hom}_{\mathsf{C}}(X, Y) \to \mathrm{Hom}_{\mathsf{D}}(F(X), F(Y))$;

such that:

(i) F preserves identities, ie, for every object $X \in \mathsf{C}$, we have $F(Id_X) = Id_{F(X)}$;

(ii) F preserves compositions, ie, $F(g \circ f) = F(g) \circ F(f)$for all morphisms $f : X \to Y$ and $g : Y \to Z$ in C

Example IV.1.16. Let C be a category. Then the identity assignment sending an object X to itself and a morphism $f : X \to Y$ to itself, defines a functor $\mathsf{C} \to \mathsf{C}$. We denote it by Id_{C} and call it the identity functor.

Example IV.1.17. Let C be a category. The datum of an object X in C is equivalent to the data of a functor $[0] \to \mathsf{C}$. The datum of a morphism $f : X \to Y$ in C is equivalent to the specification of a functor $d : [1] \to \mathsf{C}$, where $d(0) = X$, $d(1) = Y$, $d(0 \to 1) = f$.

Remark IV.1.18. Functors can be composed. If $F : \mathsf{C} \to \mathsf{D}$ and $G : \mathsf{D} \to \mathsf{E}$ are functors, then the compositions of the maps $G \circ F$ on objects and on morphisms defines a new functor $\mathsf{C} \to \mathsf{E}$.

Example IV.1.19. Let C be a category with X an object. The assignment $\mathsf{C}_{X/.} \to \mathsf{C}$ sending an object of $\mathsf{C}_{X/.}$ given by $X \to Y$, to the object Y of C, defines a functor $t : \mathsf{C}_{X/.} \to \mathsf{C}$ called the target functor.

Exercice IV.1.20. Show that the collection of all categories together with functors as morphisms forms itself a category, denoted CATS.

Example IV.1.21. The assignment sending a topological space (X, τ) to its underlying set X by forgetting the topology, defines a functor

$$\mathsf{TOP} \to \mathsf{SETS}$$

called the *forgetful functor*.

Example IV.1.22. The assignment sending a group to its underlying monoid forgetting the existence of inverses, defines a functor

$$\mathsf{GROUPS} \to \mathsf{MONOIDS}$$

also called the *forgetful functor*.

Example IV.1.23. The assignment sending a pointed space (X, x) to its underlying space X forgetting the marked point defines a functor

$$\mathsf{TOP}_* \to \mathsf{TOP}$$

Example IV.1.24. The assignment sending a commutative ring $(R, ., +)$ to its underlying abelian group $(R, +)$ by forgetting the ring structure, defines a functor

$$\mathsf{CRINGS} \to \mathsf{ABGROUPS}$$

also called the *forgetful functor*.

Example IV.1.25. Let M be an associtative monoid. Then we can construct a category $\mathsf{B}M$ as follows: $\mathsf{B}M$ contains a single object, denote $\bullet_M$ and endomorphisms given by

$$\mathrm{Hom}_{\mathsf{B}M}(\bullet_M, \bullet_M) := M$$

Compositions are defined by the associative monoid law in M. The unit of M plays the role of the identity morphism of $\bullet_M$.

The construction sending $M \mapsto \mathsf{B}M$ is functorial, ie, if $M \to N$ is a monoid homomorphism, then we have a well-induced functor $\mathsf{B}M \to \mathsf{B}N$ of categories. Moreover, this assignment sends compositions of monoid homomorphisms to compositions of functors. Overall, this means that the construction $M \mapsto \mathsf{B}M$ is part of a functor

$$\mathsf{B} : \mathsf{MONOIDS} \to \mathsf{CATS}$$

Example IV.1.26. Let TOP^{lpc} denote the category of locally path-connected spaces. The assignment sending a locally path connected space (X, τ) to its set of connected components $\pi_0(X)$ defines a functor

$$\pi_0 : \mathsf{TOP}^{lpc} \to \mathsf{SETS}$$

This is the Construction **III.2.9**.

IV.2. Isomorphisms

Definition IV.2.1. Let C be a category. A morphism $f : X \to Y$ in C is said to:

(i) have a left inverse if there exists $r : Y \to X$ such that $r \circ f = \mathsf{id}_X$. In this case we call r a retraction of f.

(ii) have a right inverse if there exists $s : Y \to X$ such that $f \circ s = \mathsf{id}_Y$. In this case we call s a section of f.

(iii) be an isomorphism if there exists another morphism $g : Y \to X$ in C such that $f \circ g = id_Y$ and $g \circ f = id_X$, ie, f has a right and left inverse (which necessarily coincide).

Example IV.2.2. Isomorphisms of topological spaces are precisely homeomorphisms as defined in Definition **II.2.15**.

Example IV.2.3. Isomorphisms in the category $\mathsf{HO}(\mathsf{TOP})$ are homotopy equivalences.

Example IV.2.4. Let us denote by J the category with two objects 0 and 1, their identities and a unique map $0 \to 1$, a unique map $1 \to 0$ such that the compositions $0 \to 1 \to 0$ and $1 \to 0 \to 1$ are the identity morphisms. For any category C, the data of a functor $J \to \mathsf{C}$ is equivalent to the data of a isomorphism $f : X \to Y$ in C.

Exercise IV.2.5. Let C be a category. Show that following are equivalent conditions for a morphism $f : X \to Y$ in C:

(i) $f : X \to Y$ is an isomorphism;

(ii) for every object $Zin\mathsf{C}$, the composition maps

$$\mathrm{Hom}_{\mathsf{C}}(Y, Z) \to \mathrm{Hom}_{\mathsf{C}}(X, Z)$$

sending $u : Y \to Z$ to $u \circ f$, is a bijection.

(iii) for every object $Zin\mathsf{C}$, the composition maps

$$\mathrm{Hom}_{\mathsf{C}}(Z, X) \to \mathrm{Hom}_{\mathsf{C}}(Z, Y)$$

sending $u : Z \to X$ to $f \circ u$, is a bijection

Definition IV.2.6. Let C be a category and X an object in C. An isomorphism $f : X \to X$ is called an automorphism. We denote by $\mathrm{Aut}_{\mathsf{C}}(X)$ the subset of $\mathrm{Hom}_{\mathsf{C}}(X, X)$ spanned by automorphisms.

Remark IV.2.7. Automorphisms form a group under composition.

Remark IV.2.8. Let C be a category and let x and y be two objects in C and $f : x \to y$ an isomophism. Then f induces an isomorphism of groups

$$\mathrm{Aut}_{\mathsf{C}}(x, x) \to \mathrm{Aut}_{\mathsf{C}}(y, y)$$

sending an automorphism g of x to the conjugation $(f \circ g \circ f^{-1})$. Its inverse sends an automorphism u of y to the conjugation $(f^{-1} \circ u \circ f)$.

This is a map of groups with respect to the composition in C: indeed, if g_1 and g_2 are automorphisms of x, then

$$f \circ (g_1 \circ g_2) \circ f^{-1} = (f \circ g_1 \circ f^{-1}) \circ (f \circ g_2 \circ f^{-1})$$

Moreover, given the identity id_x of x, we have

$$f \circ (id_x) \circ f^{-1} = f \circ f^{-1} = id_y$$

Exercise IV.2.9. We say that a functor $f : \mathsf{C} \to \mathsf{D}$ is conservative if whenever a morphism $u : X \to Y$ in C is such that $f(u) : F(X) \to F(Y)$ is an isomorphism in D, then u is an isomorphism in C.

(i) Show that the functor from commutative rings to abelian groups that forgets the ring structure is conservative.

(ii) Is the functor from topological spaces to sets that forgets the topology conservative? Hint: check the Example **II.2.20**.

IV.3. Natural transformations

Motivation IV.3.1. Let $\mathsf{VECT}_{\mathbb{C}}^{fin}$ denote the category of finite dimensional complex vector spaces. The *dual* of a vector space V is the $\mathbb{C}$-vector space $V^\vee$ of linear forms $\lambda : V \to \mathbb{C}$. A linear map $V \to W$ induces a linear map in the opposite direction $W^\vee \to V^\vee$ and this is compatible with compositions. When V is finite dimensional, the dual of the dual of V, ie $(V^\vee)^\vee$ is *isomorphic* , but not equal, to V: the map

$$\eta_V : V \to (V^\vee)^\vee$$

sending $v \in V$ to the linear form $\eta_V(v) : V^\vee \to \mathbb{C}$ given by $\lambda \mapsto \eta_V(v)(\lambda) := \lambda(v)$, defines an isomorphism of $\mathbb{C}$-vector spaces. Indeed, we check that

(i) η_V is indeed $\mathbb{C}$-linear: $\eta_V(v_1 + v_2)(\lambda) = \lambda(v_1 + v_2) = \lambda(v_1) + \lambda(v_2)$ for all $v_1, v_2 \in V$ and $\eta_V(a.v)(\lambda) = \lambda(a.v) = a\lambda(v)$ for all $a \in \mathbb{C}$ and $v \in V$;

(ii) If V is finite dimensional, then $dim_{\mathbb{C}} V = dim_{\mathbb{C}} V^\vee$. This a well-known result in linear algebra. if $\{v_1, \cdots, v_n\}$ forms a basis of V, there exists a dual basis $\lambda_1, \cdots, \lambda_n : V \to \mathbb{C}$ with the property $\lambda_i(v_j) = \delta_{i,j}$ and extended by linearity. In particular,

$$dim_{\mathbb{C}} V = dim_{\mathbb{C}} V^\vee = dim_{\mathbb{C}} (V^\vee)^\vee$$

(iii) η_V is injective: Let $v \in V$ such that $\eta_V(v) = 0$. Assume that $v \neq 0$. Then we can complete $\{v\}$ to a basis $\{v, v_2, \cdots, v_n\}$. By the same argument as in the previous item, this admits a dual basis $\lambda_1, \cdots, \lambda_n$, with $\lambda_1(v) = \eta_V(v)(\lambda) = 1$. This is a contradiction.

Therefore η_V is an isomorphism.

Definition IV.3.2. Let C be a category. We define its *opposite category* C^{op} to be the category with the same objects but with the direction and composition of morphisms reversed:

$$\mathrm{Hom}_{\mathsf{C}^{\mathsf{op}}}(X, Y) := \mathrm{Hom}_{\mathsf{C}}(Y, X)$$

Compositions are defined using the law in C.

Remark IV.3.3. Notice that $(\mathsf{C}^{\mathsf{op}})^{\mathsf{op}} = \mathsf{C}$.

Example IV.3.4. The construction sending a finite dimensional $\mathbb{C}$-vector space V to its dual $V^\vee$ defines a functor

$$((-)^\vee)^\vee : \mathsf{VECT}_{\mathbb{C}}^{fin} \to \left(\mathsf{VECT}_{\mathbb{C}}^{fin}\right)^{\mathsf{op}}$$

In particular, taking duals twice, we obtain a composite functor

$$\mathsf{VECT}_{\mathbb{C}}^{fin} \to \left(\mathsf{VECT}_{\mathbb{C}}^{fin}\right)^{\mathsf{op}} \to \left(\left(\mathsf{VECT}_{\mathbb{C}}^{fin}\right)^{\mathsf{op}}\right)^{\mathsf{op}} = \mathsf{VECT}_{\mathbb{C}}^{fin}$$

sending $V \mapsto (V^\vee)^\vee$. This is not the identity functor of $\mathsf{VECT}_{\mathbb{C}}^{fin}$. The following definition provides a way of saying that this composite functor is *isomorpic* to the identity functor.

Definition IV.3.5. Let $F, G : \mathsf{C} \to \mathsf{D}$ be functors. A *natural transformation* $\eta : F \to G$ consists of a collection of morphisms $\eta_X : F(X) \to G(X)$in D, one for every object $X \in \mathsf{C}$, such that for every morphism $f : X \to Y$ in C, the diagram

$$\begin{array}{ccc} F(X) & \xrightarrow{\eta_X} & G(X) \\ \downarrow{\scriptstyle F(f)} & & \downarrow{\scriptstyle G(f)} \\ F(Y) & \xrightarrow{\eta_Y} & G(Y) \end{array}$$

commutes.

We say that η is a *natural isomorphism* if the morphisms η_X are isomorphisms for every X.

Example IV.3.6. Let us come back to the Example **IV.3.4**. The maps η_V of the Motivation **IV.3.1** define a natural isomorphism of functors

$$Id \to ((-)^\vee)^\vee :$$

Let us check this: given a map of vector spaces $f : V \to W$ we need to check the commutativity of the diagram

$$\begin{array}{ccc} V & \longrightarrow & (V^\vee)^\vee \\ \downarrow & & \downarrow \\ W & \longrightarrow & (W^\vee)^\vee \end{array}$$

One composition gives

$$\begin{array}{ccc} v & \longrightarrow & [\eta_V(v) : \lambda \in V^\vee \mapsto \lambda(v)] \\ & & \downarrow \\ & & [\ell \in W^\vee \mapsto \ell \circ f \in V^\vee \mapsto (\ell \circ f)(v) = \ell(f(v))] \end{array}$$

The other composition gives

$$\begin{array}{c} v \\ \downarrow \\ f(v) \longrightarrow [\eta_W(v) : \ell \in W^\vee \mapsto \ell(f(v))] \end{array}$$

so they agree.

Remark IV.3.7. Let C and D be categories and consider functors $F, G, H : \mathsf{C} \to \mathsf{D}$ together with natural transformations $\eta : F \to G$ and $\psi : G \to H$. Then we can form the composed natural transformation $\psi \circ \eta : F \to H$ given for each $X \in \mathsf{C}$ by the map $\psi_X \circ \eta_X$.

Definition IV.3.8. Let C and D be categories. The collection of functors $\mathsf{C} \to \mathsf{D}$ forms a category with morphisms given by natural transformations of functors. Compositions are defined as in the Remark **IV.3.7**. This category, denoted $\mathsf{Fun}(\mathsf{C}, \mathsf{D})$ is called the *functor category.*

IV.4. Equivalences of Categories

Definition IV.4.1. Let C and D be categories. An *equivalence* of categories consists of the following data:

- a functor $F : \mathsf{C} \to \mathsf{D}$
- a functor $G : \mathsf{D} \to C$
- Natural isomorphisms $\eta : Id_{\mathsf{C}} \simeq G \circ F$ and $\psi : Id_{\mathsf{D}} \simeq F \circ G$

Terminology IV.4.2. We say that $F : \mathsf{C} \to \mathsf{D}$ is an equivalence of categories if there exists $G : \mathsf{D} \to \mathsf{C}$ and natural isomorphisms η and ψ as in the Definition **IV.4.1**.

Definition IV.4.3. We say that a functor $F : \mathsf{C} \to \mathsf{D}$ is *fully faithful* if for every pair of objects X and Y in C, the map

$$\mathrm{Hom}_{\mathsf{C}}(X, Y) \to \mathrm{Hom}_{\mathsf{D}}(F(X), F(Y))$$

is a bijection.

Definition IV.4.4. We say that a functor $F : \mathsf{C} \to \mathsf{D}$ is essentially surjective if for every object $Y \in \mathsf{D}$ there exists an object X in C and an isomorphism $F(X) \simeq Y$ in D.

Theorem IV.4.5. (*) *A functor $F : \mathsf{C} \to \mathsf{D}$ is an equivalence of categories if and only if F is fully faithful and essentially surjective.*

We will not provide proof for this theorem here.

(*) Uses the axiom of choice.

Example IV.4.6. Consider the category J of the Example **IV.2.4**. The functor $i_0 : [0] \to J$ selecting the object $0 \in J$ defines an equivalence of categories. Indeed,

$$\{Id_0\} = \mathrm{Hom}_{[0]}(0,0) \to \mathrm{Hom}_J(i_0(0), i_0(0)) = \{Id_0\}$$

is bijective meaning that i_0 is fully faithful. Moreover, i_0 is essentially surjective:

- $0 = i_0(0)$;
- $1 \simeq i_0(0)$ via the unique isomorphism in J.

A similar argument can be used to show that the functor $i_1 : [0] \to J$ selecting the object 1 in J and its identity, is also an equivalence of categories.

Exercise IV.4.7. If $F : \mathsf{C} \to \mathsf{D}$ and $G : \mathsf{D} \to \mathsf{E}$ are equivalences of categories, then the composition $G \circ F : \mathsf{C} \to \mathsf{E}$ is an equivalence of categories.

Exercise IV.4.8. Let $F : \mathsf{C} \to \mathsf{D}$, $G : \mathsf{D} \to \mathsf{C}$ form an equivalence of categories. Let E be a category. Show that composition with F and G induce an equivalence of functor categories

$$\mathsf{Fun}(\mathsf{C}, \mathsf{E}) \to \mathsf{Fun}(\mathsf{D}, \mathsf{E})$$

Definition IV.4.9. Let C be a category. The isomorphism relation defines an equivalence relation on the set of objects of C. We denote by $\pi_0(\mathsf{C})$ the quotient set.

Remark IV.4.10. Let $F : \mathsf{C} \to \mathsf{D}$ be an equivalence of categories. Then F induces a bijection $\pi_0(\mathsf{C}) \simeq \pi_0(\mathsf{D})$.

Remark IV.4.11. Equivalences of categories are something else than isomorphisms in CATS. By definition $F : \mathsf{C} \to \mathsf{D}$ is an isomorphism of categories if there exists $G : \mathsf{D} \to \mathsf{C}$ such that $G \circ F$ and $F \circ G$ *are equal* to the identity functors. This is equivalent to ask for F to induce a bijection both on objects and morphisms.
The notion of equivalence of categories is more flexible in the sense that the composites $G \circ F$ and$F \circ G$ do not have to be the identities on the nose, but rather isomorphic. The example Example **IV.4.6** illustrates this. Indeed, notice that the two categories $[0]$ and J are equivalent but not isomorphic since $[0]$ has a smaller set of objects.

We will now show that natural isomorphisms are to categories what homotopies are to topological spaces, with the role of the closed interval $[0,1]$ played by the category J. In particular, equivalences of categories are similar to homotopy equivalences of spaces. To explore this link, we start with the following observation:

Construction IV.4.12. Let C and D be categories. Then we have a natural evaluation functor

$$\mathsf{ev} : \mathsf{C} \times \mathsf{Fun}(\mathsf{C}, \mathsf{D}) \to \mathsf{D}$$

defined on objects by sending an object $(X, F : \mathsf{C} \to \mathsf{D})$ in the product to the object $F(X)$ in D. On morphisms, if $f : X \to Y$ is a morphism in C and $\eta : F \to G$

is a natural transformation, $\mathsf{ev}(f, \eta)$ is defined to be the composite $G(f) \circ \eta_X = \eta_Y \circ F(f) : F(X) \to G(Y)$. We leave it as an exercise to verify that this defines a functor.

Proposition IV.4.13. *Let* C, D, *and* E *be categories. Then the assignment*

$$\mathrm{Hom}_{\mathsf{CATS}}(\mathsf{E}, \mathsf{Fun}(\mathsf{C}, \mathsf{D})) \to \mathrm{Hom}_{\mathsf{CATS}}(\mathsf{E} \times \mathsf{C}, \mathsf{D})$$

defined by sending

$$\mathsf{E} \xrightarrow{\alpha} \mathsf{Fun}(\mathsf{C}, \mathsf{D}) \qquad \mapsto \qquad \mathsf{E} \times \mathsf{C} \xrightarrow{\alpha \times Id_{\mathsf{C}}} \mathsf{Fun}(\mathsf{C}, \mathsf{D}) \times \mathsf{C} \xrightarrow{\mathsf{ev}} \mathsf{D}$$

is a bijection.

Proof. Left as an exercise. □

Corollary IV.4.14. *Let* $\mathsf{E} = [1]$ *in the Proposition* ***IV.4.13****. Then we get a bijection*

$$\mathrm{Hom}_{\mathsf{CATS}}([1], \mathsf{Fun}(\mathsf{C}, \mathsf{D})) \to \mathrm{Hom}_{\mathsf{CATS}}([1] \times \mathsf{C}, \mathsf{D})$$

Let $F, G : \mathsf{C} \to \mathsf{D}$ *be functors. This bijection establishes a 1-to-1 correspondence between the following data:*

- *A natural transformation* $\eta : F \to G$*;*
- *A functor* $H : [1] \times \mathsf{C} \to \mathsf{D}$ *such that the composition*

  $$[0] \times \mathsf{C} \xrightarrow{i_0 \times Id_{\mathsf{C}}} [1] \times \mathsf{C} \xrightarrow{H} \mathsf{D}$$

  is equal to F *and the composition*

  $$[0] \times \mathsf{C} \xrightarrow{i_1 \times Id_{\mathsf{C}}} [1] \times \mathsf{C} \xrightarrow{H} \mathsf{D}$$

  is equal to G.

Using a similar argument, we show that

Corollary IV.4.15. *Let* $\mathsf{E} = J$ *in the Proposition* ***IV.4.13****. Then we get a bijection*

$$\mathrm{Hom}_{\mathsf{CATS}}(J, \mathsf{Fun}(\mathsf{C}, \mathsf{D})) \to \mathrm{Hom}_{\mathsf{CATS}}(J \times \mathsf{C}, \mathsf{D})$$

Let $F, G : \mathsf{C} \to \mathsf{D}$ *be functors. This bijection establishes a 1-to-1 correspondence between the following data:*

- *A natural isomorphism* $\eta : F \to G$*;*
- *A functor* $H : J \times \mathsf{C} \to \mathsf{D}$ *such that the composition*

  $$[0] \times \mathsf{C} \xrightarrow{i_0 \times Id_{\mathsf{C}}} J \times \mathsf{C} \xrightarrow{H} \mathsf{D}$$

  is equal to F *and the composition*

  $$[0] \times \mathsf{C} \xrightarrow{i_1 \times Id_{\mathsf{C}}} J \times \mathsf{C} \xrightarrow{H} \mathsf{D}$$

  is equal to G.

In particular, we see that the category J plays the role of the closed interval $[0, 1]$ in the definition of homotopy for topological spaces.

IV.5. Constructions: Products, Quotients, Pushouts, Internal-Homs

In the previous lecture we discussed some operations on topological spaces, such as products, quotients, gluings, etc. All these make sense in a general category and it is sometimes better to understand them in their full generality. To understand that, we will need to become familiarized with the terminology of *universal properties*. In vague terms, the universal property of a construction is whatever remains if we manage to formulated it using exclusively the language of objects, morphisms and compositions:

Products.

Example IV.5.1. Given two sets X and Y we can form the product set $X \times Y$. By definition, this is the set of pairs (x, y) where $x \in X$ and $y \in Y$. But how can this formulated using only categorical terminology (objects, morphisms, etc)? To start with, the set $X \times Y$ comes equipped with two natural morphisms $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ sending respectively $(x, y) \mapsto x$ and $(x, y) \mapsto y$.

What we want to isolate now is the defining properties of $(X \times Y, \pi_X, \pi_Y)$ in *in the eyes* of all other sets, ie, as an object of SETS

Notice that if Z is another set with maps $f : Z \to$X and $g : Z \to Y$, we can define a map $\Psi_{f,g} : Z \to X \times Y$ via the formula

$$z \mapsto (f(z), g(z))$$

The composition $\pi_X \circ \Psi_{f,g}$ recovers f and the composition $\pi_Y \circ \Psi_{f,g}$ recovers g. Inversely, given any map $\Psi : Z \to X \times Y$ we can define maps f and g by composing with the two projections. These procedures are inverse to each other and establish a bijection

$$\mathrm{Hom}_{\mathsf{SETS}}(Z, X \times Y) \simeq \mathrm{Hom}_{\mathsf{SETS}}(Z, X) \times \mathrm{Hom}_{\mathsf{SETS}}(Z, Y)$$

Remark IV.5.2. It seems that in the previous example we are just walking in circles since the we ended up with a formulation of the product of sets that also uses the product of sets. Indeed, for sets this is redundant, but in general categories, is very useful as the following definition shows.

Definition IV.5.3. Let C be a category and X and Y objects. We say that the product of X and Y *exists in* C if there exists an object $X \times Y$ and morphisms $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ such that for any third object $Z \in \mathsf{C}$, composition with π_X and π_Y induces a bijection of sets

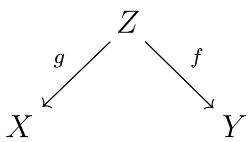
$$\mathrm{Hom}_{\mathsf{C}}(Z, X \times Y) \simeq \mathrm{Hom}_{\mathsf{C}}(Z, X) \times \mathrm{Hom}_{\mathsf{C}}(Z, Y)$$

defined by

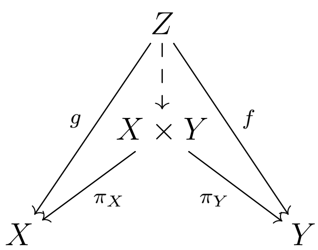
$$f \mapsto (\pi_X \circ f, \pi_Y \circ f)$$

Under this bijection, by construction if $Z = X \times Y$, the identity $Id_{X \times Y}$ is sent to the pair consisting of the two projections (π_X, π_Y).

Remark IV.5.4. Diagrammatically, the bijection in Definition **IV.5.3** reads as follows: a pairs of functions (f, g)



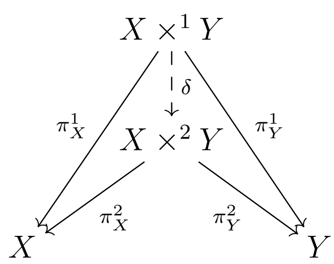
corresponds to a unique dotted map that renders the diagram commutative



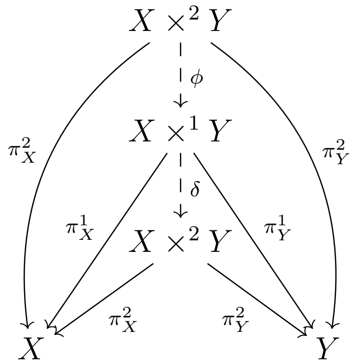
Remark IV.5.5. Notice that to formulate the notion of products in a category C we need to know a priori that products of sets exist.

Proposition IV.5.6. *Let C be a category and X and Y objects. If the product of X and Y exists in C then it is unique up canonical to isomorphism.*

Proof. Indeed, suppose that we have two candidates for the product, say $(X \times^1 Y, \pi^1_X, \pi^1_Y)$ and $(X \times^2 Y, \pi^2_X, \pi^2_Y)$, with the respective projections. Then in particular, since we are assuming that $(X \times^2 Y, \pi^2_X, \pi^2_Y)$ has the property defining the product as explained in the Remark **IV.5.4**, we have a canonical factorization δ



But since we are also assuming that $(X \times^1 Y, \pi^1_X, \pi^1_Y)$ is a product, the symmetry of the argument gives us a factorization ϕ the other way



It is now clear from this diagram that under the bijection in Definition **IV.5.3** the composition $\delta\circ\phi$ is sent to the two projections (π^2_X, π^2_Y) and by the bijection formula defining the product., corresponds to identity map. The symmetry of the argument interchanging the role of product 1 and product 2 implies that $\phi\circ\delta$ must also be the identity.

□

Remark IV.5.7. It is important to remark that the product, if it exists, is not simply an object of C. The data of the two projection maps is equally important as part of the definition, without whom the definition does not make sense.

Proposition IV.5.8. TOP *admits finite products.*

Proof. This is an easy consequence of the Exercise **II.5.5**. Indeed, this result tells us that the bijection given by the universal property of products of sets

$$\mathrm{Hom}(\mathsf{SETS})(Z, X\times Y) \simeq \mathrm{Hom}_{\mathsf{SETS}}(Z, X)\times\times\mathrm{Hom}_{\mathsf{SETS}}(Z, Y)$$

is still a bijection when restricted to the subsets spanned by those maps that are continuous:

$$\begin{array}{ccc} \mathrm{Hom}(\mathsf{TOP})(Z, X\times Y) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathsf{TOP}}(Z, X)\times\times\mathrm{Hom}_{\mathsf{TOP}}(Z, Y) \\ \big\downarrow & & \big\downarrow \\ \mathrm{Hom}(\mathsf{SETS})(Z, X\times Y) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathsf{SETS}}(Z, X)\times\times\mathrm{Hom}_{\mathsf{SETS}}(Z, Y) \end{array}$$

□

Remark IV.5.9. By construction, the forgetful functor $\mathsf{TOP}\to\mathsf{SETS}$ sends products of spaces to products of sets. We say that it *commutes* with products.

Exercise IV.5.10. Show that he category CATS has products. If C and D are categories, the product category $\mathsf{C}\times\mathsf{D}$ has objects given by $Obj(\mathsf{C})\times Obj(\mathsf{D})$ and morphisms given by

$$\mathrm{Hom}_{\mathsf{C}\times\mathsf{D}}((X_1, Y_2), (X_2, Y_2)) := \mathrm{Hom}_{\mathsf{C}}(X_1, X_2)\times\mathrm{Hom}_{\mathsf{D}}(Y_1, Y_2)$$

Compositions are defined coordinatewise.

The projection functors $\mathsf{C} \times \mathsf{D} \to \mathsf{C}$ and $\mathsf{C} \times \mathsf{D} \to \mathsf{D}$ are defined in the obvious way. We leave it as an exercise to complete the proof that this indeed verifies the universal property of products.

Coproducts. We now turn to the notion of *coproducts* which capture the properties of disjoint unions of sets:

Definition IV.5.11. Let C be a category and X,Y objects in C. The *coproduct* of X and Y, if it exists, consists of an object $X \coprod Y$ of C together with two morphisms $i_X : X \to X \coprod Y$ and $i_Y : Y \to X \coprod Y$ satisfying the following property: for every object Z in C the composition with the two maps

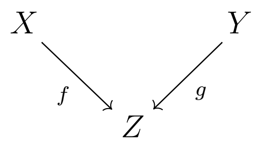
$$(\Psi : X \coprod Y \to Z) \mapsto (\Psi \circ i_X, \Psi \circ i_Y)$$

defines a bijection

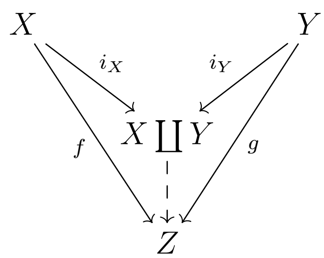
$$\mathrm{Hom}_{\mathsf{C}}(X \coprod Y, Z) \simeq \mathrm{Hom}_{\mathsf{C}}(X, Z) \times \mathrm{Hom}_{\mathsf{C}}(Y, Z)$$

Example IV.5.12. The category SETS admits coproducts given by disjoint unions of sets. See Reminder **II.5.13**.

Remark IV.5.13. Diagrammatically, the bijection in Definition **IV.5.11** reads as follows: a pairs of functions (f, g)



corresponds to a unique dotted map that renders the diagram commutative



Exercise IV.5.14. Show that coproducts are unique up to canonical isomorphism.

Proposition IV.5.15. *The category* TOP *has finite coproducts given by disjoint unions. Moreover, the forgetful functor* $\mathsf{TOP} \to \mathsf{SETS}$ *commutes with coproducts.*

PROOF. This is a consequence of the definition for the disjoint union topology of the Construction **II.5.14**. For the universal property we argue as in Proposition **IV.5.8** using Exercise **II.5.16**.

□

Here's another important example whose relevance will become clear later:

Proposition IV.5.16. *The category* GROUPS *admit coproducts.*

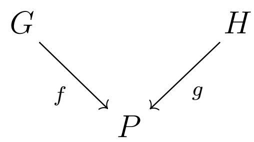
PROOF. Suppose G and H are two groups. To construct their coproduct in GROUPS we need to produce three items:

- a new group $G\coprod H$;
- a group homomorphism $i_G : G \to G\coprod H$
- a group homomorphism $i_H : G \to G\coprod H$

all of these, satisfying the universal property defining coproducts in Definition **IV.5.11**: given any third group P the composition with i_G and I_H

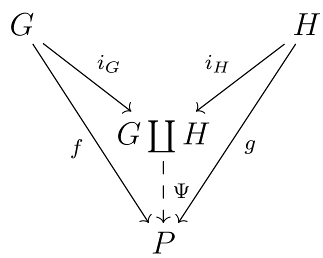
$$\mathrm{Hom}_{\mathsf{GROUPS}}(G\coprod H, P) \to \mathrm{Hom}_{\mathsf{GROUPS}}(G, P) \times \mathrm{Hom}_{\mathsf{GROUPS}}(H, P)$$

must be a bijection. Diagrammatically, as in Remark **IV.5.13**, this means that given group homomorphisms f and g



we must be able to construct a unique dotted group homomorphism rendering the commutativity of

(2)



In order to motivate the construction of $G\coprod H$, let us observe that whatever P is, in the situation above we can multiply in P elements coming from G (via f) with elements coming from H (via g). For instance, if a_1 and a_2 are elements of G and b_1 and b_2 elements of H, in P we can form the product

$$f(a_1).g(b_1).f(a_2).g(b_2)$$

There is a priori no reason why we should be able to simplify expressions such as this one, since there is a priori no relation between f and g. However, not all expressions are of this form. For instance, the product

$$f(a_1).f(a_2).g(b_1).g(b_2)$$

can be simplified as

$$f(a_1.a_2).g(b_1.b_2)$$

Following this observation start by considering the set of all words in the alphabet using the symbols given by the elements of G and H

$$W(G,H) := \{x_1x_2\cdots x_n : x_i \in G \text{ or } x_i \in H\}$$

By definition, a word is finite. We can consider the quotient set $W(G,H)/\sim$ under the equivalence relation defined as follows: whenever two elements of G (resp. H) appear consecutively in a word, we replace them by the value of their respective multiplication in G (resp. H). For instance, the word with three letters

$$a_1a_2b_1$$

with a_1, a_2 in G and $b_1$in H is equivalent to the word with two letters

$$(a_1 ._G a_2)b_1$$

where $a_1 ._G a_2$ is the multiplication in G.

To conclude, we must also force the equivalence of the empty word, with the word consisting of the neutral element of G and with the word consisting of the neutral element of H:

$$\emptyset \sim e_G \sim e_H$$

We leave it as an exercise to show that this defines an equivalence relation.

The set $W(G,H)/\sim$ carries a natural group structure given by word concatenation. We leave this as an exercise to the reader. Showing the associativity of concatenations is the most tedious part. Finally, w set

$$G \coprod H := (W(G,H)/\sim, \text{ concatenation})$$

We now define i_G to be the map $G \to G \coprod H$ sending an element a of G to the words with a single letter a By the nature of the concatenation law, and the equivalence relation on words, this defines a group homomorphism. Similarly for i_H.

Finally, it remains to check that the triple $(G \coprod H, i_G, i_H)$ satisfies the universal property of coproducts. Back to the diagram (2) we can now define the dotted map Ψ: on a word $x_1x_2\cdots x_n$ we define

$$\Psi(x_1x2\cdots x_n) := (f \text{ or } g)(x_1)._P(f \text{ or } g)(x_2)._P\cdots(f \text{ or } g)(x_n)$$

using f or g depending if the letter comes from an element of G or H, respectively. This defines a group homomorphism with the required universal property.

□

Terminology IV.5.17. The coprodut in GROUPS is also called the *free product* of groups and sometimes denoted as $*$

Exercise IV.5.18. Describe the free product $\mathbb{Z} * \mathbb{Z}$

Exercise IV.5.19. Show that ABGROUPS admits products and coproducts and that they are naturally isomorphic, given by direct sums $\bigoplus$.

Exercise IV.5.20. Show that in the CRINGS coproducts are given by tensor products over $\mathbb{Z}$. Use this to show that the forgetful functor of the Example **IV.1.24**, does not preserve coproducts.

Quotients relations.

Example IV.5.21. One can reformulate the universal property of the quotient of a set X by an equivalence relation $\mathscr{R}$ of the Reminder **II.5.18** in the following form: for any set Z, composition with the quotient map $\pi : X \to X/\mathscr{R}$ induces a bijection

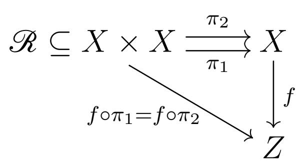
$$\mathrm{Hom}_{\mathsf{SETS}}(X/\mathscr{R}, Z) \simeq \mathrm{Hom}^{\mathscr{R}}_{\mathsf{SETS}}(X, Z)$$

where $\mathrm{Hom}^{\mathscr{R}}_{\mathsf{SETS}}(X, Z)$ is the subset of $\mathrm{Hom}_{\mathsf{SETS}}(X, Z)$ of all set-theoretic maps identifying two points in the equivalence relation, ie, $f(x) = f(y)$ if $(x, y) \in \mathscr{R}$.

