Mémoire

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présenté par

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Titre du mémoire:

Applications de la géométrie algébrique dérivée en géométrie énumérative et invariants des singularités.

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Applications of derived algebraic geometry in enumerative geometry and invariants of singularities.

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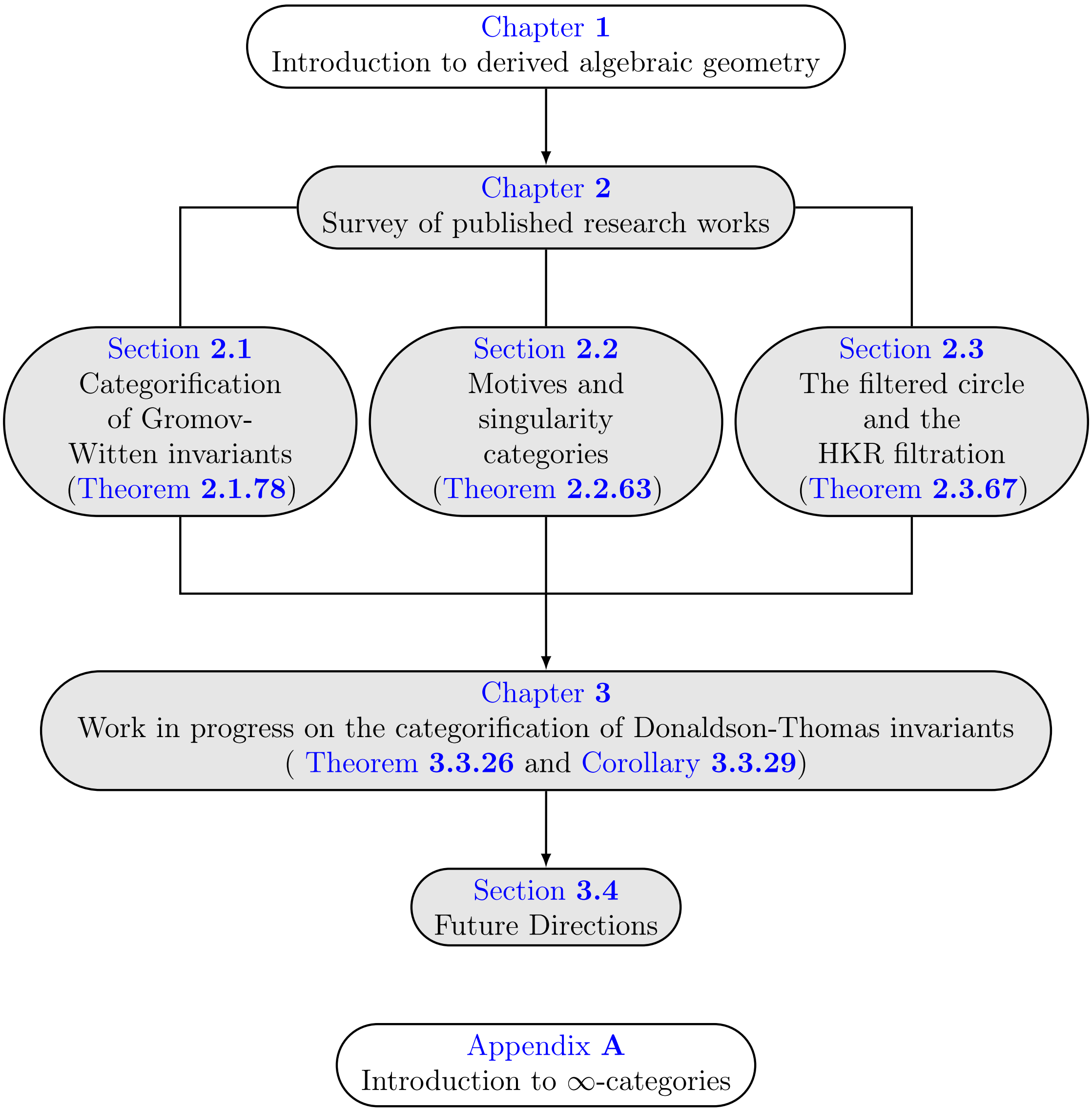
Summary

My research work uses ideas and tools from derived algebraic geometry, homotopy theory and higher categories, to address problems in enumerative algebraic geometry and theory of singularities.

This document surveys the results of three papers: [MR18; BRTV18; MRT22], obtained in different collaborations, *after* the conclusion of my Phd Thesis. For each topic we present a self-contained discussion intended as a roadmap to the subject, accessible to non-experts. The document is organized as follows:



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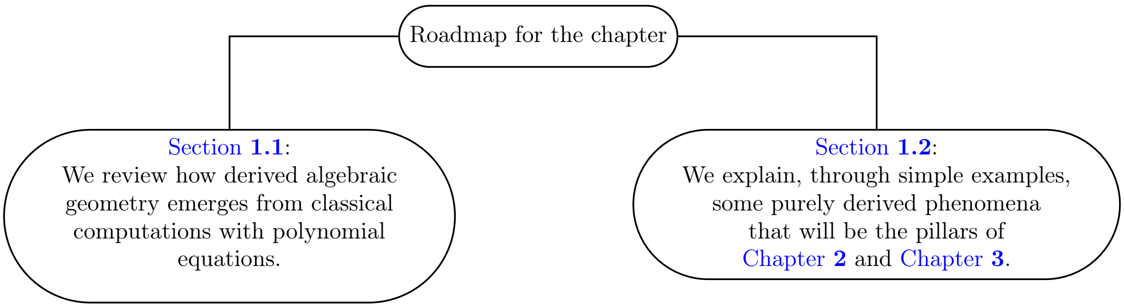
Contents

CHAPTER 1

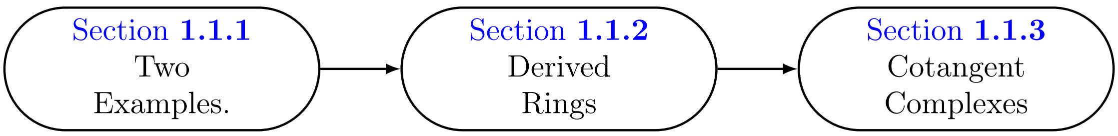
Prelude

The goal of this chapter is to explain, focusing on simple examples, the basic ideas of derived algebraic geometry.



1.1. Derived Algebraic Geometry: an appetizer

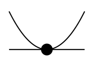
Warning 1.1.1. This section is *neither* an extensive introduction to derived algebraic geometry nor a review of its historical origins. We have merely collected some foundational examples motivating the subject, referring the reader to [PV21; Toë14; EP21] and [Lu-SAG:§0] for an introduction and [Toë09; Lur04; Lu-SAG] for the technical details.



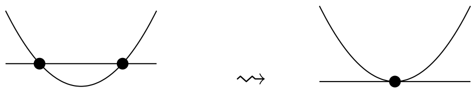
1.1.1. Two Examples. Let us start with a simple example working over the field of complex numbers $\mathbb{C}$.

Example 1.1.2. Consider the intersection in the plane $\mathbb{A}^2_{\mathbb{C}}$ given by

$$(y = x^2) \cap (y = 0)$$



Informally, one sees that this intersection is not transverse, and that its tangency multiplicity should be 2, as the following small perturbation suggests:



From the perspective of differential geometry, this intersection consists of a single point. But an algebraic geometer wants to distinguish the naive single point from a *point with double multiplicity* as is this one. This is the root of the theory of schemes. To intrinsically distinguish the two, one turns to their functions: on a simple single point the only functions are constants. But by construction, on the *double point* we have a nilpotent function that remembers the double multiplicity, expressed by the relation $x^2 = 0$. More precisely, we *define* functions on the double point by the *tensor product* of the respective rings of functions on the line and the parabola:

$$\mathbb{C}[x,y]/(y-x^2) \underset{\mathbb{C}[x,y]}{\otimes} \mathbb{C}[x,y]/(y-0) \simeq \mathbb{C}[x]/(x^2)$$

The result is the ring of dual numbers $\mathbb{C}[x]/(x^2)$ where the class of x squares to zero. In the language of schemes, the naive point is given by the affine scheme $\mathsf{Spec}(\mathbb{C})$ and the *double point* is $\mathsf{Spec}(\mathbb{C}[x]/(x^2))$. Both have the same underlying topological space, but different rings of functions. The geometric multiplicity is confirmed by the following algebraic computation of the dimensions:

$$\dim_{\mathbb{C}}[\mathbb{C}[x,y]/(y-x^2) \underset{\mathbb{C}[x,y]}{\otimes} \mathbb{C}[x,y]/(y-0)] \simeq \dim_{\mathbb{C}} \mathbb{C}[x]/(x^2) =$$

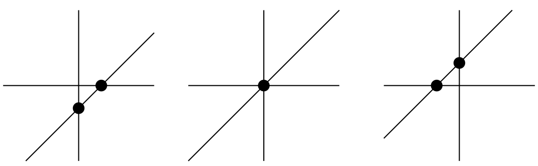
$$= \dim_{\mathbb{C}} (\mathbb{C} \oplus \mathbb{C}x) = 2$$

The algebro-geometric computation of the Example **1.1.2** fails in higher dimensions as the following example due to Serre, shows:

Example 1.1.3. [Har77:Appendix A, Example 1.1.1] Consider the intersection of the axis in 4-dimensions (x, y, z, w), with the diagonal:

$$\text{Axis} := \begin{cases} xz = 0 \\ xw = 0 \\ yz = 0 \\ yw = 0 \end{cases} \qquad \text{Diagonal} := \begin{cases} x - z = 0 \\ y - w = 0 \end{cases}$$

This intersection should have geometric multiplicity $m = 2$ as the picture suggests by moving the diagonal



However, the algebraic computation of the Example **1.1.2** gives:

$$\mathbb{C}[x,y,z,w]/(xz,xw,yz,yw) \underset{\mathbb{C}[x,y,z,w]}{\otimes} \mathbb{C}[x,y,z,w]/(x-z,y-w)$$

$$\simeq \mathbb{C}[x,y]/(x^2,xy,y^2)$$

so that we obtain a ring whose complex dimension is 3 instead of 2.

This shows that the use of tensor products is flawed with respect to multiplicities. The reason is that it does not account for redundancies, as we should now explain: consider the function

$$f := xw - yz = w(x-z) - z(y-w)$$

The element f belongs simultaneously to the two ideals

$$f \in \underbrace{(x-z, y-w)}_{I_{Diag}} \quad , \quad f \in \underbrace{(xz, xz, yz, yw)}_{I_{Axis}}$$

and therefore, belongs to their intersection

$$f \in \underbrace{(x-z, y-w)}_{I_{Diag}} \cap \underbrace{(xz, xz, yz, yw)}_{I_{Axis}}$$

One can rephrase this by saying that f vanishes for <u>two reasons</u> on the intersection.