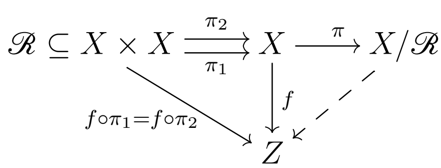
We can re-write this diagrammatically as follows: consider the diagram

$$\mathscr{R} \subseteq X \times X \underset{\pi_1}{\overset{\pi_2}{\rightrightarrows}} X$$

The condition that for a map $f : X \to Z$ to be in $\mathrm{Hom}^{\mathscr{R}}_{\mathsf{SETS}}(X, Z)$, can be re-written as the fact that $f \circ \pi_1 = f \circ \pi_2$, ie, f *equalizes* the two projections.



The universal property of the quotient set then says that there exists a unique dotted arrow



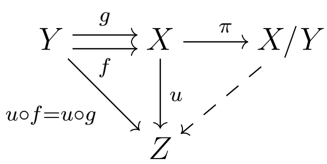
Definition IV.5.22. Let C be a category and consider a diagram of the form

$$Y \underset{f}{\overset{g}{\rightrightarrows}} X$$

Let Z be an object in C. We denote by $\mathrm{Hom}_{\mathsf{C}}^{f,g}(X, Z)$ the set of morphisms $u : X \to Z$ such that $u \circ f = u \circ g$. We say the diagram admits a coequalizer in C if there exists an object X/Y together with a map $\pi : X \to X/Y$ with $\pi \circ f = \pi \circ g$ and such that for any object Z in C composition with π induces a bijection

$$\mathrm{Hom}_{\mathsf{C}}(X/Y, Z) \simeq \mathrm{Hom}_{\mathsf{C}}^{f,g}(X, Z)$$

Remark IV.5.23. Diagrammatically, this means



Fiber products.

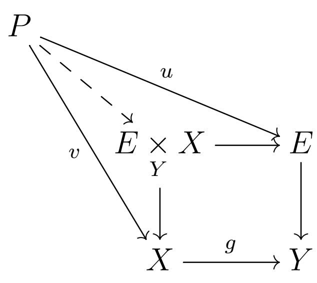
Example IV.5.24. Let

$$\begin{array}{ccc} & & E \\ & & \downarrow f \\ X & \xrightarrow{g} & Y \end{array}$$

be maps of sets. We denote by $E \underset{Y}{\times} X$ the subset of the product set $X \times E$ formed by those elements (x, e) such that $f(e) = g(x)$. For any commutative square of sets

$$\begin{array}{ccc} P & \xrightarrow{u} & E \\ \downarrow v & & \downarrow f \\ X & \xrightarrow{g} & Y \end{array}$$

the commutativity of the diagram guarantees that the induced map $P \to X \times E$ factors through the subset $E \underset{Y}{\times} X$ in a unique way



Equivalently, the composition with the two inclusions $E \underset{Y}{\times} X \to E$ and $E \underset{Y}{\times} X \to X$ induces a bijection

$$\mathrm{Hom}_{\mathsf{SETS}}(P, E \underset{Y}{\times} X) \simeq \mathrm{Hom}_{\mathsf{SETS}}(P, E) \underset{\mathrm{Hom}_{\mathsf{SETS}}(P,Y)}{\times} \mathrm{Hom}_{\mathsf{SETS}}(P, X)$$

Definition IV.5.25. Let C be a category. Let

$$\begin{array}{ccc} & & E \\ & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{g} & Y \end{array}$$

be a diagram in C. We say that the pullback exists if there exists an object $E \underset{Y}{\times} X$ together with morphisms $p_X : E \underset{Y}{\times} X \to X$ and $p_E : E \underset{Y}{\times} X \to E$ rendering the diagram

$$\begin{array}{ccc} E \underset{Y}{\times} X & \xrightarrow{p_E} & E \\ \downarrow{\scriptstyle p_X} & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{g} & Y \end{array}$$

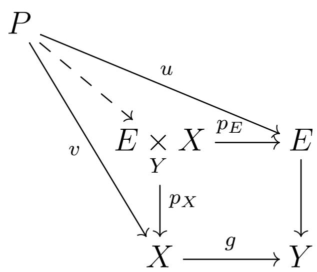
commutative and such that for any object P in C, composition with p_E and p_X induce a bijection of sets

$$\mathrm{Hom}_{\mathsf{C}}(P, E \underset{Y}{\times} X) \simeq \mathrm{Hom}_{\mathsf{C}}(P, E) \underset{\mathrm{Hom}_{\mathsf{C}}(P,Y)}{\times} \mathrm{Hom}_{\mathsf{C}}(P, X)$$

Remark IV.5.26. In other words, for every commutative square

$$\begin{array}{ccc} P & \xrightarrow{u} & E \\ \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{g} & Y \end{array}$$

there exists a unique morphism rendering the commutativity of



Example IV.5.27. The category SETS has pullbacks. This is a reformulation of the Example **IV.5.24**.

Example IV.5.28. Pullbacks exists in TOP. Indeed, given a diagram of continuous maps

$$\begin{array}{ccc} & & E \\ & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{g} & Y \end{array}$$

the set-theoretic pullback $E \underset{Y}{\times} X$ can be endowed with the subspace topology inside the product topology on $E \times X$. We leave the details as an exercise to verity that this has the required universal property.

Pushouts.

Definition IV.5.29. Let C be a category and consider a pair of morphisms in C as

$$\begin{array}{ccc} Z & \xrightarrow{g} & Y \\ \downarrow{\scriptstyle f} & & \\ X & & \end{array}$$

We say that the pushout exists, if there exists an object $X \coprod_Z Y$ and morphisms $i_X : X \to X \coprod_Z Y$ and $i_Y : X \to X \coprod_Z Y$ in C, rendering the diagram commutative

$$\begin{array}{ccc} Z & \xrightarrow{g} & Y \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle i_Y} \\ X & \xrightarrow{i_X} & X \coprod_Z Y \end{array}$$

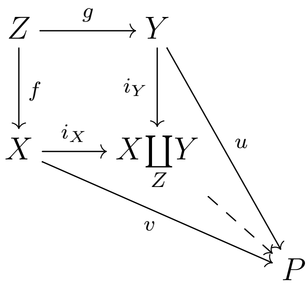
and satisfying the following universal property: for any object P, composition with i_X and i_Y induces a bijection

$$\mathrm{Hom}_{\mathsf{C}}(X \coprod_Z Y, P) \simeq \mathrm{Hom}_{\mathsf{C}}(X, P) \underset{\mathrm{Hom}_{\mathsf{C}}(Z,P)}{\times} \mathrm{Hom}_{\mathsf{C}}(Y, P)$$

Remark IV.5.30. Diagrammatically, this means that for every object P and every commutative diagram

$$\begin{array}{ccc} Z & \xrightarrow{g} & Y \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle u} \\ X & \xrightarrow{v} & P \end{array}$$

there exists a unique morphism $X \coprod_Z Y \to P$ making the diagram



commute

Example IV.5.31. The category TOP has all pushouts. This has been shown in the Proposition **II.5.43** as a consequence of the existence of quotient spaces and disjoint unions.

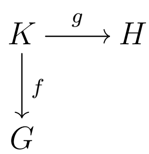
In fact, the proof that TOP has pushouts is a consequence of a more general fact:

Exercise IV.5.32. Let C be a category that admits finite coproducts and coequalizers. Then C admits pushouts.

Exercise IV.5.33. Prove that the category of pointed spaces TOP_* (see the Example **IV.1.6**) has coproducts given by the wedge sum of pointed spaces (Definition **II.5.50**).

Proposition IV.5.34. *The category* GROUPS *admits pushouts.*

PROOF. Consider a diagram of groups



We obtain its pushout as the quotient of the coproduct in GROUPS (see Proposition **IV.5.16**) $G \coprod H$ by the equivalence relation generated by the identification of two words if one is obtained from the other by replacing a letter $f(k)$ by a letter $g(k)$ for $k \in K$.

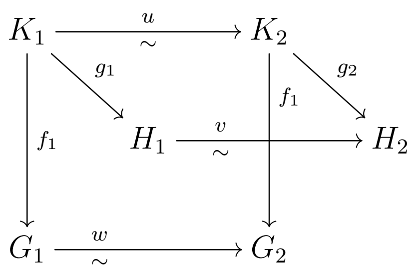
The quotient $G\coprod_K H := G \coprod H/\sim$ acquires a group structure given by concatenation of words and the quotient map is a map of groups.

The two canonical maps $i_G : G \to G\coprod_K H$ and $i_H : H \to G\coprod_K H$ are given the composition of the two inclusions $G \to G \coprod H$ and $H \to G \coprod H$ with the quotient map.

We leave it as an exercise to show that this defines the universal property, as a combination of the universal property of the quotient group and the universal property of coproducts.

□

Exercise IV.5.35. Let C be a category. Consider a commutative diagram in C of the form



where u, v, w are isomorphisms. Show that the pushouts of the two diagrams

$$\begin{array}{ccc} K_1 & \xrightarrow{g_1} & H_1 \\ \downarrow{\scriptstyle f_1} & & \\ G_1 & & \end{array} \qquad \begin{array}{ccc} K_2 & \xrightarrow{g_2} & H_2 \\ \downarrow{\scriptstyle f_2} & & \\ G_2 & & \end{array}$$

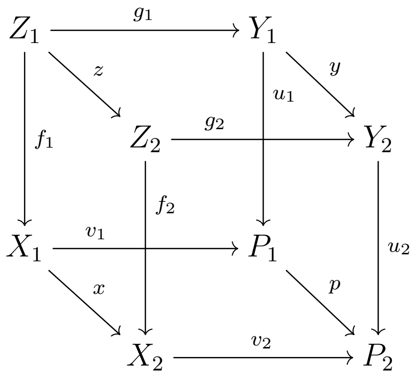
are isomorphic.

TD, Exo 5.1, Feuille 5 **Exercise IV.5.36.** Compute the pushout $\mathbb{Z}/3 \underset{\mathbb{Z}}{*} \mathbb{Z}/2$ in GROUPS.

Definition IV.5.37. Let

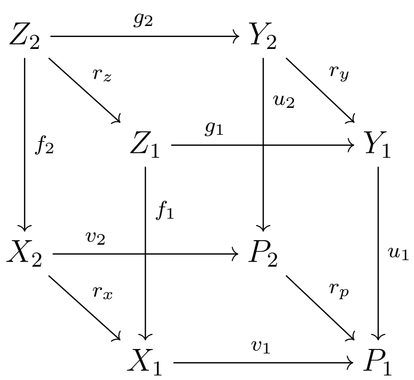
$$\begin{array}{ccc} Z_1 & \xrightarrow{g_1} & Y_1 \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle u_1} \\ X_1 & \xrightarrow{v_1} & P_1 \end{array} \quad , \quad \begin{array}{ccc} Z_2 & \xrightarrow{g_2} & Y_2 \\ \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle u_2} \\ X_2 & \xrightarrow{v_2} & P_2 \end{array}$$

be commutative squares in a category C. A *morphism* between commutative squares, is the data of morphisms $z : Z_1 \to Z_2$, $y : Y_1 \to Y_2$, $x : X_1 \to X_2$ and $p : P_1 \to P_2$ such that the diagram



commutes

Definition IV.5.38. Consider a map of commutative squares as in the Definition **IV.5.37**. We say that it is a retract, if each map z, x, y, p has a retract, respectively, r_z, r_x, r_y and r_p and together they form a morphism of commutative squares, ie, the diagram



also commutes. In this case we say that the first square is a retract of the second.

Exercise IV.5.39. Let C be a category. Show that if a commutative square is a retract of a pushout square in C, then it is also a pushout square.

Internal-Homs.

As in Reminder **II.5.54** and Proposition **II.5.64** we would like to say that in a general category C, given two objects X and Y there exists an object $\mathrm{Map}(X, Y)$ representing morphisms from X to Y.

Definition IV.5.40. Let C be a category with finite products. Let X and Y be objects in C. We say that the internal-hom of X and Y exists in C, if there exists an object $\mathrm{Map}(X, Y)$ in C together with an evaluation map $E : \mathrm{Map}(X, Y) \times X \to \mathrm{Y}$ satisfying the following universal property: for every object $Z \in \mathsf{C}$, the composition with the evaluation map induces a bijection of sets

$$\mathrm{Hom}_{\mathsf{C}}(Z, \mathrm{Map}(X, Y)) \simeq \mathrm{Hom}_{\mathsf{C}}(Z \times X, Y)$$

Example IV.5.41. The category SETS admits all internal-homs. This is the Reminder **II.5.54**.

Warning IV.5.42. The category TOP does not admit internal-homs in general. We have seen that $\mathrm{Map}(X, Y)$ has the correct universal property when X is locally compact. In general in algebraic topology one works with a class of spaces so-called compactly generated that enlarges that of locally-compact spaces and where internal-homs always exists. From a point of view of homotopy theory, there is no loss of information in restricting to such spaces [†]. One of the advantages of working with compactly generated spaces is that we eliminate the pathological behavior of the Warning **II.5.35**: a product of quotient maps of compactly generated spaces is again a quotient map.

Example IV.5.43. The category ABGROUPS admits internal-homs $\mathrm{Map}(M, N)$ given by endowing the set of group homomorphisms $\mathrm{Hom}(M, N)$ with the addition law induced from N: $(f + g)(m) := f(m) + g(m)$.

Example IV.5.44. The category CATS admits internal-homs given by functor category $\mathsf{Fun}(\mathsf{C}, \mathsf{D})$. This is the Proposition **IV.4.13**.

[†]See Hovey's book on Model Categories.

CHAPTER V

Fundamental Group

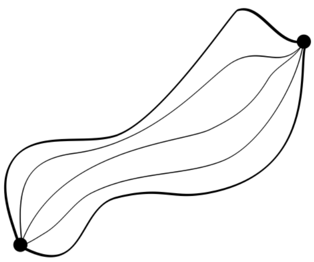
Goal V.0.1. In this chapter we return to topology and introduce the fundamental group of a space and prove the Van Kampen theorem. The categorical abstraction of the previous chapter will finally pay-off.

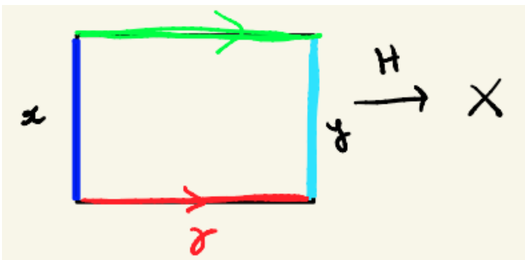
V.1. The fundamental group

Definition V.1.1. Let $\gamma, \beta : I \to X$ be two paths in X with $\gamma(0) = \beta(0) = x$ and $\gamma(1) = \beta(1) = y$. We say that they are *homotopic* as paths if there exists a homotopy $H : I \times I \to X$ with :

- $H(t, 0) = \gamma(0), H(t, 1) = \beta(t)$ for all $t \in I$
- $H(0, s) = x$ and $H(1, s) = y$ for all $s \in I$.

The second condition means that the endpoints of the path remain fixed throughout time. We call such H a homotopy of paths.





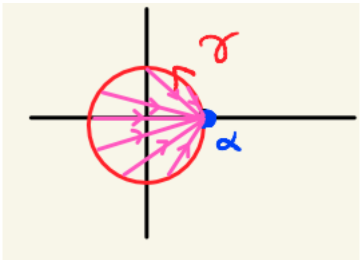
Example V.1.2. Consider $X = \mathbb{R}^2$, γ the path of the Example **III.1.3** $\gamma(t) = (\cos(2\pi t), \sin(2\pi t))$ and α the constant path at $(1, 0)$. Then the map $H : I \times I \to X$ given by

$$H(t, s) = (1 - s)\gamma(t) + s.\alpha(t)$$

gives an homotopy of paths between γ and α since

$$H(0, s) = (1 - s)(\gamma(0)) + s.\alpha(0) = (1 - s)(1, 0) + s.(1, 0) = (1, 0)$$

$$H(1,s) = (1-s)(\gamma(1)) + s.\alpha(1) = (1-s)(1,0) + s.(1,0) = (1,0)$$



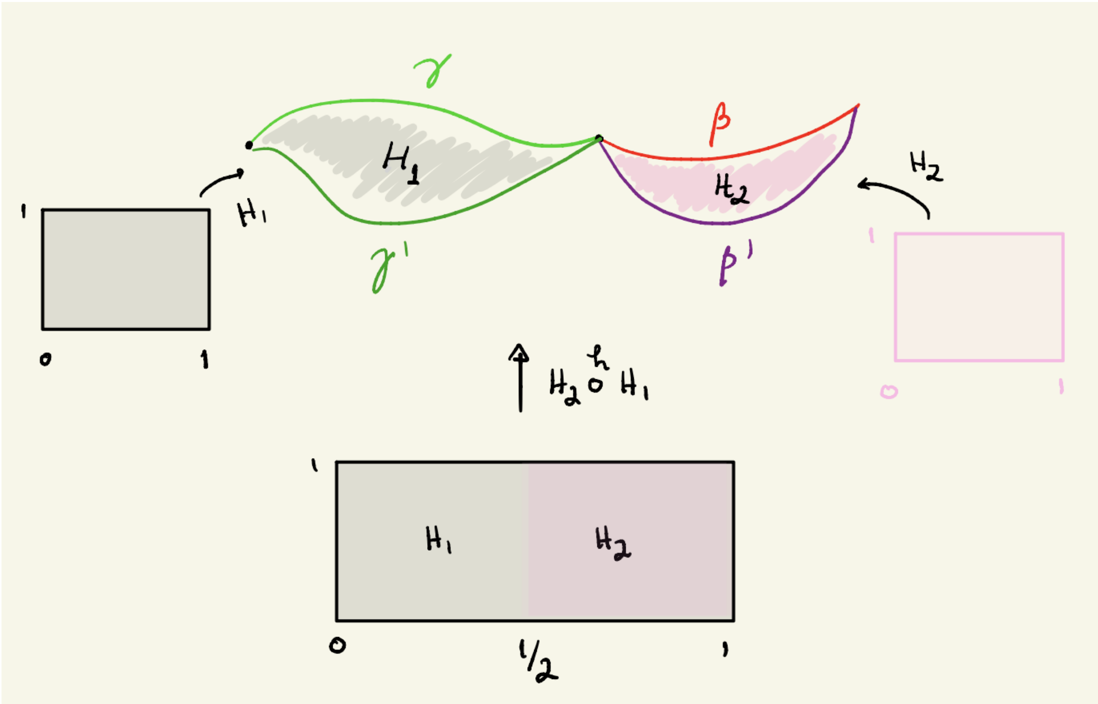
Remark V.1.3. The notion of homotopy of paths coincides with the notion of homotopy relatively to the boundary ∂I as in Definition **III.3.11**. In particular, the homotopy relation on paths forms an equivalence relation: reversing homotopies gives the symmetry and vertical concatenation of homotopies gives transitivity.

Notation V.1.4. Let X be a topological space and $x, y \in X$. We write $hPath(x,y)$ to describe the set of equivalence classes of paths starting at x and finishing at y under the homotopy equivalence relation:

$$hPath(x,y) := \{\gamma : I \to X : \gamma \text{ is continuous, and } \gamma(0) = x, \gamma(1) = y\}/ \sim^{homotopy}$$

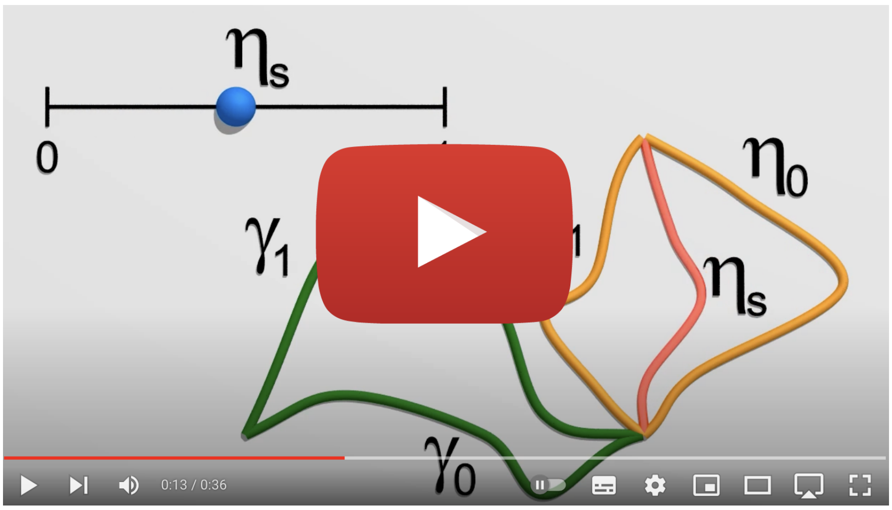
Given a path $\gamma : I \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$, we write $[\gamma]$ for its representative in $hPath(x,y)$

Construction V.1.5 (Horizontal concatenation of homotopies)**.** Let X be a topological space and consider paths $\gamma, \gamma', \beta, \beta' : I \to X$ with $\gamma(1) = \beta(0)$ and $\gamma'(1) = \beta'(0)$. Assume that γ and γ' are homotopic and that β and β' are homotopic. Then the paths $\beta' * \gamma'$ and $\beta * \gamma$ are homotopic via the horizontal concatenation of homotopies



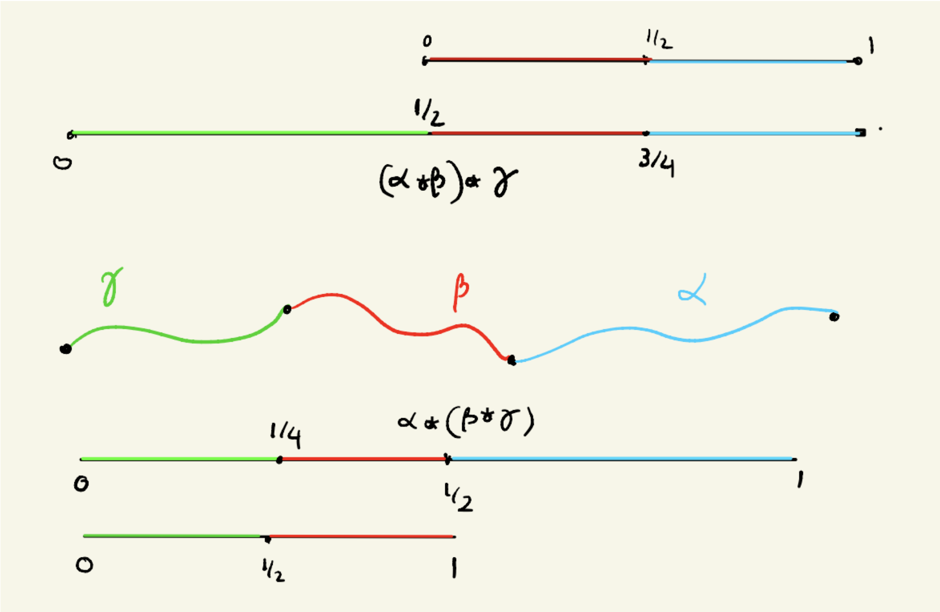
$$H_2 \circ^h H_1(t,s) = \begin{cases} H_1(2t,s) & (t,s) \in [0,\frac{1}{2}] \times [0,1] \\ H_2(2t-1,s) & (t,s) \in [\frac{1}{2},1] \times [0,1] \end{cases}$$

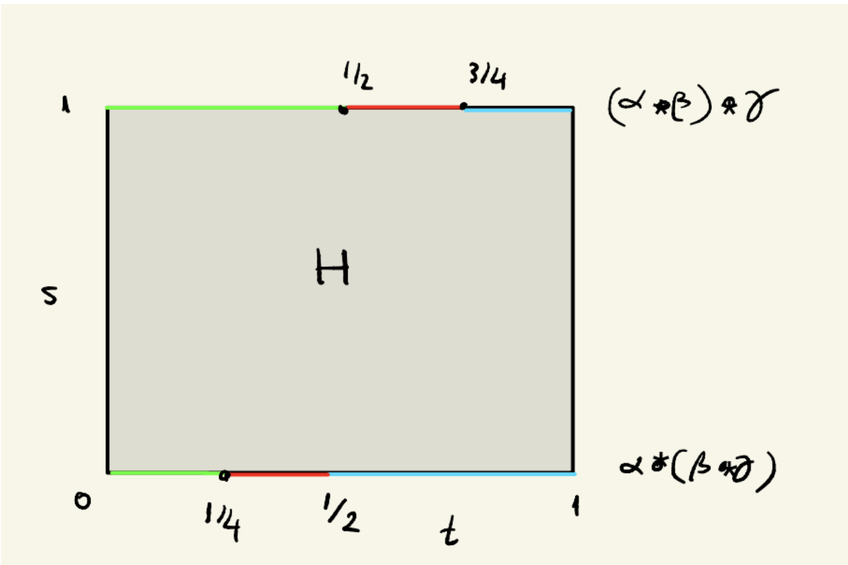
Here's a video illustrating this:

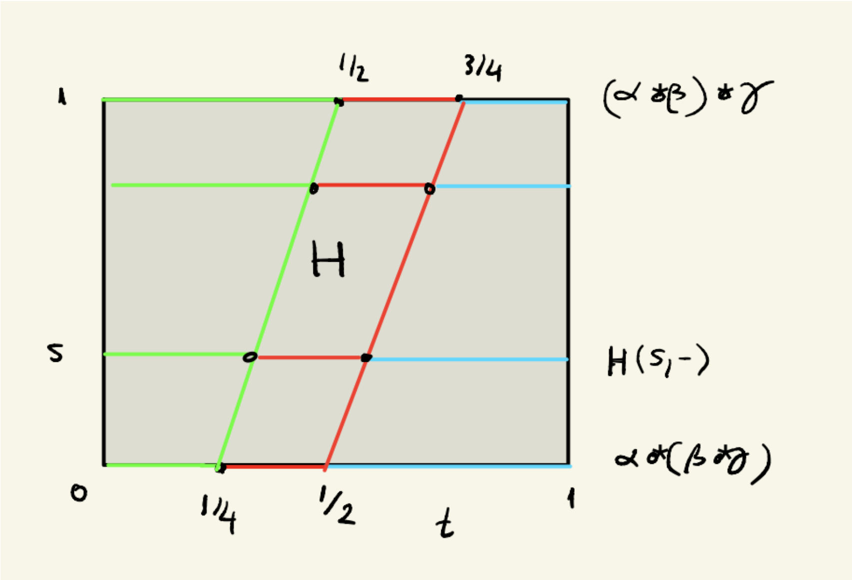


Lemma V.1.6. *Let γ, β, α be paths on X such that $\gamma(1) = \beta(0)$ and $\beta(1) = \alpha(0)$. Then the paths $\alpha * (\beta * \gamma)$ and $(\alpha * \beta) * \gamma$ are homotopic.*

PROOF. We sketch the construction of the homotopy:


0
1/2
1
1/2
0
3/4
(α∗β)∗γ
γ
β
α
1/4
α∗(β∗γ)
0
1/2
1
0
1/2
1


1/2
3/4
1
(α∗β)∗γ
s
H
α∗(β∗γ)
0
1/4
1/2
t
1



To make this precise, we construct a function $\Psi : I \to I$ such that

$$(\alpha * (\beta * \gamma))(\Psi(t)) = ((\alpha * \beta) * \gamma)(t)$$

together with a homotopy $\Psi \sim \mathsf{id}_I$. Composing with $(\alpha * \beta) * \gamma$ thus gives us the required homotopy.

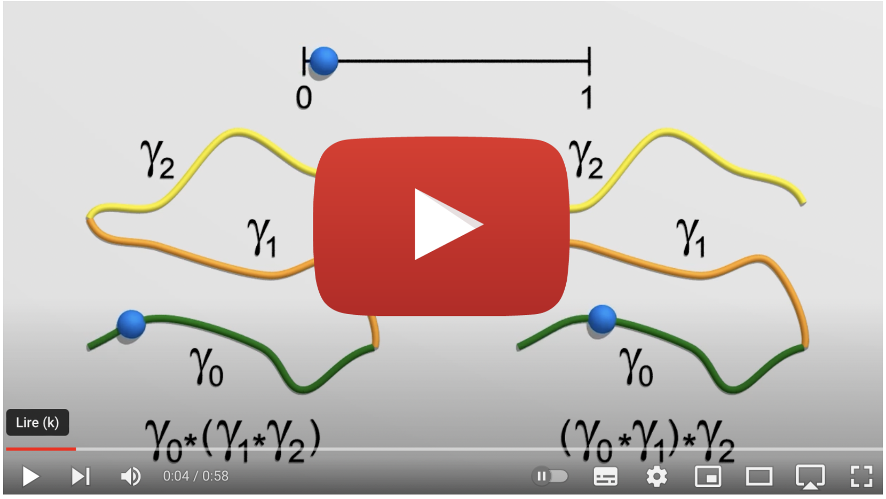
Following the picture, we want to go twice faster from 0 to $\frac{1}{4}$, same speed from $\frac{1}{2}$ to $\frac{3}{4}$ and twice slower from $\frac{3}{4}$ to 1

$$\Psi(t) = \begin{cases} 2t & t \in [0, \frac{1}{4}] \\ t + \frac{1}{4} & t \in [\frac{1}{4}, \frac{1}{2}] \\ \frac{1}{2}t + \frac{1}{2} & t \in [\frac{1}{2}, 1] \end{cases}$$

It now follows that $\Psi : I \to I$ is homotopic to the identity via the linear interpolation

$$H(s, t) = (1 - s)\mathsf{id}(t) + s.\Psi(t)$$

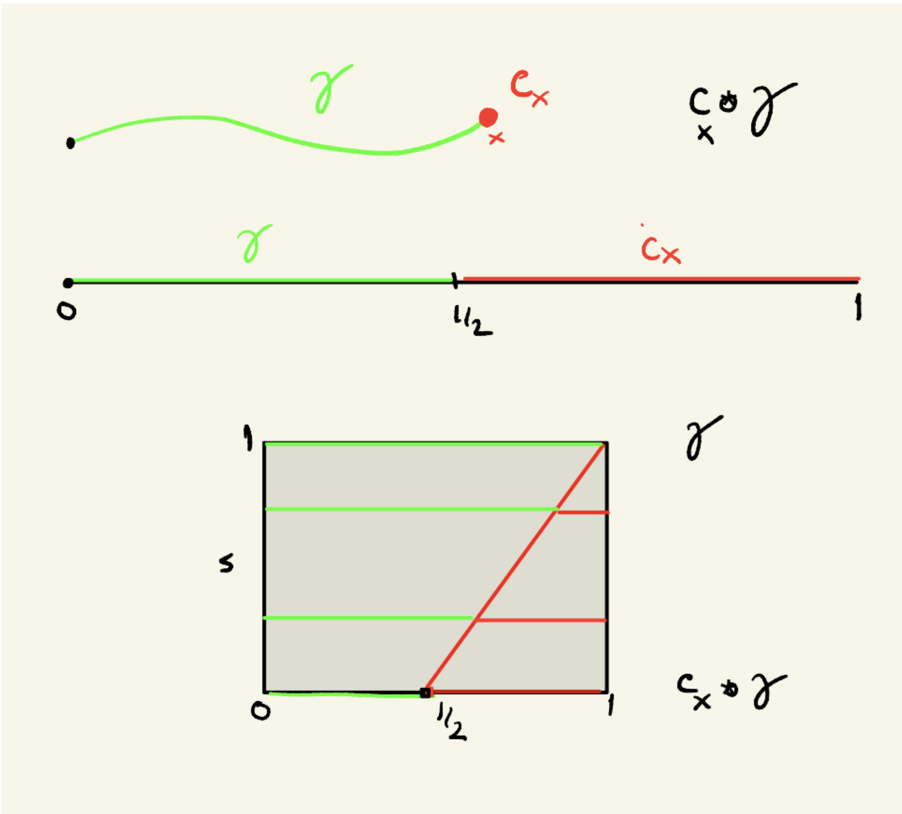
Here's a video illustrating this idea:

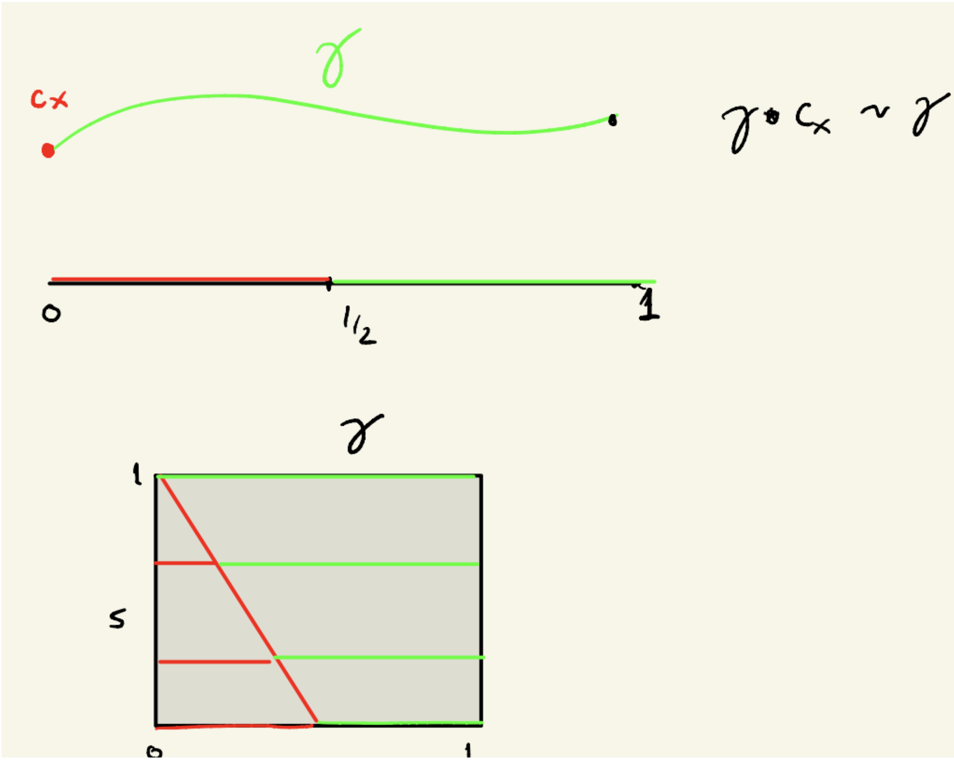


□

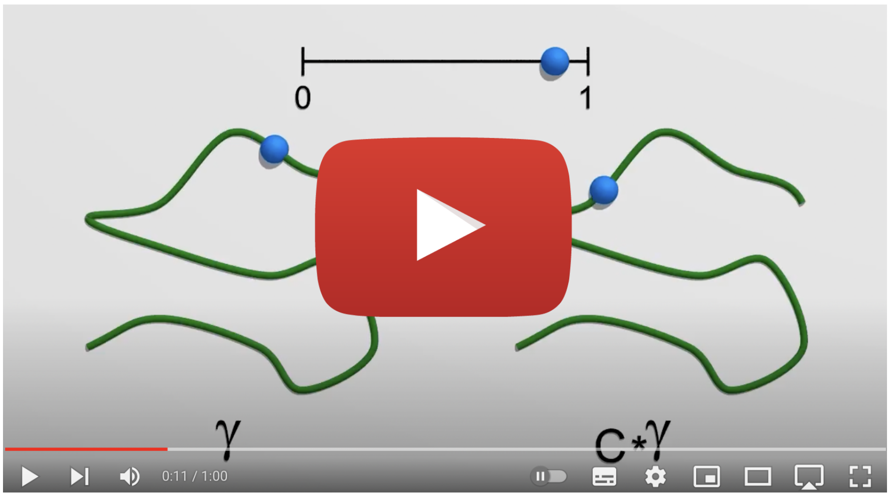
Lemma V.1.7. *Let γ be a path. Let $x = \gamma(0)$ and $y = \gamma(1)$. Let c_x denote the constant path at x and c_y the constant path at y. Then $\gamma * c_x$ is homotopic to γ and $c_y * \gamma$ is homotopic to γ.*

PROOF. We sketch the construction of the homotopies:





Here's a video illustration:



□

Proposition V.1.8. *Let X be a topological space. Then the points of X are the objects of a category $\Pi_1(X)$ where*

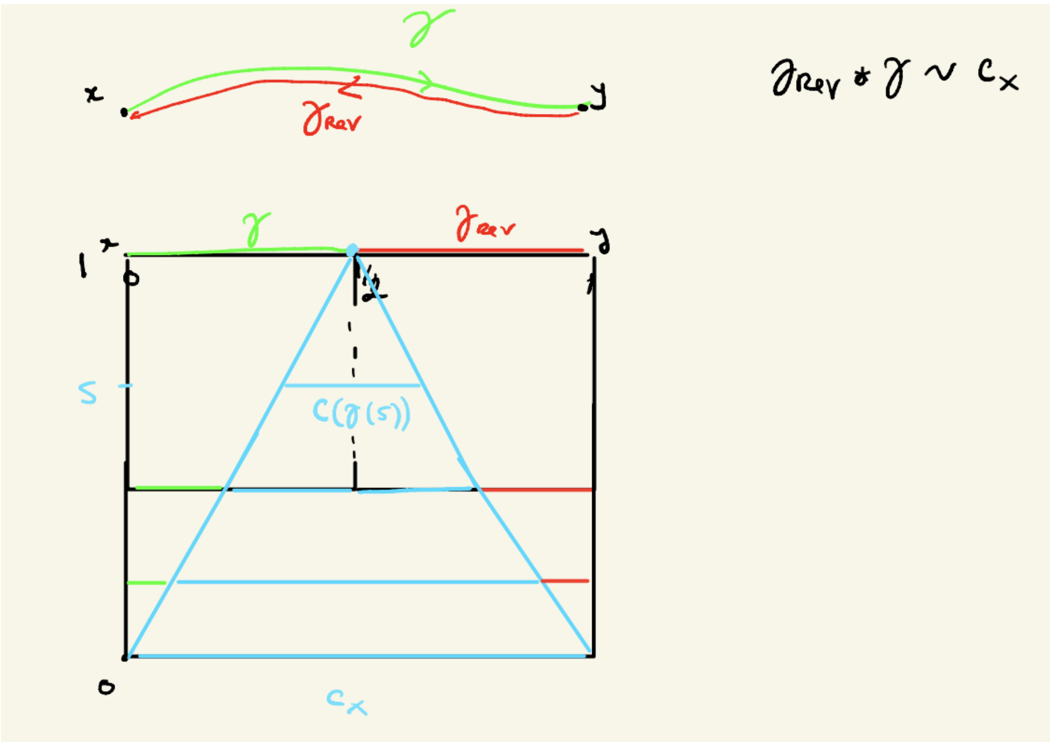
$$\Pi_1(X)(x, y) := hPath(x, y)$$

and composition is given by concatenation of paths.

PROOF. The existence of a well-defined composition law is established by the Construction **V.1.5**. The associativity of compositions is a consequence of Lemma **V.1.6**. Finally, identities are given by constant paths as shown in Lemma **V.1.7**. □

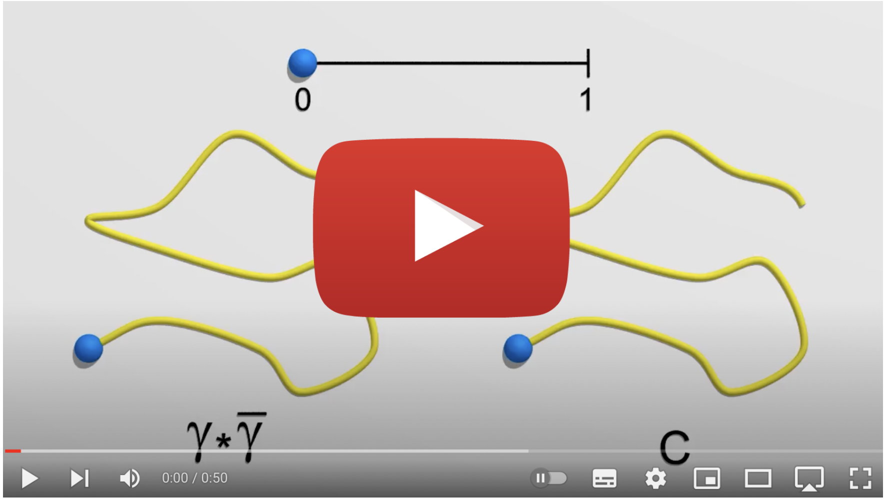
Lemma V.1.9. *Let X be a topological space. Then every morphism in $\Pi_1(X)$ is an isomorphism.*

PROOF. We show that γ and γ_{rev} provide inverse morphisms in $\Pi_1(X)$. Namely, that $\gamma * \gamma_{rev}$ is homotopic to the constant path with value $\gamma(1)$ and $\gamma_{rev} * \gamma$ is homotopic to the constant path with value $\gamma(0)$. We sketch the construction of the homotopy in the second case:



The first case follows from a similar construction.

Here's a video illustrating the situation:



□

Definition V.1.10. A category C where every morphism is an isomorphism is called a groupoid.

Definition V.1.11. The groupoid $\Pi_1(X)$ associated to a topological space X is called the *fundamental groupoid* of X.

Remark V.1.12. We have a canonical isomorphism of set $\pi_0(\Pi_1(X)) \simeq \pi_0(X)$.

Notation V.1.13. The collection of groupoids forms a category: objects are groupoids and morphisms are given by functors between groupoids. We denoted it by $\mathsf{GROUPOIDS}$

Example V.1.14. Let $X = *$ be the point. Then $\Pi_1(*) = [0]$.

Terminology V.1.15. Let X be a topological space and $x \in X$. A path $\gamma : I \to X$ with $\gamma(0) = \gamma(1) = x$ is called a based loop at x.

Remark V.1.16. If $\gamma : I \to X$ is a loop based at $x \in X$ then γ factors through the quotient $[0,1]/(0 \sim 1) \simeq \mathsf{S}^1 \to X$ sending $1 \in \mathsf{S}^1$ to x. See the Example **II.5.33**.

Corollary V.1.17. *Let X be a topological space and $x \in X$. The set of homotopy equivalence classes loops based at x forms a group under the concatenation law for paths.*

Proof. Indeed, it remains to observe that if C is a groupoid, and x is an object in C, then the inclusion $\mathrm{Aut}_{\mathsf{C}}(x) \subseteq \mathrm{Hom}_{\mathsf{C}}(x,x)$ is an equality and therefore $\mathrm{Hom}_{\mathsf{C}}(x,x)$ is group since every morphism has an inverse with respect to composition. Applying this to $x \in \Pi_1(X)$ we get the result. □

Definition V.1.18. The group obtained in the Corollary **V.1.17** is called the *fundamental group of* X at x and denoted by $\pi_1(X, x)$.

Example V.1.19. Let $X = *$, then $\pi_1(*, *) = \{0\}$ is the trivial group.

Definition V.1.20. A groupoid C is said to be *connected* if for every pair of objects X and Y there exists a morphism $X \to Y$. Equivalently, $\pi_0(\mathsf{C}) \simeq *$.

Remark V.1.21. Since in Definition **V.1.20**, C is assumed to be a groupoid, the condition is equivalent to asking for the existence of a morphism $Y \to X$: since any morphism is an isomorphism, we can take its inverse. In particular, for any two objects X and Y in C, the groups of automorphisms $\mathrm{Aut}_{\mathsf{C}}(X)$ and $\mathrm{Aut}_{\mathsf{C}}(Y)$ are isomorphic as groups, via the conjugation of the Remark **IV.2.8**.

Proposition V.1.22. *Let X be a topological space. Then $\Pi_1(X)$ is connected as a groupoid if and only if X is path-connected.*

Proof. Obvious from the definition of morphisms in $\Pi_1(X)$ as homotopy classes of paths. □

Corollary V.1.23. *When X is path connected, the fundamental groups $\pi_1(X, x)$ do not depend on the choice of the base points x.*

Proposition V.1.24. *The construction of the fundamental groupoid of Proposition **V.1.8** defines a functor*

$$\Pi_1 : \mathsf{TOP} \to \mathsf{GROUPOIDS}$$

Proof. If $f : X \to Y$ is a continuous map, then we have a well-defined functor $\Pi_1(f) : \Pi_1(X) \to \Pi_1(Y)$ defined on objects by $x \in X \mapsto f(x) \in Y$ and on morphisms by sending a homotopy class of paths $[\gamma] : x \to y$ to the homotopy class of $[f \circ \gamma] : f(x) \to f(y)$. This is well-defined on homotopy classes because a continuous map can be composed with homotopies as in Proposition **III.3.9**. Compatibility with compositions is a consequence of compatibility with concatenations. Compatibility with units comes from the fact f sends to constant path to the constant path, and again, is compatible with homotopies.
It remains to check that if $f : X \to Y$ and $g : Y \to Z$ are continous maps, we have $\Pi_1(g) \circ \Pi_1(f) = \Pi_1(g \circ f)$. This as an exercise. □

Proposition V.1.25. *The functor $\Pi_1 : \mathsf{TOP} \to \mathsf{GROUPOIDS}$ commutes with products.*

Proof. This follows because a continuous path on the product space $X \times Y$ is by universal property of products, a path on X and a path on Y. By the same universal property, homotopies of paths on $X \times Y$ are also defined coordinate-wise (see the Proposition **III.3.10**) so the result follows. More precisely, this implies that the canonical map

$$\Pi_1(X \times Y) \to \Pi_1(X) \times \Pi_1(Y)$$

induced by the universal property of products in GROUPOIDS is an isomorphism of groupoids. □

To conclude this section we want to show that homotopy equivalent spaces have isomorphic fundamental groups. The proof of this result rests on the following construction and lemma:

Construction V.1.26. Let us consider the functor $\Psi : J \to \Pi_1(I)$ given by sending 0 to the point 0 in I , 1 to the point 1 in I, the unique morphism $0 \to 1$ in J to the homotopy class of the continuous path from 0 to 1 given by $\gamma = Id_I$ and the unique morphism $1 \to 0$ in J to the same path traveled in reverse sense. This is a well-defined functor because $\Pi_1(I)$ is a groupoid as seen in the proof of Lemma **V.1.9**. We are merely selecting an isomorphism and its inverse.

Lemma V.1.27. *The functor $\Psi : J \to \Pi_1(I)$ is an equivalence of categories.*

Proof. We first show that the functor is fully faithful. This is to say that there is a unique homotopy class of paths in the interval, ie, if $\gamma_1, \gamma_2 : I \to I$ are two paths in the interval with $\gamma_1(0) = \gamma_2(0) = 0$ and $\gamma_1(1) = \gamma_2(1) = 1$. Then the homotopy $H(s,t) := (1-s)\gamma_1(t) + s.\gamma_2(s)$ defines a homotopy between the two paths, fixing the endpoints. This shows that the functor is fully faithful.

To see that it is essentialy surjective, it is enough to observe that the interval is path-connected, and therefore every point in the interval can be connected to 0 by a path.

□

Remark V.1.28. The composite functor $\Psi : [0] \to \Pi_1(I)$ selecting the object $0 \in I$ is an equivalence of categories. This follows from the Example **IV.4.6**, together with the Exercise **IV.4.7**.

Proposition V.1.29. *The fundamental groupoid functor*

$$\Pi_1 : \mathsf{TOP} \to \mathsf{GROUPOIDS}$$

sends homotopies of continuous maps to natural isomorphisms of functors. In particular, it sends homotopy equivalences of spaces to equivalences of groupoids.

Proof. Using the Remark **V.1.28** we see that if $H : I \times X \to Y$ is a homotopy between two maps $f, g : X \to Y$, then the induced functor $\Pi_1(H) : \Pi_1(I \times X) \to \Pi_1(Y)$, thanks to the Proposition **V.1.25** is the same as a functor

$$\Pi_1(I) \times \Pi_1(X) \to \Pi_1(Y)$$

The composition with the functor $J \to \Pi_1(I)$ of Remark **V.1.28**, produces a composite functor

$$J \times \Pi_1(X) \to \Pi_1(I) \times \Pi_1(X) \to \Pi(Y)$$

But this is precisely the data of a natural isomorphism between the functors $\Pi_1(H(0,-))$ and $\Pi_1(H(1,-))$ as explained in the Corollary **IV.4.15** □

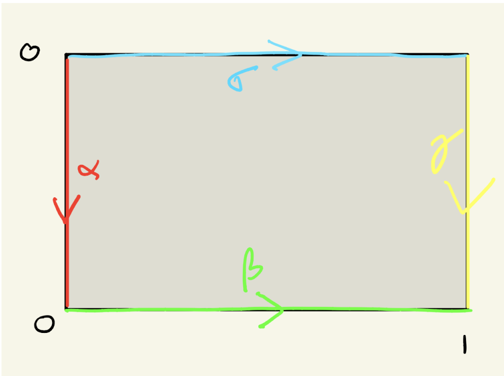
Corollary V.1.30. *The fundamental groupoid functor* $\Pi_1 : \mathsf{TOP} \to \mathsf{GROUPOIDS}$ *sends homotopy equivalences of spaces to equivalences of categories. In particular, homotopy equivalent spaces have isomorphic fundamental groups.*

In order to give an explicit description of how the Corollary **V.1.30** applies to fundamental groups concretely, we need the following easy lemma:

Remark V.1.31 (Square Lemma)**.** Let $I \times I = [0,1] \times [0,1]$ be the square. Consider the paths obtained by traveling along the four sides of the square:

$$\alpha : [0,1] \to I \times I \quad t \mapsto \alpha(t) := (0, 1-t) \quad , \quad \beta : [0,1] \to I \times I \quad t \mapsto \beta(t) := (t, 0)$$

$$\gamma : [0,1] \to I \times I \quad t \mapsto \gamma(t) := (1, 1-t) \quad , \quad \sigma : [0,1] \to I \times I \quad t \mapsto \sigma(t) := (t, 1)$$



Then, the two concatenations $\beta * \alpha$ and $\gamma * \sigma$ are homotopic. We extract this as a consequence of the isomorphism of groupoids of the Proposition **V.1.25**:

$$\Pi_1(I \times I) \simeq \Pi_1(I) \times \Pi_1(I)$$

Let us denote by $i : I \to I$ the path in I from 0 to 1 given by the identity: $t \mapsto t$ and by $i_{inv} : I \to I$ its inverse path, ie, $t \mapsto (1-t)$. Let us also write c_0 and c_1 to denote the constant paths respectively at 0 and 1 in I. Paths in a product space are determined coordinate-wise. It follows that coordinate-wise, the paths γ,β, σ and α are given by

$$\alpha = (c_0, i_{inv}) \quad \beta = (i, c_0)$$

$$\gamma = (c_1, i_{inv}) \quad \sigma = (i, c_1)$$

Therefore, their concatenations, coordinate-wise are given by

$$\gamma * \sigma = (c_1 * i, i_{inv} * c_1) \qquad \beta * \alpha = (i * c_0, c_0 * i_{inv})$$

But in the proof of the isomorphism of groupoids $\Pi_1(I \times I) \simeq \Pi_1(I) \times \Pi_1(I)$ we used that homotopies in the product are also determined coordinate-wise. And we know that coordinate-wise we have $c_1 * i \sim i$ and $i_{inv} * c_1 \sim i_{inv}$ (by the Lemma **V.1.7**) so that using this homotopies in each coordinate, we get

$$\gamma * \sigma \sim (i, i_{inv})$$

Also coordinate-wise, we have homotopies $i * c_0 \sim i$ and $c_0 * i_{inv} \sim i_{inv}$, so that

$$\beta * \alpha \sim (i, i_{inv})$$

Finally,

$$\beta * \alpha \sim (i, i_{inv}) \sim \gamma * \sigma$$

Exercise V.1.32. Use the square lemma to show that homotopy classes of paths (using homotopies that do not fix the endpoints) are in bijection with conjugacy classes of paths up to homotopy of paths fixing the endpoints. TD 3, Exo 1

Remark V.1.33. Let us unfold the content of the proof of the Corollary **V.1.30** in in concrete terms for the fundamental groups: assume $f_1, f_2 : X \to Y$ are homotopic via a homotopy H with $H_{|_0} = f_1$ and $H_{|_1} = f_2$. For each $x \in X$, the homotopy H provides a path $\alpha_x := H(-, x) : [0, 1] \to Y$ from $f_1(x)$ to $f_2(x)$. Let $[\gamma] \in \pi_1(X, x)$, represented by a loop $\gamma : [0, 1] \to X$. Then we can form the composition

$$\begin{array}{ccccc} I \times I & \xrightarrow{\mathsf{id}_I \times \gamma} & X \times I & \xrightarrow{H} & Y \\ & & & & \\ (t, s) \mapsto & & (\gamma(t), s) \mapsto & & H(s, \gamma(t)) \end{array}$$

Using the Remark **V.1.31**, this provides a homotopy between the sides of the square:

$$(f_2 \circ \gamma) \circ \alpha_x \sim \alpha_x \circ (f_1 \circ \gamma)$$

In other words, conjugation with the path α_x makes the diagram commute

$$\begin{array}{ccc} \pi_1(X, x) & \xrightarrow{(f_1)_*} & \pi_1(Y, f_1(x)) \\ \downarrow (f_2)_* & \swarrow \alpha_x \circ - \circ \alpha_x^{-1} & \\ \pi_1(Y, f_2(x)) & & \end{array}$$

Now, assume $f : X \to Y$ is a homotopy equivalence with inverse $g : Y \to X$ and homotopies H_1 between $g \circ f$ and id_X and H_2 between $f \circ g$ and id_Y. Then $(\mathsf{id}_X)_*$ differs from $(g \circ f)_*$ by a conjugation, so that $(g \circ f)_*$ is an isomorphism. The same for id_Y.

Example V.1.34. Let X be a contractible space and $x \in X$ any point. We have $\pi_1(X, x) = \{0\}$. Indeed, the homotopy equivalence between X and $*$, by the previous corollary, induces an equivalence of categories between $\Pi_1(X)$ and $\Pi_1(*) = [0]$. In particular:

- $\pi_1(\mathbb{R}^n, 0) = \{0\}$.
- $\pi_1(D^n, 1) \simeq \pi_1(D^n, 0) = \{0\}$.

Definition V.1.35. A space is said to be simply-connected if it is path-connected and $\pi_1(X, x) = \{0\}$ for all $x \in X$.

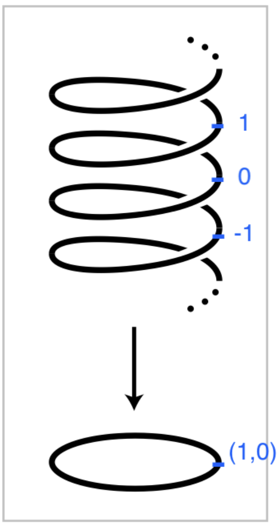
Exercise V.1.36. Show that the space in the Warning **III.2.7** is simply-connected.

Exercise V.1.37. Show that $\pi_1(X, x)$ can be identified with the set of base-point preserving homotopy classes of base-point preserving maps $(\mathsf{S}^1, 1) \to (X, x)$.

The goal of the next section is to compute the fundamental group of the circle. In particular, our computation will show that the circle is not simply-connected.

V.2. Fundamental group of the circle

We now turn to the computation of $\pi_1(\mathsf{S}^1, 1)$. The inspiration comes from the Example **I.2.12** where the circle is somehow the simplest space where every corridor (path) brings us back to the *same* room and at the extreme opposite we have $\mathbb{R}$, where every path brings us to a *different* room. These two extreme opposites are related via the exponential map $\exp : \mathbb{R} \to \mathsf{S}^1$ of the Example **II.2.19** defined on cartesian coordinates by $x \mapsto (\cos(2\pi x), \sin(2\pi x))$. We have already seen that the periodicity of the cosinus and sinus function imply that the exponential map is a quotient map Example **II.5.25** and Exercise **II.5.32**. We can visualize it as wrapping the real line around over the circle:



Picture from Hatcher's book.

Construction V.2.1. For each $n \in \mathbb{Z}$, we denote by s_n the path $[0,1] \to \mathbb{R}$ given by

$$s_n(t) = tn$$

We define a path on the circle

$$\gamma_n := \exp \circ s_n$$

explicitly, it is given by

$$\gamma_n(t) = (\cos(2\pi n t), \sin(2\pi n t))$$

Notice that since $\gamma_n(0) = e^0 = 1$ and $\gamma_n(1) = e^{2\pi i.1.n} = 1$, γ_n defines a loop based at $1 \in \mathsf{S}^1$.

We remark that the concatenation $\gamma_1^n := \underbrace{\gamma_1 * \cdots * \gamma_1}_{n}$ is homotopic to γ_n. To check why this is true, let us first denote by $s_{n+1}^n : [0,1] \to \mathbb{R}$ the path from n to $n+1$ given by $s_{n+1}^n(t) = (1-t)n + t(n+1) = n + t$. Notice that the exponential function makes

$$\exp \circ s_1 = \exp \circ s_{n+1}^n$$

because

$$e^{2\pi i t} = e^{2\pi i t}.1 = e^{2\pi i t + 2\pi i n} = e^{2\pi i (t+n)}$$

for every $n \in \mathbb{Z}$. Therefore, the concatenation γ_1^n can also be written as

$$\gamma_1^n = \exp \circ s_n^{n-1} * \cdots * \exp \circ s_2^1 * \exp \circ s_1$$

which is the same as concatenating first the paths s_{k+1}^k in $\mathbb{R}$ and then applying the exponential map

$$\gamma_1^n = \exp \circ (s_n^{n-1} * \cdots * s_2^1 * s_1)$$

The path $s_n^{n-1} * \cdots * s_2^1 * s_1$ is a path from 0 to n. Since $\mathbb{R}$ is contractible, every two paths with the same end-points are homotopic (its fundamental groupoid is equivalent to the trivial category). Therefore, $s_n^{n-1} * \cdots * s_2^1 * s_1$ is path-homotopic to s_n. If we want a precise formula, we can use the homotopy $H(\lambda, t) = (1-\lambda)s_n(t) + \lambda(s_n^{n-1} * \cdots * s_2^1 * s_1)(t)$. Applying the exponential map, we deduce that γ_n is homotopic to γ_1^n.

It follows that $\gamma_n * \gamma_m$ is homotopic to $\gamma_1^n * \gamma_1^m = \gamma_1^{n+m}$ which is then homotopic to γ_{n+m}. This implies that the assignment

$$\Psi : \mathbb{Z} \to \pi_1(\mathsf{S}^1, 1)$$

sending $n \mapsto [\gamma_n]$ is well-defined and is a group homomorphism with respect the additive law in $\mathbb{Z}$.

Theorem V.2.2: Fundamental Group of the Circle

The group homomorphism

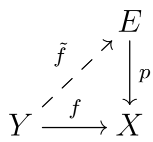
$$\Psi : \mathbb{Z} \to \pi_1(\mathsf{S}^1, 1)$$

$$n \mapsto [\gamma_n]$$

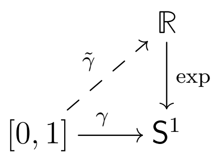
is an isomorphism of groups.

Since the paths γ_n were constructed using the exponential map, the proof of this theorem relies on some fundamental properties of the exponential map. The first concerns the surjectivity of Ψ, telling us that a loop in the circle is always homotopic to one of the γ_n's:

Definition V.2.3. Let $p : E \to X$ be a morphism in a category C. A lift of $f : Y \to X$ along p is the data of a morphism $\tilde{f} : Y \to E$ making the diagram commute



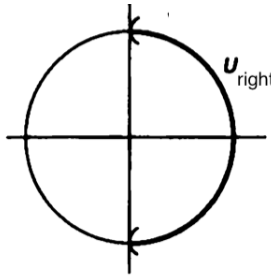
Lemma V.2.4. *Every path γ on the circle can be lifted along the exponential map to a path in $\mathbb{R}$.*



This lifting is unique if we fix the starting point in $\mathbb{R}$.

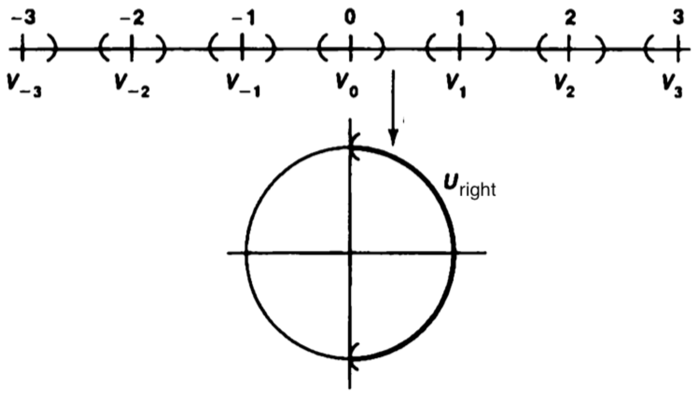
In order to prove this result we will need to delve deeper in the properties of the exponential map. Motivated by the picture above, we will explore the fact that the exponential function is, at least locally, a homeomorphism.

Example V.2.5. Let us consider U_{right} the right side of the circle, ie, those points $(x, y) \in \mathsf{S}^1 \subseteq \mathbb{R}^2$ with $x > 0$. The subset U_{right} is the intersection of the circle with the open subset of $\mathbb{R}^2$ with strictly positive x-coordinate. and therefore, by definition of the subspace topology, U_{right} is open in S^1.



(Picture take from Munkres, Topology)

The pre-image $\exp^{-1}(U_{right}) = \{x \in \mathbb{R} : \cos(2\pi x) > 0\}$ consists of a disjoint union of open intervals $V_n :=]n - \frac{1}{4}, n + \frac{1}{4}[$



(Picture take from Munkres, Topology)

and restriction of exp to each of this intervals induces a homeomorphism

$$\exp_{|V_n} : V_n \to U_{right}$$

Indeed, restrict to each closure $\overline{V_n}$ the function $x \in \overline{V_n} \mapsto \sin(2\pi x)$ is injective (since it is strictly decreasing in these intervals). Moreover, as a map $\overline{V_n} \to \overline{U}$ it is surjective. We use the criterium of the Proposition **II.3.22**. to deduce that $\overline{V_n} \to \overline{U}$ is a homeomorphism. Indeed, we just checked it is a continuous bijection. It remains to observe that $\overline{U}$ is Hausdorff as a subspace of S^1 which is Hausdorff, and $\overline{V_n}$ is a closed interval, therefore quasi-compact. It restricts to a homemorphism V_n to U.

In more concrete terms, the local inverse of $\exp_{|V_n}$ is given by a local branch of the complex logarithm: recall that our exponential function is defined as the restriction of the complex exponential $\mathbb{C} \to \mathbb{C}^*$ sending $z \mapsto e^z$ to the imaginary line $2\pi i.\mathbb{R} \subseteq \mathbb{C}$. The complex logarithm log is defined on every open subset of $\mathbb{C}$ obtained by removing half a line. In our case,

$$U_{right} \subseteq \mathsf{S}^1 \subseteq \mathbb{C} \setminus \{z : Re(z) < 0\}$$

so on U_{right} the complex logarithm

$$\log_{right} : U_{right} \to \mathbb{C}, \quad w \mapsto \ln(|w|) + \frac{arg(w)}{2\pi}$$

with $arg(w) \in]-\pi, \pi[$, provides a local inverse to $V_0 \to U_{right}$. Changing the range of the angle by adding $2\pi n$ we find the explicit local inverses to $V_n \to U_{right}$.

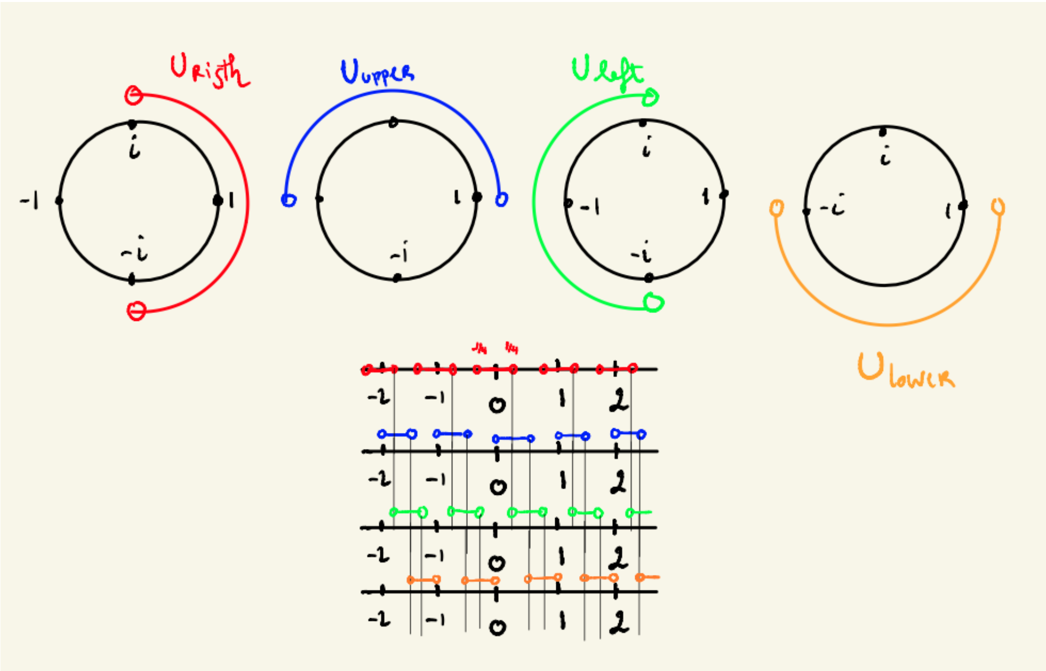
One can now apply similar arguments to the open subsets of the half-circles

$$U_{lower} = \{(x, y) \in \mathsf{S}^1 : y < 0\}$$

$$U_{left} = \{(x, y) \in \mathsf{S}^1 : x < 0\}$$

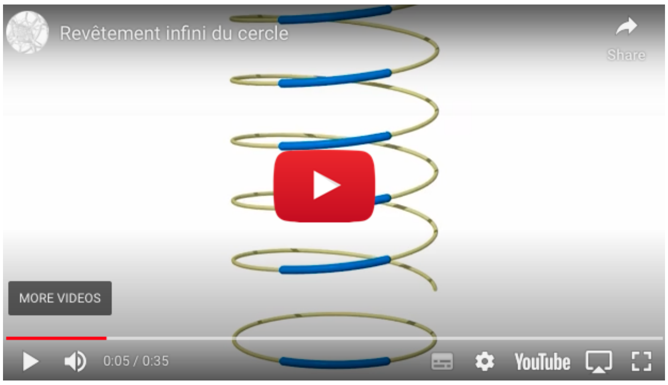
$$U_{upper} = \{(x, y) \in \mathsf{S}^1 : y > 0\}$$

Here's a picture of what we get



using the sinus or cosinus functions respectively.

Here's a video illustrating this discussion:



Let us resume the content of the Example **V.2.5** as follows:

Proposition V.2.6. *Consider the exponential map* $\exp : \mathbb{R} \to \mathsf{S}^1$*. Then for every point in the circle* x *there exists an open neighborhood* $U \subseteq \mathsf{S}^1$ *such that* $\exp^{-1}(U)$ *is equal to a disjoint union of open subsets* V_n *in* $\mathbb{R}$ *such that* $\exp_{|V_n} : V_n \to U$ *is a homeomorphism.*

Proof. Take the open subsets $U_{right}, U_{left}, U_{upper}, U_{lower}$. Any point in the circle belongs to one of these subsets and all of them satisfy the required property. □

Proof of the Lemma **V.2.4**. Let $\gamma : [0,1] \to \mathsf{S}^1$ be a path. The starting point $\gamma(0)$ is contained in at least one of the four $U_{right}, U_{left}, U_{upper}, U_{lower}$ of the Proposition **V.2.6**. Without loss of generality suppose $\gamma(0) \in U_{right}$. As we follow $t \in [0,1]$, γ will travel between the different $U_{right}, U_{left}, U_{upper}, U_{lower}$ so we can choose a partition of the interval $[0,1]$ as a union of finitely many closed sub-intervals

$$[0, a_1], [a_1, a_2], \cdots, [a_n, 1]$$

such that each one of the $\gamma([a_i, a_{i+1}])$ is fully contained in one of the four U_{right}, U_{left}, U_{upper} or U_{lower} (*).

For instance, since we have assumed $\gamma(0) \in U_{right}$ we can assume $\gamma([0,a_1]) \subseteq U_{right}$.

Fix x_0 a lift of γ_0. Then x_0 belongs to one and only one of the V_n's. Suppose without loss of generality it belongs to the V_0. The map $\exp_{|V_0} : V_0 \to U_{right}$ has an inverse $\exp^{-1}_{|V_0} : U_{right} \to V_0$. Since $\gamma([0,a_1]) \subseteq U_{right}$, we can form the composition $\exp^{-1}_{|V_0} \circ \gamma_{|[0,a_1]} : [0,a_1] \to V_0$. This gives us a path in V_0 from $x_0 = \exp^{-1}_{|V_0}(\gamma(0))$ to $x_1 = \exp^{-1}_{|V_0}(\gamma(a_1))$ in V_0.

Now $\gamma(a_1)$ belongs to one of the U_{right}, U_{left}, U_{upper} or U_{lower}. Say it is U_{down} and assume again, without loss of generality that this choice is also such that $\gamma([a_1, a_2]) \subseteq U_{down}$. Write $\exp^{-1}(U_{down}) = \coprod_n W_n$ as a disjoint union of open

(*) In fact a careful proof of the existence of this partition requires the Lebesgue covering dimension of Example **V.3.11** below.

subsets, each mapping homeomorphically to U_{down}. Let W_1 be the unique one containing x_1.

Notice that W_1 intersects V_0 (x_1 belongs to both).

Then $\exp_{|W_1} : W_1 \to U_{down}$ is a homeomorphism with inverse $\exp^{-1}_{|W_n} : U_{down} \to W_1$. Consider the composition $\exp^{-1}_{|W_1} \circ \gamma_{|_{a_1,a_2}} : [a_1, a_2] \to W_1$. This a path in W_1 from x_1 to a point $x_2 := \exp^{-1}_{|W_1} \circ \gamma_{|_{[a_1,a_2]}}(a_2)$.

The pasting lemma (Exercise **II.2.14**) applied to decomposition of the interval $[0, a_2]$ as a union of the closed sub-intervals $[0, a_1]$ and $[a_1, a_2]$, allows to glue the first lifted path from x_0 to x_1 with this new lifted path from x_1 to x_2 to form a continuous path $[0, a_2] \to \mathbb{R}$.