Using Grobner basis, one can explicitly compute generators for the intersection of the two ideals, and find

$$I_{Diag} \cap I_{Axis} = (wy[w-y],\ \ yz[y-w],\ \ f,\ \ [x-z]yz,\ \ xz[x-z])$$

to conclude that all the generators of this intersection, except f, are actually elements of the product ideal $I_{Diag} \bullet I_{Axis}$. Since f does not belong to the product ideal one can say that the two reasons why f vanishes on the intersection, are <u>different</u>. In particular, the class of f in the quotient abelian group

$$0 \neq [f] \in \frac{I_{Diag} \cap I_{Axis}}{I_{Diag} \bullet I_{Axis}} \tag{1}$$

is non-zero. As a consequence of our computation, we also see that this quotient is actually of dimension 1 generated by the class $[f]$. This gives a direct hint on how to correct the algebraic formula in the Example **1.1.3**: the correct multiplicity should discount the redundancy generated by f:

$$\underbrace{\text{Correct Multiplicity}}_{2} = \underbrace{\text{Algebraic multiplicity}}_{3} - \underbrace{\text{redundancies}}_{1} \tag{2}$$

Remark 1.1.4. A beginner's exercise in homological algebra[(*)], tells us that the redundancy quotient (1) is actually the first Tor group of the intersection:

$$\frac{I_{Diag} \cap I_{Axis}}{I_{Diag} \bullet I_{Axis}} \simeq \mathsf{Tor}^1_{\mathbb{C}[x,y,z,w]}(\mathbb{C}[x,y,z,w]/I_{Diag}\ ,\ \mathbb{C}[x,y,z,w]/I_{Axis})$$

This group is actually defined as the first homology group of a chain complex, namely, the *derived tensor product of rings*

$$\mathbb{C}[x,y,z,w]/I_{Diag} \overset{\mathbb{L}}{\underset{\mathbb{C}[x,y,z,w]}{\otimes}} \mathbb{C}[x,y,z,w]/I_{Axis} \tag{3}$$

Remark 1.1.5. Following the Remark **1.1.4**, what the Example **1.1.2**, Example **1.1.3**, and Remark **1.1.4** show is that even when solving classical systems of polynomial equations, algebraic redundancies hidden in the homological algebra contribute to geometry. The formula (2) is a particular example of the *Serre intersection formula* [Ser65a] at a point x belonging to the intersection of two ideals I and J,

$$\sum_{i\geq 0}(-1)^i \mathsf{length}_{\mathscr{O}_{X,x}} \mathsf{Tor}_i^{\mathscr{O}_{X,x}}(\mathscr{O}_{X,x}/I_x\ ,\ \mathscr{O}_{X,x}/J_x)$$

Remark 1.1.6. The classical tensor product of commutative rings is again a commutative ring. This fact is one of the building blocks of scheme theory, that allows us in particular to define fiber products of schemes:

$$\mathsf{Spec}(\mathbb{C}[x,y,z,w]/I_{Diag}) \underset{\mathbb{A}^4}{\times} \mathsf{Spec}(\mathbb{C}[x,y,z,w]/I_{Axis}) \simeq \mathsf{Spec}\Big(\mathbb{C}[x,y,z,w]/I_{Diag} \underset{\mathbb{C}[x,y,z,w]}{\otimes} \mathbb{C}[x,y,z,w]/I_{Axis}\Big)$$

Meanwhile, the *derived* tensor product of commutative rings is not even an abelian group, but rather a complex of abelian groups, so it cannot be a ring in the most naive way.

Derived algebraic geometry starts with the observation that the commutative ring structures on each of the inputs in (3), compound, and induce on the resulting chain complex a structure of commutative differential graded algebra. This is not obvious since the result depends on a *choice* of resolutions not simply as modules but as algebras. Before coming back to the Example **1.1.3**, let us illustrate this with a simpler example:

Example 1.1.7. Let us compute the self-intersection of the point 0 in the affine line $\mathbb{A}^1_{\mathbb{C}}$. If t denotes the coordinate in $\mathbb{A}^1_{\mathbb{C}}$, then we are looking at the system of equations

$$\begin{cases} t = 0 \\ t = 0 \end{cases}$$

[(*)] See [Wei94:Exercise 3.1.3]

From a geometric point of view, the solution is a point, 0 in $\mathbb{A}^1_{\mathbb{C}}$. But by looking at the same equation twice, we have generated a redundancy as in the Example **1.1.3**: the function t now vanishes for two reasons, in the sense that

$$t \in (t) \cap (t) = (t)$$

but these two reasons are different, in the sense that

$$t \notin (t).(t) = (t^2)$$

So the class $\epsilon := [t]$ does not vanish in the quotient:

$$0 \neq [\epsilon] \in \frac{(t) \cap (t)}{(t).(t)} = \frac{(t)}{(t^2)} \simeq \mathsf{Tor}^1_{\mathbb{C}[t]}(\mathbb{C}, \mathbb{C}) \simeq \mathbb{C}$$

From the viewpoint of derived geometry, what we are saying is that the correct "ring" of functions on the self-intersection of 0 in $\mathbb{A}^1_{\mathbb{C}}$ is the derived tensor product

$$\mathbb{C} \overset{\mathbb{L}}{\underset{\mathbb{C}[t]}{\otimes}} \mathbb{C}$$

To compute it explicitly, we *choose* the Koszul algebra resolution of $\mathbb{C}$ as a $\mathbb{C}[t]$-algebra, namely, the commutative differential graded algebra

$$[\underbrace{0}_{\deg 2} \longrightarrow \underbrace{\mathbb{C}[t]}_{\deg 1} \xrightarrow{.t} \underbrace{\mathbb{C}[t]}_{\deg 0} \longrightarrow \underbrace{0}_{\deg -1}]$$

where by definition, every element in degree 1 squares to zero. Notice that we are using homological conventions and will continue to do so throughout the text. As a commutative differential graded algebra, this is the symmetric(*) algebra $\mathsf{Sym}_{\mathbb{C}[t]}(\mathbb{C}[t][1])$ with differential given by multiplication by t. If we pick $P(t)$ and $Q(t)$ two polynomials in the copy of $\mathbb{C}[t]$ in degree 1, we have the graded Leibniz rule $t.(Q * P) = (t.Q)P + (-1)^1 t.P.Q = t.Q.P - t.P.Q = 0$ which coincides with the square zero multiplication law on the differential graded Sym. Finally, notice that the map

$$\mathbb{C} \to (\mathsf{Sym}_{\mathbb{C}[t]}(\mathbb{C}[t][1]), t) \tag{4}$$

given by the inclusion of $\mathbb{C}$ in degree zero, is a morphism of commutative differential graded algebras. This is of course not an isomorphism but it induces an isomorphism on homology groups, namely, because $\mathsf{H}^1((\mathsf{Sym}_{\mathbb{C}[t]}(\mathbb{C}[t][1]), t)) = 0$ and $\mathbb{C} \to \mathsf{H}^0((\mathsf{Sym}_{\mathbb{C}[t]}(\mathbb{C}[t][1]), t))$ is an isomorphism. Therefore, the map (4) is a quasi-isomorphism of commutative differential graded algebras and $\mathsf{Sym}_{\mathbb{C}[t]}(\mathbb{C}[t][1])$ is a resolution of $\mathbb{C}$. Using this explicit resolution, we obtain a natural structure of commutative differential graded algebra structure on the tensor product

$$\mathbb{C} \overset{\mathbb{L}}{\underset{\mathbb{C}[t]}{\otimes}} \mathbb{C} \underbrace{\simeq}_{q.iso} \mathbb{C} \underset{\mathbb{C}[t]}{\otimes} \big(\mathsf{Sym}_{\mathbb{C}[t]}(\mathbb{C}[t][1], t)\big) \underbrace{\simeq}_{iso} [\underbrace{0}_{\deg 2} \longrightarrow \underbrace{[\epsilon].\mathbb{C}}_{\deg 1} \xrightarrow{0} \underbrace{\mathbb{C}}_{\deg 0} \longrightarrow \underbrace{0}_{\deg -1}]$$

(*)as graded

where we see the class of ϵ appearing as a generator for the copy of $\mathbb{C}$ in degree 1. The algebra structure is the usual multiplication in degree 0 and is determined by the condition $[\epsilon]^2 = 0$ in degree 1. In other words, this is the free commutative differential graded algebra with a generator in homological degree 1:

$$\mathbb{C} \overset{\mathbb{L}}{\underset{\mathbb{C}[t]}{\otimes}} \mathbb{C} \simeq \mathbb{C}[\epsilon]$$

Notice that the function ϵ does not exist in the classical schematic self-intersection of 0 in $\mathbb{A}^1$. It appears here as a *witness* of the two copies of 0.

Example 1.1.8. Let us return to the Example **1.1.3**. To describe the commutative differential algebra structure on the derived tensor product (3) we also use Koszul resolutions [Vez20:Section 2]: given A a commutative ring and f an element in A, we form the Koszul complex $\mathsf{K}(A, f) := (\mathsf{Sym}_A(A[1]), f)$ with underlying complex given by

$$[\underbrace{0}_{\text{deg } 2} \longrightarrow \underbrace{A}_{\text{deg } 1} \xrightarrow{.f} \underbrace{A}_{\text{deg } 0} \longrightarrow \underbrace{0}_{\text{deg } -1}]$$

where two elements in degree 1 multiply to zero. When f is a non-zero divisor this is a resolution of $A/(f)$ over A. We apply this to $A := \mathbb{C}[x, y, z, w]$ of the Example **1.1.3** and form the Koszul commutative differential graded algebras

$$\mathsf{K}(A, xz), \quad \mathsf{K}(A, xw), \quad \mathsf{K}(A, yz), \quad \mathsf{K}(A, yw), \quad \mathsf{K}(A, x - y) \quad \mathsf{K}(A, y - w)$$

The tensor product of commutative differential graded algebras over A

$$\mathsf{K}(A, xz) \underset{A}{\otimes} \mathsf{K}(A, xw) \underset{A}{\otimes} \mathsf{K}(A, yz) \underset{A}{\otimes} \mathsf{K}(A, yw) \underset{A}{\otimes} \mathsf{K}(A, x - y) \underset{A}{\otimes} \mathsf{K}(A, y - w)$$

gives an explicit model for the commutative differential graded algebra structure on the derived tensor product in (3).

1.1.2. Derived Rings. What we learn from the examples in Section **1.1.1** is that in order to cast an explicit form of the commutative differential graded structure on the derived tensor product of classical rings, one must first choose algebra resolutions. These choices are *quasi-isomorphic* but not isomorphic. To solve this ambiguity one must work in a setting where quasi-isomorphic objects become identified as if they were isomorphic. After the progress in the last decades with the foundational works of Quillen [Qui67], Grothendieck [Gro22], Dwyer and Kan [DK80c; DK80a; DK80b], Simpson [Sim97], Rezk [Rez01] and many others, it became clear that homotopy theories obtained by forcing quasi-isomorphisms to become *de facto* (weakly) isomorphisms, are *examples* of ∞*-categories*. It is beyond the scope of this text to give a full account of the theory of ∞-categories but for the reader's convenience we included a minimal toolkit in Appendix **A** using Joyal's quasi-categories as models. The collection of commutative differential positively graded $\mathbb{C}$-algebras up to quasi-isomorphisms forms an ∞-category which we denote as [(*)]

$$\mathsf{cdga}_{\mathbb{C}}^{\geq 0}$$

Experimental evidence shows that this is the correct place to deal with the Example **1.1.3** and Example **1.1.7**.

Remark 1.1.9. The ∞-category $\mathsf{cdga}_{\mathbb{C}}^{\geq 0}$ contains the 1-category of classical commutative $\mathbb{C}$-algebras $\mathsf{CRings}_{\mathbb{C}}$ as the full subcategory spanned by commutative differential graded algebras A with $\mathsf{H}^i(A) = 0$, $\forall i \neq 0$. Moreover, from every object $A \in \mathsf{cdga}_{\mathbb{C}}^{\geq 0}$ one can extract a commutative ring by taking the zeroth-homology $\mathsf{H}^0(A)$. The functor $\mathsf{H}^0 : \mathsf{cdga}_{\mathbb{C}}^{\geq 0} \to \mathsf{CRings}_{\mathbb{C}}$ provides a left adjoint to the inclusion $\mathsf{CRings}_{\mathbb{C}} \subseteq \mathsf{cdga}_{\mathbb{C}}^{\geq 0}$.