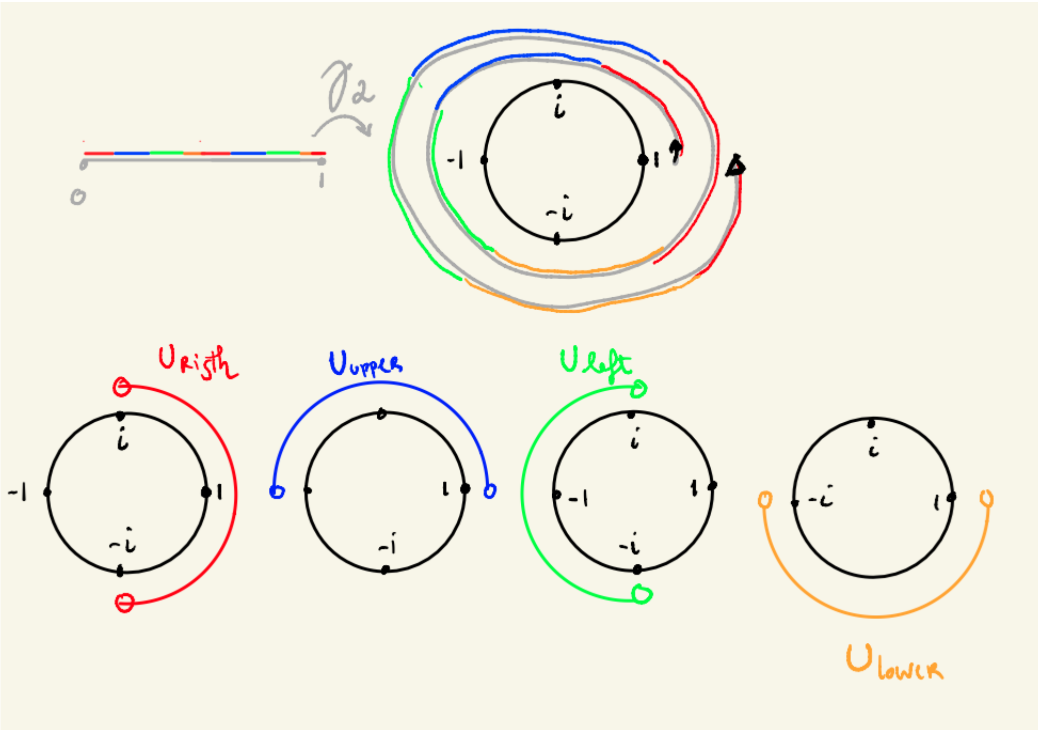
We now proceed by induction until reaching the last interval $[a_n, 1]$. The resulting path $\widetilde{\gamma} : [0, 1] \to \mathbb{R}$ obtained from the pasting lemma is a lifting of γ. Indeed $p \circ \widetilde{\gamma}$ can be checked locally to match with γ, as the construction so forces.

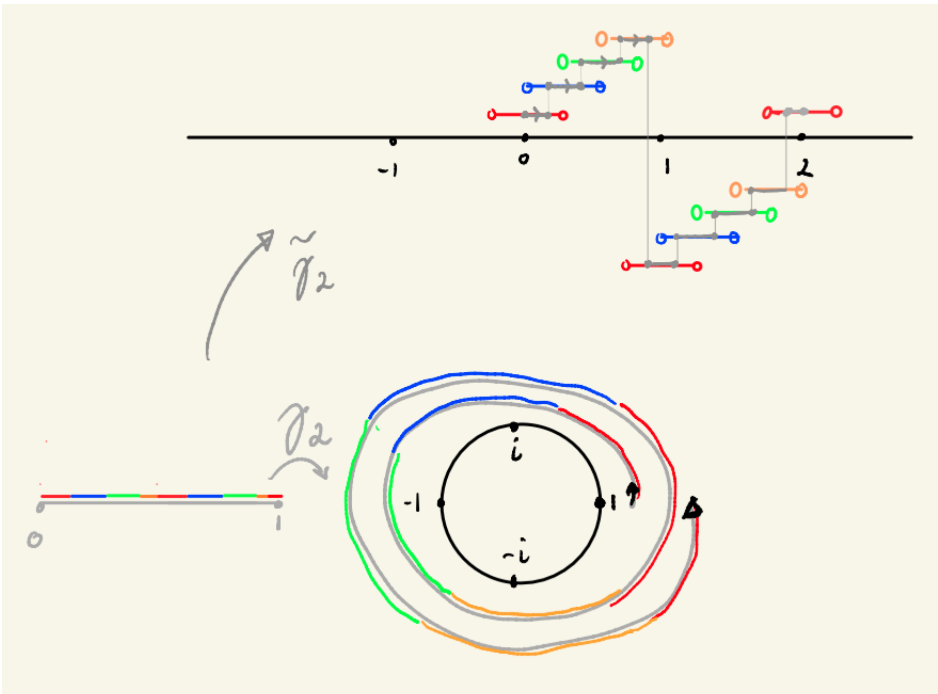
Notice that the resulting lift does not depend on the fact that we used the covering $\{U_{right}, U_{left}, U_{lower}, U_{upper}\}$. Any open covering satisfying the property of the Proposition **V.2.6** will allow us to construct a lift. The fact that the local inverses to the exponential function are unique (given by the logarithm), implies that the lift is unique.

□

Remark V.2.7. The proof of the Lemma **V.2.4** provides liftings for loops based at $1 \in \mathsf{S}^1$. If $\gamma : [0, 1] \to \mathsf{S}^1$ is a loop based at 1, there is a unique lift $\widetilde{\gamma} : [0, 1] \to \mathbb{R}$ starting at $0 \in \mathbb{R}$. Since $\exp(\widetilde{\gamma}(1)) = \gamma(1) = \gamma(0) = 1$, we have $\widetilde{\gamma}(1) \in \mathbb{Z}$. More generally, for any lift $\widetilde{\gamma}$ of a loop γ based at 1, independently of the choice of the starting point, the difference $\widetilde{\gamma}(1)-\widetilde{\gamma}(0)$ belongs to $\mathbb{Z}$ because $\exp(\widetilde{\gamma}(1)) = \exp(\widetilde{\gamma}(0))$.

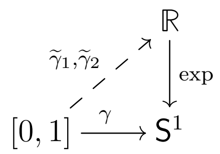
Example V.2.8. Here's an example of what we get for constructing the lift of γ_2 in S^1:





Proposition V.2.9. *Let $\gamma : [0,1] \to \mathsf{S}^1$ be a loop based at $1 \in \mathsf{S}^1$. Then every two lifts of γ differ by a translation by an integer in $\mathbb{R}$.*

Proof. Let $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ be lifts of γ



Observe that both $\widetilde{\gamma}_2(0)$ and $\widetilde{\gamma}_1(0)$ are integers since being a lift of 1, we have $1 = \exp(2\pi i\,\widetilde{\gamma}_2(0)) = \exp(2\pi i\,\widetilde{\gamma}_1(0))$. Therefore, the difference

$$n := \widetilde{\gamma}_2(0) - \widetilde{\gamma}_1(0)$$

is an integer. We now claim that

$$\widetilde{\gamma}_2(t) - \widetilde{\gamma}_1(t) = n$$

for all $t \in [0,1]$. Clearly, since $\exp(2\pi i\,\widetilde{\gamma}_2(t)) = \exp(2\pi i\widetilde{\gamma}_1(t))$ for all $t \in [0,1]$, we have $\exp(2\pi i\,(\widetilde{\gamma}_2(t) - \widetilde{\gamma}_1(t))) = 1$ for all $t \in [0,1]$. Therefore, the function $[0,1] \to \mathbb{R}$ given by $t \mapsto \widetilde{\gamma}_2(t) - \widetilde{\gamma}_1(t)$ takes values in $\mathbb{Z} \subseteq \mathbb{R}$. By construction this function is continuous since it is given by the difference of two continuous functions. Since $[0,1]$ is connected, this function must be constant (Remark **II.6.3**). But we already know that at 0 its value is n.

□

Construction V.2.10. Let $\gamma : [0,1] \to \mathsf{S}^1$ be a loop based at $1 \in \mathsf{S}^1$. Let $\widetilde{\gamma}$ be any lift of γ. The integer number

$$\mathsf{deg}_\gamma := \widetilde{\gamma}(1) - \widetilde{\gamma}(0)$$

is well-defined, ie, it is independent of the choice of lift $\widetilde{\gamma}$. We call it the degree of γ.

Example V.2.11. The based loops $\gamma_n = \exp \circ s_n$ have s_n has a lift and therefore degree $n = s_n(1) - s_n(0)$.

Proposition V.2.12. *If γ and β are two base loops at 1 in S^1 and $\gamma * \beta$ is their concatenation, then*

$$\mathsf{deg}_{\gamma*\beta} = \mathsf{deg}_\gamma + \mathsf{deg}_\beta$$

Proof. Let x_0 be a lift of $\beta(0)$ and let $\widetilde{\beta}$ be the unique lift of β starting at x_0. Since $\beta(1) = \gamma(0) = 1$, take the unique lift $\widetilde{\gamma}$ of γ starting at $x_1 := \widetilde{\beta}(1)$. The concatenation $\widetilde{\gamma} * \widetilde{\beta}$ is a lifting of the concatenation $\gamma * \beta$. Since the degree does not depend on the choice of the lifting, we have

$$\mathsf{deg}_{\gamma*\beta} = \widetilde{\gamma} * \widetilde{\beta}(1) - \widetilde{\gamma} * \widetilde{\beta}(0) = \widetilde{\gamma}(1) - \widetilde{\beta}(0)$$

But since $\widetilde{\gamma}(0) = \widetilde{\beta}(1)$, we have

$$\widetilde{\gamma}(1) - \widetilde{\beta}(0) = \widetilde{\gamma}(1) - \widetilde{\gamma}(0) + \widetilde{\beta}(1) - \widetilde{\beta}(0) = \mathsf{deg}_\gamma + \mathsf{deg}_\beta$$

□

Proposition V.2.13. *If γ and β are two based loops at 1 in S^1 that are homotopic, then*

$$\mathsf{deg}_\gamma = \mathsf{deg}_\beta$$

The proof of this result relies on the observation that the the strategy used to lift paths also works to lift homotopies:

Lemma V.2.14. *Let $H : I \times I \to \mathsf{S}^1$ be a homotopy of paths, between γ and β having the same endpoints. Fix $x_0 \in \mathbb{R}$ a lift of $\exp(x_0) = \gamma(0) = \beta(0)$. Then the homotopy H admits a unique lifting to a path homotopy $\widetilde{H} : I \times I \to \mathbb{R}$ between the unique lifting $\widetilde{\gamma}$ of γ at x_0 and the unique lifting $\widetilde{\beta}$ of β at x_0.*

Proof. Choose a subdivision of the square $I \times I$ in small enough squares such that for each small square K_i, the image of $H(K_i)$ is fully contained in one of the open intervals $U_{right}, U_{left}, U_{upper}, U_{lower}$. This is possible again by the Lebesgue covering argument that we will see in the next section.

Rename U_1 the open where $H(K_1)$ is fully contained. The point x_0 is contained in only and only one of the disjoint open intervals in $\mathbb{R}$ in $\exp^{-1}(U_1)$. Name it V_1 and use the local inverse of $\exp_{V_1}^{-1} : U_1 \to V_1$ to define the composition

$$exp_{V_1}^{-1} \circ H_{|K_1} : K_1 \to U_1 \to V_1$$

One now proceeds by induction using the pasting lemma, gluing this homotopy along the squares K_i.

The result is a continuous map $\widetilde{H} : I \times I \to \mathbb{R}$. It remains to check this is a homotopy of paths between $\widetilde{\gamma}$ and $\widetilde{\beta}$. But $\widetilde{H}(0, -) : I \to \mathbb{R}$ provides by construction another lifting of γ, starting at x_0. By the unicity of the lifting once a starting point is fixed, we must have $\widetilde{H}(0, -) = \widetilde{\gamma}$. The same argument shows that $\widetilde{H}(1, -) = \widetilde{\beta}$. To show that $\widetilde{H}$ is a homotopy of paths, it remains to explain why the endpoints of $\widetilde{H}(s, -)$ remain fixed for all $s \in I$. But notice that by construction $\widetilde{H}(-, 0) : I \to \mathbb{R}$ provides a lifting for the constant path with value $\gamma(0) = \beta(0)$. This is because by assumption H is a path homotopy and therefore $H(, -0)$ is constant equal to $\gamma(1) = \beta(1)$. By uniqueness of the lifting, $\widetilde{H}(-, 0) : I \to \mathbb{R}$ must therefore be the constant path (since the constant path also provides a lifting). The same argument for $\widetilde{H}(-, 1) : I \to \mathbb{R}$ explains why it must also be a constant path. □

Proof of the Proposition **V.2.13**. Consider β and γ as in the statement of the Proposition **V.2.13** and H a homotopy of paths between them. Since the degree does not depend on the choice of lifting, fix the unique liftings $\widetilde{\gamma}$ and $\widetilde{\beta}$ starting at $x = 0$, together with the unique path homotopy $\widetilde{H}$ given by the Lemma **V.2.14**. Since $\widetilde{H}$ is a homotopy of paths, the endpoint $\widetilde{H}(s, 1)$ is independent of the parameter s in the homotopy. Therefore,

$$\mathsf{deg}_\gamma = \widetilde{H}(0, 1) = \widetilde{H}(s, 1) = \widetilde{H}(1, 1) = \mathsf{deg}_\beta$$

□

Corollary V.2.15. *The map sending an homotopy class of based loop at $1 \in \mathsf{S}^1$, $[\gamma]$ to its degree, defines a group homomorphism*

$$\mathsf{deg} : \pi_1(\mathsf{S}^1, 1) \to \mathbb{Z}$$

which is an inverse of Ψ.

Proof. The fact that $\mathsf{deg} \circ \Psi = \mathsf{id}_{\mathbb{Z}}$ is a consequence of the Example **V.2.11** and the fact that the degree does not depend on the choice of the lifting or the representative of the homotopy class. This implies that Ψ is injective. Since we already knew it Ψ was surjective thanks to the lifting property for paths (Lemma **V.2.4**), the Theorem **V.2.2** is therefore proved, as is this corollary. □

Example V.2.16. There can be no retract for the inclusion $\mathsf{S}^1 \hookrightarrow D^2$. Indeed, if there was a retract, we would have also a retraction for the induced map of groups $\mathbb{Z} \simeq \pi_1(\mathsf{S}^1, 1) \to \pi_1(D^2, 1) = \{0\}$. But clearly this is not possible since the identity of $\mathbb{Z}$ cannot factor through 0.

Exercise V.2.17.

(i) Let

$$X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots \subseteq X := \bigcup_{i \geq 0} X_i$$

be a sequence of inclusions of Hausdorff spaces where X is equipped with the inductive limit topology: a subset $U \subseteq X$ is open if and only if each intersection $U \cap X_i$ is open in X_i.

Show that a compact subset $K \subseteq X$ is contained in some X_i.

(ii) Use (i) to show that for every n and choice of $x_0 \in X_0$, the canonical map

$$\mathsf{colim}_n\, \pi_1(X_i, x_0) \to \pi_1(X, x_0)$$

is an isomorphism of abelian groups. [(†)]

(iii) Use the sequence of inclusions along the equator

$$\mathsf{S}^1 \subseteq \mathsf{S}^2 \subseteq \mathsf{S}^3 \subseteq \cdots \subseteq \bigcup_{i \geq 0} \mathsf{S}^i =: \mathsf{S}^\infty$$

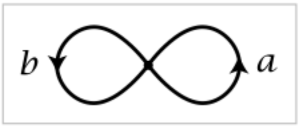
to show that S^∞ is simply-connected.

[(†)] We have not defined general colimits/inductive limits in these notes. So in this exercise we assume this is known from elsewhere.

Exercise V.2.18. Show that if the circle S^1 had a trivial fundamental group, then all topological spaces would have a trivial fundamental group.

V.3. Van Kampen theorem

The Theorem **V.2.2** finally gives a precise sense to the claim in the Example **I.2.12** that the group associated to the circle is the free group in one generator (ie, $\mathbb{Z}$). We would like now to finally prove the claim that the group associated to the space $\mathsf{S}^1 \vee \mathsf{S}^1$



(Picture taken from Hatcher's book)

is the free group with two generators a and b as claimed in the Remark **I.2.11**. Proving this is the goal of this chapter. The general phenomena behind is the content of the following theorem
Our main goal in this section is to prove the following theorem:

Theorem V.3.1. *Let X be a topological space obtained as a union of two open subsets $X = U \bigcup V$. Then, the functor Π_1 sends the pushout diagram in* TOP *of the Example **II.5.48***

$$\begin{array}{ccc} U \bigcap V & \longrightarrow & V \\ \downarrow & & \downarrow \\ U & \longrightarrow & X \end{array}$$

to a pushout diagram in GROUPOIDS

$$\begin{array}{ccc} \Pi_1(U \bigcap V) & \longrightarrow & \Pi_1(V) \\ \downarrow & & \downarrow \\ \Pi_1(U) & \longrightarrow & \Pi_1(X) \end{array} \tag{3}$$

We proceed in several steps.

Construction V.3.2. Consider a diagram of groupoids

$$\begin{array}{ccc} \mathsf{E} & \xrightarrow{G} & \mathsf{D} \\ \downarrow{\scriptstyle F} & & \\ \mathsf{C} & & \end{array}$$

such that both functors F and G are injective on objects. Let us denote by $\mathsf{C}\coprod_{\mathsf{E}}\mathsf{D}$ the category defined by:

- objects given by the pushout in sets $Ob(\mathsf{C}) \coprod_{Ob(\mathsf{E})} Ob(\mathsf{D})$, which since both functors are injective on objects, is imply the union $Ob(\mathsf{C}) \bigcup Ob(\mathsf{D})$.
- A morphism $X \to Y$ consists of a *formally composable string*,

  $$f_0\, f_1\, f_2 \cdots f_{n-1}$$

  where each f_i is a morphism either in C or in D and the target object of f_i is the source object of f_{i+1}, the source of f_0 is X and the target of f_{n-1} is Y.(‡) We consider the equivalence relation on strings generated by the conditions
  - Whenever f_i is an identity morphism, we can omit it;
  - If f_i and f_{i+1} are both either in C or in D, we can compose it in that category and replace their two letters by the letter given by their composition.
  - $F(u) = G(u)$ for every morphism u in E.
- Compositions of strings are given by concatenation.

Under this construction $\mathsf{C}\coprod_{\mathsf{E}}\mathsf{D}$ defines a category which in particular is a groupoid, since every string admits an inverse string obtained using the inverses from C and D.

Moreover, it comes equipped with two natural inclusion functors $\mathsf{C} \to \mathsf{C}\coprod_{\mathsf{E}}\mathsf{D}$ and $\mathsf{D} \to \mathsf{C}\coprod_{\mathsf{E}}\mathsf{D}$.

Finally, given a commutative diagram of functors

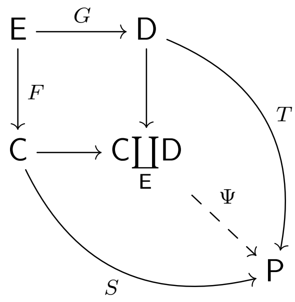
$$\begin{array}{ccc} \mathsf{E} & \xrightarrow{G} & \mathsf{D} \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle T} \\ \mathsf{C} & \xrightarrow{S} & \mathsf{P} \end{array}$$

we define a functor Ψ rendering the commutativity of the diagram

(‡)Formally we can picture it as

$$X = X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} X_{n-1} \xrightarrow{f_{n-1}} X_n = Y$$

although the compositions a priori do not make sense neither in C or in D.



defined on objects by

$$\Psi(X) = \begin{cases} S(X) & X \in \mathsf{C} \\ T(X) & X \in \mathsf{D} \end{cases}$$

(this is well defined on objects because $T \circ G = S \circ F$) . We define it the same away on morphisms by applying S or T to each letter in a word, depending if it corresponds to a morphism on C or D, and composing the morphisms in P. In case the letter comes from E, again the fact that $T \circ G = S \circ F$ garantees the procedure is well-defined under the equivalence relation.

Exercise V.3.3. Check that the Construction **V.3.2** satisfies the universal property of pushouts in $\mathsf{GROUPOIDS}$

Remark V.3.4. The two functors

$$\Pi_1(U \bigcap V) \to \Pi_1(U) \text{ and } \Pi_1(U \bigcap V) \to \Pi_1(V)$$

are injective of objects (objects being the points of X). So we fall under the conditions of the Construction **V.3.2** and have an explicit description of the pushout

$$\Pi_1(U) \coprod_{\Pi_1(U \bigcap V)} \Pi_1(V)$$

Its objects are the points of X and as explained in Construction **V.3.2**, a morphism in $\Pi_1(U) \coprod_{\Pi_1(U \bigcap V)} \Pi_1(V)$ is a formal string of composable morphisms

$$f_0\, f_1\, f_2 \cdots f_{n-1}$$

where each f_i is a morphism either in $\Pi_1(U)$ or in $\Pi_1(V)$. Therefore, each f_i is of the form $[\gamma_i]$ a homotopy class of a path γ_i either fully contained in U or in V. The same way, if $f_i = [\gamma_i] = [\gamma_i']$, then the homotopy H rendering γ_i and γ_i' homotopic must itself also be defined either fully in U or in V.

Following this remark let us provide an equivalent description for the sets of morphisms in $\Pi_1(U) \coprod_{\Pi_1(U \bigcap V)} \Pi_1(V)$.

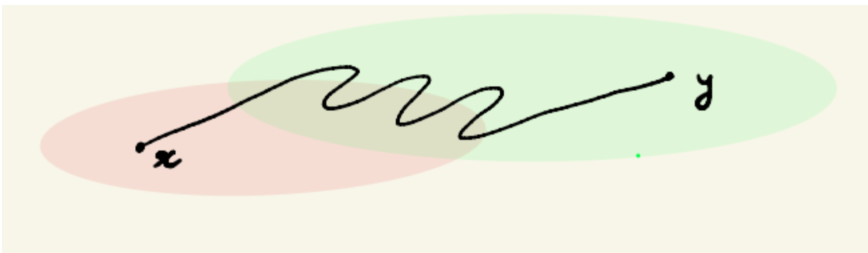
Definition V.3.5. Let X be a topological space obtained as in the pushout (II.5.48). A *partitioned path* is a path $\gamma : [0,1] \to X$ together with a finite partition of $[0,1]$ by closed sub-intervals I_i, and such that the for each i, $\gamma(I_i)$ is either contained in U or in V.

Given two partitioned paths $(\alpha, \{I_i\})$ and $(\beta, \{J_j\})$ such that the two partitions $\{I_i\}$ and $\{J_j\}$ coincide, a *partitioned homotopy* is a family of homotopies $\{H_i : [0,1] \times I_i \to X\}_{i \in I}$ where each H_i is a homotopy between α restricted to $I_i$and β restricted to I_i, that fixes the endpoints of I_i and whose image $H_i([0,1] \times [0,1])$ is completely inside either U or V, depending on where the path lies.

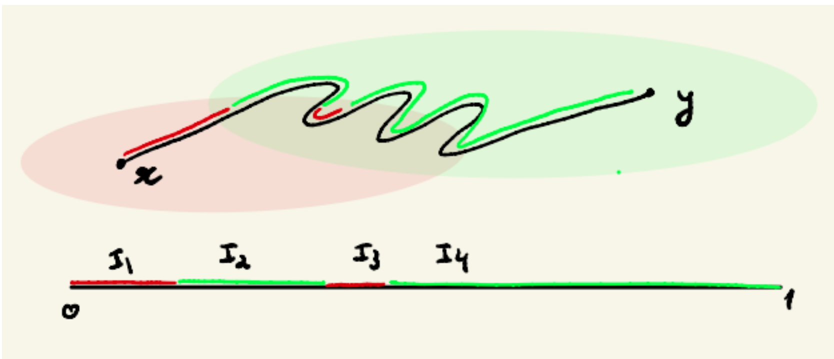
We consider the equivalence relation on partitioned paths generated by forcing $(\alpha, \{I_i\}) \sim (\beta, \{J_j\})$ if

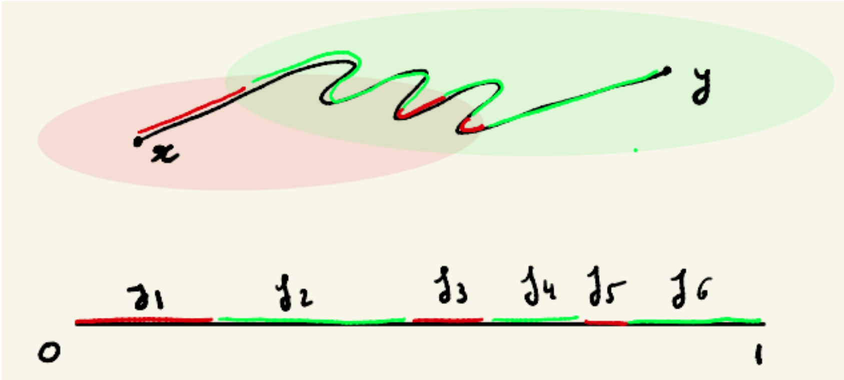
- α and β coincide as paths, or,
- The two partitions $\{I_i\}$ and $\{J_j\}$ agree and there exists a partitioned homotopy.

Example V.3.6. Here is a path

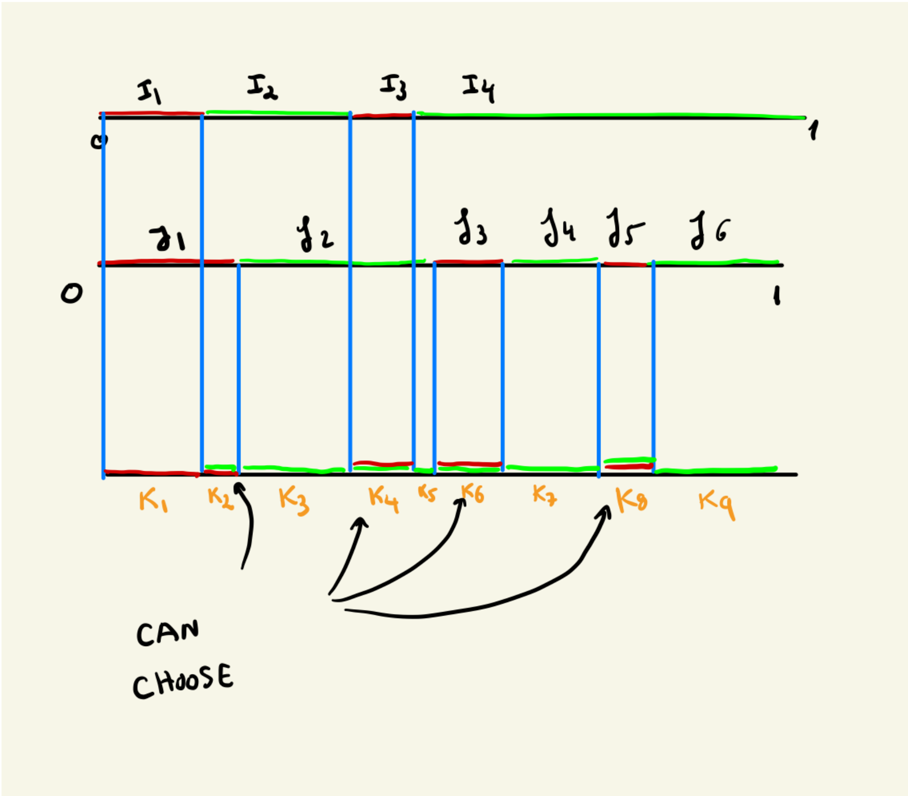


With two different partitions:





Remark V.3.7. Notice that two partitions $\{I_i\}$ and $\{I_j\}$ of the same path α, have a common refinement by taking intersections and choosing if want to see the image in U or in V whenever it belongs to the intersection.



Construction V.3.8. Assume the context of Definition **V.3.5**. Every partitioned path $(\alpha, \{I_i\})$ provides a morphism in $\Pi_1(U) \coprod_{\Pi_1(U \bigcap V)} \Pi_1(V)$, namely, the string given by

$$[\alpha_{|_{I_n}}][\alpha_{|_{I_{n-1}}}][\alpha_{|_{I_{n-2}}}] \cdots [\alpha_{|_{I_1}}]$$

where each chunk of the path is either in U or in V. By the nature of the equivalence relation on partitioned paths, this descends to the quotient.

Lemma V.3.9. *The induced map from equivalence classes of partitioned paths to morphisms in the pushout, is a bijection.*

Proof. Indeed, to find the inverse just concatenate all the paths in the word as paths in X and endowed it with the induced partition coming from the concatenation. $\square$

We can finally turn to the proof of the Van Kampen theorem: Here is the main lemma that will guarantee that enough partitions will both for paths and for homotopies:

Lemma V.3.10 (Lebesgue number lemma). *Let (K, d) be a compact metric space and let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover. Then there exists a constant $\delta > 0$ such that every subspace $A \subseteq K$ of diameter less than δ, is contained in one of the U_i's.*

Proof. For each $x \in K$ pick a radius $\epsilon(x)$ small enough such that ball $B(x, 2\epsilon(x))$ is contained in some of the U_i. In particular, the smaller ball $B(x, \epsilon(x))$ is still contained in U_i. This is possible because the metric topology has a basis given by open balls.
Now, the collection $\{B(x, \epsilon(x))\}_{x \in K}$ forms an open cover of K. Since K is compact this cover admits a finite subcover, say $\{B(x_1, \epsilon(x_1)), \cdots, B(x_n, \epsilon(x_n))\}$.
Take $\delta = \inf\{\epsilon(x_1), \cdots \epsilon(x_n)\}$.
Now, let A be a subspace of diameter smaller than δ. Let $a \in A$. Then a belongs to at least one of the $B(x_i, \epsilon(x_i))$'s, say $B(x_1, \epsilon(x_1))$. We claim that in this case the whole A is contained in $B(x_1, 2\epsilon(x_1))$. Indeed, let $a' \in A$. Then by definition of diameter we have $d(a, a') \leq \delta$. But by the triangle inequality, we have

$$d(a', x_1) \leq d(a', a) + d(a, x_1) \leq \delta + \epsilon(x_1) \leq 2\epsilon(x_1)$$

Since the ball $B(x_1, 2\epsilon(x_1))$ is by construction contained in one of the U_i, so is A.

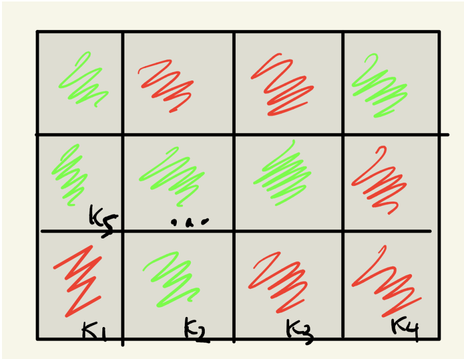
$\square$

We will apply this in two ways. First with paths:

Example V.3.11. Let X be as in Definition **V.3.5**. Then $\{U, V\}$ forms an open cover of X. Let $\gamma : [0,1] \to X$ be a path. Then $\{\gamma^{-1}(U), \gamma^{-1}(V)\}$ forms an open cover of $K = [0,1]$ which is a compact metric space. Take the constant δ in Lemma **V.3.10** and pick a partition of $[0,1]$ by subintervals I_i of size smaller than δ. It follows that the restriction of γ to each of the I_i's has image either in U or in V.

Then with homotopies

Example V.3.12. Let X be as in Definition **V.3.5** with $\{U, V\}$ an open cover. Let $H : K = [0,1] \times [0,1] \to X$ be a continuous map. Then $\{H^{-1}(U), H^{-1}(V)\}$ forms an open covering of K which is a metric space. Since the metric topology in this case has a basis by boxes as in the Example **II.5.9**, the Lemma **V.3.10** guarantees the existence of a partition of K in a finite number small enough squares, $K_1, \cdots K_n$ such that $H(K_i)$ is fully contained in U or V. By reducing the size of the squares if necessary, we can suppose they are all of the same size:



Proof of Theorem V.3.1. We must show that the induced functor

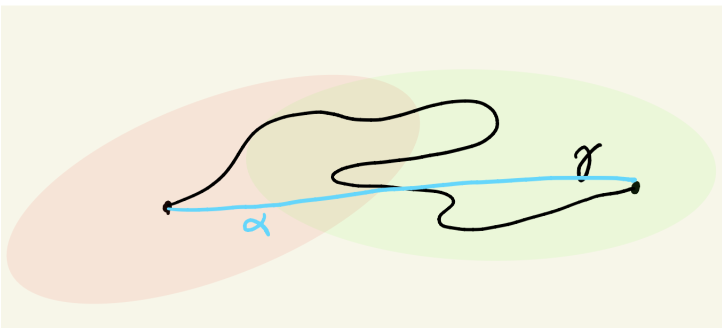
$$\Psi : \Pi_1(U) \coprod_{\Pi_1(U \bigcap V)} \Pi_1(V) \to \Pi_1(X)$$

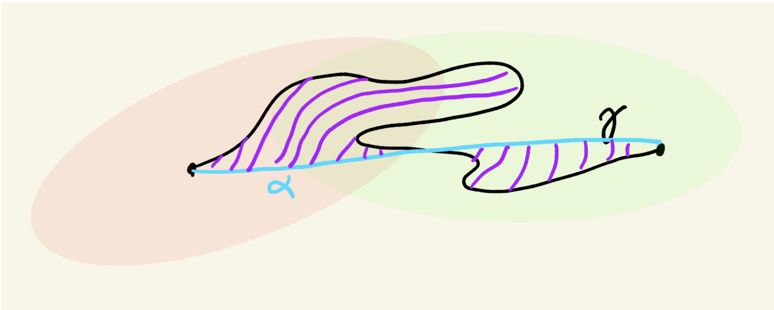
is fully faithful and essentially surjective. The second is automatic since the functor is by construction bijective on objects. It remains to prove fully faithfulness. For this purpose we will use the description of morphisms in the pushout as equivalence classes of partitioned paths (Lemma **V.3.9**.). In practice, what we have to show is that, given two points $x, y \in X$, then the map sending a partitioned path $(\gamma, \{I_i\})$ with $\gamma(0) = x$, and $\gamma(1) = y$ to the path γ, sends homotopies of partitioned paths to homotopies of paths and induces a bijection on the quotient sets

$$\{(\gamma, \{I_i\}) : \gamma(0) = x, \gamma(1) = y)\}/\sim \longrightarrow \{\gamma : \gamma(0) = x, \gamma(1) = y\}/\sim$$

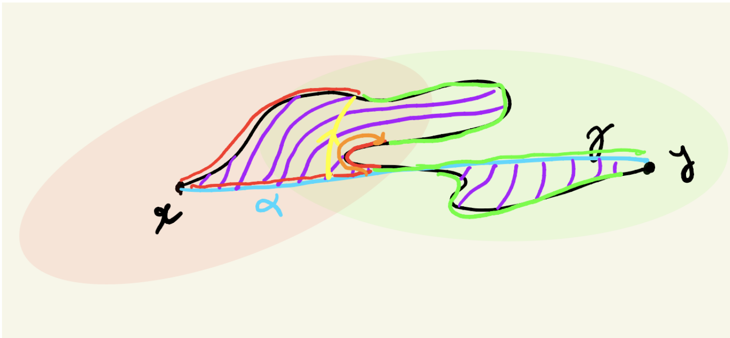
The surjectivity of this map comes the existence of partitions for a given path. This is the Example **V.3.11** as a consequence of Lemma **V.3.10**.

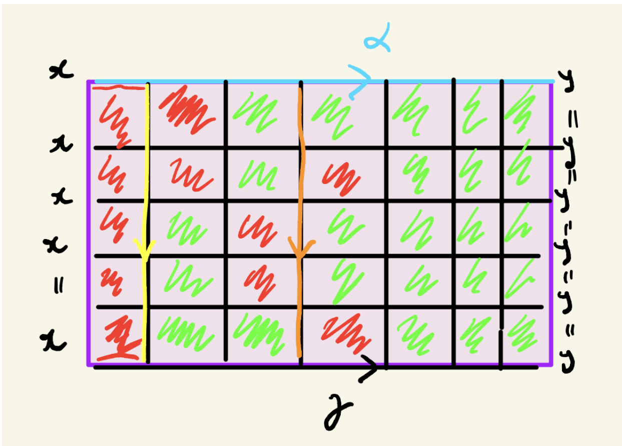
Injectivity: Consider two partitioned paths $(\alpha, \{I_i\})$ and $(\beta, \{J_j\})$ and assume they are homotopic as paths, ie, α and β are homotopic via a homotopy H preserving the endpoints.



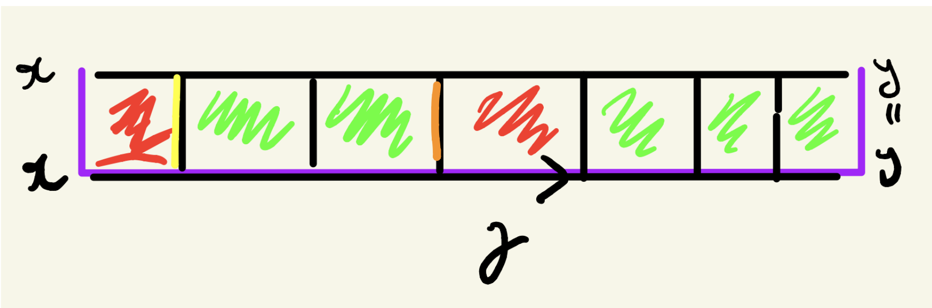


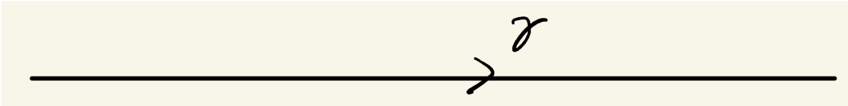
We must show that it is possible to partition H. But this is exactly the Example **V.3.12**.

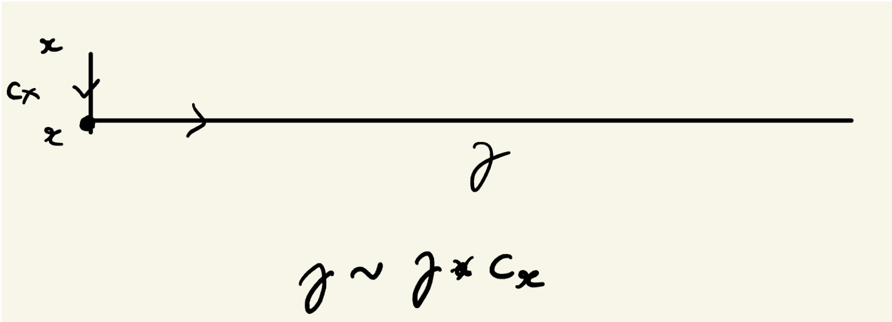


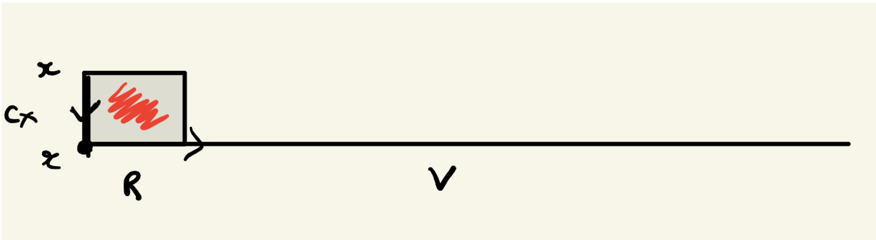


We must now show that this gives us a partitioned homotopy. For this we use the square lemma to produce homotopies of paths respecting the partitions, block by block, starting from the bottom left square





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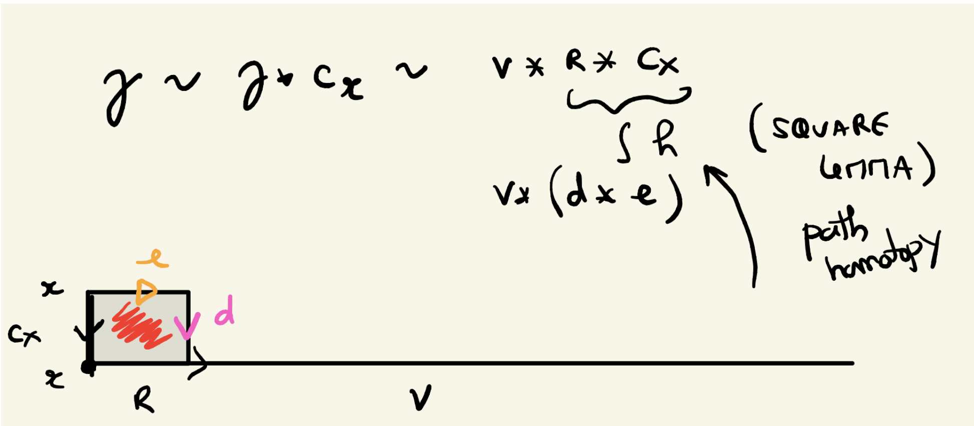
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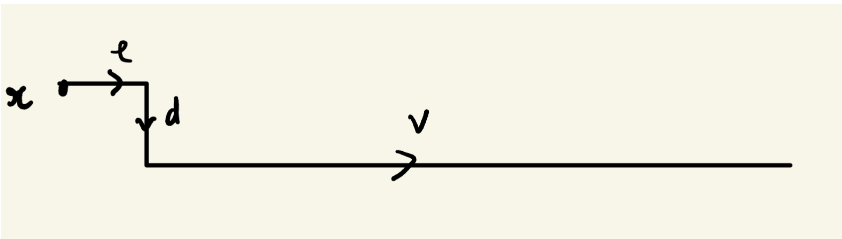
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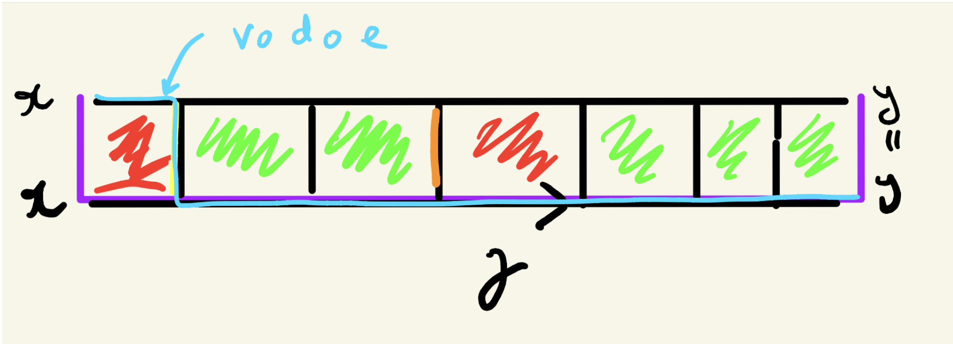
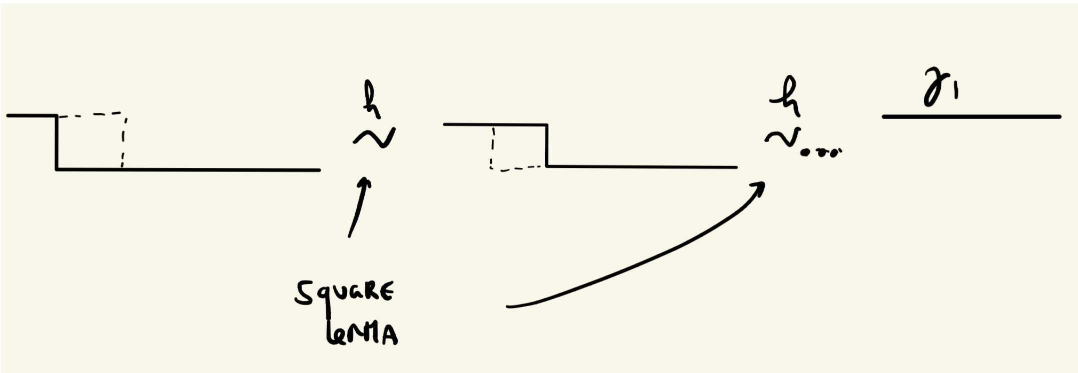
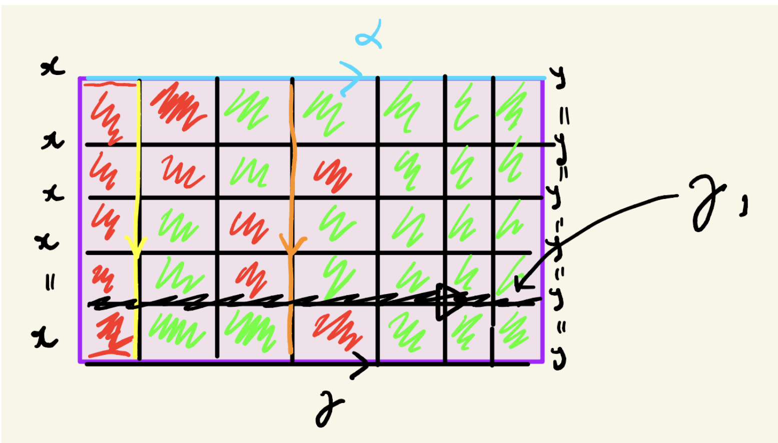
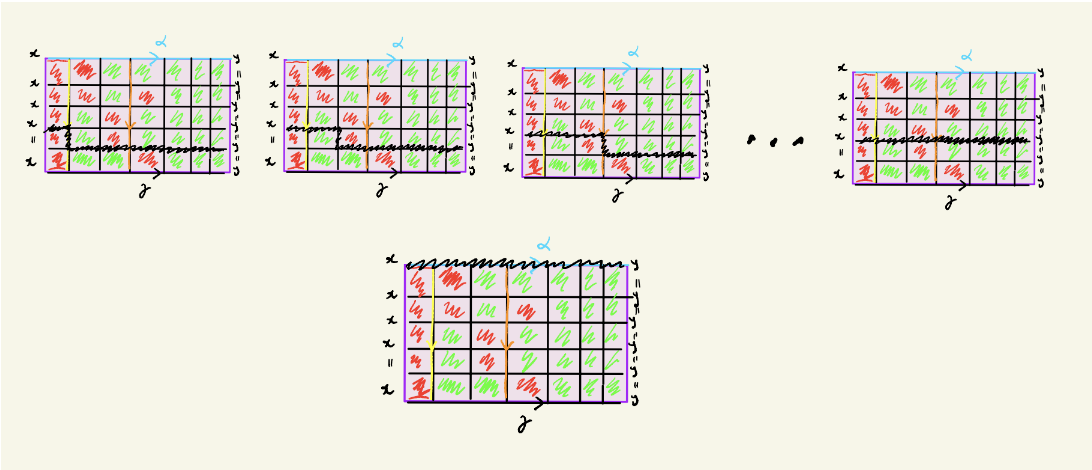


By induction:





Continuing this process by induction, we a sequence of path homotopies respecting the partitions



□

Corollary V.3.13. *Let X be a topological space obtained as a disjoint union of two spaces $X \simeq U \coprod V$. Then $\Pi_1(X) \simeq \Pi_1(U) \coprod \Pi_1(V)$.*

The version of the Van-Kampen theorem of the Theorem **V.3.1** is not convenient for many computations. For all the practical purposes, we will instead work with the following version:

Theorem V.3.14: Van-Kampen for groups

Let X be a topological space. Assume that X is obtained as a union of two non-empty open subsets U and V such that all X, U, V and $U \bigcap V$ are path-connected. Let $x \in U \bigcap V$. Then the functor $\pi_1(-,x)$ sends the pushout of spaces

$$\begin{array}{ccc} U \bigcap V & \longrightarrow & V \\ \downarrow & & \downarrow \\ U & \longrightarrow & X \end{array}$$

to a pushout diagram in GROUPS

$$\begin{array}{ccc} \pi_1(U \bigcap V, x) & \longrightarrow & \pi_1(V, x) \\ \downarrow & & \downarrow \\ \pi_1(U, x) & \longrightarrow & \pi_1(X, x) \end{array} \tag{4}$$

ie

$$\pi_1(X,x) \simeq \pi_1(V,x) \underset{\pi_1(U \bigcap V, x)}{*} \pi_1(U,x)$$

Remark V.3.15. There is also a version of the van Kampen theorem obtained by considering closed subsets instead of open subsets. You can check the proof here. We will need need in this this course.

We will deduce Theorem **V.3.14** from Theorem **V.3.1**. This requires some formal steps:

Construction V.3.16. Let G be a group. In particular, forgetting that the operation has inverses, G is an associative monoid. So we can form the category $\mathsf{B}G$ of the Example **IV.1.25**. Since G is a group, this category is now a groupoid. We have therefore upgraded the functor $\mathsf{B} : \mathsf{MONOIDS} \to \mathsf{CATS}$, to a functor

$$\mathsf{B} : \mathsf{GROUPS} \to \mathsf{GROUPOIDS}$$

Exercise V.3.17. Show that the functor $\mathsf{B} : \mathsf{GROUPS} \to \mathsf{GROUPOIDS}$ of the Construction **V.3.16** preserves pushouts. Use the explicit description of pushouts of groups in the Proposition **IV.5.34** and pushouts of groupoids in the Construction **V.3.2**.

Proposition V.3.18. *Let C be a connected groupoid. Let c be an object in C and $G := \mathrm{Aut}_{\mathsf{C}}(c)$ its group of automorphisms. Then the functor*

$$\mathsf{B}_c : \mathsf{B}\,\mathrm{Aut}_{\mathsf{C}}(c) \to \mathsf{C}$$

sending $\bullet_G \mapsto c$ and defined by the identity on morphisms, defines an equivalence of categories.

The data of an inverse Ω_c to B_c is determined by the choice of a collection of morphisms $\{\alpha_{c'} : c \to c'\}_{c' \in \mathsf{C}}$ for every object $c' \in \mathsf{C}$. In particular, it is possible to choose Ω_c such that $\Omega_c \circ \mathsf{B}_c = \mathsf{id}$.

Proof. Indeed, being the identity on morphisms, the functor is automatically fully faithful. It remains to show that it is essentially surjective. But since C is assume to be connected, every object c' in C is isomorphic to c so B_c is essentially surjective.

An explicity inverse to B_c can be constructed by picking a collection of morphisms $\{\alpha_{c'} : c \to c'\}_{c' \in \mathsf{C}}$ for every object $c' \in \mathsf{C}$: indeed, set $\Omega_c(c') = \bullet_c$ and if $f : c_1 \to c_2$ is a morphism in C, set $\Omega_c(f) = \alpha_{c_2}^{-1} \circ f \circ \alpha_{c_1}$. By choosing $\alpha_c = \mathsf{id}_c$, this defines a functor Ω_c with $\Omega_c \circ \mathsf{B}_c = \mathsf{id}$.

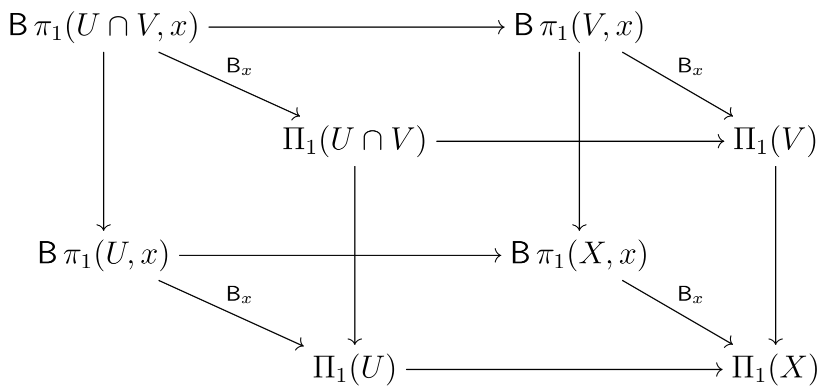
To show that B_c and Ω_c form an equivalence of categories, it remains to exhibit a natural isomorphism of functors $\lambda : \mathsf{id}_{\mathsf{C}} \simeq \mathsf{B}_c \circ \Omega_c$. For each $c' \in \mathsf{C}$, define $\lambda_{c'} = \alpha_{c'}^{-1}$. Then we have for any morphism $f : c_1 \to c_2$ in C, a commutative diagram

$$\begin{array}{ccc} c_1 & \xrightarrow{\alpha_{c_1}^{-1}} & c = \mathsf{B}_c\Omega_c(c_1) \\ \big\downarrow f & & \big\downarrow \mathsf{B}_c\Omega_c(f) := \alpha_{c_2}^{-1} f \alpha_{c_1} \\ c_2 & \xrightarrow{\alpha_{c_2}^{-1}} & c = \mathsf{B}_c\Omega_c(c_2) \end{array}$$

□

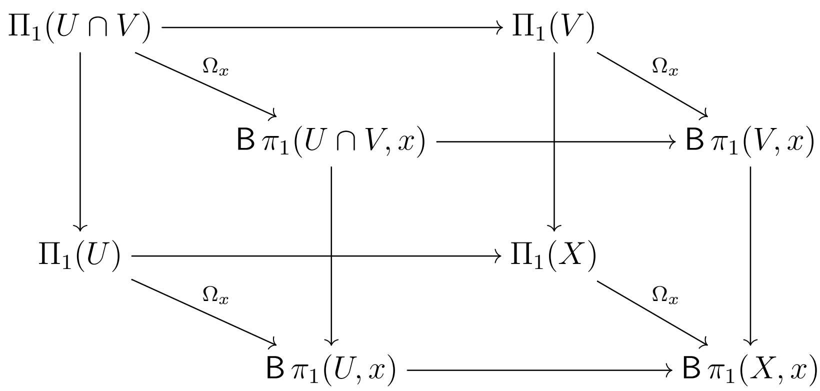
Proof of the Theorem **V.3.14**. Let X be a topological space obtained as a union of two non-empty open subsets U and V such that all X, U, V and $U \bigcap V$ are path-connected. Let $x \in U \bigcap V$. Then the functors B_x of the Proposition **V.3.18** provide a commutative diagram of groupoids

(5)



This is automatic from the definitions and the fact the base point x belongs to the intersection.

Now, since all U, V and $U \cap V$ are path-connected we can choose for every $y \in X$ a path $\alpha_y : x \to y$ such that if y is in U then α_y is a path in U and if y is in V, α_y is a path in V and if y is in $U \cap V$ the two choices of paths coincide. In this case the associated retraction functors Ω_x of the Proposition **V.3.18** provide a retraction of commutative squares (5) as in the Definition **IV.5.38**



So, by the Exercise **IV.5.39** applied to $\mathsf{C} = \mathsf{GROUPOIDS}$, we deduce that

$$\begin{array}{ccc} \mathsf{B}\,\pi_1(U \bigcap V, x) & \longrightarrow & \mathsf{B}\,\pi_1(V, x) \\ \downarrow & & \downarrow \\ \mathsf{B}\,\pi_1(U, x) & \longrightarrow & \mathsf{B}\,\pi_1(X, x) \end{array}$$

is a pushout of groupoids. It follows now from the explicit description of pushouts of groupoids in the Construction **V.3.2** and the explicit description of pushouts of groups in Proposition **IV.5.34**, that the diagram of groups

(6)
$$\begin{array}{ccc} \pi_1(U \bigcap V, x) & \longrightarrow & \pi_1(V, x) \\ \downarrow & & \downarrow \\ \pi_1(U, x) & \longrightarrow & \pi_1(X, x) \end{array}$$

is a pushout.

□

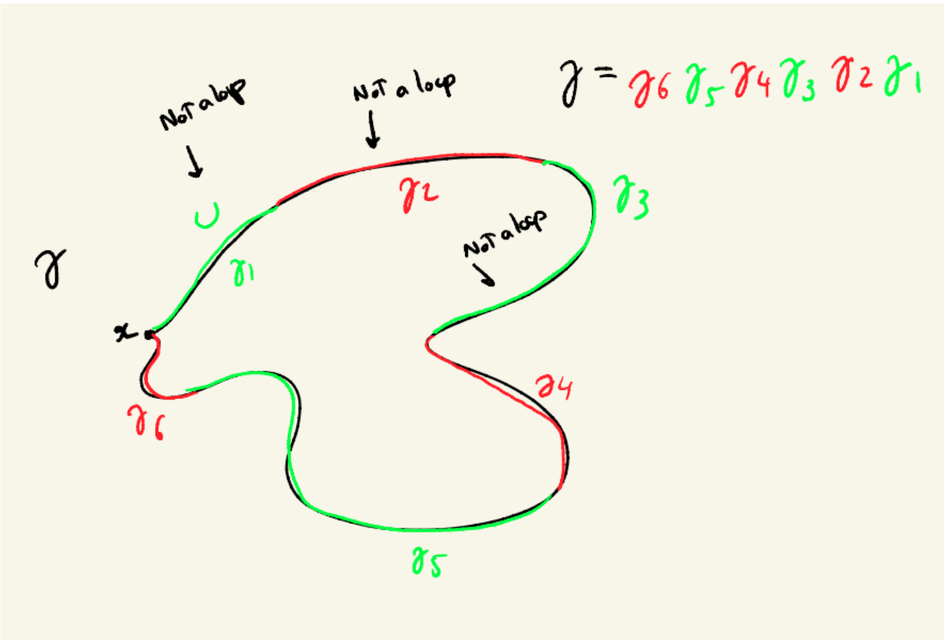
Remark V.3.19. In fact, what we have used in the last part of the proof of the Proposition **V.3.18** when comparing the explicit description of pushouts of groupoids in the Construction **V.3.2** and the explicit description of pushouts of groups in Proposition **IV.5.34**, is that the functor $\mathsf{B} : \mathsf{GROUPS} \to \mathsf{GROUPOIDS}$ preserves pushouts.

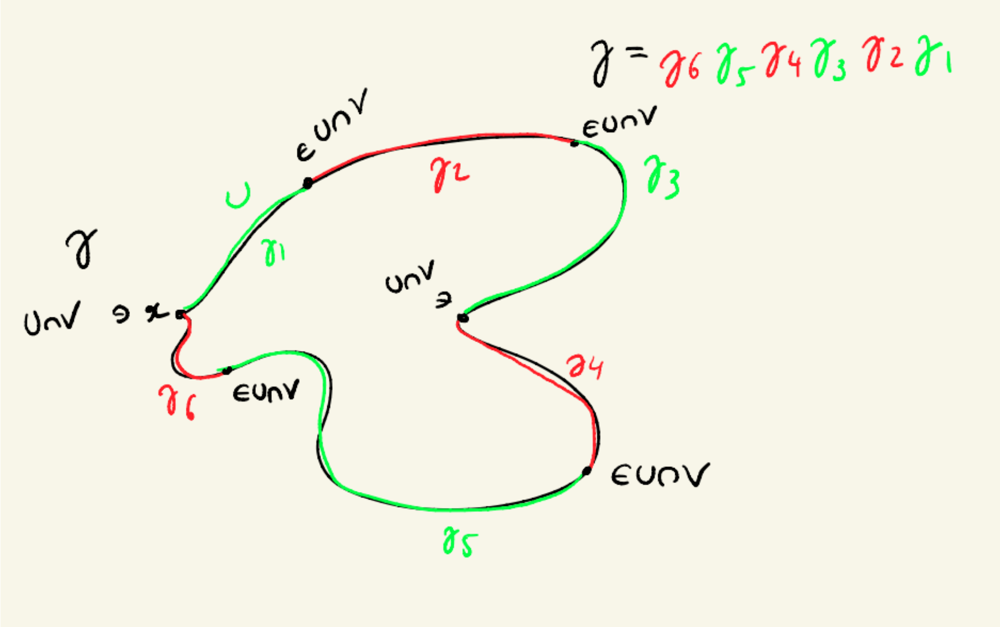
Remark V.3.20. Let us describe in more explicit terms what we have done in the proof of the Theorem **V.3.14**. We wanted to show that the concatenation map

$$\pi_1(U, x) \underset{\pi_1(U \cap V, x)}{*} \pi_1(V, x) \to \pi_1(X, x)$$

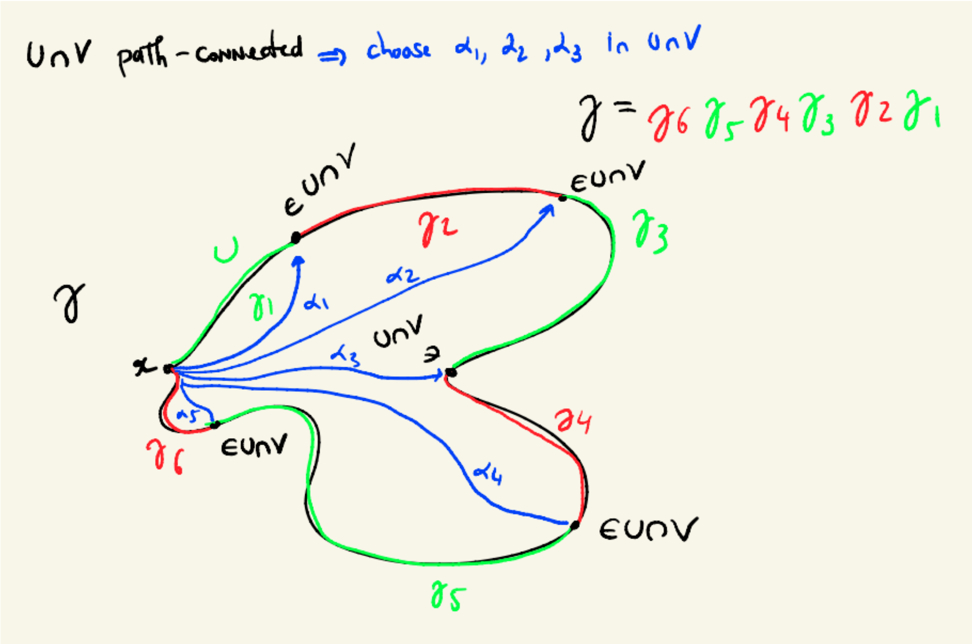
is both injective and surjective. Injectivity follows from the square lemma and the lebesgue number lemma applied to the square.

To show surjectivity we had to use the fact that $U \cap V$ is path-connected. Indeed, if $\gamma : I \to X$ is a loop in X, the partition of γ does not exhibit γ as a concatenation of loops, but rather paths, as shown in the picture:





In the proof of the Theorem **V.3.14** we made a choice of paths $\alpha_1, \alpha_2, \alpha_3$



that allow us to write

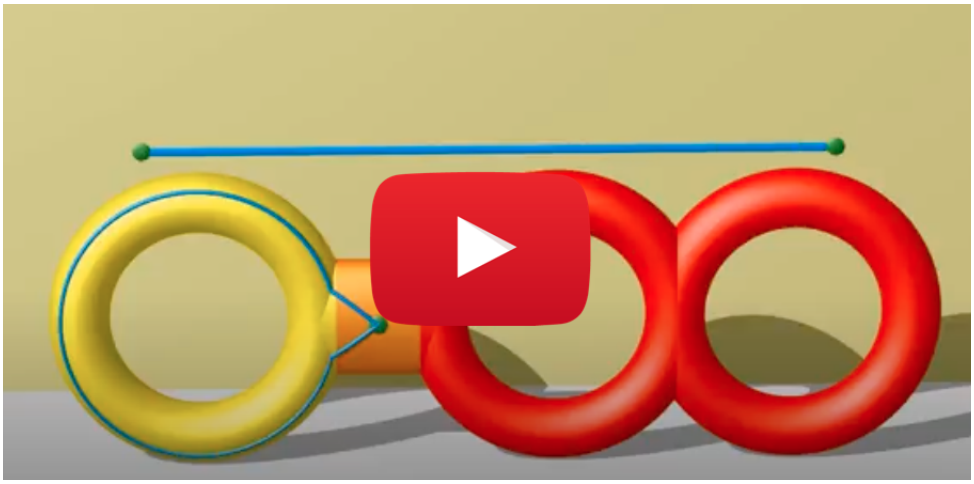
$$\gamma = \gamma_6 * \gamma_5 * \gamma_4 * \gamma_3 * \gamma_2 * \gamma_1$$

as

$$\gamma = \gamma_6 * \alpha_5 * \alpha_5^{-1} * \gamma_5 * \gamma_4 * \alpha_4 * \alpha_4^{-1} * \gamma_3 * \alpha_3 * \alpha_3^{-1} * \alpha_2 * \alpha_2^{-1} * \gamma_2 * \alpha_1 * \alpha_1^{-1} * \gamma_1$$

and therefore, as a concatenation of loops based at x.

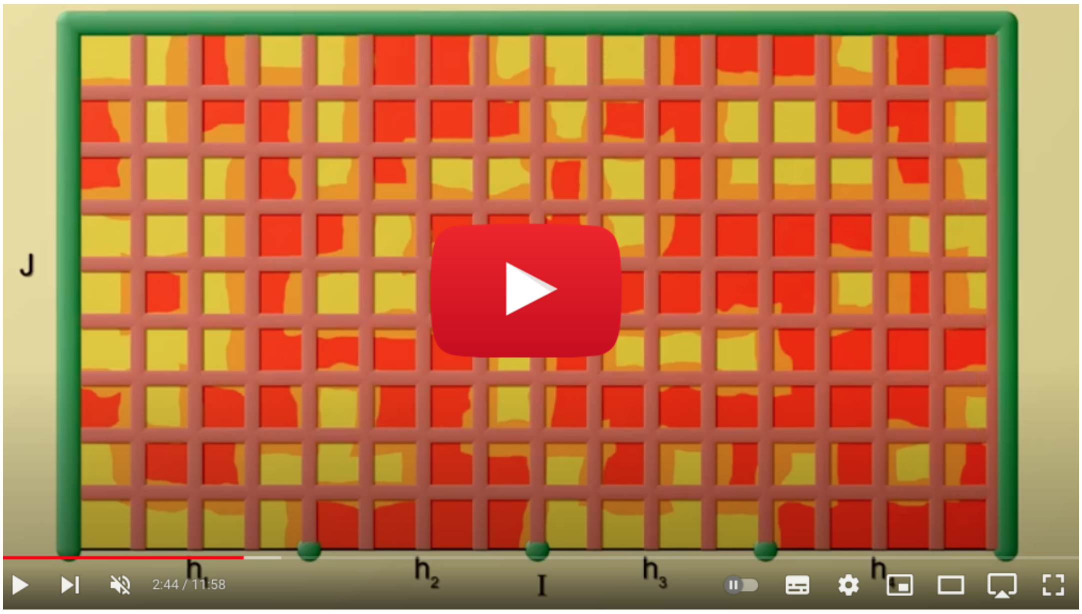
Here's a video illustrating this



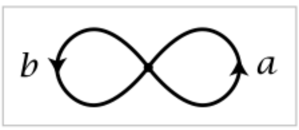
To prove that the morphism

$$\pi_1(U,x) \underset{\pi_1(U\cap V,x)}{*} \pi_1(V,x) \to \pi_1(X,x)$$

is injective, we used the square lemma, the Lebesgue number lemma applied to the square and the fundamental groupoid. In class, we gave a different more direct proof that avoids the use of the fundamental groupoid. Here's a video illustrating a direct argument for the proof of injectivity without using the language of groupoids:

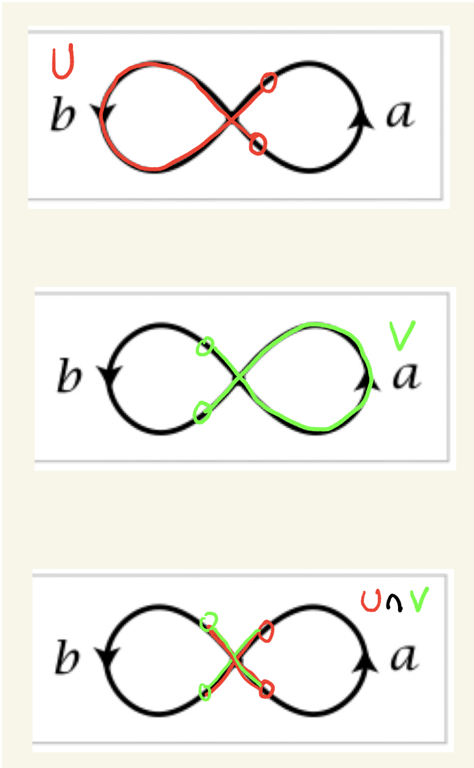


Example V.3.21. We can now finally compute the fundamental group of the wedge of circles $S^1 \vee S^1$ and compare it to the discussion in the Solution **I.2.2**.



(Picture taken from Hatcher's book)

We consider two open subsets U and V of the form



As shown in the picture, their intersection is path-connected. Moreover, the open subset U deformation retracts to the circle b and the open subset V deformation retracts to the circle a. Moreover, the intersection $U \cap V$ deformation retracts to the middle node x.

It follows that

$$\pi_1(U, x) \simeq .b\mathbb{Z}$$

$$\pi_1(V, x) \simeq .a\mathbb{Z}$$

$$\pi_1(U \cap V, x) \simeq \{0\}$$

By the Van-Kampen theorem for groups Theorem **V.3.14**, we have a coproduct of groups

$$\pi_1(\mathsf{S}^1 \vee \mathsf{S}^1, x) \simeq a\mathbb{Z} * b\mathbb{Z}$$

which, by definition is exactly the free group on two generators.

Example V.3.22. As an example, let us use Theorem **V.3.14** to compute the fundamental group of the 2-sphere S^2. Let $N = (0, 0, 1)$ and $S = (0, 0, -1)$ denote respectively the northern and southern poles. Consider the open subsets

$$U_N := S^2 \setminus \{S\} \qquad U_S := S^2 \setminus \{N\}$$

U_N is an open subset contained the northern hemisphere and U_S is an open subset containing the southern hemisphere.

All U_N, U_S and the intersection $U_N \cap U_S = S^2 \setminus \{N, S\}$ is path-connected so can apply the Theorem **V.3.14** to

(7)
$$\begin{array}{ccc} U_N \bigcap U_S & \longrightarrow & U_S \\ \downarrow & & \downarrow \\ U_N & \longrightarrow & S^2 \end{array}$$

Let $x = (1, 0, 0)$. The fundamental group of S^2 is then isomorphic to the pushout of groups

$$\pi_1(S^2, x) \simeq \pi_1(U_N, x) \underset{\pi_1(U_N \bigcap U_S, x)}{*} \pi_1(U_S, x)$$

The stereographic projection from the north pole gives a homeomorphism

$$\phi_N : U_S \underset{\sim}{\rightarrow} \mathbb{R}^2$$

Its restriction to the intersection $U_S \cap U_N$ gives one identification $\phi_S : U_S \cap U_N \underset{\sim}{\rightarrow} \mathbb{R}^2 \setminus \{0\}$.

The stereographic projection from the south pole gives a homeomorphism

$$\phi_S : U_N \underset{\sim}{\rightarrow} \mathbb{R}^2$$

and another identification $\phi_N : U_N \cap U_S \underset{\sim}{\rightarrow} \mathbb{R}^2 \setminus \{0\}$.

The fundamental groups are thus given by

$$\pi_1(U_N, x) \simeq \{0\} \qquad \pi_1(U_S, x) = \{0\} \qquad \pi_1(U_N \cap U_S, x) \simeq \mathbb{Z}$$

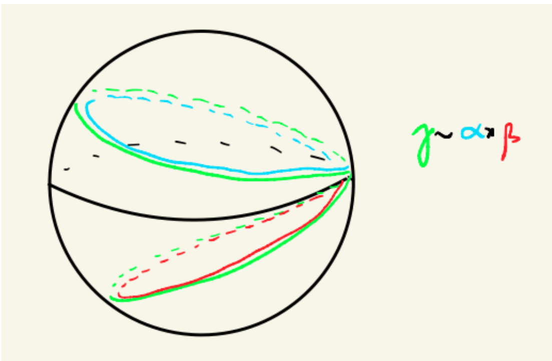
Since $\phi_N^{-1} \circ \phi_S$ is the identity on the unit circle in $\mathbb{R}^2 \setminus \{0\}$, the induced diagram of groups

(8)
$$\begin{array}{ccc} \pi_1(U_N \bigcap U_S, x) & \longrightarrow & \pi_1(U_S, x) \\ \downarrow & & \downarrow \\ \pi_1(U_N, x) & \longrightarrow & \pi_1(X, x) \end{array}$$

is isomorphic to

(9)
$$\begin{array}{ccc} \mathbb{Z} & \longrightarrow & \{0\} \\ \downarrow & & \downarrow \\ \{0\} & \longrightarrow & \pi_1(X, x) \end{array}$$

So, by the Exercise **IV.5.35**, the pushouts are isomorphic
$$\pi_1(S^2, x) \simeq \pi_1(U_N, x) \underset{\pi_1(U_N \bigcap U_S, x)}{*} \pi_1(U_S, x) \simeq \{0\} \underset{\mathbb{Z}}{*} \{0\}$$



But by the explicit description of the pushout of groups in the Proposition **IV.5.34** as the free groups generated by words, the result is the trivial group:

$$\pi_1(S^2, x) \simeq \{0\}$$

Remark V.3.23. As already mentioned in the Example **III.4.15**, the sphere S^2 is not contractible. But the computation in the Example **V.3.22** shows that S^2 is simply-connected, ie, $\pi_1(S^1, x) = \{0\}$. In the same way that to prove the circle S^1 is not contractible we had to introduce the machinery of fundamental groups, to show that S^2 is non-contractible one needs a new kind of machinery, namely, that of second homotopy groups $\pi_2(X, x)$. One can show that $\pi_2(S^2) \simeq \mathbb{Z}$. More generally, one can associate higher homotopy groups $\pi_n(X)$ for all $n \geq 1$. For the sphere we have $\pi_n(S^n) \simeq \mathbb{Z}$ showing that none of the higher dimensional spheres are contractible. These theorems are beyond the scope of this introductory course and are really the beginning of the subject of algebraic topology.