Remark 1.1.10. We can give another construction of the ∞-category $\mathsf{cdga}_{\mathbb{C}}^{\geq 0}$ by first taking the ∞-category $\mathsf{Mod}_{\mathbb{C}}$ underlying the Quillen model structure on chain complexes of $\mathbb{C}$-modules up to quasi-isomorphisms (cf. Construction **A.5.23**). This model structure is compatible with the symmetric tensor product of complexes [Lu-HAlg: 7.1.2.8, 7.1.2.13] and makes $\mathsf{Mod}_{\mathbb{C}}$ a symmetric monoidal ∞-category where by definition the tensor structure is now the derived tensor product. One can recover $\mathsf{cdga}_{\mathbb{C}}^{\geq 0}$ as the full subcategory $\mathsf{CAlg}^{\mathsf{cn}}$ of *connective* $\mathsf{E}_{\infty}^{\otimes}$-algebras in $\mathsf{Mod}_{\mathbb{C}}$, $\mathsf{CAlg}_{\mathsf{k}} := \mathsf{CAlg}(\mathsf{Mod}_{\mathbb{C}})$ [Lu-HAlg: 7.1.4.7, 7.1.4.11].

Remark 1.1.11. For classical rings, tensor products are pushouts. This fact generalizes to the ∞-category $\mathsf{CAlg}_R := \mathsf{CAlg}(\mathsf{Mod}_R)$ for any classical ring R: derived tensor products coincide with homotopy pushouts [Lu-HAlg: 3.2.4.7]. In particular, as a consequence of the Remark **1.1.10**, the same holds for $\mathsf{cdga}_{\mathbb{C}}^{\geq 0}$ because connective objects in $\mathsf{Mod}_{\mathbb{C}}$ are stable under derived tensor products [Lu-HAlg: 7.1.1.7.].

Remark 1.1.12. Commutative differential graded algebras are one specific choice to represent derived tensor products of rings. There are other possibilities. Let k be a base ring:

- A less explicit but more universal possibility is to start with the 1-category $\mathsf{Poly}_{\mathsf{k}}$ of *polynomial rings* of the form $\mathsf{k}[x_1, \cdots, x_n]$ and *formally complete it under all relative derived tensor products*[(†)]

$$\mathsf{k}[x_1, \cdots, x_n] \overset{\mathbb{L}}{\underset{\mathsf{k}[y_1, \cdots, y_m]}{\otimes}} \mathsf{k}[z_1, \cdots, z_l]$$

(*) More precisely, the 1-category of commutative differential positively graded $\mathbb{C}$-algebras with weak-equivalences given by quasi-isomorphisms, forms a Quillen model category [Lu-HAlg: 7.1.4.10, 7.1.4.20] with cofibrant objects given by retracts of semi-free commutative differential graded algebras. In particular, the underlying chain complex of cofibrant objects are projective. We define $\mathsf{cdga}_{\mathbb{C}}^{\geq 0}$ to be the *underlying* ∞-category of this model category. See Appendix **A**.

(†) More precisely, we should complete it under all sifted colimits.

We denote by $\mathsf{dRings}_\mathsf{k}$ the resulting ∞-category.(*). We call an object of $\mathsf{dRings}_\mathsf{k}$ a *derived ring*. See [Lu-SAG:§25.1.1].

- Another possibility would be to take *simplicial resolutions* and obtain instead simplicial commutative rings. The 1-category of simplicial commutative rings admits a Quillen model structure. We denote by SCR_k its underlying ∞-category. See the Example **A.4.15**.

- The ∞-categories $\mathsf{dRings}_\mathsf{k}$ and SCR_k are always equivalent, independently of the characteristic of k [Lu-HAlg: 25.1.1.5].

- For any base ring k, the Dold-Kan normalization functor provides a *forgetful* map $\Theta : \mathsf{SCR}_\mathsf{k} \to \mathsf{CAlg}(\mathsf{Mod}_\mathsf{k})$ [Lu-SAG: 25.1.2.2]. When k is a $\mathbb{Q}$-algebra, Θ induces an equivalence of SCR_k with $\mathsf{cdga}_\mathsf{k}^{\geq 0}$.

In particular, this shows that away from characteristic zero, the simplicial approach is the only that remains correct.

Terminology 1.1.13. Following the Remark **1.1.12** we will allow ourselves throughout this text to work with the ∞-category of derived rings $\mathsf{dRings}_\mathsf{k}$ without specifying the model (commutative differential graded algebras or simplicial algebras) unless it becomes strictly necessary for computations.

Assumption 1.1.14. From now on, we fix k a base ring without further assumptions about its characteristic.

Notation 1.1.15. We denote by $\pi_n(R)$ the higher homotopy groups of a derived ring R. If R is modeled by a positively-graded commutative differential graded algebra, this coincides via the Dold-Kan construction with the homology groups H_n.

Underlying the language of schemes is the principle that every element in a commutative ring R can be interpreted as a *function* on the space $\mathsf{Spec}(R)$.

> *Derived algebraic geometry*, following Toën-Vezzosi in [HAG-II] and J. Lurie [Lur04; Lu-SAG], extends the role of commutative rings in the foundations of algebraic geometry by allowing derived rings as rings of functions.

Construction 1.1.16. For any $R \in \mathsf{dRings}_\mathsf{k}$, there is an *affine derived scheme* $\mathsf{Spec}(R)$ with R as a *derived ring* of global functions. The derived scheme $\mathsf{Spec}(R)$ has a *classical truncation* given by $\mathsf{t}_0(\mathsf{Spec}(R)) := \mathsf{Spec}(\pi_0(R))$ the classical affine scheme of the commutative ring $\pi_0(R)$. The truncation map $R \to \pi_0 R$ defines a closed immersion of derived schemes $\mathsf{t}_0\,\mathsf{Spec}(R) \hookrightarrow \mathsf{Spec}(R)$ which induces an isomorphism on the underlying topological spaces. The new extra data are functions

(*)In the notations of [Lu-HTT:§25.1.1] this is $\mathscr{P}^\Sigma(\mathsf{N}(\mathsf{Poly}_\mathsf{k}))$

living in the higher homological degrees of R. See [Toë14:§2.2] for details.

The collection of affine derived schemes forms an ∞-category $\mathsf{dAff_k}$, equivalent to $(\mathsf{dRings_k})^{\mathsf{op}}$ via the functor Spec. Combined with the Remark **1.1.11** this equivalence implies that derived fiber products of derived schemes correspond to derived tensor products. This was taken as an implicit fact in our computations above.

Review 1.1.17. The notions of schemes, flat, étale and smooth morphisms , 1-stacks [LMB18] and Simpson's higher stacks in classical algebraic geometry [SH01; Sim96a] can be extended to the framework of derived geometry, providing notions of *derived schemes* (Zariski gluings of affine derived schemes) and *derived stacks*, ie ∞-functors

$$\mathsf{dRings_k} \to \mathcal{S}$$

with values in the ∞-category of spaces - $\mathcal{S}$ (see Example **A.4.13**) - and satisfying the ∞-categorical version of étale hyperdescent. The collection of derived schemes and derived stacks form, respectively, ∞-categories $\mathsf{dSch_k}$ and $\mathsf{dSt_k}$. Assigning to a derived scheme its functor of points, provides a fully faithful embedding $\mathsf{dSch_k} \subseteq \mathsf{dSt_k}$ which fits in a commutative diagram:

$$\begin{tikzcd} \mathsf{dAff_k} \arrow[r, hook] & \mathsf{dSch_k} \arrow[r, hook] & \mathsf{dSt_k} \\ \mathsf{Aff_k} \arrow[r, hook] \arrow[u, hook] & \mathsf{Sch_k} \arrow[r, hook] \arrow[u, hook] & \mathsf{St_k} \arrow[u, hook] \end{tikzcd} \tag{5}$$

Similarly to the classical setting, there is also a notion of derived *geometric* stacks. The reader can consult [PV21; Toë14] for details.

Remark 1.1.18. Since, in general, derived tensor products do not agree with naive tensor products unless one of the maps is flat, the vertical inclusions in (5) do not preserve homotopy fiber products. Of course, when k is a field, everything is flat over k and therefore the vertical inclusions at least commute with finite products.

Remark 1.1.19. The construction of classical truncations t_0 for derived rings can be Kan extented to all derived schemes and derived stacks. The result is a right-adjoint to the inclusion $\mathsf{St_k} \subseteq \mathsf{dSt_k}$. In particular, and in opposition to the discussion in Remark **1.1.18**. t_0 commute with all limits, ie, the classical truncation of a derived fiber product is the classical fiber product.

Let us come back to examples.

Example 1.1.20. In the Example **1.1.3** we have implicitly defined the derived intersection as Spec of the derived tensor product. Via the equivalence between derived schemes and derived rings, this becomes a consequence of the canonical equivalence

$$\mathsf{Spec}(\mathbb{C}[x,y,z,w]/I_{Diag}) \overset{\mathsf{dAff}_{\mathbb{C}}}{\underset{\mathbb{A}^4}{\times}} \mathsf{Spec}(\mathbb{C}[x,y,z,w]/I_{Axis}) \simeq \mathsf{Spec}\Big(\mathbb{C}[x,y,z,w]/I_{Diag} \overset{\mathbb{L}}{\underset{\mathbb{C}[x,y,z,w]}{\otimes}} \mathbb{C}[x,y,z,w]/I_{Axis}\Big)$$

which is a consequence of the fact that derived tensor products of derived rings are homotopy pushouts (cf.Remark **1.1.11**), and therefore, homotopy pullbacks of affine derived schemes.

Example 1.1.21. In the same vein as the Example **1.1.20**, the derived tensor product of the Example **1.1.7** is the derived self-intersection in 0 in $\mathbb{A}^1_{\mathbb{C}}$ taken in the ∞-category $\mathsf{dAff}_{\mathbb{C}}$

$$\begin{tikzcd} \mathsf{Spec}(\mathbb{C}[\epsilon]) \arrow[r] \arrow[d] & \mathsf{Spec}(\mathbb{C}) \arrow[d, "0"] \\ \mathsf{Spec}(\mathbb{C}) \arrow[r, "0"] & \mathbb{A}^1_{\mathbb{C}} \end{tikzcd}$$

In a sense, this is the most degenerate possibility for a non-transverse intersection. Topologically $\mathsf{Spec}(\mathbb{C}[\epsilon])$ consists of a single point $\mathsf{Spec}(\mathsf{H}^0(\mathbb{C}[\epsilon])) = \mathsf{Spec}(\mathbb{C}) = *$. Notice the similarity with the Example **1.1.2** where the resulting intersection also consists of a single point with the nilpotent ring $\mathbb{C}[x]/(x^2)$ as functions.

What derived geometry teaches is that if we want to systematically correct non-transversal intersections one must also allow nilpotents placed in higher homological degrees.