Exercise V.3.24. Let $n \geq 2$. Show that $\pi_1(S^n) = 0$ TD, Exo 3, Feuille 3

Exercise V.3.25. Let X be a path-connected space. Compute the fundamental group of the suspension $S(X)$. Hint: use the Exercise **II.5.52**.

Exercise V.3.26. Show that $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^n$ for $n \neq 2$ TD, Exo 4, Feuille 3

Exercise V.3.27 (Eckmann-Hilton). TD, Exo 13, Feuille 3

(i) Let M be a set. We suppose M is equipped with two product laws

$$* : M \times M \to M \quad , \quad \bullet : M \times M \to M$$

verifying the following properties:
- Each law has a unit element, respectively 1_* et $1_\bullet$.
- The map $* : M \times M \to M$ is compatible with the operation $\bullet$, i.e.,

$$(x \bullet x') * (y \bullet y') = (x * y) \bullet (x' * y')$$

  (a) Show thate $1_* = 1_\bullet$.
  (b) Show that the two laws $*$ et $\bullet$ are equal.
  (c) Show that the two product laws define a single monoid structure on M that is abelian.

(ii) Let $(G, *, e)$ be a path-connected topological group.
  (a) Show that the group law of G induces an extra group law on $\pi_1(G, e)$: If $\alpha, \beta \colon [0, 1] \to G$ are two loops in G at e, we denote by $\alpha * \beta$ the new loop defined by pointwise multiplication in G, $\alpha * \beta(t) = \alpha(t) * \beta(t)$ for all $t \in [0, 1]$.
  (b) If $\alpha, \beta \colon [0, 1] \to G$ are two loops in G at e, we denote by $\alpha \circ \beta$ their path-concatenation. Show that $M = \pi_1(G, e)$ with the two product laws $*$ et $\circ$ fits in the situation of question (1) above, i.e., the law $*$ is compatible with $\circ$.
  (c) Show that $\alpha \circ \beta, \beta \circ \alpha$ et $\alpha * \beta$ are path homotopic.
  (d) Conclude that $\pi_1(G, e)$ is abelian.

Exercise V.3.28. Show that any continuous map $f : D^2 \to D^2$ admits a fixed point. TD, Exo 5, Feuille 3

Exercise V.3.29. Show that any complex polynomial which is non-constant, admits at least one root. TD, Exo 6, Feuille 3

Exercise V.3.30. Let Σ_2 be the compact connected surface obtained by connected sum of two torus. Show that its fundamental group admits a presentation as TD, Exo 7, Feuille 3

$$< a_1, b_1, a_2, b_2, ..., a_n, b_n | [a_1, b_1].[a_2, b_2] = 1 >$$

TD, Exo 8, Feuille 3

Exercise V.3.31. Show using the Van-Kampen theorem that $\pi_1(\mathbb{R}P^2) \simeq \mathbb{Z}/2$.

TD, Exo 6, Feuille 5

Exercise V.3.32. Use the Van-Kampen theorem to compute the fundamental group of the Klein bottle K (see Exercise **II.5.40**).

Exercise V.3.33. Let X be a topological space. Show that $S(X)$ is simply-connected.

TD, Exo 16, Feuille 3

Exercise V.3.34. [Borsuk-Ulam theorem in dimension 2] Let $f : S^2 \to \mathbb{R}^2$ be a continuous function. Show that there exists a point $x \in S^2$ such that $f(x) = f(-x)$.

Exercise V.3.35. Give an example of a path-connected topological space whose fundamental group is isomorphic to the free product $\mathbb{Z} * \mathbb{Z}/2$.

Exercise V.3.36. Let X be the space obtained by removing the three positive half-axis from the 3-dimensional disk $D^3 \subseteq \mathbb{R}^3$. Compute the fundamental group of X.

CHAPTER VI

Covering Spaces and their classification

We start by axiomatizing the key property of the exponential map seen in the Proposition **V.2.6**:

Goal VI.0.1. In this chapter we will finally explain the mathematics of the Exercise **I.2.10**.

Warning VI.0.2. The approach taken in this chapter is far from being the fastest. In fact, it is the most exhaustive, as it tries to distillate as much as possible the topological side from the algebraic side of the theory. In the main lectures we will short-circuit it and go directly from covers to subgroups. These notes are intended as side material in case you want to see what is really going on behind the scenes.

The main reference for the lectures are the notes of Ilia Itenberg.

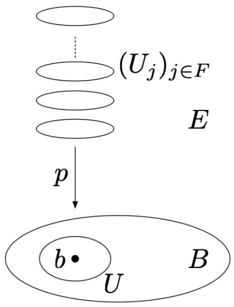
VI.1. Covering Spaces

Definition VI.1.1. Let $p : E \to X$ be a continuous map. We say that p is a covering map (revêtement en français) (or that E is a covering space of X) if:

(i) The map p is surjective;

(ii) for every $b \in X$ there exists an open neighborhood U of b such that $F := p^{-1}(U)$ consists of a disjoint union of open subsets V_j of E, with $j \in p^{-1}(\{b\})$, and such that each restriction $p_{|V_j} : V_j \to U$ is a homeomorphism.

We say that X is the base of the covering space and E is the total space. An open neighborhood U satisfying the condition in (ii) is called a trivializing neighborhood.



Remark VI.1.2. Let $p : E \to X$ be a covering map. Let $b \in X$ and U a trivializing neighborhood. Then the map $p_{|V_i} : V_i \to U$ admits a canonical section given by the inverse $s_i = p_{|V_i}^{-1}$. This section is uniquely determined and verifies $V_i := s_i(U)$.

Proposition VI.1.3. *A map $p : E \to X$ is a covering map if and only if for every point $b \in X$ there exists an open neighborhood U of b in X, a discrete space F and a homeomorphism $\phi : p^{-1}(U) \to U \times F$ such that the diagram commutes:*

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{\ \phi\ } & U \times F \\ & {\scriptstyle p}\searrow \quad \swarrow & \\ & U & \end{array}$$

The map ϕ is called a local trivialization of p.

Example VI.1.4. Let X be a topological space and F a discrete space. Then the projection $\pi_X : X \times F \to X$ is a covering map. Indeed, it satisfies the condition in the Proposition **VI.1.3** with ϕ being the identity.

Example VI.1.5. The exponential map $\exp : \mathbb{R} \to \mathsf{S}^1$ is a covering map. This is the Proposition **V.2.6**.

Question VI.1.6. Is the projection map $\mathbb{R}^2 \to \mathbb{R}$ sending $(x, y) \mapsto x$ a covering map?

We now state a series of general properties of covering spaces.

Proposition VI.1.7. *Let $p : E \to X$ be a covering map. Then:*

(i) If $E' \to X'$ is another covering map, then the product $E' \times E \to X' \times X$ is a covering map.
(ii) Given any $f : Y \to X$ continuous map, the pullback $E \underset{X}{\times} Y \to Y$ is a covering map.

(iii) If X is Hausdorff then so is E.
(iv) If X is connected, then all the fibers are isomorphic to the same discrete space F.
(v) If both E and X are compact, then all the fibers are finite.
(vi) p is a local homeomorphism.

Proof.

- (i) and (ii) follow directly from the definition of a cover using local trivializations.
- (iii) Let $e, e' \in E$ and choosing separating open neighborhoods of $p(e)$ and $p(e')$ in X, U and V. Find trivializing neighborhoods S and W for $p(e)$ and $p(e')$ and take $A = U \cap S$ and $B = V \cap W$. Then $p^{-1}(A)$ and $p^{-1}(B)$ are separating open neighborhoods for e and e'.
- Using the local trivializations, the cardinality of the fibers form a locally constant function. Since X is assume to be connected, the Proposition **II.6.22** establishes the result.
- Let $x \in X$. Then $p^{-1}(\{x\}) \subseteq E$ is a closed subset of E which is compact. By the Exercise **II.3.19**, $p^{-1}(\{x\})$ must be compact. But thanks to the definition of a covering map, $p^{-1}(\{x\})$ is discrete with respect to the subspace topology. Therefore it must be finite.
- Follows directly from the local trivializations.

□

Remark VI.1.8. It follows from the Proposition **VI.1.7**-(vi) that if $p : E \to X$ is a covering map and X is locally "bla" then E is locally "bla".

Definition VI.1.9. Let $p : E \to X$ be a covering map with X connected. The degree of p is the cardinality of its fibers.

Definition VI.1.10. Let X be a topological space endowed with an action of a (discrete) group G with unit element $e \in G$. We say that the action of G on X is a covering action if every $x \in X$ there exists an open neighborhood U such that all the images $g(U)$ for varying $g \in G$ are disjoint, ie $g_1(U) \cap g_2(U) = \emptyset$ for all $g_1, g_2 \in G$. TD, Exo 1, Feuille 4

Proposition VI.1.11. *Let X be a topological space endowed with an action of a (discrete) group G with unit element $e \in G$ If the the action is a covering action as in the Definition **VI.1.10**, then the quotient map $X \to X/G$ is a covering map.*

Proof. Recall from Exercise **II.5.41** that the quotient map $p : X \to X/G$ is an open map: if U is an open in X then $p^{-1}(p(U)) = \bigcup_{g \in G} g(U)$. If U is as in the statement of the proposition, the assumption of a covering action implies that this is a disjoint union $p^{-1}(p(U)) = \coprod_{g\in G} g(U)$ and that U satisfies the covering condition. □

Exercise VI.1.12. Let G be a (discrete) group acting on space X. We say that the action is properly discontinuous if every point $x \in X$ admits an open neighborhood U such that the subset $\{g \in G : g(U) \cap U \neq \emptyset\} \subseteq G$ is finite. Show that if X is Hausdorff and the action is properly discontinuous then the action is a covering action.

Exercise VI.1.13. Let G be a (discrete group) acting on a Hausdorff space X. Show that if the action is free and the group is finite then the action is a covering action.

TD, Exo 1, Feuille 4

Exercise VI.1.14. In this exercise we extend the results of the Proposition **VI.1.11** to include "topological groups", ie, we assume that G is a topological group and that the action of G on X is continuous. Assume both X and G are locally compact spaces and consider the following assertions:

(i) The pre-image of every compact along the action map $G \times X \to X \times X$ defined by $(g, x) \mapsto (x, gx)$, is compact.

(ii) For every K compact in X, the subset of G given by $\{g \in G : gK \cap K \neq \emptyset\}$ is compact in G.

(iii) For every K compact in X, the subset of G given by $\{g \in G : gK \cap K \neq \emptyset\}$ is compact in G.

(iv) Every point $x \in X$ admits a compact neighborhood K such that $\{g \in G : gK \cap K \neq \emptyset\} = \{e\}$

Show that:

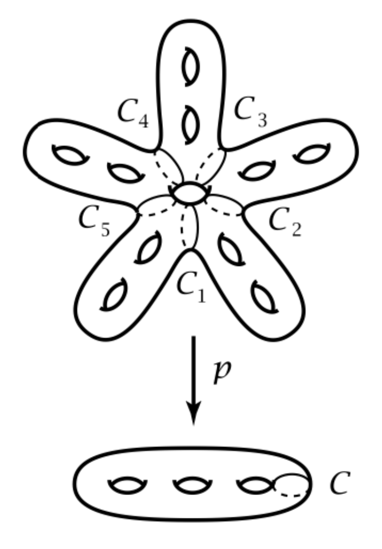
(i) (i) and (ii) are equivalent.
(ii) If G has the discrete topology, then (i), (ii) and (iii) are equivalent
(iii) If G has the discrete topology and the action is free, then all (iii) implies (iv).
(iv) Show that for X locally compact and G discrete, (4) implies that the quotient map $X \to X/G$ is a covering map and that the quotient X/G is Hausdorff.

Exercise VI.1.15. Show that the maps $\mathsf{S}^1 \to \mathsf{S}^1$ given by $z \mapsto z^n$ are covering maps for all $n \geq 1$

Exercise VI.1.16. Show that the quotient map $S^n \to \mathbb{R}P^n$ is a covering map of degree 2.

Exercise VI.1.17. Show that the quotient map $\mathbb{R}^2 \to K$ of the Exercise **II.5.40** defining the Klein bottle, is a covering map.

Example VI.1.18. Here's a 5-sheeted cover of a surface of genus 3, $\Sigma_{11} \to \Sigma_3$ obtained by quotient of Σ_{11} by the action of $\mathbb{Z}/5$



(Picture taken from Hatcher's book, Example 1.41)

Exercise VI.1.19. Let $X \to Y$ be a local homeomorphism and U an open in X. Show that the restriction of f to U still defines a local homeomorphism. Use this to construct an example of a local homeomorphism that is not a covering map. TD, Exo 5, Feuille 4

Exercise VI.1.20. Show that a local homeomorphism $f : X \to Y$ between compact Hausdorff spaces is a covering space.

Exercise VI.1.21. TD, Exo 7, Feuille 4

(i) Show that the exponential map $\exp : \mathbb{C} \to \mathbb{C}^*$ is a covering map.
(ii) Show that the map $\mathbb{C}^* \to \mathbb{C}^*$ sending $z \mapsto z^2$ is a covering map. Is the same thing true for the same formula seen as a map $\mathbb{C} \to \mathbb{C}$?

Exercise VI.1.22. Construct a 2-sheet covering map $\mathsf{S}^1 \times \mathsf{S}^1 \to K$. TD, Exo 8, Feuille 4

Definition VI.1.23. Let $p_1 : E_1 \to X$ and $p_2 : E_2 \to X$ be covering maps with the same base space. A morphism of coverings of X, $(E_1, p_1) \to (E_2, p_2)$ is a map $\xi : E_1 \to E_2$ such that the diagram commutes

$$\begin{array}{ccc} E_1 & \xrightarrow{\ \xi\ } & E_2 \\ & {\scriptstyle p_1}\searrow \quad \swarrow{\scriptstyle p_2} & \\ & X & \end{array}$$

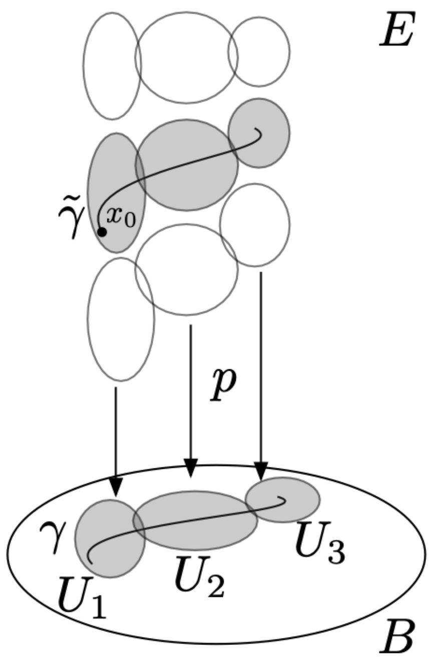
Definition VI.1.24. Let X be a topological space. Morphisms of covering spaces of X can be composed. We denote by COV_X the category whose objects are covering spaces with base X and morphisms are morphisms of coverings in the sense of Definition **VI.1.23**.

VI.2. Covering Spaces and Fundamental group

We start this section by specifying the lifting properties of covering maps that make them behave like the exponential map $\exp : \mathbb{R} \to \mathsf{S}^1$:

Lemma VI.2.1. *Let $p : E \to X$ be a covering map. Then any path in X admits a lifting. Moreover, the lifting is unique if the starting point is fixed.*

Proof. This is mutatis-mutandis the proof of the Lemma **V.2.4** replacing that specific trivializing cover by a general one.



(Picture taken from le Polycopie d'Itenberg)

□

Lemma VI.2.2. *Let $p : E \to X$ be a covering map. Let $H : I \times I \to X$ be a homotopy of paths, between γ and β in X having the same endpoints. Fix $x_0 \in E$ a lift of $p(x_0) = \gamma(0) = \beta(0)$. Then the homotopy H admits a unique lifting to a path homotopy $\widetilde{H} : I \times I \to E$ between the unique lifting $\widetilde{\gamma}$ of γ at x_0 and the unique lifting $\widetilde{\beta}$ of β at x_0.*

Proof. Apply mutatis-mutandis the proof of the Lemma **V.2.14** replacing that specific trivializing cover by a general one. ☐

Proposition VI.2.3. *Let $p : E \to X$ be a covering map. Then the functor $\Pi_1(f) : \Pi_1(E) \to \Pi_1(X)$ satisfies the following properties:*

(i) $\Pi_1(p)$ is surjective on objects;

(ii) For every morphism $f : x \to y$ in $\Pi_1(X)$ and for every object $e \in \Pi_1(E)$ with $p(e) = x$, there exists a unique lift of f to $\Pi_1(E)$.

Proof. This a direct consequence/reformulation of the lifting of paths and homotopies of paths of Lemma **VI.2.1** and Lemma **VI.2.2**.. ☐

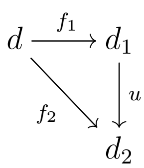
Let us axiomatize this

Definition VI.2.4. Let $F : \mathsf{C} \to \mathsf{D}$ be a functor between groupoids. We say that F is a covering map of groupoids if it satisfies the following two properties:

(i) F is surjective on objects;

(ii) For every morphism $x \to y$ in D and every $c \in \mathsf{C}$ with $F(c) = x$, there exists a unique lift $c \to d$ in C of the morphism $x \to y$.

Example VI.2.5. Let D be a connected groupoid. Fix $d \in \mathsf{D}$ an object. Then the target functor $t : \mathsf{D}_{d/.} \to \mathsf{D}$ of the Example **IV.1.19** is a covering map of groupoids. Indeed, given $u : d_1 \to d_2$ and an object $f_1 : d \to d_1$ in $\mathsf{D}_{d/.}$, there exists a unique lifting of u starting at f_1, namely, the morphism



with $f_2 := u \circ f_1$.

Remark VI.2.6. Let $F : \mathsf{C} \to \mathsf{D}$ be a covering map of groupoids. If C is connected then D is connected. Indeed, let d_1 and d_2 in D. Since F is surjective on objects, $d_1 = F(c_1)$ and $d_2 = F(c_2)$. But C is connected, so there exists an isomorphism $c_1 \to c_2$ in C. Therefore, $F(c_1) \to F(c_2)$ is an isomorphism, showing that D is connected.

We now extract some consequences of this general definition.

Proposition VI.2.7. *Let $F : \mathsf{C} \to \mathsf{D}$ be a covering map of groupoids. Then:*

(i) Let $c \in \mathsf{C}$ and $x = F(c) \in \mathsf{D}$. Then the map of groups

$$F : \mathrm{Hom}_{\mathsf{C}}(c, c) = \mathrm{Aut}_{\mathsf{C}}(c) \to \mathrm{Hom}_{\mathsf{D}}(x, x) = \mathrm{Aut}_{\mathsf{D}}(x)$$

is injective. In particular it defines a subgroup of $\mathrm{Aut}_{\mathsf{D}}(x)$, which we will simply denote by $F(\mathrm{Aut}_{\mathsf{C}}(c))$.

(ii) If c_1 and c_2 belong to the same connected component of C (ie, represent the same class in $\pi_0(\mathsf{C})$) with $F(c_1) = F(c_2) = x$ then the two subgroups

$$F(\mathrm{Aut}_{\mathsf{C}}(c_1)) \hookrightarrow Aut_{\mathsf{D}}(x)$$

$$F(\mathrm{Aut}_{\mathsf{C}}(c_2)) \hookrightarrow \mathrm{Aut}_{\mathsf{D}}(x)$$

are conjugated.

(iii) Conversely, let $c \in \mathsf{C}$ and $x = F(c)$ in D. Then, for any morphism $\alpha : x \to x$ in D there exists an object c' in C such that the conjugation $\alpha.F(\mathrm{Aut}_{\mathsf{C}}(c)).\alpha^{-1}$ is equal to the subgroup $F(\mathrm{Aut}_{\mathsf{C}}(c'))$

Proof.

(i) Since the map is automatically a map of groups because F is a functor, it is enough to show that if $F(f : c \to c) = \mathsf{id}_x$ then $f = \mathsf{id}_c$. But since f provides a lifting for the identity of x, by the property (ii) of Definition **VI.2.4**, f must be the identity of c.

(ii) If c_1 and c_2 belong to the same connected component, by definition, it means there exists an isomorphism $u : c_1 \to c_2$ in C. As in the Remark **IV.2.8**, u induces an isomorphism of groups through conjugation

$$\mathrm{Aut}_{\mathsf{C}}(c_1) \to \mathrm{Aut}_{\mathsf{C}}(c_2)$$

$$g \mapsto ugu^{-1}$$

Since $F(u) : F(c_1) = x \to F(c_2) = x$, $F(u)$ is an automorphism of x in C. Functoriality tells us that $F(\mathrm{Aut}_{\mathsf{C}}(c_1))$ is conjugated to $F(\mathrm{Aut}_{\mathsf{C}}(c_2))$ via conjugation with $\alpha = F(u)$.

(iii) Given $\alpha : x \to x$, since c is a lift of x, property (ii) in Definition **VI.2.4** guarantees the existence and uniqueness of a lifting of $\alpha : c \to c'$ with $F(c') = x$. We now conclude by applying (iii) above.

□

Corollary VI.2.8. *Let $p : E \to X$ be a covering map. Let $x \in X$.*

(i) Let $e \in p^{-1}(\{x\})$. Then the induced map $\pi_1(E, e) \to \pi_1(X, x)$ is injective.

(ii) If $e, e' \in p^{-1}(\{x\})$ are in the same path-connected component of E, then the subgroups $p_*(\pi_1(E, e))$ and $p_*(\pi_1(E, e'))$ are conjugated.

(iii) Any subgroup of $\pi_1(X, x)$ conjugated to $p_*(\pi_1(E, e))$ is of the form $p_*(\pi_1(E, e'))$ for some $e' \in p^{-1}(\{x\})$ with e' in the same path-connnected component of e in E.

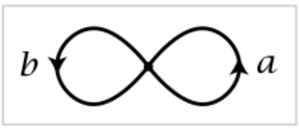
What this corollary shows is that for any topological space X with a choice of base point x we have a well-defined map of sets

$$\{ \text{ Iso. class of (connected) covering maps of } X\} \to \{\text{Conjugacy class. of subgroups of } \pi_1(X, x) \ \}$$

sending a (connected) covering map $p : E \to X$ to the conjugacy class of the subgroup $p_*(\pi_1(E, e))$.

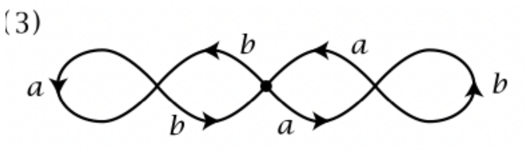
The main goal of this chapter is to provide a proof that this assignment is a bijection: there are as many covering maps as conjugacy classes of subgroups of $\pi_1(X, x)$.

Example VI.2.9. The Corollary **VI.2.8** finally gives a first precise sense to the phenomena illustrated for the labyrinth of the Remark **I.2.11**. In this language, the labyrinth is the choice of the covering map and the subgroup is given by $p_*(\pi_1(E, e)) \subseteq \pi_1(X, x)$. More concretely, by taking the space $X = \mathsf{S}^1 \vee \mathsf{S}^1$ with x the point at the intersection we have seen in the Example **V.3.21** that $\pi_1(X, x)$ is the free group with two generators $< a, b >$.



(Picture taken from Hatcher's book)

In this case the labyrinth E given by the graph



(Picture taken from Hatcher's book)

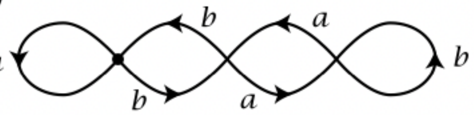
is a covering map $p : E \to X$ by declaring that p sends the nodes to the unique node in X, all paths named a to a in X and all paths named b to b in X.

Let us choose the darker node as a base point in E, denote it by e. Let us denote by x the node in X so that $x = p(e)$. The Corollary **VI.2.8** explains why $p_*(\pi_1(E, e))$ is a subgroup of $\pi_1(X, x) =< a, b >$. As a subgroup we have

$$p_*(\pi_1(E, e)) =< a^2, b^2, aba^{-1}, bab^{-1} >$$

Now, we could pick another point e' in the fiber of x, such as

(4)



(Picture taken from Hatcher's book)

In this case we have

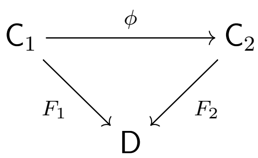
$$p_*(\pi_1(E, e')) =< a, b^2, ba^2b^{-1}, baba^{-1}b^{-1} >$$

Now the Corollary **VI.2.8** explains why the two subgroups are conjugated: since E is path-connected, and $p(e) = p(e')$, any path between e and e' in E gives us a way to conjugate the subgroups. For instance, take the path b^{-1} from e' to e. In this case we find

$$b.p_*(\pi_1(E, e)).b^{-1} = p_*(\pi_1(E, e'))$$

Before proceeding to the next section, it will be convenient to isolate the nature of what we did in this section, passing from a covering map of spaces to a covering of groupoids:

Definition VI.2.10. Let D be a groupoid. We denote by $\mathsf{COV_D}$ the category whose objects are functors $F : \mathsf{C} \to \mathsf{D}$ that are coverings maps of groupoids and morphisms given commutative diagrams

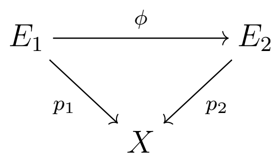


Exercise VI.2.11. Let X be a topological space. Show that the Π_1 extends to a functor

$$\mathsf{COV}_X \to \mathsf{COV}_{\Pi_1(X)}$$

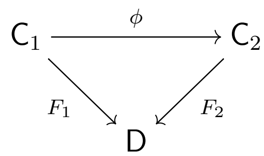
sending a covering map $p : E \to X$ to the covering map of groupoids $\Pi_1(E) \to \Pi_1(X)$.

Exercise VI.2.12. Consider a morphism of covering spaces



with E_2 path-connected and locally connected. Show that ϕ is itself a covering map.

Exercise VI.2.13. Consider a map of covering groupoids.



Show that if ϕ is surjective on objects then ϕ is itself a covering map of groupoids.

Proposition VI.2.14. *Let $F : \mathsf{C} \to \mathsf{D}$ be a covering of groupoids. Show that if F is bijective on objects then F is an isomorphism of groupoids.*

Proof. We need to show that F is also bijective on morphisms, ie, for any pair of objects $x, y \in \mathsf{C}$, the map

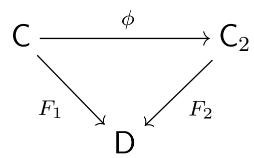
$$F : \mathrm{Hom}_{\mathsf{C}}(x, y) \to \mathrm{Hom}_{\mathsf{D}}(F(x), F(y))$$

is a bijection. The fact that it is surjective is the existence of liftings. The fact that it is injective, is the uniqueness of liftings.

□

Combining these two exercices we obtain

Corollary VI.2.15. *Let*



be a morphism of covering maps of groupoids. Then ϕ is an isomorphism if and only if it induces a bijection on objects.

Proof. Thanks to the Exercise **VI.2.13**, ϕ is itself a covering of groupoids. Since it is a bijection on objects, we use Proposition **VI.2.14** to conclude. □

VI.3. Classification Theorem - The algebraic side

We will first deal with the non-topological aspects of the theorem, ie, we separate as much as possible the topology (covering maps) from everything that is of algebraic nature (coverings of groupoids).

First we exhibit the information contained in a covering map of groupoids, in a different way:

Construction VI.3.1 (Fiber functor)**.** Let $F : \mathsf{C} \to \mathsf{D}$ be a covering map of groupoids. We associate to F a functor

$$\mathsf{Fib}_F : \mathsf{D} \to \mathsf{SETS}$$

defined as follows:

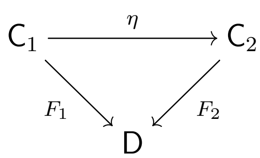
- For every object $x \in \mathsf{D}$, we define $\mathsf{Fib}_F(x) := F^{-1}(\{x\})$.
- For every morphism $f : x \to y$ in D we define $\mathsf{Fib}_F(f) : F^{-1}(\{x\}) \to F^{-1}(\{y\})$ by sending $c \in F^{-1}(\{x\})$ to the target of the unique morphism $\tilde{f}$ lifting f and starting at c.
- if $f : x \to y$ and $g : y \to z$ are morphism in D the composition

$$F^{-1}(\{x\}) \to F^{-1}(\{y\}) \to F^{-1}(\{z\})$$

is well-defined and associative thanks to the uniqueness property of liftings in (ii) in Definition **VI.2.4**. The same argument guarantees compatibility with identities.

The functor Fib_F is called the Fiber functor associated to the covering map of groupoids.

Construction VI.3.2. The construction of the Fiber functor sending a covering map of groupoids $F : \mathsf{C} \to \mathsf{D}$ to $\mathsf{Fib}_F : \mathsf{D} \to \mathsf{SETS}$, is functorial, ie, given a morphism of covering maps of groupoids,



we have an associated natural transformation of functors $\mathsf{Fib}_\eta : \mathsf{Fib}_{F_1} \to \mathsf{Fib}_{F_2}$ defined on each object $x \in \mathsf{D}$ by the map

$$\mathsf{Fib}_{F_1}(x) = F_1^{-1}(\{x\}) \xrightarrow{\eta} \mathsf{Fib}_{F_2}(x) = F_2^{-1}(\{x\})$$

sending $c_1 \mapsto \eta(c_1)$. This is well-defined because $F_2 \circ \eta = F_1$.

We now check that this is a natural transformation. We have to explain why for each morphism $f : d \to d'$ in D, the diagram

$$\begin{array}{ccc} F_1^{-1}(d) & \xrightarrow{\eta_d} & F_2^{-1}(d) \\ \downarrow{\scriptstyle \mathsf{Fib}_{F_1}(f)} & & \downarrow{\scriptstyle \mathsf{Fib}_{F_2}(f)} \\ F_1^{-1}(d') & \xrightarrow{\eta_{d'}} & F_2^{-1}(d') \end{array}$$

commutes. Given $c_1 \in F_1^{-1}(d)$, the object $\mathsf{Fib}_{F_1}(f)(c_1)$ is the target of the unique lifting $\tilde{f}$ of f along F_1 starting at c_1, $\tilde{f} : c_1 \to \mathsf{Fib}_{F_1}(f)(c_1)$. Applying η we get $\eta(\tilde{f}) : \eta(c_1) \to \eta(\mathsf{Fib}_{F_1}(f)(c_1))$. This is a lifting of f along F_2 starting at $\eta(c_1)$. Since such a lifting is unique, we must have $\eta(\mathsf{Fib}_{F_1}(f)(c_1)) = \mathsf{Fib}_{F_2}(f)(\eta(c_1))$.

Finally, one must establish that given two composable maps of coverings of groupoids

$$\begin{array}{ccccc} \mathsf{C}_1 & \xrightarrow{\eta_1} & \mathsf{C}_2 & \xrightarrow{\eta_2} & \mathsf{C}_3 \\ & {\scriptstyle F_1}\searrow & \downarrow{\scriptstyle F_2} & \swarrow{\scriptstyle F_3} & \\ & & \mathsf{D} & & \end{array}$$

we have $\mathsf{Fib}(\eta_2 \circ \eta_1) = \mathsf{Fib}(\eta_2) \circ \mathsf{Fib}_{\eta_1}$. This follows automatically from the definitions of $\mathsf{Fib}(\eta_i)$ and compositions of natural transformations. This establish functoriality.

We denote by

$$\mathsf{Fib} : \mathsf{COV}_\mathsf{D} \to \mathsf{Fun}(\mathsf{D}, \mathsf{SETS})$$

the resulting functor.

We now show that there is essentially no less of information passing from covering maps of groupoid to fiber functors. In other words, it is merely a repacking of the same information.

Proposition VI.3.3. *The functor* Fib *is an equivalence of categories.*

PROOF. In order to prove this we will construct an inverse functor

$$\mathsf{COV}_\mathsf{D} \leftarrow \mathsf{Fun}(\mathsf{D}, \mathsf{SETS}) : \Gamma$$

and natural isomorphisms $\sigma : \Gamma \circ \mathsf{Fib} \simeq \mathsf{id}$ and $\xi : \mathsf{Fib} \circ \Gamma \simeq \mathsf{id}$.

Step 1: The construction of Γ.

We start by defining that Γ does on objects.

Let $\mathscr{F} : \mathsf{D} \to \mathsf{SETS}$ be an object of $\mathsf{Fun}(\mathsf{D}, \mathsf{SETS})$. We define a category $\Gamma(\mathscr{F})$ as follows:

- Objects of $\Gamma(\mathscr{F})$ are pairs (d, s) where d is an object in D and $s \in \mathscr{F}(d)$;
- A morphism $(d_1, s_1) \to (d_2, s_2)$ is a morphism $f : d_1 \to d_2$ in D such that $s_2 = \mathscr{F}(f)(s_1)$.
- Compositions and identities are well-defined since they are inherited from D.

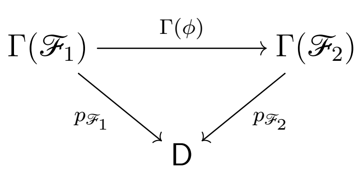
The category $\Gamma(\mathscr{F})$ is a groupoid as a consequence of D being a groupoid and F being a functor and therefore compatible with compositions.

The category $\Gamma(\mathscr{F})$ comes with a canonical functor $p_{\mathscr{F}} : \Gamma(\mathscr{F}) \to \mathsf{D}$ by sending $(d, s) \mapsto d$ on objects and a morphism $(d_1, s_1) \to (d_2, s_2)$ to its underlying morphism $d_1 \to d_2$ in D. By definition of compositions in $\Gamma(\mathscr{F})$, $p_{\mathscr{F}}$ is compatible with compositions and therefore defines a functor.

We claim that $p_{\mathscr{F}}$ is in fact a covering of groupoids: (i) it is surjective on objects by default and (ii) given a morphism $f : d_1 \to d_2$ in D, and an object (d_1, s_1) over d_1, the morphism f lifts in a unique way to a morphism in $\Gamma(\mathscr{F})$ by setting $(d_1, s) \to (d_2, s_2 := \mathscr{F}(f)(s_1))$. By the definition of morphisms in $\Gamma(\mathscr{F})$ this is the unique possibility for a lift of f.

We now establish the functoriality of the assignment $\mathscr{F} \mapsto [\Gamma(\mathscr{F}) \to \mathsf{D}]$.

If $\phi : \mathscr{F}_1 \to \mathscr{F}_2$ is a natural transformation defining a morphism in $\mathsf{Fun}(\mathsf{D}, \mathsf{SETS})$ we define a map of covering groupoids



by setting

$$\Gamma(\phi)(d, s) := (d, \phi_d(s))$$

on objects. Given a morphism $f : (d_1, s_1) \to d_2, s_2)$ in $\Gamma(\mathscr{F}_1)$, we define

$$\Gamma(\phi)[f : (d_1, s_1) \to d_2, s_2)] := [f : (d_1, \phi_{d_1}(s_1)) \to (d_2, \phi_{d_2}(s_2))]$$

on morphisms. $\Gamma(\phi)$ is automatically compatible with compositions because ϕ is natural transformation.

One must also check that given $\phi : \mathscr{F}_1 \to \mathscr{F}_2$, $\psi : \mathscr{F}_2 \to \mathscr{F}_3$ we have $\Gamma(\psi \circ \phi) = \Gamma(\psi) \circ \Gamma(\phi)$. This is again a tedious but trivial exercise.

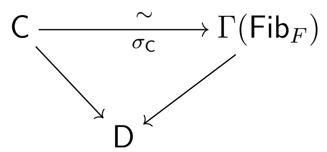
Finally, we have constructed a functor

$$\mathsf{COV}_\mathsf{D} \leftarrow \mathsf{Fun}(\mathsf{D}, \mathsf{SETS}) : \Gamma$$

We must now exhibit natural isomorphisms $\sigma : \Gamma \circ \mathsf{Fib} \simeq \mathsf{id}$ and $\xi : \mathsf{Fib} \circ \Gamma \simeq \mathsf{id}$.

Step 2: The construction of $\sigma : \Gamma \circ \mathsf{Fib} \simeq \mathsf{id}_{\mathsf{COV}_D}$.

One must specify for each covering map of groupoids $F : \mathsf{C} \to \mathsf{D}$ an *isomorphism* of coverings of groupoids

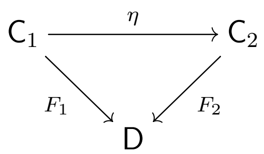


natural in $F : \mathsf{C} \to \mathsf{D}$. We start with the construction of the functor σ_C:

- On objects $c \in \mathsf{C}$, we set $\sigma_\mathsf{C}(c) := (F(c), c)$.
- On morphisms $u : c_1 \to c_2$ in C, we define $\sigma_\mathsf{C}(u) = F(u) : (F(c_1), c_1) \to (F(c_2), c_2)$.
- σ_C is compatible with compositions because of unique lifting property of F.

Therefore σ_C defines a morphism covering maps of groupoids over D.

We now claim that σ defines a natural transformation. More precisely, we have to check that for every morphism of coverings of groupoids



the diagram of functors

$$\begin{array}{ccc} \mathsf{C}_1 & \xrightarrow{\sigma_{\mathsf{C}_1}} & \Gamma(\mathsf{Fib}_{F_1}) \\ \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \Gamma(\mathsf{Fib}(\eta))} \\ \mathsf{C}_2 & \xrightarrow{\sigma_{\mathsf{C}_2}} & \Gamma(\mathsf{Fib}_{F_2}) \end{array}$$

commutes. We check this directly on objects and morphisms. Let $f : c \to c'$ be a morphism in C_1. We have

$$\begin{array}{ccc} [f : c \to c'] & \longrightarrow & F_1(f) : (F_1(c), c) \to (F_1(c'), c') \\ & & \downarrow \\ & & F_1(f) : (F_1(c), \eta(c)) \to (F_1(c'), \eta(c')) \end{array}$$

which since $F_2 \circ \eta = F_1$, coincides with

$$\begin{array}{ccc} [f : c \to c'] & & \\ \downarrow & & \\ \eta(f) : \eta(c) \to \eta(c') & \longrightarrow & F_2(\eta(f)) : (F_2(\eta(c)), \eta(c)) \to (F_2(\eta(c')), \eta(c')) \end{array}$$

Finally, by construction, σ_{C} is a bijection on objects and because of the condition (ii) in Definition **VI.2.4**, it is also bijective on morphisms (see Corollary **VI.2.15**), so it defines a natural isomorphism.

Step 3: The construction of $\xi : \mathsf{Fib} \circ \Gamma \simeq \mathsf{id}_{\mathsf{Fun}(\mathsf{D},\mathsf{SETS})}$.

One must specify for each functor $\mathscr{F} : \mathsf{D} \to \mathsf{SETS}$ an isomorphism of functors

$$\mathsf{Fib}(\Gamma(\mathscr{F})) \to \mathscr{F}$$

natural in $\mathscr{F}$. Notice that by definition of Fib and Γ, for each object $d \in \mathsf{D}$, we have a canonical bijection

$$\mathsf{Fib}(\Gamma(\mathscr{F}))(d) = \mathscr{F}(d)$$

Therefore we define $\xi_{\mathscr{F}}$ to be given by the identity map for each d.

Functoriality and naturality here are obvious as is the fact that ξ is an isomorphism □

Remark VI.3.4. The equivalence of categories of the Proposition **VI.3.3** is a particular case of a general construction for categories, called the Grothendieck construction.

Corollary VI.3.5. *Let* D *be a connected groupoid. Let* $d \in \mathsf{D}$ *and consider the equivalence of categories* $\mathsf{B}_d : \mathsf{B}(\mathrm{Aut}_{\mathsf{D}}(d)) \to \mathsf{D}$ *with inverse* $\Omega_d : \mathsf{D} \to \mathsf{B}(\mathrm{Aut}_{\mathsf{D}}(d))$ *of the Proposition* ***V.3.18*** *. Then, composition with* B_d *and* Ω_d *induces an equivalence of categories*

$$\mathsf{COV}_{\mathsf{D}} \xrightarrow{\mathsf{Fib}} \mathsf{Fun}(\mathsf{D}, \mathsf{SETS}) \xrightarrow{-\circ \mathsf{B}_d} \mathsf{Fun}(\mathsf{B}\ \mathrm{Aut}_{\mathsf{D}}(d), \mathsf{SETS})$$

with inverse

$$\mathsf{COV}_{\mathsf{D}} \xleftarrow[\Gamma]{} \mathsf{Fun}(\mathsf{D}, \mathsf{SETS}) \xleftarrow[-\circ \Omega_d]{} \mathsf{Fun}(\mathsf{B}\ \mathrm{Aut}_{\mathsf{D}}(d), \mathsf{SETS})$$

PROOF. Use the Exercise **IV.4.8**. □

Example VI.3.6. Consider a connected groupoid D, with $d \in \mathsf{D}$. Then the equivalence of categories given by composition with $\mathsf{B}_d : \mathsf{B}\,\mathrm{Aut}_{\mathsf{D}}(d) \to \mathsf{D}$

$$\mathsf{Fun}(\mathsf{D}, \mathsf{SETS}) \to \mathrm{Aut}_{\mathsf{D}}(d) - \mathsf{SETS}$$

sends the functor $\mathrm{Hom}_{\mathsf{D}}(d, -) : \mathsf{D} \to \mathsf{SETS}$ to the $\mathrm{Aut}_{\mathsf{D}}(d)$-set $\mathrm{Aut}_{\mathsf{D}}(d)$ with its left action.

More generally, let $H \subseteq \mathrm{Aut}_{\mathsf{D}}(d)$ be a subgroup. Then then image of the $\mathrm{Aut}_{\mathsf{D}}(d)$-set $\mathrm{Aut}_{\mathsf{D}}(d)/H$ along the equivalence of categories is the functor

$$F_H : \mathsf{D} \to \mathsf{SETS}$$

defined by sending $d' \in \mathsf{D}$ to the quotient set $\mathrm{Hom}_{\mathsf{D}}(d, d')/H$ where two morphisms $f : d \to d'$ and $g : d \to d'$ are declared equivalent if $g^{-1} \circ f \in H$.

Definition VI.3.7. Let G be a group. A $G-$*set* is a set F together with an action $G \times F \to F$. A morphism of G-sets is a morphism of sets $F \to F'$ compatible with the G-actions. The category of G-sets will be denoted by $G - \mathsf{SETS}$.

Example VI.3.8. Let G be a discrete group and $H \subseteq G$ a subgroup. Then the quotient set G/H is a G-set with action defined by the formula $g.[g'] := [g.g']$ for every $g \in G$ and $[g'] \in G/H$.

Construction VI.3.9. Let G be a group. Every G-set F determines a functor $\mathscr{F} : \mathsf{B}G \to \mathsf{SETS}$ defined on objects by sending the unique object $\bullet_G$ to the set F and on morphisms, by the action map $G \to \mathrm{Aut}_{\mathsf{SETS}}(F)$. If $f : F \to F'$ is a map of G-sets, we can define a natural transformation $\mathscr{F} \to \mathscr{F}'$ by $\mathscr{F}(\bullet_G) = F \to \mathscr{F}'(\bullet_G) = F'$. The fact that f is compatible with the G-actions implies that is a natural transformation. This construction defines a functor

$$G - \mathsf{SETS} \to \mathsf{Fun}(\mathsf{B}G, \mathsf{SETS})$$

Proposition VI.3.10. *The functor*

$$G - \mathsf{SETS} \to \mathsf{Fun}(\mathsf{B}G, \mathsf{SETS})$$

is an isomorphism *of categories.*

PROOF. An explicit inverse is given by the evaluation functor sending $\mathscr{F} : \mathsf{B}G \to \mathsf{SETS}$ to $F := \mathscr{F}(\bullet_G)$. □

Corollary VI.3.11. *Let* D *be a connected groupoid. Let* $d \in \mathsf{D}$ *and* $G := \mathrm{Aut}_{\mathsf{D}}(d)$ *be the group of automorphisms of* d*. Then the composition functor*

$$\mathsf{Fib}_d : \mathsf{COV}_{\mathsf{D}} \to G - \mathsf{SETS}$$

is an equivalence of categories.

Remark VI.3.12. More precisely, the functor $\mathsf{Fib}_d : \mathsf{COV}_{\mathsf{D}} \to G - \mathsf{SETS}$ sends a covering map of groupoids $F : \mathsf{C} \to \mathsf{D}$ to the set $F^{-1}(d)$ equipped with the G-action where $g \in G$ acts by $s \in F^{-1}(d) \mapsto gs := s'$ where $s \to s'$ is the unique lift of g. The inverse functor sends a G-set F to the covering morphism of groupoids $\Gamma(\Omega_d(F)) \to \mathsf{D}$.

What this result shows is that the theory of coverings of groupoids, is essentially, the theory of G-sets.

Definition VI.3.13. Let F be a G-set. We say that F is *connected* if the action is transitive.

Construction VI.3.14. Let F be a connected G-set. Then the choice of an element $x \in F$ determines an isomorphism of G-sets

$$f_x : G/H \to F$$

where H denote the stabilizer subgroup of x and f_x sends $g \mapsto g.x$. In particular, every connected G-set is isomorphic to one of the form G/H via the choice of an element.

The G-set G/H itself has a canonical choice of element given by the unit in G.

Exercise VI.3.15. For a G-set F and a subgroup $H \subseteq G$, let us write F^H for the set of points fixed by H

$$F^H := \{x \in F : h.x = x\,,\ \forall h \in H\}$$

(i) Show that the map of sets

$$\mathrm{Hom}_{G-\mathsf{SETS}}(G/H, F) \to F^H$$

sending $[f : G/H \to F]$ to the element $f([e]) \in F$, defines a bijection with inverse sending $x \in F^H$ to the map $G/H \to F$ sending $[g] \mapsto g.x$.

(ii) Show that when $H = \{0\}$ is the trivial subgroup

$$\mathrm{Hom}_{G-\mathsf{SETS}}(G, F) \to F$$

is a bijection.

Proposition VI.3.16. *Let G be a group and $H, K \subseteq G$ subgroups. A map of G-sets $f : G/H \to G/K$ has the form $f(g.H) = \alpha.gH$ for some $\alpha \in G$ satisfying $\alpha^{-1}.H.\alpha \subseteq K$. In particular two G-sets G/H and G/K are isomorphic through an isomorphism of G-sets if and only if H and K are conjugated subgroups.*

Proof. Thanks to the Exercise **VI.3.15**, we know that a map of G-sets $f : G/H \to G/K$ is determined by an element $[\alpha] \in (G/K)^H$ and is of the form $f : g.H \mapsto g\alpha.K$ with $\alpha.K$ fixed by H, ie, $h.\alpha K = \alpha.K$ for every $h \in H$. In particular, $\alpha^{-1}h\alpha \in K$ for every $h \in H$, so $\alpha^{-1}H\alpha \subseteq K$.

Now if f is an isomorphism the same argument applies for the inverse morphism $f^{-1} : G/K \to G/H$: it is of the form $g.H \mapsto g.\beta H$ for some class $\beta.H$ fixed by K, ie, for all $k \in K$ we have $\beta^{-1}k.\beta \in H$. Now, the fact that f and f^{-1} are inverse, implies that $\alpha.\beta \in H$ and $\beta.\alpha \in K$. It follows that

$$K = \alpha^{-1}.\beta^{-1}.K\beta.\alpha \subseteq \alpha^{-1}H\alpha.$$

□

Corollary VI.3.17. *Let G be a group and H a subgroup. Then*

$$\mathrm{Aut}_{G-\mathsf{SETS}}(G/H) = N(H)/H$$

where $N(H) := \{\alpha \in G : \alpha^{-1}.H.\alpha = H\}$ is the normalizer of H.

Proof. By the argument above, an automorphism of G/H as a G-set gives an element $\alpha.H$ in G/H such that for any $h \in H$, $\alpha^{-1}.h.\alpha$ is in H. The fact that the map is automorphism gives $\alpha^{-1}.H.\alpha = H$ so by definition $\alpha.H$ belongs the quotient $N(H)/H$. □

Corollary VI.3.18. *Let G be a group. Then, the assignment $H \mapsto G/H$ establishes a bijection*

$$\{\text{ conjugacy classes of subgroupds of } G\} \simeq \{\text{iso. classes of connected } G-\text{sets}\}$$

with inverse given by taking stabilizer subgroups.

We can now analyse how connected G-sets are described via the equivalence of the Corollary **VI.3.11**:

Proposition VI.3.19. *Let $F : \mathsf{C} \to \mathsf{D}$ be a covering map of groupoids. Let $d \in \mathsf{D}$ and set $G := \mathrm{Aut}_{\mathsf{D}}(d)$.*

(i) The stabilizer subgroup of $c \in F^{-1}(d)$ is the subgroup $F(\mathrm{Aut}_{\mathsf{C}}(c)) \subseteq \mathrm{Aut}_{\mathsf{D}}(d)$.

(ii) Assume that D is connected. Then action of G on $F^{-1}(d)$ is transitive if and only if the groupoid C is connected.

(iii) Assume that C is connected groupoid . Then the choice of an element $c \in F^{-1}(d)$ induces and isomorphism of G-sets

$$G/F(\mathrm{Aut}_{\mathsf{C}}(c)) \simeq F^{-1}(d)$$

In particular, the degree of F as a covering is equal to the index of the subgroup $F(\mathrm{Aut}_{\mathsf{C}}(c))$ in G.

Proof.

(i) Write $H \subseteq G$ for the stabilizer of $c \in F^{-1}(d)$. We want to show that $H = F(\mathrm{Aut}_{\mathsf{C}}(c))$. We show both that both inclusions hold:

- $H \subseteq F(\mathrm{Aut}_{\mathsf{C}}(c))$: Let $h \in \mathrm{Aut}_{\mathsf{D}}(d)$ stabilize c. This means that the unique isomorphism $\tilde{h}$ in C lifting h has c as an endpoint. It is therefore an automorphism of c and lives in the $F(\mathrm{Aut}_{\mathsf{C}}(c))$.
- The inclusion $F(\mathrm{Aut}_{\mathsf{C}}(c)) \subseteq H$ is clear since all automorphisms of c have c as target object.

(ii) Assume the action is transitive and let c_1 and c_2 be objects of C. By assumption, D is connected so there exists an isomorphisms $u : F(c_1) \simeq d$ and $v : F(c_2) \simeq d$ in D. By definition of covering map of groupoids, there exists unique liftings $\tilde{u} : c_1 \to c_1'$ and $\tilde{v} : c_2 \to c_2'$ in C with $c_1', c_2' \in F^{-1}(d)$. Since the action is transitive there exists an element $g \in G = \mathrm{Aut}_{\mathsf{D}}(d)$ whose unique lift $\tilde{g}$ is of the form $\tilde{g} : c_1' \to c_2'$. The composition $\tilde{v}^{-1} \circ \tilde{g} \circ \tilde{u} : c_1 \to c_2$ gives an isomorphism between c_1 and c_2. This argument shows that C is connected as a groupoid.

Conversely, assume that C is connected as a groupoid. Then for every two objects in the fiber $c_1, c_2 \in F^{-1}(d)$ there exists an isomorphism α relating them. $F(\alpha)$ is an automorphism of d in D and by the unique lifting property the action of $F(\alpha)$ on c_1 is c_2.

(iii) This last statement is the Construction **VI.3.14** and the fact that if C is connected then D is connected.

□

Corollary VI.3.20. *Let D be a connected groupoid and let $F : \mathsf{C} \to \mathsf{D}$ be a covering map of groupoids. Let $c \in \mathsf{C}$ and $d := F(c)$. Then the equivalence of categories of the Corollary **VI.3.11** provides an isomorphism of groups*

$$\mathrm{Aut}_{\mathsf{COV}_{\mathsf{D}}}(\mathsf{C}) \simeq N(F(\mathrm{Aut}_{\mathsf{C}}(c)))/F(\mathrm{Aut}_{\mathsf{C}}(c))$$

PROOF. Follows from the fully faithfulness of the equivalence of categories, together with the Corollary **VI.3.17**. □

Here's a recap of what we did: we converted information of conjugacy classes of subgroups into covering maps of groupoids.

VI.4. Classification Theorem of covering spaces

We can finally state the main technical result underlying the dictionnary between labyrinths (coverings spaces) and subgroups. We have seen how coverings give subgroups. In order to be able to reverse the dictionnary we will need an extra assumption on X:

Definition VI.4.1. Let X be a topological space. We say that X is locally simply-connected if for every point $x \in X$ and for every open neighbourhood U of x, there exists an open neighbouhood V with $x \in V \subseteq U$ and V simply-connected.

Theorem VI.4.2. *Let X be locally path-connected, semi-locally simply connected topological space. Then the functor*

$$\mathsf{COV}_X \xrightarrow{\Pi_1} \mathsf{COV}_{\Pi_1(X)}$$

defines an equivalence of categories between covering maps of spaces and covering maps of groupoids.

In order to prove this theorem, we will first need two technical lemmas about liftings of maps along covering spaces and liftings of functors along covering maps of groupoids.

Lemma VI.4.3. *Let $F : \mathsf{C} \to \mathsf{D}$ be a covering map of groupoids and $T : \mathscr{P} \to \mathsf{D}$ any functor with $\mathscr{P}$ a connected groupoid . Let $p \in \mathscr{P}$ and $c \in \mathsf{C}$ with $d := T(p) = F(c)$. Then there exists a lifting $\tilde{T} : \mathsf{C} \to \mathscr{P}$ of T sending p to c if and only if we have an inclusion*

$$T(\mathrm{Aut}_{\mathscr{P}}(p))) \subseteq F(\mathrm{Aut}_{\mathsf{C}}(c))$$

in $\mathrm{Aut}_{\mathsf{D}}(d)$. Moreover, when such a lift exists, it is unique with this property.

PROOF. It is clear that if such a lifting exists, the commutativity $F \circ \tilde{T} = T$ forces the inclusion on automorphisms groups.

It remains to check that the condition is enough.

Assume the condition. We want to construct the functor $\tilde{T}$.

- On objects: Let $q \in \mathscr{P}$. Since $\mathscr{P}$ is connected, there exists an isomorphism $f : p \to q$ in $\mathscr{P}$. By the unique lifting property of F, there exists a unique lifting for the morphism $T(f) : T(p) \to T(q)$ starting at $c \in \mathsf{C}$. Let us call it $\tilde{f} c \to c_{q,f}$. We define $\tilde{T}(q)$ as the target of this morphism $c_{q,f}$ which a priori depends on q and on the choice of f. In fact, it does not depend on f: if g was a second choice of isomorphism $g : p \to q$, then $\beta := g^{-1} \circ f : p \to p$ defines an automorphism of p. By the assumption, $T(\beta)$ is in $F(\mathrm{Aut}_{\mathsf{C}}(c))$ meaning that it has a unique lifting to an automorphism $\tilde{\beta} : c \to c$. Let $\tilde{g} : c \to c_{d,g}$ be the unique lifting of $T(g) : T(p) \to T(q)$. In this case we notice that

$$F(\tilde{g} \circ \tilde{\beta}) = T(g) \circ T(g^{-1} \circ f) = T(f) = F(\tilde{f})$$

  therefore $\tilde{g} \circ \tilde{\beta}$ and $\tilde{f}$ are two different liftings of the same morphism. By unicity of the lift, they must coincide. It follows that $c_{q,f} = c_{q,g}$.

- On morphisms: given $u : q_1 \to q_2$, we use the same argument: choose an isomorphism $f : p \to q_1$ and define $\tilde{T}(q_1) : c \to c_{q,f}$ as above. Now define $\tilde{T}(u)$ as the unique lifting of $T(u)$ starting at $c_{q,f}$. The uniqueness of liftings as above shows that this does not depend on the choice of f.

- $\tilde{T}$ is compatible with compositions. But again, this follows from the unicity of liftings.

Finally, It remains to argue about the uniqueness of such functor $\tilde{T}$. But we see here that the uniqness of liftings along F strongly restrains the possibilities of $\tilde{T}$: there is only one as soon as the condition $p \mapsto c$ is fixed. ☐

The same lifting lemma has a topological version.

Lemma VI.4.4. *Let $p : E \to X$ be a covering map and let $f : Y \to X$ be a continuous map with Y path-connected and locally path-connected* [(*)]*. Let $y \in Y$ and $e \in E$ with $f(y) = p(e) = x$. Then the map f admits a continuous lift $\tilde{f} : Y \to E$ with $p \circ \tilde{f} = f$ and sending y to e, if and only if we have an inclusion of subgroups*

$$f_*(\pi_1(Y, y)) \subseteq p_*(\pi_1(E, e))$$

in $\pi_1(X, x)$. Moreover, if such a lift exists, it is unique with this property.

[(*)]Recall the Warning **III.2.7** that a path-connected space is not necessarily locally path-connected

Proof. If the lifting exists, then the condition is automatic. It remains to prove the converse. Applying the functor $\Pi_1(-)$ to this topological situation, we recover the context of the Lemma **VI.4.3**. Therefore, the condition guarantees the existence of a lifting at the level of fundamental groupoids

$$\begin{array}{ccc} & & \Pi_1(E) \\ & \overset{\tilde{T}}{\nearrow} & \Big\downarrow{\scriptstyle F:=\Pi_1(p)} \\ \Pi_1(Y) & \xrightarrow[T:=\Pi_1(f)]{} & \Pi_1(X) \end{array}$$

Since the objects of the fundamental groupoids are the points of the spaces, at the set-level, we have our candidate for $\tilde{f}$, namely

$$Y \to E \qquad , y \mapsto \tilde{f}(y) := \tilde{T}(y)$$

The unicity of the functor $\tilde{T}$ guaranteed by the Lemma **VI.4.3** also tells us that this is the only possible $\tilde{f}$.

It remains to show that $\tilde{f}$ defined this way is a continuous map. This is a consequence of the fact p is a local homeomorphism and our assumption that X is locally path-connected. Indeed, take U an open in E. We want to show that $\tilde{f}^{-1}(U)$ is open in Y. Let $y_0 \in Y$. It will be enough to show that there exists an open subset of V of Y with $y_0 \in V \subseteq \tilde{f}^{-1}(U)$. Choose a trivializing neighborhood of p, W, around $f(x_0) = p(\tilde{f}(y_0))$ and consider U' the unique connected component of $p^{-1}(W)$ that contains $\tilde{f}(y_0)$. Take the intersection $U \cap U'$. Since $p_{|_{U'}} : U' \to W$ is a homeomorphism, $p(U \cap U')$ of x_0 is an open neighborhood of $f(y_0)$ in X and therefore $f^{-1}(p(U \cap U'))$ is an open neighborhood of y_0 in Y. Now since Y is locally path-connected, take V any path-connected neighborhood of y_0 in Y such that $V \subseteq f^{-1}(p(U \cap U'))$. We claim that V is contained in $\tilde{f}^{-1}(U \cap U')$. Indeed, by construction of the map $\tilde{f}$ in the Lemma **VI.4.3**, we used path-connectedness to construct $\tilde{f}$ by unique lifting for paths.

□

Example VI.4.5. The assumption in Lemma **VI.4.4** that Y in is locally path-connected is crucial. Indeed, let us take Y to be the topologist's circle from the Warning **III.2.7**. Let us consider the map $f : Y \to \mathsf{S}^1$ sending the all points of the form $[(0, y)]$ to $1 \in \mathsf{S}^1$. The space Y is simply connected (Exercise **V.1.36**) but the map f does not admit a lifting $\tilde{f}$ along the covering map $\exp : \mathbb{R} \to \mathsf{S}^1$.

Exercise VI.4.6. Show that if a path-connected, locally path-connected space X has $\pi_1(X)$ finite, then every map $X \to \mathsf{S}^1$ is nullhomotopic. Hint: use the exponential map.

We can now turn to the proof of the Theorem **VI.4.2**. Let us remark that the Lemma **VI.4.4** together with Lemma **VI.4.3** already implies that our functor is fully faithful among path-connected covers:

We now want to show that this is an equivalence of categories. We first need to explain how to go back. For this purpose we will construct a functor in the opposite direction

$$\mathsf{COV}_X \xleftarrow{\mathsf{Obj}} \mathsf{COV}_{\Pi_1(X)}$$

The idea is that if $\mathsf{C} \to \Pi_1(X)$ is a covering map of groupoids, the associated covering space will have as points exactly the objects of C. The technicality is in the definition of the topology. The insight comes from the following remark:

Remark VI.4.7. Let $p : E \to X$ be a covering space. Then the topology on E admits a canonical basis obtained by liftings of open subsets in X. Let V be an open subset of E and $e \in E$. Let U be a trivializing neighbourhood of $x = p(e)$ for the covering p and let W denote the connected component of $p^{-1}(U)$ that contains e. Then $W \cap V$ is an open neighborhood of e contained in V. Moreover, since the restriction $p_{|W} : W \to U$ is a homeomorphism, we have $W \cap V = s(p(W \cap V))$ with s the local inverse of p. In other words, $W \cap V$ is the image of an open subset of x through the section s.

It is clear from this construction that the collection of all such open subsets of E obtained via the image of local sections, form a basis for the topology of E.

Construction VI.4.8. Let us start with a covering map of groupoids $F : \mathsf{C} \to \Pi_1(X)$. We want to construct a topology τ_{C} on the set of objects $\mathsf{Obj}(\mathsf{C})$, together with a covering map of spaces $\mathsf{Obj}(\mathsf{C}) \to X$.

Let $c \in \mathsf{Obj}(\mathsf{C})$ with $F(c) = x$. For each simply-connected neighborhood U in X with $x \in U$, the Lemma **VI.4.3** guarantees the existence of a unique lifting s_c sending x to c

$$\begin{array}{ccc} & & \mathsf{C} \\ & \overset{s_c}{\nearrow} & \downarrow \\ \Pi_1(U) & \xrightarrow{inc} & \Pi_1(X) \end{array}$$

Following the Remark **VI.4.7** we define a basic open neighborhood $\mathscr{O}(c, U)$ of c as the image $s_c(\mathsf{Obj}(\Pi_1(U))) = s_c(U)$, ie, all objects in C obtained from the unique lifting property of paths in U starting at c:

$$\mathscr{O}(c, U) := \{c' \in Obj(\mathsf{C}) : \text{there exists a morphism } f : c \to c' \text{ in } \mathsf{C} \text{ lifting a path } \gamma : x \to y \text{ in } U\}.$$

We denote by β_{C} the collection of all subsets of the form $\mathscr{O}(c,U)$ running over the objects of C and the simply-connected neighborhoods U in X.

We now show that β_{C} satisfies the requirements of the Proposition **II.1.14** and forms a basis of a topology. This is where the hypothesis that X is locally simply-connected plays a role:

(i) Every $c \in \mathsf{Obj}(\mathsf{C})$ is contained in some element of β_{C}. Since X is locally simply-connected there exists an open neighborhood U of $x = F(c)$ which is simply connected. Since $\mathsf{id}_c : c \to c$ is a lift of the constant path at x, therefore contained in U, we find $c \in \mathscr{O}(c,U)$.

(ii) Let $c \in \mathscr{O}(c_1,U_1) \cap \mathscr{O}(c_2,U_2)$ with $x = F(c)$, $x_1 = F(c_1)$ and $x_2 = F(c_2)$. Then by definition, there is a morphism $c_1 \to c$ lifting some path γ_1 from x_1 to x contained in U_1 and $c_1 \to c$ lifting some path γ_2 from x_2 to x contained in U_2. By definition of a locally simply-connected space, there exists U an open neighborhood of x, contained in $U_1 \cap U_2$ with U simply-connected. Again $\mathsf{id}_c : c \to c$ is a lifting of the constant path in U and we have $c \in \mathscr{O}(c,U)$.

Finally, if $c \to c'$ is a morphism lifting a path $x \to x'$ in U, by uniqueness of the lifting, this morphism is also the unique lifting of the path seen in U_1 or in U_2. In other words $\mathscr{O}(c,U) \subseteq \mathscr{O}(c_1,U_1) \cap \mathscr{O}(c_2,U_2)$.

We consider the topology τ_β generated by the basis β_{C} on $\mathsf{Obj}(\mathsf{C})$.

Remark VI.4.9. The subsets $\mathscr{O}(c,U)$ and $\mathscr{O}(c',U)$ are disjoint for $c \neq c'$, $F(c) = F(c') = x$. This is a consequence of the fact that if two liftings agree on a pair of points, then they must agree everywhere by unicity of the lift (see Lemma **VI.4.3**).

Proposition VI.4.10. *The map of sets $p_{\mathsf{C}} : \mathsf{Obj}(\mathsf{C}) \to X$ sending $c \in Obj(\mathsf{C})$ to $F(c)$ is continuous with respect to the topology of Construction **VI.4.8**.*

Proof. Indeed, since X is locally-simply connected, it is enough to test that $p_{\mathsf{C}}^{-1}(U)$ is open for U a simply-connected neighborhood in X. Let $c \in p_{\mathsf{C}}^{-1}(U)$. Then $c \in \mathscr{O}(c,U)$ (via the constant path) and $\mathscr{O}(c,U) \subseteq p_{\mathsf{C}}^{-1}(U)$ by the definition of $\mathscr{O}(c,U)$ as the image of a local section over U. The Remark **II.2.5** allows us to conclude that p_F is continuous.

□

Proposition VI.4.11. *For each covering map of groupoids $F : \mathsf{C} \to \Pi_1(X)$, the continuous map $p_{\mathsf{C}} : \mathsf{Obj}(\mathsf{C}) \to X$ is a covering map.*

Proof. Take $x \in X$ and U any simply-connected neighborhood x in X, then

$$p_{\mathsf{C}}^{-1}(U) = \coprod_{c \in F^{-1}(x)} \mathscr{O}(c, U)$$

This a consequence of the fact all the $\mathscr{O}(c, U)$ are disjoint for $c \in F^{-1}(x)$- see Remark **VI.4.9**.

Now notice that each restriction $(p_{\mathsf{C}})_{|\mathscr{O}(c,U)} : \mathscr{O}(c, U) \to U$ is a bijection. This is a consequence of the description of $\mathscr{O}(c, U)$ as the image of a section.

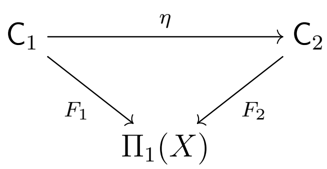
We conclude by arguing that p_{C} is an open map. Indeed, it is enough to test on the basis elements (see Exercise **II.2.6**). But in this case we have $p_{\mathsf{C}}(\mathscr{O}(c, U)) = U$.

□

Proposition VI.4.12. *The construction sending a covering map of groupoids* $F : \mathsf{C} \to \Pi_1(X)$ *to the covering space* $\mathsf{Obj}(\mathsf{C}) \to X$ *defines a functor*

$$\mathsf{COV}_X \xleftarrow{\mathsf{Obj}} \mathsf{COV}_{\Pi_1(X)}$$

Proof. We start by explaining what Obj does on morphisms of coverings of groupoids. Let

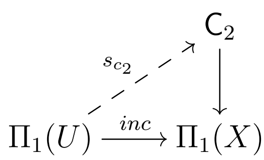


As a map of sets, $\mathsf{Obj}(\eta) : \mathsf{Obj}(\mathsf{C}_1) \to \mathsf{Obj}(\mathsf{C}_2)$ is defined by what η does on objects, so the compatibility with compositions is automatic. It remains to confirm continuity.

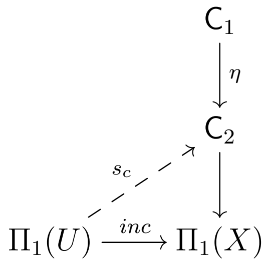
By definition of a basis, it suffices to check that $\eta^{-1}(\mathscr{O}(c_2, U))$ is open in C_1 for each $c_2 \in \mathsf{C}_2$ and U a simply-connected neighborhood in X and $x = F_2(c_2)$.
Notice that:

$$c \in \eta^{-1}(\mathscr{O}(c_2, U)) \Leftrightarrow \eta(c) \in \mathscr{O}(c_2, U) \Leftrightarrow \eta(c) = s_{c_2}(y) \text{ for some } y \in U$$

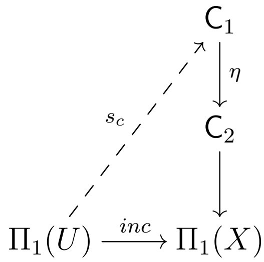
.
where s_{c_2} is the unique lifting guaranteed by the Lemma **VI.4.3** using the fact U is simply-connected.



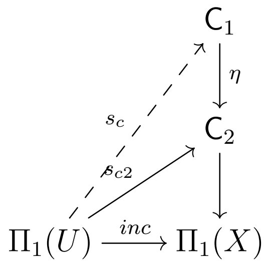
Consider now the diagram of groupoids



The same lifting lemma applied to $F_1 = F_2 \circ \eta$ guarantees the existence of a unique lifting



with the property that $s_c(y) = c$. By construction, we have $\eta \circ s_c(y) = \eta(c)$ so the two liftings s_{c_2} and $\eta \circ s_c$ must agree by the unicity property, ie, the diagram



commutes.

In particular, $c \in \mathcal{O}(c, U)$ and by construction $\mathcal{O}(c, U) \subseteq \eta^{-1}(\mathcal{O}(c_2, U))$ since

$$\eta(\mathcal{O}(c, U)) = \eta \circ s_c(U) = s_{c_2}(U) = \mathcal{O}(c_2, U)$$

□

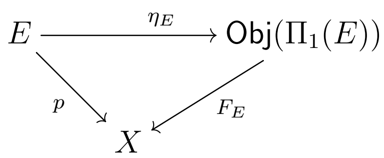
We now finally turn to the proof of our main result in this section:

Proof of the Theorem **VI.4.2**.

We construct natural isomorphisms $\eta : \mathcal{O} \circ \Pi_1 \simeq \mathsf{id}_{\mathsf{COV}_X}$ and $\lambda : \Pi_1 \circ \mathcal{O} \simeq \mathsf{id}_{\mathsf{COV}_{\Pi(X)}}$.

Step 1: Construction of $\eta : \mathsf{Obj} \circ \Pi_1 \simeq \mathsf{id}_{\mathsf{COV}_X}$.

Let $p : E \to X$ be a covering map. We want to construct an isomorphism of covering spaces



natural in $E \to X$. As a map of sets, this is clear: the points of $\mathsf{Obj}(\Pi_1(E))$ are the objects of $\Pi_1(E)$ which are exactly the points of E. In this case we set η_E to be the identity map at the level of sets. Commutativity of the diagram is automatic.

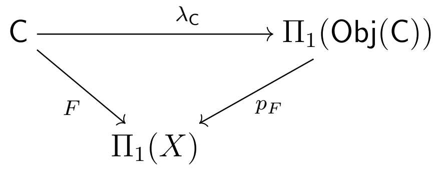
It remains to show that η_E is a homeomorphism. But this is a consequence of the Remark **VI.4.7** that explains how the the topology on E is already the one obtained through liftings of open subsets in X via local sections. The two topologies are defined by the same basis.

Finally, the fact that at the level of underlying sets η is the identity, it is automatic that it defines a natural isomorphism.

In fact, we have shown that $\mathsf{Obj} \circ \Pi_1 = \mathsf{id}$.

Step 2: Construction of $\lambda : \mathsf{id}_{\mathsf{Cov}_{\Pi(X)}} \simeq \Pi_1 \circ \mathsf{Obj}$.