Remark 1.1.22. In classical set theory, the self-intersection of an element e in a set E is always trivial. To get a feeling of how self-intersections can become non-trivial, one has to work in a homotopical context. Let us illustrate this with a simple example: consider a diagram of groupoids

$$\begin{tikzcd} & \mathsf{E} \arrow[d, "G"] \\ \mathsf{C} \arrow[r, "F"] & \mathsf{D} \end{tikzcd}$$

An *homotopy pullback* (or homotopy fiber product) for the diagram, if it exists, is a groupoid P together with maps $u : \mathsf{P} \to \mathsf{E}$ and $v : \mathsf{P} \to \mathsf{C}$ and a natural isomorphism of functors $\alpha : F \circ v \to G \circ u$ making the diagram

$$\begin{tikzcd} \mathsf{P} \arrow[r] \arrow[d] & \mathsf{E} \arrow[d, "G"] \\ \mathsf{C} \arrow[r, "F"] & \mathsf{D} \end{tikzcd}$$

commute and universal with respect to this property. Here's an explicit description of P: it classifies triples (X, Y, α) consisting of an object X in C, an object $Y \in \mathsf{E}$

and the data of an isomorphism in D, $\alpha : F(X) \simeq G(Y)$.

Finally, here's the simplest example of a non-trivial self-intersection: let G be a discrete group and let $\mathsf{B}G$ be its classifying groupoid with $* \to \mathsf{B}G$ the unique object $\bullet$ in $\mathsf{B}G$. Then the commutative diagram

$$
\begin{tikzcd}
G \arrow[r] \arrow[d] & * \arrow[d] \\
* \arrow[r] & BG
\end{tikzcd}
\tag{6}
$$

is an homotopy pullback diagram, where G is seen as a discrete category: indeed, G can be identified with the category of triples $(\bullet, \bullet, \alpha)$ where $\alpha : \bullet \simeq \bullet$ is an isomorphism in $\mathsf{B}G$ given by an element of the group. See Appendix **A.5**.

Construction 1.1.23. Every derived ring R admits a derived ∞-category of module objects in $\mathsf{Mod}_{\mathsf{k}}$ (cf. Construction **A.5.23**),

$$\mathsf{Mod}_R := \mathsf{Mod}_{\Theta(R)}(\mathsf{Mod}_{\mathsf{k}})$$

with a symmetric monoidal structure given by relative derived tensor products over R [Lu-HAlg:§3.3.3, 3.4.1.9, 4.4.1.4, 4.4.1.6, 4.5.2.1] and Θ the Dold-Kan construction. Given a derived stack X, we define its ∞-category of quasi-coherent sheaves $\mathsf{QCoh}(X)$ to be the limit in the ∞-category of ∞-categories

$$\mathsf{QCoh}(X) := \lim_{\mathsf{Spec}(R)\to X} \mathsf{Mod}_R$$

indexed by the system of all affine derived schemes over X [Lu-SAG: 6.2.2.1, 6.2.2.7, 6.2.3.4] (*) As a limit of symmetric monoidal ∞-categories, it inherits a symmetric monoidal structure [Lu-HAlg: 3.2.2.1, 3.2.2.5, 4.8.1.9] (†). By construction, any map of derived stacks $f : X \to Y$ admits a pullback $f^* : \mathsf{QCoh}(Y) \to \mathsf{QCoh}(X)$ and a right adjoint pushfoward f_*.

In this setting, we define a perfect complex on X as a dualizable object in $\mathsf{QCoh}(X)$ and denote by $\mathsf{Perf}(X) \subseteq \mathsf{QCoh}(X)$ the full subcategory of perfect complexes [TT90; BZFN10]. By [Lu-HAlg:Proposition 4.6.1.11, 4.6.1.12], it follows that an object $E \in \mathsf{QCoh}(X)$ is dualizable if and only if the pullback u^*E is dualizable in Mod_R for every map $\mathsf{Spec}(R) \to X$. In particular, perfect complexes are stable under general pullbacks.

Construction 1.1.24. The ∞-category $\mathsf{Mod}_{\mathsf{k}}$ admits a t-structure compatible with the symmetric monoidal structure, where M is *connective* if and only if $\mathsf{H}_n(M)$ vanishes for strictly negative n's [Lu-HAlg: 7.1.1.10].

(*) In particular, when $X = \mathsf{Spec}(R)$ is an affine derived scheme, $\mathsf{QCoh}(X) \simeq \mathsf{Mod}_R$.

(†) When X is classical scheme, the homotopy category of $\mathsf{QCoh}(X)$ coincides with derived category of $\mathscr{O}_X$-modules with quasi-coherent cohomology.

This induces a t-structure on Mod_A for every derived ring A [Lu-HAlg: 7.1.1.10, 7.1.1.13.], with connective objects given by those A-modules M with $\mathsf{H}_n(M) = 0$ for $n < 0$.

Finally, under some mild conditions, $\mathsf{QCoh}(X)$ admits a t-structure defined by the condition that $F \in \mathsf{QCoh}(X)$ is connective if and only if for every map from an affine derived scheme $u : \mathsf{Spec}(A) \to X$, the pullback $u^*(F)$ is connective in Mod_A. See [Lu-SAG:6.2.5.7, 6.2.5.8, 6.2.5.9, 6.2.3.4-(3)]. In particular, for every map $f : X \to Y$, the pullback $f^* : \mathsf{QCoh}(Y) \to \mathsf{QCoh}(X)$ is right t-exact (ie, preserves ≥ 0 (homological convention) and the right adjoint $f_* : \mathsf{QCoh}(X) \to \mathsf{QCoh}(Y)$ is left t-exact (ie, preserves ≤ 0). See [Lu-HAlg: 1.3.3.1]. In particular, this implies that t-structure on $\mathsf{QCoh}(X)$ is compatible with the symmetric monoidal structure in the sense that the tensor product of connective objects remains connective.

1.1.3. Cotangent Complexes. As a reward from all this heavy machinery, elements such as $[f]$ in Example **1.1.2** and ϵ in Example **1.1.7** can be interpreted as *homologically shifted functions*. Therefore, their *de Rham differentials* $\mathsf{d}_{\mathsf{dR}}([f])$ and $\mathsf{d}_{\mathsf{dR}}(\epsilon)$ are naturally homologically shifted differential forms. To accommodate this, one must expand the classical construction of tangent bundles and Kähler differentials, to the notion of tangent/cotangent *complex* - $\mathbb{L}_X$ - first discovered by Illusie [Ill71; Ill72]. What derived geometry brings new to Illusie's construction is the reason why it exists, namely, as differentials of functions in higher homological degrees, invisible through classical scheme theory. By construction $\mathbb{L}_X$ is an object in the derived category of quasi-coherent sheaves $\mathsf{QCoh}(X)$ and as consequence of the ∞-categorical setting, it behaves functorially.

We now define $\mathbb{L}_X$ independently of the choice of resolutions, using a universal property similar to Kähler differentials via derivations:

Construction 1.1.25. Let $A \in \mathsf{dRings}_{\mathsf{k}}$ and let $M \in \mathsf{Mod}_A^{\geq 0}$. Then ∞-functor sending M to the space of derivations

$$\mathsf{Map}_{\mathsf{k}/\mathsf{dRings}_{\mathsf{k}}/A}(A, A \oplus M)$$

with $A \oplus M$ the trivial square zero extension of A by M, is representable, ie, there exists $\mathbb{L}_{A/\mathsf{k}} \in \mathsf{Mod}_A^{\geq 0}$ and a natural equivalence

$$\mathsf{Map}_{\mathsf{Mod}_A}(\mathbb{L}_{A/\mathsf{k}}, M) \simeq \mathsf{Map}_{\mathsf{dRings}_{\mathsf{k}}/A}(A, A \oplus M)$$

One way to prove the existence of cotangent complexes is by taking Kähler differentials on cofibrant resolutions (see [PV21] and [Lu-SAG: 25.3.1.1, 25.3.1.4, 25.3.1.5, 25.3.1.8]).

Remark 1.1.26. For a general derived stack the notion of tangent complex as dual to cotangent complex might not exist. However, this is not an issue for a wide class of derived stacks, namely, those that are locally of finite presentation over k: in this case $\mathbb{L}_X$ is a perfect complex on X and we define $\mathbb{T}_X := \mathbb{L}_X^\vee$ as its dual. See Construction **1.1.23**, [HAG-II: 2.2.2.4] and [Lu-SAG: 17.4.2.3]).

For the purposes of this introduction, we will need two basic properties relevant for computations:

Proposition 1.1.27.

(i) the tangent complex of a derived intersection is the derived intersection of respective tangent complexes of each term;

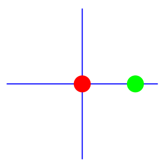
(ii) the tangent complex of a smooth scheme X coincides with its classical tangent bundle seen as a complex concentrated in degree zero - $(\Omega_X^1)^\vee$.

To illustrate, let us discuss some simple examples where the derived intersection actually coincides with the classical one. Already in this case the formalism of derived geometry gives non-trivial tangent complexes:

Example 1.1.28. Let $\mathsf{k} = \mathbb{C}$ and consider the zero locus V of the function $f : \mathbb{A}^2 \to \mathbb{A}^1$ given by $f(x,y) := x.y$:

(7)
$$\begin{tikzcd} V = \{f(x,y) := x.y = 0\} \arrow[r] \arrow[d] & \mathbb{A}^2_{\mathsf{k}} \arrow[d, "f"] \\ 0 \arrow[r] & \mathbb{A}^1_{\mathsf{k}} \end{tikzcd}$$

It consists of the union of the two axis:



In this example, V happens to be automatically a derived intersection because f is a non-zero divisor. Indeed, if we compute the derived fiber product of the intersection, using the Koszul resolution of k over $\mathsf{k}[t]$ as in the Example **1.1.7**, we find

$$\mathsf{k} \overset{\mathbb{L}}{\underset{\mathsf{k}[t]}{\otimes}} \mathsf{k}[x,y] \simeq [0 \to \mathsf{k}[x,y] \xrightarrow{.f} \mathsf{k}[x,y] \to 0]$$

with

$$\pi_0(\mathsf{k} \overset{\mathbb{L}}{\underset{\mathsf{k}[t]}{\otimes}} \mathsf{k}[x,y]) \simeq \mathsf{k}[x,y]/(f)$$

$$\pi_1(\mathsf{k} \overset{\mathbb{L}}{\underset{\mathsf{k}[t]}{\otimes}} \mathsf{k}[x,y]) \simeq \{P \in \mathsf{k}[x,y] : P.f = 0\} = \{0\}$$

By the Proposition **1.1.27**, the *tangent complex* of this derived intersection is the derived fiber of the derivative of f. In concrete terms, this derived fiber is given by the construction of the cone, that automatically plugs information in a new homological degree: at a point $p \in V$, we have

$$\mathbb{T}_{V,p} \simeq [\, 0 \longrightarrow \underbrace{\mathbb{T}_{\mathbb{A}^2,p} \simeq \mathsf{k}.\frac{\partial}{\partial x} \oplus \mathsf{k}.\frac{\partial}{\partial y}}_{\deg 0} \xrightarrow{Df_{|p}} \underbrace{\mathbb{T}_{\mathbb{A}^1,p} \simeq \mathsf{k}}_{\deg -1} \longrightarrow 0\,]$$

When $p = \bullet$ we get $\mathsf{H}^0 = \mathsf{k}=$ usual cotangent space at p of dimension 1 and $\mathsf{H}^1 = 0$. However, when $p = \bullet$ we get $\mathsf{H}^0 = \mathsf{k} \oplus \mathsf{k}$ of dimension 2 and , $\mathsf{H}^1 \simeq \mathsf{k}$. This extra copy of k sitting in degree 1 is a witness of the non-smooth nature of the intersection at the origin.