Let $F : \mathsf{C} \to \mathsf{D}$ be a covering map of groupoids. We want to construct an isomorphism of coverings of groupoids



natural in C. We defined the functor λ_{C} as follows:

- By construction, the objects of $\Pi_1(\mathsf{Obj}(\mathsf{C}))$ are the objects of C, so we define λ_{C} on objects by the identity map $c \mapsto c$;
- On morphisms, if $f : c_1 \to c_2$ is a morphism in $\Pi_1(\mathsf{Obj}(\mathsf{C}))$, we use the unique lifting property along p_F to define $\lambda_{\mathsf{C}}(f)$. Take a representative $\gamma_f : I \to X$ for $F(f)$ in $\Pi_1(X)$. γ_f is a path starting at $F(c_1)$ and ending at $F(c_2)$. The fact that I is contractible, allows us to use the Lemma **VI.4.4** to find a unique continuous lifting of the path, $\widetilde{\gamma_f}$, starting at $c_1 \in \mathsf{Obj}(\mathsf{C})$

$$\begin{array}{ccc} & & \mathsf{Obj}(\mathsf{C}) \\ & \overset{\widetilde{\gamma_f}}{\nearrow} & \downarrow \\ \Pi_1(I) & \xrightarrow{\gamma_f} & X \end{array}$$

We define

$$\lambda_{\mathsf{C}}(f) := [\widetilde{\gamma_f}]$$

the homotopy class of this path. The uniqueness of liftings guarantees that this construction is compatible with compositions and therefore defines a functor.

The Corollary **VI.2.15** concludes that λ_{C} is an isomorphism of groupoids.

The fact that λ_{C} is natural in C is again a consequence of the uniquness of liftings. We leave it as an exercise to write down the details.

□

We will now translate the content of the Theorem **VI.4.2** to the more down-to-earth dictionary between labyrinths and subgroups:.

Corollary VI.4.13. *Let X be path-connected and locally simply connected space. Let $x \in X$. Then the fiber functor at x induces an equivalence of categories*

$$\mathsf{COV}_X \to \pi_1(X,x) - \mathsf{SETS}$$

In particular, a covering space is path-connected (ie, its total space is path-connected) if and only if the corresponding $\pi_1(X,x)$-set is transitive.

The following theorem summarizes the main results proved so far:

Theorem VI.4.14: Classification of Covering Spaces

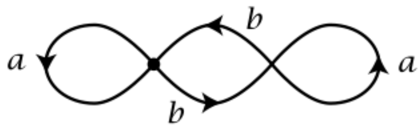
Let X be path-connected and locally simply connected. Then the assignment sending a connected covering $p : E \to B$ to the conjugation class of the subgroup $p_*(\pi_1(E,b)) \subseteq \pi_1(X,p(b))$ establishes a bijection

$$\{\text{iso. cl. of connected coverings of } X\} \simeq \{\text{ conj. cl. of subgroups of } \pi_1(X,x)\}$$

VI.5. Galois Coverings

Throughout this section we assume that X is path connected and locally simply connected.

Example VI.5.1. In the Remark **I.2.11** we have also looked at another example of a covering map of the wedge of two circles given by $E =$



(Picture taken from Hatcher's book)

In this case the subgroup $p_*(\pi_1(E,e)) =< a, b^2, bab^{-1} >\subseteq \pi_1(X,x) =< a,b >$ is a normal subgroup. This can be check showing that all conjugations $a.p_*(\pi_1(E,e)).a^{-1}$, $a^{-1}.p_*(\pi_1(E,e)).a$, $b.p_*(\pi_1(E,e)).b^{-1}$ and $b^{-1}.p_*(\pi_1(E,e)).b$ are subsets of $p_*(\pi_1(E,e))$.

The Proposition **VI.2.7** and the Example **VI.5.1** suggest that some covering maps exhibit a distinguished feature, namely the associated subgroup is normal. We isolate this feature in the following definition:

Definition VI.5.2. Let $F : \mathsf{C} \to \mathsf{D}$ be a covering map of groupoids with C connected (and therefore D connected). We say that F is normal (régulier en français) if for any $d \in D$ and $c \in F^{-1}(d)$, the subgroup $F(\mathrm{Aut}_{\mathsf{C}}(c)) \subseteq \mathrm{Aut}_{\mathsf{D}}(d)$ is a normal subgroup.

Definition VI.5.3. Let $p : E \to X$ be a covering map with E path-connected.. We say that p is normal if the associated covering map of groupoids $\Pi_1(E) \to \Pi_1(X)$ is normal.

Proposition VI.5.4. *Let $F : \mathsf{C} \to \mathsf{D}$ be a covering map of groupoids with C connected. Let $d \in \mathsf{D}$. The following conditions are equivalent:*

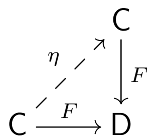
(i) $F : \mathsf{C} \to \mathsf{D}$ is normal;

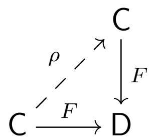
(ii) For every pair of objects $c_1, c_2 \in F^{-1}(c)$ there exists an automorphism of covering groupoids $\eta : \mathsf{C}_1 \to \mathsf{C}_2$ sending c_1 to c_2.

PROOF. Assume $F : \mathsf{C} \to \mathsf{D}$ is normal. Then the subgroups $F(\mathrm{Aut}_{\mathsf{C}}(c_1))$ and $F(\mathrm{Aut}_{\mathsf{C}}(c_2))$ must coincide since they are conjugated, ie,

$$F(\mathrm{Aut}_{\mathsf{C}}(c_1)) = F(\mathrm{Aut}_{\mathsf{C}}(c_2))$$

But in this case the lifting lemma Lemma **VI.4.3** guarantees the existence of η and ρ





with $\eta(c_1) = c_2$ and $\rho(c_2) = c_1$. By unicity of the liftings, one must have $\rho \circ \eta = \mathsf{id}$ and $\eta \circ \rho = \mathsf{id}$.

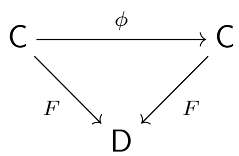
Conversely, if the automorphism exists for any pair of points, it is automatic by the commutativity of the diagram that

$$F(\mathrm{Aut}_{\mathsf{C}}(c_1)) = F(\mathrm{Aut}_{\mathsf{C}}(c_2))$$

□

Construction VI.5.5. Let $\mathsf{C} \to \mathsf{D}$ be a covering map of groupoids and let $d \in \mathsf{D}$. Then the group $\mathrm{Aut}_{\mathsf{COV}_{\mathsf{D}}}(\mathsf{C})$ acts on the fiber of F at d, $F^{-1}(d)$: if $\eta : \mathsf{C} \to \mathsf{C}$ is an automorphism of F, and $c \in F^{-1}(d)$, then $\eta(c) \in F^{-1}(d)$.

Definition VI.5.6. Let $F : \mathsf{C} \to \mathsf{D}$ be a covering map of groupoids. Then we say that F is Galois if the action of the Construction **VI.5.5** is transitive, ie, for any pair of objects c_1, c_2 in the fiber of d, there exists an automorphism of covering groupoids $\phi \in \mathrm{Aut}_{\mathsf{COV}_{\mathsf{D}}}(F)$



such that $\phi(c_1) = c_2$.

Definition VI.5.7. We say that a covering space $p : E \to X$ is Galois if its groupoid covering $\Pi_1(E) \to \Pi_1(X)$ is Galois.

Remark VI.5.8. The Proposition **VI.5.4** can now be reformulated by saying that a covering of groupoids is Galois if and only if it is normal.

Corollary VI.5.9. *Let X be path connected and locally simply connected. Let $p : E \to X$ be a covering map with E connected, $e \in E$ and $x := p(e)$. Then, the fully faithfulness of the equivalence of categories of Corollary **VI.4.13** gives us an isomorphism of groups*

$$\mathrm{Aut}_{\mathsf{Cov}_X}((E,p)) \simeq N(p_*(\pi_1(E,e))/p_*(\pi_1(E,e))$$

In particular, if $p : E \to X$ is normal (or Galois), we get an isomorphism of groups

$$\mathrm{Aut}_{\mathsf{Cov}_X}((E,p)) \simeq \pi_1(X,x)/p_*(\pi_1(E,e)$$

Proof. Follows from the Corollary **VI.3.20**. □

We now give a third description of Galois coverings:

Proposition VI.5.10. *Let X be a path-connected and locally simply-connected topological space. Let $p : E \to X$ be a covering map with E path-connected. Then the following are equivalent:*

(i) *$p : E \to X$ is a Galois covering.*

(ii) *The action of $G := \mathrm{Aut}_{\mathsf{Cov}_X}(E)$ (with the discrete topology) on E is properly discontinuous and the quotient map*

$$E/G \to X$$

is a homeomorphism.

Proof. Assume first that the covering is Galois. Let $e \in E$ and let U be a trivializing neighborhood of $p(x)$. Let V be the connected component of $p^{-1}(U)$ containing e. Then the Galois condition applied implies that the action of the group of automorphism will shuffle the different connected components, but never intersect them. By the Proposition **VI.1.11**, the quotient map $E \to E/G$ is a covering map.

Let $f : E/G \to X$ be the map induced by the universal property of the quotient. Since the action of the G is transitive, it follows that f is injective. It is surjective because p is surjective. Therefore, f is a bijection.
We now observe that f is an open map, since p is an open map (local homeomorphisms are open maps).

Let us now prove the converse statement. Assume that the action of G is property discontinuous and that the quotient map $E/G \to X$ is a homeomorphism. One must show that the action of G is transitive on each fiber of the map $E \to X$: but this is automatic since if e_1 and e_2 are in the fiber of x, we must have $f([e_1]) = f([e_2])$ but since f is injective, one has $[e_1] = [e_2]$, so there is an element $g \in G$ with $g(e_1) = e_2$. □

Exercise VI.5.11. Let X be a topological space with a properly discontinuos action of a (discrete) group G. Show that:

(i) The quotient map $\pi : X \to X/G$ is a Galois covering.

(ii) Assume X is path-connected. Show that G is the group of automorphisms of π as a covering space of X/G.

Exercise VI.5.12. Show that the map $\mathbb{C}^* \to \mathbb{C}^*$ given by $z \to z^3$ is a Galois covering map. Compute its group of automorphisms. TD, Exo 2.1, Feuille 5

Remark VI.5.13. The form of the Theorem **VI.4.14** (through the equivalence of categories of the Corollary **VI.4.13**) appears in other areas of mathematics, manifesting the same basic principle: objects being classifying by conjugacy classes of subgroups of a certain group.

- Let k be a field and let L be a finite Galois extension of k with Galois group G. Then the assignment sending an intermediate extension $k \subseteq M \subseteq L$ to the group of automorphisms of L over M - $\text{Aut}(L/M)$ - defines a bijection between intermediate extensions of $k \subseteq L$ and subgroups of G with inverse given by sending a subgroup H to the subfield $L^H \subseteq L$ of those elements fixed by H. Moreover, the intermediate extension is Galois if and only if the subgroup is normal.

- In the theory of Riemann surfaces that you will see next month, you will be led to study spaces $E \to X$ that behave like covering spaces everywhere except at a finite number of points of X. These are called branched covers of X. When X is a Riemann surface, the classification of branched covers can be made via algebraic techniques and proved to be equivalent to the theory of field extensions of the field of meromorphic functions on X.

For more on this, check the book T. Szamuely Galois groups and fundamental groups

VI.6. Universal Covers

Definition VI.6.1. Let E be a path-connected space. We say that $p : E \to X$ is a universal cover if E is connected and simply-connected[(†)].

(†)In particular, it is path-connected

Remark VI.6.2. Let X be a path-connected, locally path-connected topological space, and assume it admits a universal cover $p : E \to X$. Then since E is also locally path-connected (cf. Remark **VI.1.8**), the necessary requirement of the Lemma **VI.4.4** for the existence of a lifting to any other covering map is automatic. Fix a point $e \in E$ and denote $x := p(e)$. Then for any covering map $p' : E' \to X$ and choice of point $e' \in E'$ there exists a lifting

$$\begin{array}{ccc} & & E' \\ & \nearrow & \downarrow{\scriptstyle p'} \\ \mathsf{E} & \xrightarrow[p]{} & X \end{array}$$

sending e to e'. In other words, if a universal cover exists, it dominates all other covers. Hence the terminology universal.

Proposition VI.6.3. *Let X be a path-connected and locally path-connnected space and assume $p_1 : E_1 \to X$ and $p_2 : E_2 \to X$ are two universal covers. Then E_1 and E_2 are isomorphic.*

Proof. Since both E_1 and E_2 are locally path-connected, by the Lemma **VI.4.4**, both satisfy the requirement for liftings of the Lemma **VI.4.4**: given e_1 and e_2 with $p_1(e_1) = p_2(e_2)$, there exists unique liftings

$$\begin{array}{ccc} & & E_2 \\ & \overset{\Psi}{\nearrow} & \downarrow{\scriptstyle p_2} \\ E_1 & \xrightarrow[p_1]{} & X \end{array}$$

$$\begin{array}{ccc} & & E_1 \\ & \overset{\Phi}{\nearrow} & \downarrow{\scriptstyle p_1} \\ E_2 & \xrightarrow[p_2]{} & X \end{array}$$

sending e_1 to e_2 and e_2 to e_2. By unicity of liftings, one must have $\Phi \circ \Psi = \mathsf{id}$ and $\Psi \circ \Phi = \mathsf{id}$.

□

Remark VI.6.4. Let $p : E \to X$ be universal cover, with $e \in E$.Let $x = p(e)$.

- The subgroup $p_*(\pi_1(E, e)) \subseteq \pi_1(X, x)$ is the trivial subgroup, since $\pi_1(E, e) = 0$.

- $p : E \to X$ is a Galois covering and the associated $\pi_1(X, x)$-set is the underlying set of $\pi_1(X, x)$ with the action via left multiplication.

- Since E is path-connected, the action is transitive (Proposition **VI.3.19**), and therefore the choice of an element $e \in E$ over x induces a bijection

$\pi_1(X,x)/\{0\} \simeq \pi_1(X,x) \simeq p^{-1}(x)$.

- By the Corollary **VI.5.9**, the choice of an element $e \in p^{-1}(\{x\})$ induces an isomorphism of groups

$$\mathrm{Aut}_{\mathsf{Cov}_{\Pi_1(X)}}(E) \simeq \pi_1(X,x)$$

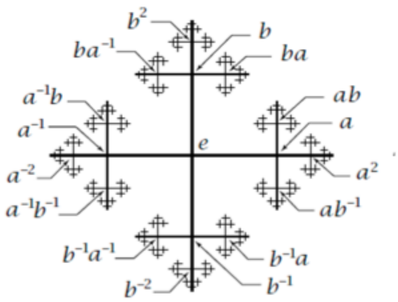
- By the Proposition **VI.5.10**, the action of $\mathrm{Aut}_{\mathsf{Cov}_{\Pi_1(X)}}(E) \simeq \pi_1(X,x)$ on X is properly discontinuous and we have a homeomorphism

$$E/\pi_1(X,x) \simeq X$$

Example VI.6.5.

- The exponential map $\exp : \mathbb{R} \to \mathsf{S}^1$ is a universal cover.
- The complex exponential map $\exp : \mathbb{C} \to \mathbb{C}^*$ is a universal cover.
- The quotient map $\mathbb{R}^2 \to \mathbb{T}$ is a universal cover for the torus.

- The identity map $S^2 \to S^2$ is a universal cover.
- The graph

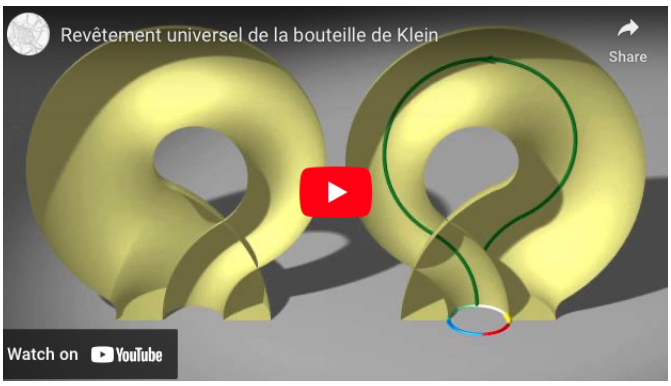


is a universal cover of the wedge of two circles $\mathsf{S}^1 \vee \mathsf{S}^1$.

- The quotient map $\mathbb{R}^2 \to K$ defining the Klein bottle, is a universal cover (Exercise **II.5.40**)
- The universal cover of the cylinder $\mathsf{S}^1 \times \mathbb{R}$:

- The universal cover of the Mobius band:

- The universal cover of the Klein bottle:



Proposition VI.6.6. *Let X path connected and locally simply connected. Then it admits a universal cover.*

PROOF. This is a direct consequence of the equivalence of categories in Corollary **VI.4.13**. Namely since X is path-connected, let us fix $x \in X$ a base point and take the covering $\widetilde{X}_x := \mathsf{Obj}(\Gamma(- \circ \Omega_x(\pi_1(X,x)))) \to X$ associated to the $\pi_1(X,x)$-set given by $\pi_1(X,x)$ acting on itself via left multiplication. The covering map $\widetilde{X}_x \to X$ obtained this way is indeed a universal cover:

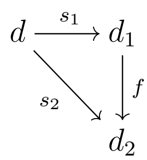
- It is path-connected since the action of $\pi_1(X,x)$ on itself is transitive (Proposition **VI.3.19**-(ii));
- It is simply-connected: since it is a covering map, its fundamental group is isomorphic to the subgroup $p_*(\pi_1(\widetilde{X}_x, e))$ for any point e lying over x. But by construction we have asked for this subgroup to be the trivial subgroup.
- Comes with a canonical point $\tilde{x}$ lying over x given by using $0 \in \pi_1(X,x)$ to generate all other elements under the transitivity of the action.

□

Remark VI.6.7. For clarity's sake, we outline the construction of $\widetilde{X}_x$ in more detail, following the steps of the equivalence Corollary **VI.4.13**.

As explained in the Example **VI.3.6**, we have $\widetilde{X}_x \simeq \mathsf{Obj}(\Gamma(\mathrm{Hom}_{\mathsf{D}}(d,-)))$.

Now, the category $\Gamma(\mathrm{Hom}_{\mathsf{D}}(d,-))$ is by definition the category of pairs $(d' \in \mathsf{D}, s \in \mathrm{Hom}_{\mathsf{D}}(d,d'))$ and morphisms $(d_1, s_1 : d \to d_1) \to (d_2, s_2 : d \to d_2)$ are morphisms $f : d_1 \to d_2$ in D such that the diagram commutes



This is precisely the comma category $\mathsf{D}_{d/.}$ of objects under d of the Example **VI.2.5**.

Applying this to our case, $\mathsf{D} = \Pi_1(X)$ and $d = x \in X$, the objects of this category are homotopy classes of paths $[\gamma] : x \to x'$ with starting point x.

Therefore, as a set, what we really have is

$$\widetilde{X}_x = \{[\gamma] : \gamma \text{ is a path out of } x \text{ in } X\}$$

so it is canonically pointed by the class $\tilde{x} := [c_x]$ of the constant path at x. Moreover, the map $\widetilde{X}_x \to X$ is the evaluation at the endpoint.

Tracing back the construction of the topology $\widetilde{X}_x$ in Construction **VI.4.8** the basis subsets $\mathscr{O}((x', [\gamma] : x \to x'), U)$ are precisely those pairs $(x'', [\alpha] : x \to x'')$ such that there exists a morphism

$$\eta : (x', [\gamma] : x \to x') \to (x'', [\beta] : x \to x'')$$

corresponding to a commutative diagram

$$\begin{array}{ccc} x & \xrightarrow{[\beta]} & x' \\ & \underset{[\gamma]}{\searrow} & \downarrow{\scriptstyle [\eta]} \\ & & x'' \end{array}$$

such that $[\eta] : x' \to x''$ is the homotopy class of a path in a simply connected open in X. In other words, it is the set

$$\{[\gamma'] : \text{ there exists a path } \eta \text{ in } U \text{ such that } [\gamma'] = [\eta] * \gamma]\}$$

corresponding to "small" contractible continuations of γ along its endpoint.

Moreover, under the isomorphism of groups $\mathrm{Aut}_{\mathsf{COV}_X}(\tilde{X}_x) \simeq \pi_1(X, x)$ obtained by using the base point $\tilde{x}$, the action is given by sending $[\alpha] \in \pi_1(X, x)$, and $[\gamma]$ a path starting at x, to the concatenation $[\gamma * \alpha]$[‡]

Remark VI.6.8. As explained above, by construction, the universal cover $\widetilde{X}_x$ comes with the point $\tilde{x}$ in the fiber of x corresponding to 0 in the $\pi_1(X, x)$. The fully faithfulness of the equivalence of categories, gives for any other path-connected covering map $q : E \to X$, a bijection

$$\mathrm{Hom}_{\mathsf{COV}_X}(\widetilde{X}_x, E) \simeq \mathrm{Hom}_{\pi_1(X,x)-\mathsf{SETS}}(\pi_1(X, x), \mathsf{Fib}_x(E)) \simeq \mathsf{Fib}_x(E)$$

by the Exercise **VI.3.15**-(ii). This bijection quantifies the Remark **VI.6.2** by expressing exactly how many liftings exist - one for each point e in the fiber $p^{-1}(\{x\})$ of E, sending $\tilde{x} \mapsto e$.

Remark VI.6.9. One can also use the Example **VI.3.6** to trace back the covering space E_H associated to a subgroup $H \subseteq \pi_1(X, x)$, namely, its points are the objects of $\Gamma(F_H)$, ie,

$$E_H = \{[\gamma] : \gamma \text{ is a path out of } x \text{ in } X\}/([\gamma_1] \sim [\gamma_2] \text{ iff } [\gamma_2^{-1} * \gamma_1] \in H)$$

Remark VI.6.10. Let X be a path-connected and locally simply-connected space, $x \in X$ and $p : \tilde{X}_x \to X$ a universal cover. Let also $\tilde{x} \in \tilde{X}_x$ be the point corresponding to the constant path at x as explained in the Remark **VI.6.7**.

[‡]which is the natural action on the functor $\mathrm{Hom}_{\Pi_1(X)}(x, -)$.

The choice of $\tilde{x}$ determines an isomorphism of groups $G := \mathrm{Aut}_{\mathsf{COV}_X}(\tilde{X}_x) \simeq \pi_1(X, x)$. Let $H \subseteq G$ be a subgroup. Consider the action of H on $\tilde{X}_x$ via automorphisms. Since the action of G is properly discontinuous (as the universal cover is Galois and Proposition **VI.5.10**), so is the restricted action by elements of H. In particular, the quotient map

$$\pi : \tilde{X}_x \to \tilde{X}_x/H$$

is a covering map. The action of G descends to an action of G/H on $\tilde{X}_x/H$ which is still property discontinuous so the further quotient

$$q : \tilde{X}_x/H \to (\tilde{X}_x/H)/(G/H) \simeq \tilde{X}_x/G \simeq X$$

is a covering map of X:

$$\begin{array}{ccc} \tilde{X}_x & \xrightarrow{\pi} & \tilde{X}_x/H \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & X & \end{array}$$

Using the explicit description of points in the universal cover as homotopy classes of paths starting at x - Remark **VI.6.7** - we see that the quotient $\tilde{X}_x/H$ is identifying two homotopy classes of paths $[\gamma_1]$ and $[\gamma_2]$ starting at x iff $[\gamma_2] = [\gamma_1 * \alpha]$ for $[\alpha] \in H \subseteq \pi_1(X, x)$. In equivalent terms, if $[\gamma_2^{-1} * \gamma_1] \in H$

At the same time, the canonical map of G-sets $G \to G/H$ induces, via the functoriality of the equivalence of categories in Corollary **VI.4.13** a map of covering spaces $\tilde{X}_x \to E_H$ where E_H is as described in the Remark **VI.6.9**.

$$\begin{array}{ccc} \tilde{X}_x & \longrightarrow & E_H \\ & {\scriptstyle p}\searrow \quad \swarrow & \\ & X & \end{array}$$

But the explicit description in the Remark **VI.6.9** tells us that E_H is defined exactly as the quotient $\tilde{X}_x/H$: the map $\tilde{X}_x \to E_H$ descends to a continuous map in quotient

$$\tilde{X}_x/H \to E_H$$

which is a bijection on points (by inspection). Moreover, this an homeomorphism since E_H is path-connected and locally path-connected so by (Exercise **VI.2.12**) the map $\tilde{X}_x \to E_H$ is itself a covering map, and therefore an open map.

In particular, this shows that all path-connected covering spaces associated to a subgroup H can be realized as explicit quotients of the universal cover.

VI.7. Examples of the classification theorem

Example VI.7.1. Let us classify all covering spaces of the circle S^1.

Its fundamental group is $\mathbb{Z}$ which is an abelian group. In particular, all its subgroups are normal.

The classification of subgroups of $\mathbb{Z}$ is easy: if $H \subseteq \mathbb{Z}$ is a subgroup, then if $H \neq \{0\}$, H must contain a smallest positive element

$$n := \min\{h \in H : h > 0\}$$

Since H is a subgroup, H must contain all multiples of n so $n\mathbb{Z} \subseteq H$. Conversely, if $h \in H$ is non-zero, then $h \geq n$. So take the euclidean division $h = a.n + r$ where both $a, r < n$. But in this case $r = a.n - h$ and since $a.n$ and h are in H, r is in H. We have thus constructed an element of H which is strictly smaller than n. The only possibility is $r = 0$. But then $h = a.n$ so $H \subseteq n\mathbb{Z}$.

Summarizing, all subgroups of $\mathbb{Z}$ are of the form $n\mathbb{Z}$ for $n \in \mathbb{N}$.

Since $\exp : \mathbb{R} \to \mathbb{Z}$ is the universal cover of S^1, we have shown (see Remark **VI.6.10**) that all (connected) covering spaces of S^1 are (up to isomorphism) obtained as a quotient of the universal cover $\mathbb{R}$ by the action of the subgroup $n\mathbb{Z}$. The explicit description of the action given in the Remark **VI.6.10**, seeing 0 in $\mathbb{R}$ as the canonical point representing the constant path at $1 \in \mathsf{S}^1$, tells us that $n\mathbb{Z}$ acts on $\mathbb{R}$ by a shifted by n. The quotient space $\mathbb{R}/n\mathbb{Z}$ is itself homeomorphic to the circle via the map $\mathbb{R}/n\mathbb{Z} \simeq \mathsf{S}^1$ sending $[x] \mapsto \exp(\frac{2\pi i x}{n})$. In this case the cover we obtain

$\mathbb{R}/n\mathbb{Z} \simeq \mathsf{S}^1 \to \mathsf{S}^1$ is given by $z \mapsto z^n$. These correspond precisely to the different labyrinths in the Example **I.2.12**.

Example VI.7.2. Let us classify all (connected) covering spaces of $\mathbb{R}P^2$. By the Exercise **V.3.31** we have $\pi_1(\mathbb{R}P^2, x) \simeq \mathbb{Z}/2$. This is an abelian group with only itself and $\{0\}$ as subgroups. We have seen above that the quotient map $S^2 \to \mathbb{R}P^2$ is a covering map and we know that S^2 is simply-connected. So this is the universal cover. The only other covering is the one corresponding to the whole $\mathbb{Z}/2$, ie, the identity map $\mathbb{R}P^2 \to \mathbb{R}P^2$.

Example VI.7.3. Let us classify all covering spaces of the torus $\mathsf{S}^1 \times \mathsf{S}^1$. Its fundamental group is $\mathbb{Z} \oplus \mathbb{Z}$ which is an abelian group. It follows that all subgroups are normal and all connected covers are Galois.

We need to list the subgroups of $\mathbb{Z} \oplus \mathbb{Z}$. Let H be a subgroup and consider the short exact sequence of abelian groups

$$0 \longrightarrow \mathbb{Z} \xrightarrow{i_2} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{p_1} \mathbb{Z} \longrightarrow 0$$

Since p_1 is a map of abelian groups, $p_1(H)$ is a subgroup of $p_1(\mathbb{Z} \oplus \mathbb{Z}) = \mathbb{Z}$. Therefore $p_1(H)$ is of the form $n\mathbb{Z}$ for some $n \in \mathbb{N}$. Pick an element $x \in H$ such that $p_1(x) = n$, ie , $x = (n, a) \in H$.

Now $i_2^{-1}(H)$ is a subgroup of the second copy of $\mathbb{Z}$ so $i_2^{-1}(H)$ must be of the form $m\mathbb{Z}$ for some $m \in \mathbb{N}$.

Let $h \in H$. In this case $p_1(h) = a_h.n$ for a unique a_h. Consider the element $h - a_h.x$ in H. It follows that $p_1(h - a_h.x) = p_1(h) - a_h.p(x) = a_h.n - a_h.n = 0$. So $h - a_h.x$ belongs to the image of i_2 so it is of the form $k_h.(0, m)$. Therefore, any $h \in H$ can be written as $h = a_h.x + k_h.m = (a_h).(n, a) + k_h.(0, m)$.

In other words, all subgroups are of the form,

$$H = (n, a).\mathbb{Z} \oplus (0, m).\mathbb{Z}$$

Now $\mathbb{Z} \oplus \mathbb{Z}$ acts on the universal cover $\mathbb{R}^2$ by translations: if $(n, m) \in \mathbb{Z} \oplus \mathbb{Z}$ and $(x, y) \in \mathbb{R}^2$, $(n, m).(x, y) = (x + n, y + m)$. In particular, the cover associated to the subgroup H is obtained by identifying $(x, y) \sim (x', y')$ iff there exists $u, v \in \mathbb{Z}$ such that $x' = x + u.n$ and $y' = y + u.a + v.m$.

- if $n = m = a = 0$, ie, $H = \{0\}$ we get the universal cover of the torus, ie, the plane $\mathbb{R}^2$;

- if $n = a = 0$, and $m \neq 0$, ie, $H = (0, m).\mathbb{Z}$ we get the equivalence relation $(x, y) \sim (x', y')$ iff $y' - y \in m\mathbb{Z}$. It follows that the quotient $\mathbb{R}^2/H$ is homeomorphic to $\mathbb{R} \times \mathbb{R}/m\mathbb{Z}$ which is homeomorphic to $\mathbb{R} \times \mathsf{S}^1$ via the map

$$\mathbb{R} \times \mathbb{R}/m\mathbb{Z} \to \mathbb{R} \times \mathsf{S}^1$$

sending $[(x, y)] \mapsto (x, \exp(\frac{2\pi iy}{m}))$. The associated covering map is then the map

$$\mathbb{R} \times \mathsf{S}^1 \to \mathsf{S}^1 \times \mathsf{S}^1$$

given by

$$(x, z) \mapsto (\exp(2\pi ix), z^n)$$

- if $a = 0$ and $n \neq 0$, $m \neq 0$, ie, $H = (n, 0).\mathbb{Z} \oplus (0, m).\mathbb{Z}$, the quotient space $\mathbb{R}^2/H$ is isomorphic to the product of the two quotients $\mathbb{R}/n\mathbb{Z} \times \mathbb{R}/m\mathbb{Z}$. This is homeomorphic to a product of two circles via the map

$$\mathbb{R}/n\mathbb{Z} \times \mathbb{R}/m\mathbb{Z} \simeq S^1 \times S^1$$

sending

$$[(x, y)] \mapsto (\exp(\frac{2\pi.ix}{n}), \exp(\frac{2\pi iy}{m}))$$

and the associated covering map $\mathbb{R}/H \to \mathsf{S}^1 \times \mathsf{S}^1$ is given by

$$\mathsf{S}^1 \times \mathsf{S}^1 \to \mathsf{S}^1 \times \mathsf{S}^1$$

sending

$$(z, w) \mapsto (z^n, z^m)$$

ie, the product of two covers of the circle.

- if $m = 0$, and $n, a \neq 0$, ie, $H = (n, a).\mathbb{Z}$, we get a quotient of $\mathbb{R}^2$ under the equivalence relation $(x, y) \sim (x', y')$ iff there exists $k \in \mathbb{Z}$ such that $x' - x = k.n$ and $y' - y = ka$. Using the change of coordinates $\mathbb{R}^2 \to \mathbb{R}^2$ sending (x, y) to (u, v) given by

$$\begin{pmatrix} u \\ v \end{pmatrix} := \frac{1}{n^2 + a^2} \begin{pmatrix} n & a \\ -a & n \end{pmatrix} . \begin{pmatrix} x \\ y \end{pmatrix}$$

the quotient $\mathbb{R}^2/H$ becomes homeomorphic to the cylinder $\mathbb{R}/\mathbb{Z} \times \mathbb{R}$. The associated covering space is given by

$$\mathbb{R} \times \mathsf{S}^1 \to \mathsf{S}^1 \times \mathsf{S}^1$$

$$(z, t) \mapsto (z^n. \exp(2\pi it)^{-a}, z^a. \exp(2\pi it)^n)$$

- Finally, if $a, n, m \neq 0$, $H = (n, a).\mathbb{Z} \oplus (0, m).\mathbb{Z}$ we get the covering space

$$\mathsf{S}^1 \times \mathsf{S}^1 \to \mathsf{S}^1 \times \mathsf{S}^1$$

$$(z, w) \mapsto (z^n, z^a.w^m)$$

Exercise VI.7.4. Let X be a path-connected, locally simply-connected space with fundamental group isomorphic to the symmetric group S_3. Its order is $3! = 6 = 2.3$. By the first Sylow theorem S_3 admits subgroups of order 3 and subgroups of order 2.

By the third Sylow theorem, the number of subgroups of order 2 must be 3 (corresponding to the three transpositions (1)(23), (12)(3) and (13)(2)) and the number of subgroups of order 3 is 1 corresponding to the, corresponding to cyclic subgroup generated by the permutation (123).

By the second Sylow theorem, all 2-Sylow subgroups are conjugated.

By definition, order 2 means index 3 and vice-versa, subgroups of order 3 corresponds to subgroups of index 2.

Since the degree of the cover coincides with the index of the subgroup we find that there are precisely 1 isomorphism classes of 3-sheeted path-connected covering spaces and 1 isomorphism class of 2-sheeted path-connected covering spaces.

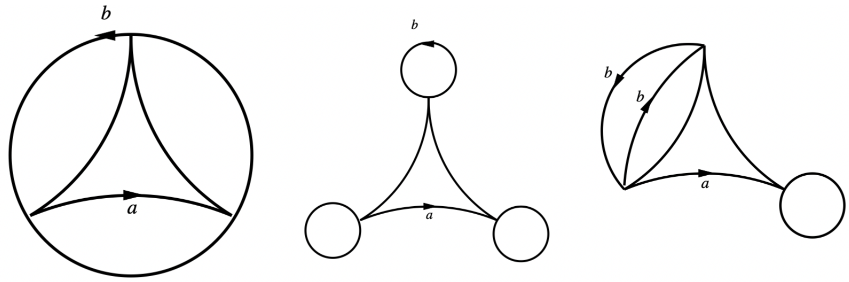
Example VI.7.5. Describe all covering spaces or $\mathbb{R}P^2 \vee \mathbb{R}P^2$.

Exercise VI.7.6. Show that any map $S^2 \to \mathsf{S}^1$ is null-homotopic. Hint: use the lifting criterium for coverings.

Exercise VI.7.7. Construct a 3-sheeted cover of a surface of genus 2, $\Sigma_4 \to \Sigma_2$.

Exercise VI.7.8. Show that the torus is a 2-sheeted cover of the Klein bottle.

Exercise VI.7.9. For each of the following pictures of a cover of $\mathsf{S}^1 \vee \mathsf{S}^1$, write down the respective subgroups of the free group with two generators $\langle a, b \rangle$



Which are Galois covers?

Exercise VI.7.10. Consider the kernel Ker of the map of groups $\langle a, b \rangle \to \mathbb{Z}/4$ sending both a and b to 1. What is the covering space associated to Ker?

Exercise VI.7.11. Let G be a topological group with unit e. Suppose that G is path-connected and locally simply-connected.

(i) Show that the universal cover $\tilde{G}_e$ admits a unique structure of topological group such that the covering map $\tilde{G}_e \to G$ is a morphism of topological groups.

(ii) Exhibit an exact sequence of topological groups

$$1 \to \pi_1(G, e) \to \tilde{G}_e \to G \to 1$$

with $\pi_1(G, e)$ endowed with the discrete topology.

(iii) Show that the image of $\pi_1(G, e)$ is contained in the center of $\tilde{G}_e$.