Remark 1.1.29. By dualizing, the cotangent complex of the Example **1.1.28** at a point p is given by the two-term complex

$$\mathbb{L}_{V,p} \simeq [\, 0 \longrightarrow \underbrace{\mathsf{k}}_{\deg 1} \xrightarrow{(\mathsf{D}f_{|p})^\vee} \underbrace{\mathsf{k} \oplus \mathsf{k}}_{\deg 0} \longrightarrow 0\,]$$

Examples such as Example **1.1.28** where the cotangent complex is concentrated in Tor-amplitudes[(*)] $[1, 0]$ are important to this thesis. We isolate this property in the following definition:

Definition 1.1.30. Let X be a derived scheme or stack over k locally of finite presentation. We say that X is *quasi-smooth* if its cotangent complex is perfect, in Tor-amplitude[(†)] $[1, 0]$ (we use homological conventions).

Remark 1.1.31. Zariski-locally every quasi-smooth derived scheme is of the form $\mathsf{Spec}(\mathsf{K}(\mathsf{k}[x_1, \cdots, x_n], f_1, \cdots, f_n))$ generalizing the Example **1.1.28**. See [AG15:§2.1].

Example 1.1.32. Let $\mathsf{k} = \mathbb{C}$. Consider the function $f : \mathbb{A}^1_{\mathsf{k}} \to \mathbb{A}^1_{\mathsf{k}}$ given by $x \mapsto f(x) = \frac{1}{3}x^3$. The *derived critical locus* of f - $\mathsf{dCrit}(f)$ - is the derived intersection of $df = x^2.dx$ with the zero section of the cotangent bundle

$$\begin{tikzcd} & \mathbb{A}^1_{\mathsf{k}} \arrow[d, "df"] \\ \mathbb{A}^1_{\mathsf{k}} \arrow[r, "0"] & \mathrm{T}^*\mathbb{A}^1_{\mathsf{k}} \simeq \mathbb{A}^2_{\mathsf{k}} \end{tikzcd}$$

It is defined as the derived scheme with derived ring of functions given by the derived tensor product

(*)[Lu-HAlg:Def. 7.2.4.21]
(†)[Lu-HAlg:Def. 7.2.4.21]

$$\mathcal{O}_{\mathsf{dCrit}(f)} := \mathcal{O}_{\mathbb{A}^1_{\mathsf{k}}} \underset{\mathcal{O}_{\mathrm{T}^*\mathbb{A}^1_{\mathsf{k}}}}{\overset{\mathbb{L}}{\otimes}} \mathcal{O}_{\mathbb{A}^1_{\mathsf{k}}} \simeq \mathsf{k}[x] \underset{\mathsf{k}[x,y]}{\overset{\mathbb{L}}{\otimes}} \mathsf{k}[x]$$

Using the Koszul resolution of $\mathsf{k}[x]$ over $\mathsf{k}[x,y]$ given by

$$(\mathsf{Sym}_{\mathsf{k}[x,y]}(\mathsf{k}[x,y][1]), y) \simeq \ [0 \to \mathsf{k}[x,y] \xrightarrow{.y} \mathsf{k}[x,y] \to 0] \tag{8}$$

we obtain an equivalence of derived rings

$$\mathcal{O}_{\mathsf{dCrit}(f)} \simeq (\mathsf{Sym}_{\mathsf{k}[x]}(\mathsf{k}[x][1]), x^2) \simeq \ [0 \to \mathsf{k}[x] \xrightarrow{.x^2} \mathsf{k}[x] \to 0] \ \simeq \mathsf{k}[x]/(x^2)$$

Not only we conclude that in this case, the derived intersection agrees with the classical intersection but also, that this derived intersection recovers precisely the scheme of the Example **1.1.2** resulting from the intersection of the parabola and the line.

As in the Example **1.1.28** and following the Proposition **1.1.27** the tangent complex of this derived intersection is the cone (aka derived intersection) of the tangent complexes. This computation gives us a 2-term complex of $\mathsf{k}[x]/(x^2)$-modules for $\mathbb{T}_{\mathsf{dCrit}(f)}$:

$$\underbrace{\left(\mathbb{T}_{\mathbb{A}^1_{\mathsf{k}}} \underset{\mathcal{O}_{\mathbb{A}^1_{\mathsf{k}}}}{\overset{\mathbb{L}}{\otimes}} \mathcal{O}_{\mathsf{dCrit}(f)}\right) \simeq \mathsf{k}[x]/(x^2)}_{\deg\ 0} \xrightarrow{\mathrm{Hess}(f)=2.x} \underbrace{\left(\mathbb{L}_{\mathbb{A}^1_{\mathsf{k}}} \underset{\mathcal{O}_{\mathbb{A}^1_{\mathsf{k}}}}{\overset{\mathbb{L}}{\otimes}} \mathcal{O}_{\mathsf{dCrit}(f)}\right) \simeq \mathsf{k}[x]/(x^2)}_{\deg\ -1} \tag{9}$$

with internal differential given by the Hessian of f. By taking the $\mathsf{k}[x]/(x^2)$-dual we obtain the cotangent complex $\mathbb{L}_{\mathsf{dCrit}(f)}$ as a two-term complex

$$\underbrace{\mathsf{k}[x]/(x^2)}_{\deg\ 1} \xrightarrow{\mathrm{Hess}(f)^\vee=2.x} \underbrace{\mathsf{k}[x]/(x^2)}_{\deg\ 0} \tag{10}$$

The fact that this complex has non-trivial homology in degree 1, $\mathsf{H}_1(\mathbb{L}_{\mathsf{dCrit}(f)}) \simeq \mathsf{k}.x$ is a measure of the non-transversality of the intersections.

Remark 1.1.33. The model for the derived critical locus in the Example **1.1.32** generalizes to any function f on a smooth scheme U given by the derived intersection

$$\begin{tikzcd} \mathsf{dCrit}(U,f) \arrow[r] \arrow[d] & U \arrow[d, "df"] \\ U \arrow[r, "0"] & \mathsf{T}^*U \end{tikzcd}$$

In general, this won't agree with the classical schematic critical locus. If $U = \mathsf{Spec}(R)$ is an affine scheme, we have an explicit model for $\mathcal{O}_{\mathsf{dCrit}(U,f)}$ given by the explicit commutative differential graded algebra

$$[\; \dots \longrightarrow \underbrace{\bigwedge^2 \mathbb{T}_U}_{\text{deg } 2} \xrightarrow{\iota_{df}} \underbrace{\mathbb{T}_U}_{\text{deg } 1} \xrightarrow{\iota_{df}} \underbrace{\mathcal{O}_U}_{\text{deg } 0} \;]$$

with differential given by contraction with df (see [Vez20:Prop. 2.6]). In particular, we get $\pi_0(\mathcal{O}_{\mathsf{dCrit}(U,f)}) \simeq \mathcal{O}_U/(\partial_1 f, \cdots \partial_n f)$ is the ring of functions on the classical critical locus.

The explicit formula for the tangent complex of (9) is recycled mutatis-mutandis:

$$\underbrace{\left(\mathbb{T}_U \overset{\mathbb{L}}{\underset{\mathcal{O}_U}{\otimes}} \mathcal{O}_{\mathsf{dCrit}(f)}\right)}_{\text{deg } 0} \xrightarrow{\text{Hess}(f)} \underbrace{\left(\mathbb{L}_U \overset{\mathbb{L}}{\underset{\mathcal{O}_U}{\otimes}} \mathcal{O}_{\mathsf{dCrit}(f)}\right)}_{\text{deg } 1} \tag{11}$$

We can now give an example of a shifted differential form:

Definition 1.1.34. Let X be a derived stack locally of finite presentation over a ring k. A (n)-shifted p-form is a global section of the derived quasi-coherent sheaf $(\Lambda^p \mathbb{L}_X)[n]$. More precisely, the space of n-shifted p-forms is the mapping space

$$\mathsf{Map}_{\mathsf{QCoh}(X)}(\mathcal{O}_X, (\Lambda^p \mathbb{L}_X)[n])$$

Example 1.1.35. When X is a smooth algebraic variety, then $\mathbb{L}_X \simeq \Omega^1_X$ and therefore n-shifted forms do not exist for $n \neq 0$. The existence of shifted forms is a measure of the singular behavior of X. We have already came across an example of a (-1)-shifted 1-form λ in the Example **1.1.32**, namely, the class of $1.x \in \mathsf{H}_1(\mathbb{L}_{\mathsf{dCrit}(f)}) \simeq \mathsf{k}.x$.

The two Examples **1.1.28** and **1.1.32** explain how derived geometry accommodates intersection singularities: the higher homological terms in the cotangent complex compensate for the singular behavior of the degree zero part. In fact, this feature also extends to quotient singularities when formulated using the language of stacks:

Example 1.1.36. Consider the action of the multiplicative algebraic group $\mathbb{G}_{m\,\mathsf{k}}$ on $\mathbb{A}^1_{\mathsf{k}}$ by

$$(\lambda, t) \mapsto \lambda.t$$

Away from $0 \in \mathbb{A}^1_{\mathsf{k}}$ this action is free and transitive but at 0 we have the full group $\mathbb{G}_{m\,\mathsf{k}}$ as stabilizer. The naive quotient of $\mathbb{A}^1_{\mathsf{k}}$ under this action contains two points: a smooth point corresponding to the class of [1] and a singular point [0]. The formalism of classical algebraic stacks incorporates stabilizers: the quotient stack

$[\mathbb{A}^1_{\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}]$ contains two points: one point $[1] : \mathsf{Spec}(\mathsf{k}) \to [\mathbb{A}^1_{\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}]$ and a stacky point $[0] : \mathsf{B}\,\mathbb{G}_{\mathrm{m}\,\mathsf{k}} := [\mathsf{Spec}(\mathsf{k})/\,\mathbb{G}_{\mathrm{m}\,\mathsf{k}}] \to [\mathbb{A}^1_{\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}]$ containing a copy of $\mathbb{G}_{\mathrm{m}\,\mathsf{k}}$ as inertia. The stack $[\mathbb{A}^1_{\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}]$ is a smooth Artin stack. Smoothness can be seen using the smooth atlas $\mathbb{A}^1_{\mathsf{k}} \to [\mathbb{A}^1_{\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}]$ but it can also be found through the lens of derived geometry by looking at $[\mathbb{A}^1_{\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}]$ as a derived stack through the inclusion $\mathsf{St}_{\mathsf{k}} \subseteq \mathsf{dSt}_{\mathsf{k}}$: for each point $[x] : \mathsf{Spec}(\mathsf{k}) \to [\mathbb{A}^1_{\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}]$ the stacky fiber recovers $\mathbb{G}_{\mathrm{m}\,\mathsf{k}}$ and its action on x:

$$\begin{tikzcd}
\mathbb{G}_{\mathrm{m}\,\mathsf{k}} \arrow[r, "\mathrm{orb}_x"] \arrow[d] & \mathbb{A}^1_{\mathsf{k}} \arrow[d] \\
\mathsf{Spec}(\mathsf{k}) \arrow[r, "{[x]}"] & {[\mathbb{A}^1_{\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}]}
\end{tikzcd}$$

with the map $\mathrm{orb}_x : \mathbb{G}_{\mathrm{m}\,\mathsf{k}} \to \mathbb{A}^1_{\mathsf{k}}$ sending $\lambda \mapsto \lambda.x$. By Proposition **1.1.27**, the tangent complex of the stacky fiber at $[x]$ is the cone of the cotangent bundles:

$$\begin{tikzcd}
\mathbb{T}_{\mathbb{G}_{\mathrm{m}\,\mathsf{k}},1} \simeq \mathsf{k} \arrow[r, "\mathsf{D}(\mathrm{orb}_x)"] \arrow[d] & \mathbb{T}_{\mathbb{A}^1_{\mathsf{k}},x} \simeq \mathsf{k} \arrow[d] \\
0 \arrow[r] & \mathbb{T}_{[\mathbb{A}^1_{\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}],[x]}
\end{tikzcd}$$

The result of the cone construction is a complex with new homological degrees 0 and 1

$$\mathbb{T}_{[\mathbb{A}^1_{\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}],[x]} \simeq [0 \to \mathfrak{g} \underset{\mathsf{D}(\mathrm{orb}_x)}{\to} \mathbb{T}_{X,x} \to 0]$$

Now, when $x = [1]$, the map $\mathrm{orb}_1 : \mathbb{G}_{\mathrm{m}\,\mathsf{k}} \to \mathbb{A}^1_{\mathsf{k}}$ is an open immersion and therefore an isomorphism on tangent spaces. In particular, we have a quasi-isomorphism of complexes $\mathbb{T}_{[\mathbb{A}^1_{\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}],[1]} \simeq 0$ so that in particular, at 1, $[\mathbb{A}^1_{\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}]$ is of dimension 0. However, then $x = 0$, orb_0 is constant and $D(\mathrm{orb}_0)$ is the zero map. In particular we have a quasi-isomorphism $\mathbb{T}_{[\mathbb{A}^1_{\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}],[0]} \simeq \mathsf{k} \oplus \mathsf{k}[1]$ which is also a chain complex of virtual dimension 0.

Allowing differentials to live through multiple homological degrees actually allows us to re-interpret the classical notions of obstruction and deformation spaces in Schlessinger's deformation theory [Har10]:

Example 1.1.37. An example that will serve as baseline for Gromov-Witten theory in Section **2.1**, is the deformation theory of curves. If C is a projective algebraic curve of genus g over a field k having at worst nodal singularities (ie, the singular points are locally modeled by the equation $xy = 0$), then, because C is generically smooth and locally a complete intersection, the cotangent complex $\mathbb{L}_C$ is equivalent (ie, quasi-isomorphic) in $\mathsf{QCoh}(C)$ to the coherent cotangent sheaf Ω^1_C (which in this case is not a vector bundle). See [DM69; Ill71]. It is a consequence of this fact that the deformation theory of such curves is controlled by the cohomology groups

$$\mathsf{H}^i(C,\mathbb{T}_C) \simeq \mathrm{Ext}^i_{\mathscr{O}_C}(\Omega_C,\mathscr{O}_C)$$

for $i=0,1,2$, where $\mathbb{T}_C$ is the tangent complex of C. The group $i=0$ classifies infinitesimal automorphisms, the $i=1$ classifies infinitesimal deformations and the $i=2$ encodes obstructions to deformations. All this becomes natural when seen through the lens of derived geometry: the moduli stack of such curves - M_g - acquires a canonical *tangent complex* given by the chain complex of derived global sections

$$\mathbb{T}_{\mathsf{M}_g,C} = \mathbb{R}\Gamma(C,\mathbb{T}_C)[1] \tag{12}$$

In particular, its homology groups recover the deformation and obstruction groups above. See the Example **1.2.12** and Example **2.1.8** below.

Example 1.1.38. In this example we want to consider the moduli stack of vector bundles of rank n, $\underline{\mathsf{Vect}}_n$. For the reader convenience, we recall here its construction and characterization as a classifying stack of the group scheme GL_n. By definition, the functor of points of $\underline{\mathsf{Vect}}_n$ is given by sending a classical ring A to the groupoid of projective A-modules of finite type, $\mathsf{Proj}^{\mathsf{ft}}_n(A)$. As in classical algebraic topology, this can be compared to the classifying stack of the group GL_n, namely, let us consider the functor of points $\mathsf{B}\left(\mathsf{GL}_n(-)\right)$ sending a commutative ring A to the classifying groupoid $\mathsf{B}\left(\mathsf{GL}_n(A)\right)$. This functor of points is not a sheaf for the étale topology. Let us denote BGL_n its stackyfication. We have a natural transformation of presheaves of groupoids

$$\eta : \mathsf{B}\left(\mathsf{GL}_n(-)\right) \to \underline{\mathsf{Vect}}_n$$

defined on a classical commutative ring A

$$\mathsf{B}\left(\mathsf{GL}_n(A)\right) \to \underline{\mathsf{Vect}}_n(A)$$

by sending the unique point $\bullet$ of $\mathsf{B}\left(\mathsf{GL}_n(A)\right)$ to the trivial A-module of rank n, $A^{\oplus n}$, and by sending an element $M \in \mathsf{GL}_n(A)$ to the corresponding automorphism of $A^{\oplus n}$. This map clearly induces an isomorphism at the level of π_1-presheaves of sets

$$\pi_1(\mathsf{B}(\mathsf{GL}_n(A)),\bullet) = \mathsf{GL}_n(A) \simeq \pi_1(\underline{\mathsf{Vect}}_n, A^{\oplus n})$$

Since $\underline{\mathsf{Vect}}_n$ is a stack, η descends to the stackyfication of $\mathsf{B}\left(\mathsf{GL}_n(-)\right)$ and induces a map of stacks

$$\tilde{\eta} : \mathsf{BGL}_n \to \underline{\mathsf{Vect}}_n$$

It remains to check that $\tilde{\eta}$ is an isomorphism on π_0-sheaves. But this follows from the fact that étale locally every vector bundle is isomorphic to the trivial module $A^{\oplus n}$.

Now that we have characterized $\underline{\mathsf{Vect}}_n$, we want to treat it as a derived stack, via the inclusion $\mathsf{St}_{\mathsf{k}} \subseteq \mathsf{dSt}_{\mathsf{k}}$. Seen through this inclusion, its tangent complex at a point corresponding to a vector bundle E, can be shown (see the Example **1.2.13** below) to be given by

$$\mathbb{T}_{\underline{\mathsf{Vect}}_n,E} = \mathbb{R}\,\mathrm{End}(E)[1]$$

recovering

$$\mathsf{H}^i(\mathbb{T}_{\underline{\mathsf{Vect}}_n,E}) = \mathrm{Ext}^{i+1}(E,E)$$

To conclude, let us remark that this formula applied to the trivial vector bundle rank n, k^n, tells us that

$$\mathbb{T}_{\mathsf{BGL}_n,\mathsf{k}^n} = \mathfrak{gl}_n(\mathsf{k})[1]$$

Remark 1.1.39. The computation of the Example **1.1.38** generalizes if we replace vector bundles by coherent sheaves. This will be the baseline for Chapter **3** on Donaldson-Thomas invariants.

Example 1.1.40. In the Example **1.1.36**, 1.1.37 and 1.1.38 we have only computed the tangent complexes at a point. In this example we compute the quasi-coherent sheaf $\mathbb{L}_{\mathsf{BGL}_n} \in \mathsf{QCoh}(\mathsf{BGL}_n)$. The first thing to observe is that BGL_n can be obtained as a colimit of a simplicial diagram in derived stacks, namely, the bar-construction

$$\begin{tikzcd} \cdots & \mathsf{GL}_n \times \mathsf{GL}_n \arrow[r, shift left=2, "p_1"] \arrow[r, "m"] \arrow[r, shift right=2, "p_2"'] & \mathsf{GL}_n \arrow[r, shift left, "p"] \arrow[r, shift right, "p"'] & \mathsf{Spec}(\mathsf{k}) \arrow[r] & \mathsf{BGL}_n \end{tikzcd}$$

where m denotes the multiplication law and p_1 and p_2 the two projections. Indeed, this is precisely the definition of the B-construction applied objectwise to the presheaf $\mathsf{B}(\mathsf{GL}_n(-))$ of the Example **1.1.38**, together with the fact stackyfication, being a left adjoint, commutes with arbitrary colimits. Moreover, one can show that this diagram is actually the Čech nerve of the smooth atlas $q : \mathsf{Spec}(\mathsf{k}) \to \mathsf{BGL}_n$. Indeed, this boils down to showing that the self-intersection $\mathsf{Spec}(\mathsf{k}) \underset{\mathsf{BGL}_n}{\times} \mathsf{Spec}(\mathsf{k})$ is the homotopy pullback

$$\begin{tikzcd} \mathsf{GL}_n \arrow[r, "p"] \arrow[d, "p"] & \mathsf{Spec}(\mathsf{k}) \arrow[d, "q"] \\ \mathsf{Spec}(\mathsf{k}) \arrow[r, "q"] & \mathsf{BGL}_n \end{tikzcd} \tag{13}$$

and that the higher iterated m^{th}-self-intersections give GL_n^m. There are several ways to show this; one is to use the equivalence $[\mathsf{GL}_n/\mathsf{GL}_n] \simeq \mathsf{Spec}(\mathsf{k})$ and the fact colimits in an ∞-topos are universal [Lu-HTT: 6.1.0.1].

Now, using the fact that QCoh transforms colimits of derived stacks in limits of ∞-categories [Lu-SAG:6.2.3.1-(b)] we obtain a limit diagram

$$\mathsf{QCoh}(\mathsf{BGL}_n) \xrightarrow{q^*} \mathsf{QCoh}(\mathsf{Spec}(\mathsf{k})) = \mathsf{Mod}_k \underset{p^*}{\overset{p^*}{\rightrightarrows}} \mathsf{QCoh}(\mathsf{GL}_n) = \mathsf{Mod}_{\mathcal{O}(\mathsf{GL}_n)} \begin{smallmatrix} \xrightarrow{p_2^*} \\ \xrightarrow{m^*} \\ \xrightarrow[p_1^*]{} \end{smallmatrix} \mathsf{QCoh}(\mathsf{GL}_n \times \mathsf{GL}_n) = \mathsf{Mod}_{\mathcal{O}(\mathsf{GL}_n) \otimes \mathcal{O}(\mathsf{GL}_n)} \cdots$$

In particular, it follows from the fact GL_n is smooth (therefore flat) that we are under the base change hypothesis of the comonadic version of [Lu-HAlg:4.7.5.2-(3)] (see [Mou19:Prop. 4.6]) so that $\mathsf{QCoh}(\mathsf{BGL}_n)$ is comonadic over Mod_k, with comonad given by $q^* q_*$.
We claim that this comonad is also the one defining the ∞-category of representations $\mathsf{Rep}(\mathsf{GL}_n) = \mathcal{O}(\mathsf{GL}_n) - \mathsf{CoMod}$: indeed, using flat base change for the pullback diagram (13), we find

$$q^* q_* \simeq p_* p^* = \mathcal{O}(\mathsf{GL}_n) \overset{\mathbb{L}}{\underset{\mathsf{k}}{\otimes}} -$$

and an equivalence of ∞-categories

$$\mathsf{QCoh}(\mathsf{BGL}_n) \simeq \mathsf{Rep}(\mathsf{GL}_n)$$

To conclude, we argue that under this equivalence, the cotangent complex $\mathbb{L}_{\mathsf{BGL}_n}$ corresponds to the adjoint representation on the shifted Lie algebra $\mathfrak{gl}_n^\vee[-1]$. For this purpose we use the fact that the cotangent complex satisfies flat descent [Bha12:§2] (see also [BMS19:Theorem 3.1]). Using the flat atlas $\mathsf{Spec}(\mathsf{k}) \to \mathsf{BGL}_n$, this implies that as an object of $\mathsf{Rep}(\mathsf{GL}_n)$, $\mathbb{L}_{\mathsf{BGL}_n}$ is given by the limit of the diagram of representations

$$0 \rightrightarrows \mathfrak{gl}_n^\vee \begin{smallmatrix} \longrightarrow \\ \longrightarrow \\ \longrightarrow \end{smallmatrix} \mathfrak{gl}_n^\vee \oplus \mathfrak{gl}_n^\vee \cdots$$

where the middle map is the diagonal (to see this use the fact that the derivative of the multiplication $m : \mathsf{GL}_n \times \mathsf{GL}_n \to \mathsf{GL}_n$ at the identity $(\mathsf{id}, \mathsf{id})$ is the addition law $+ : \mathfrak{gl}_n \oplus \mathfrak{gl}_n \to \mathfrak{gl}_n$). Then, [Lu-HAlg: 1.2.4.13] (see also [Rob15:Prop. 4.50]) concludes.

Finally, to conclude this introduction we explain in which way the derived structure is controlled by the cotangent complex:

Remark 1.1.41. The truncation $\mathsf{t}_0(X)$ of a derived scheme X (see Construction **1.1.16**) is the zero level of a *Postnikov tower* of truncations

$$\mathsf{t}_0 X := \mathsf{t}_{\leq 0}(X) \hookrightarrow \mathsf{t}_{\leq 1}(X) \hookrightarrow \mathsf{t}_{\leq 2}(X) \hookrightarrow \cdots \hookrightarrow X$$

In terms of the derived structure sheaves, this amounts to the universal property of the Postnikov truncation:

- For $i > n$, $\pi_i(\mathsf{t}_{\leq n}\mathcal{O}_X)) \simeq 0$;
- For $i \leq n$, $\pi_i(\mathsf{t}_{\leq n}\mathcal{O}_X)) \simeq \pi_i(\mathcal{O}_X)$;

A key result in the subject [Lu-HAlg: 7.4.1.28] is the fact that each morphism of derived rings in the Postnikov tower $\mathsf{t}_{\leq n+1}\mathscr{O}_X \to \mathsf{t}_{\leq n}\mathscr{O}_X$ is a *square zero extension*, determined by a derivation $\mathsf{t}_{\leq n}\mathscr{O}_X \to \mathsf{t}_{\leq n}\mathscr{O}_X \oplus \pi_{n+1}(\mathscr{O}_X)[n+2]$ (see also [HAG-II:Lemma 2.2.1.1.]). In particular, the information of $\pi_{n+1}(\mathscr{O}_X)$ is controlled by the relative cotangent complexes of the layers of the Postnikov tower, namely, we have canonical isomorphisms

$$\pi_{n+1}(\mathscr{O}_X) \simeq \pi_{n+2}(\mathbb{L}_{\mathsf{t}_{\leq n}\mathscr{O}_X/\mathsf{t}_{\leq n+1}\mathscr{O}_X})$$

See [HAG-II: 2.2.2.8]. This provides a rigorous basis for the heuristic principle:

Derived Algebraic Geometry = Classical Algebraic Geometry + Deformation theory

Remark 1.1.42. Another incarnation of this heuristic principle is the fact that a morphism of derived schemes $f : X \to Y$ is an equivalence if and only if it is étale (*) and the induced map on the classical truncations $\mathsf{t}_0 X \to \mathsf{t}_0 Y$ is an isomorphism of classical schemes. See [HAG-II:§2.2.2].

Concluding Remark 1.1.43. Although originally developed in the algebraic setting as an enrichment of classical algebraic geometry, over the last twenty years the ideas of derived geometry have been implemented in other contexts, in particular applied to $\mathsf{E}_\infty^\otimes$-rings in spectra - see [Lu-SAG]; in the case of complex analytic and rigid analytic geometry [Lur11a; PY18; PY20b; Pri18]; and in the case of C^∞-differential geometry [Joy; Spi08; CS19; Ste23]

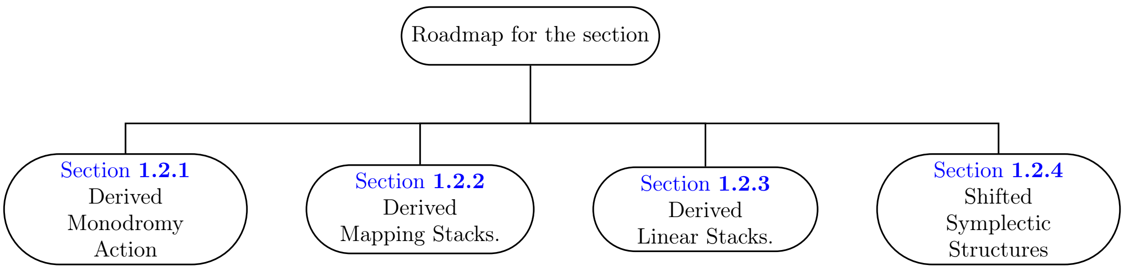
This concludes our brief overview. Next we discuss a list of geometric constructions on classical geometric objects that become accessible via derived geometry

(*) A morphism of derived schemes $f : X \to Y$ is said to be of étale if it is locally homotopically of finite presentation and the relative cotangent complex $\mathbb{L}_f$, given by the homotopy fiber of $f^*\mathbb{L}_Y \to \mathbb{L}_X$ in $\mathsf{QCoh}(X)$, vanishes

1.2. New structures found through derived algebraic geometry

In Section 1.1 we discussed derived fiber products as a mechanism to produce examples of derived objects out of classical ones. In this section we discuss four other such mechanisms.

Warning 1.2.1. The selection of topics is not intended to cover all the applications of derived algebraic geometry, leaving behind what one was one of its original historical developments, such as the link between formal deformation theory, dg-Lie algebras and deRham stacks [Lur]. See [Toë17] for an overview. Instead, we decided to focus only on the main mechanisms relevant to the research papers covered in this thesis.



1.2.1. A derived monodromy action. Working with derived intersections allows us to *see new invariants of singularities* that *cannot* be seen through classical algebraic geometry. This first example is reminiscent of a well-known situation in homotopy theory:

Construction 1.2.2. When Y is a topological space and $y \in Y$ is a point, the space of based loops $\Omega_y Y$ carries a natural group law up to homotopy given by concatenation of loops at y. Moreover, when $f : Z \to Y$ is a Serre fibration, there is a well-defined homotopy action of the group $\Omega_y Y$ on the fiber of f at y. In fact, one can drop the assumption of Serre fibration if one replaces the fiber product by the *homotopy fiber product*. In other words, if we work in the ∞-category of spaces $\mathcal{S}$ (aka ∞-groupoids), this action is always canonically defined: first the loop space $\Omega_y Y$ is given by the derived self-intersection

$$\begin{tikzcd} \Omega_y Y \arrow[r] \arrow[d] & * \arrow[d, "y"] \\ * \arrow[r, "y"] & Y \end{tikzcd}$$

Indeed, this is so because the path space of Y at y - $P_y Y$ - is contractible and the evaluation at 1 map $P_y Y \to Y$ is a Serre fibration. By taking the double iterated fiber products in the ∞-category $\mathcal{S}$

$$\begin{tikzcd}
& \Omega_y Y \arrow[rr] \arrow[dd] & & * \arrow[dd] \\
\Omega_y Y \times^h \Omega_y Y \arrow[ur] \arrow[rr] \arrow[dd] & & \Omega_y Y \arrow[ur] \arrow[dd] & \\
& * \arrow[rr] & & Y \\
\Omega_y Y \arrow[ur] \arrow[rr] & & * \arrow[ur] &
\end{tikzcd}$$

we obtain the composition of loops via the projection to the first and third component:

$$\Omega_y Y \overset{h}{\times} \Omega_y Y \simeq * \underset{Y}{\overset{h}{\times}} * \underset{Y}{\overset{h}{\times}} * \longrightarrow * \underset{Y}{\overset{h}{\times}} * \simeq \Omega_y Y$$

Similarly, the higher associativity constraints are recovered from the the higher iterated fiber products of $*$ over Y.

Now, consider a continuous map $f : Z \to Y$, and denote by $f^{-1,h}(\{y\})$ its homotopy fiber

$$\begin{tikzcd}
f^{-1,h}(\{y\}) \arrow[r] \arrow[d] & Z \arrow[d, "f"] \\
* \arrow[r, "y"] & Y
\end{tikzcd}$$

Informally we can think of the points of $f^{-1,h}(\{y\})$ as pairs (z, η) where $z \in Z$ and η is a path in Y between $f(z)$ and y. Indeed, this is precisely what we get if we replace the map $* \to Y$ by the homotopy equivalent path-space fibration $P_y Y \to Y$. We obtain this way an action of $\Omega_y Y$ in $f^{-1,h}(\{y\})$

$$\Omega_y Y \overset{h}{\times} f^{-1,h}(\{y\}) \to f^{-1,h}(\{y\})$$

by sending a loop at y, γ, and a pair (z, η), to the pair $(z, \eta \circ \gamma)$. Formally, this is obtained by taking the iterated homotopy fiber product

$$\begin{tikzcd}
& f^{-1,h}(\{y\}) \arrow[rr] \arrow[dd] & & Z \arrow[dd] \\
\Omega_y Y \times^h f^{-1,h}(\{y\}) \arrow[ur] \arrow[rr] \arrow[dd] & & f^{-1,h}(\{y\}) \arrow[ur] \arrow[dd] & \\
& * \arrow[rr] & & Y \\
\Omega_y Y \arrow[ur] \arrow[rr] & & * \arrow[ur] &
\end{tikzcd}$$

Example 1.2.3. A simple incarnation is given by $Y = \mathsf{S}^1$ and $y = 1$. In this case, $\Omega_y Y$ is homotopy equivalent to the discrete space $\mathbb{Z}$ and the group law is the addition law of integer numbers. Now take $S^1 \to S^1$ the map sending $z \mapsto z^m$. This is a covering map and in particular a Serre fibration. In particular its fiber at 1 coincides with the homotopy fiber given by a copy of μ_m, the group of m-th roots of the unit. In this case the action of $\mathbb{Z}$ on μ_m is given by $(n, i) \mapsto i^n$.

Example 1.2.4. Take G a discrete group acting on a topological manifold M and form the homotopy quotient M/G in the ∞-category $\mathcal{S}$. This is homotopy equivalent to the space of homotopy co-invariants of the G-action. Let $M \to *$ be the projection to the point. This map is G-equivariant with respect to the trivial action of G on the point. In particular, it descends to homotopy quotients $M/G \to */G$ where $*/G$ is the classifying space $\mathsf{B}G$. We have $\Omega_* \mathsf{B}G \simeq G$. The homotopy fiber of $M/G \to \mathsf{B}G$ at the unique point of $\mathsf{B}G$ recovers M, ie we have a homotopy cartesian diagram

$$\begin{tikzcd} M \arrow[r] \arrow[d] & M/G \arrow[d] \\ * \arrow[r] & \mathsf{B}G \end{tikzcd}$$

and the homotopy fiber action of the Construction **1.2.2** of $\Omega_* \mathsf{B}G \simeq G$ on M recovers the original action. Notice that this mechanism is reversible: any object Y over $\mathsf{B}G$ in $\mathcal{S}$ is the homotopy quotient of the induced action of G on the homotopy fiber

$$\begin{tikzcd} \tilde{Y} \arrow[r] \arrow[d] & Y \arrow[d] \\ * \arrow[r] & \mathsf{B}G \end{tikzcd} \tag{14}$$

ie, $\tilde{Y}/G \simeq Y$ in $\mathcal{S}$. Indeed, this holds because $* \to \mathsf{B}G$ is an effective epimorphism, and therefore so is $\tilde{Y} \to Y$ (see [Lu-HTT:Proposition 6.2.3.15], which is a direct consequence of the universality of colimits in an ∞-topos [Lu-HTT: 6.1.0.1]).

In fact, the Construction **1.2.2** works in any ∞-category with fiber products, so in particular, it also applies to the ∞-category of derived affine schemes $\mathsf{dAff}_{\mathsf{k}}$:

Example 1.2.5. Let us consider again $\mathsf{Spec}(\mathsf{k}[\epsilon])$ the derived self-intersection of 0 in $\mathbb{A}^1_{\mathsf{k}}$ of the Examples **1.1.7** and **1.1.21**. The Construction **1.2.2** perfomed in the ∞-category $\mathsf{dAff}_{\mathsf{k}}$ provides a $\mathsf{E}_1^{\otimes}$-group law on $\mathsf{Spec}(\mathsf{k}[\epsilon])$ under composition of loops. *A priori* this operation of loops is only associative and not commutative. But in this particular case, because $\mathbb{A}^1_{\mathsf{k}}$ itself is an abelian group with respect to the additive law, the loop space $\Omega_0 \mathbb{A}^1_{\mathsf{k}}$ is actually a $\mathsf{E}_{\infty}^{\otimes}$-group (Eckmann–Hilton).

In terms of functions this can be understood as a structure of *derived Hopf algebra* on $\mathsf{k}[\epsilon]$. As in the Construction **1.2.2**, this structure emerges when we take not only the double self-intersection but also the *triple*, where have a canonical equivalence of derived rings

$$\mathsf{k} \overset{\mathbb{L}}{\underset{\mathsf{k}[t]}{\otimes}} \mathsf{k} \overset{\mathbb{L}}{\underset{\mathsf{k}[t]}{\otimes}} \mathsf{k} \simeq \left(\mathsf{k} \overset{\mathbb{L}}{\underset{\mathsf{k}[t]}{\otimes}} \mathsf{k}\right) \underset{\mathsf{k}}{\otimes} \left(\mathsf{k} \overset{\mathbb{L}}{\underset{\mathsf{k}[t]}{\otimes}} \mathsf{k}\right) = \mathsf{k}[\epsilon] \underset{\mathsf{k}}{\otimes} \mathsf{k}[\epsilon]$$

Through this equivalence, the group law on $\mathsf{Spec}(\mathsf{k}[\epsilon])$ is given by the map corresponding to the inclusion of the first and third copies of k

$$\mathsf{k}[\epsilon] = \mathsf{k} \overset{\mathbb{L}}{\underset{\mathsf{k}[t]}{\otimes}} \mathsf{k} \to \mathsf{k} \overset{\mathbb{L}}{\underset{\mathsf{k}[t]}{\otimes}} \mathsf{k} \overset{\mathbb{L}}{\underset{\mathsf{k}[t]}{\otimes}} \mathsf{k} \simeq \mathsf{k}[\epsilon] \overset{\mathbb{L}}{\underset{\mathsf{k}}{\otimes}} \mathsf{k}[\epsilon]$$

The higher iterated self-intersections provide the higher coherences of the Hopf structure. See [Pre11:Construction 3.1.1 and Remark 3.1.3] for an explicit description.

Let us return to the Example **1.1.28** where we studied the derived zero locus V of the function $f(x,y) = x.y$ on $\mathbb{A}^2_{\mathsf{k}}$. Construction **1.2.2** gives us an action of $\Omega_0\mathbb{A}^1_{\mathsf{k}}$ on V. This can be seen in terms of the derived rings of functions as co-action of the derived Hopf algebra $\mathsf{k}[\epsilon]$ on $\mathscr{O}_V$ given by the triple intersection (as in Construction **1.2.2**)

$$\mathscr{O}_V := \mathsf{k} \overset{\mathbb{L}}{\underset{\mathsf{k}[t]}{\otimes}} \mathsf{k}[x,y] \underset{1,3}{\to} \mathsf{k} \overset{\mathbb{L}}{\underset{\mathsf{k}[t]}{\otimes}} \mathsf{k} \overset{\mathbb{L}}{\underset{\mathsf{k}[t]}{\otimes}} \mathsf{k}[x,y] \simeq \left(\mathsf{k} \overset{\mathbb{L}}{\underset{\mathsf{k}[t]}{\otimes}} \mathsf{k}\right) \underset{\mathsf{k}}{\otimes} \left(\mathsf{k} \overset{\mathbb{L}}{\underset{\mathsf{k}[t]}{\otimes}} \mathsf{k}[x,y]\right) = \mathsf{k}[\epsilon] \underset{\mathsf{k}}{\otimes} \mathscr{O}_V$$

For more details about the higher coherences for this action, see [Vez20:Section 3].

Remark 1.2.6. Note that at the level of classical truncations, nothing is happening: the group structure on $\mathsf{t}_0(\mathsf{Spec}(\mathsf{k}[\epsilon])) = \mathsf{Spec}(\mathsf{H}^0(\mathsf{k}[\epsilon])) = \mathsf{Spec}(\mathsf{k})$ is the identity map and the so its the action $\mathsf{t}_0(\mathsf{Spec}(\mathsf{k}[\epsilon]) \times V) = \mathsf{Spec}(\mathsf{k}) \times V = V \to \mathsf{t}_0(V) = V$.

The action in the Example **1.2.5** is an important invariant of the singularities of the zero fiber of f that cannot be seen using the classical language of schemes. We will return to this in Section **2.2**.

1.2.2. Derived Mapping Stacks.

Construction 1.2.7. If X and Y are derived stacks, we can form a *derived mapping stack* $\mathbb{R}\mathsf{Map}(X,Y) \in \mathsf{dSt}_{\mathsf{k}}$ parameterizing families of maps with functor of points given by

$$S = \mathsf{Spec}(R) \in \mathsf{dAff}_{\mathsf{k}} \mapsto \mathbb{R}\mathsf{Map}(X,Y)(\mathsf{Spec}(R)) := \mathsf{Map}_{\mathsf{dSt}_{\mathsf{k}}}(X \times S, Y)$$

where the last denotes the mapping space in the ∞-category of derived stacks.

The fact that this functor of points is representable by a derived stack is a consequence of presentability of the ∞-category of derived stacks $\mathsf{dSt}_{\mathsf{k}}$. In particular, by definition of internal-homs, we have for any derived stack S

$$\mathsf{Map}_{\mathsf{dSt}_{\mathsf{k}}}(S \times X, Y) \simeq \mathsf{Map}_{\mathsf{dSt}_{\mathsf{k}}}(S, \mathbb{R}\mathsf{Map}(X,Y))$$

Example 1.2.8. Let X be a derived stack over k. Then by definition

$$\mathbb{R}\mathsf{Map}_{\mathsf{k}}(\mathsf{Spec}(\mathsf{k}), X) \simeq X$$

.

Remark 1.2.9. If X and Y are higher stacks in the sense of Simpson [Sim96a] over a classical base ring k, seen as derived stacks via the inclusion $i : \mathsf{St}_{\mathsf{k}} \hookrightarrow \mathsf{dSt}_{\mathsf{k}}$, and if X is flat over k, then the truncation $\mathsf{t}_0\mathbb{R}\mathsf{Map}(i(X), i(Y))$ is Simpson's higher stack of maps $\mathsf{Map}(X, Y)$ given by the internal-hom in St_{k}. Indeed, the truncation being a adjoint to the inclusion, we have

$$\mathsf{Map}_{\mathsf{St}_{\mathsf{k}}}(Z, \mathsf{Map}(X, Y)) \simeq \mathsf{Map}_{\mathsf{St}_{\mathsf{k}}}(Z \times X, Y) \simeq \mathsf{Map}_{\mathsf{dSt}_{\mathsf{k}}}(i(Z \times X), i(Y)) \simeq$$

In general, the inclusion of higher stacks in derived stacks does not commute with cartesian products, but in this case it does because X is flat over k (see Remark **1.1.18**):

$$\simeq \mathsf{Map}_{\mathsf{dSt}_{\mathsf{k}}}(i(Z) \times i(X), i(Y)) \simeq \mathsf{Map}_{\mathsf{dSt}_{\mathsf{k}}}(i(Z), \mathbb{R}\mathsf{Map}(i(X), i(Y))) \simeq$$

$$\simeq \mathsf{Map}_{\mathsf{St}_{\mathsf{k}}}(Z, \mathsf{t}_0(\mathbb{R}\mathsf{Map}(i(X), i(Y))))$$

A core result in the subject of derived geometry is J. Lurie's version of Artin's representability theorem. See [Lur04] ,[Lu-SAG:Chapters 18 and 19],and also [HAG-II:2.2.6.11 and C.0.9.]. For the case of mapping stacks, this implies:

Theorem 1.2.10. *Let Y be a derived Artin stack which is locally of finite presentation over a (classical) base S, and let X be a flat and proper classical S-scheme. Then:*

(i) the derived mapping stack $\mathbb{R}\mathsf{Map}_S(X, Y)$ admits a global relative cotangent complex over S given by

$$\mathbb{L}_{\mathbb{R}\mathsf{Map}_S(X,Y)} \simeq (\pi_*\mathsf{ev}^*\mathbb{L}_Y^\vee)^\vee$$

where $\pi : \mathbb{R}\mathsf{Map}_S(X, Y) \times X \to \mathbb{R}\mathsf{Map}_S(X, Y)$ is the projection and ev *: $\mathbb{R}\mathsf{Map}_S(X, Y) \underset{S}{\times} X \to Y$ is the evaluation.*

(ii) If moreover Y admits quasi-affine diagonal over S, by (i) and Lurie's representability theorem, $\mathbb{R}\mathsf{Map}_{\mathsf{k}}(X, Y)$ is again a derived Artin stack locally of finite presentation over S.

Proof. One can consult [HLP23:§5.1.6 and 5.1.10] and [CHS21:B.10.21] for the existence of the cotangent complex given by the formula in (i). The claim in (ii) is a corollary of (i) and Lurie's representability. See [HLP23:§5.1.1]. ☐

Example 1.2.11. Fix k a field and let Y be a quasi-projective k-scheme and C a smooth projective algebraic curve over k. The derived mapping stack $\mathbb{R}\mathsf{Map}(C, Y)$ is in fact a derived scheme whose truncation is the classical scheme of maps introduced by Grothendieck (see [HAG-II: 2.2.6.14] and [FGIKNV05:Thm 5.23]). The advantage of the derived structure is that it comes with a natural deformation theory of points: the Theorem **1.2.10** tells us that its tangent complex at a closed point corresponding to a map $f : C \to Y$ is the complex $\mathbb{R}\Gamma(C, f^*\mathbb{T}_Y)$. In particular, its cohomology groups $\mathsf{H}^0(C, f^*\mathbb{T}_Y)$ and $\mathsf{H}^1(C, f^*\mathbb{T}_Y)$ thus recover the classical deformation theory of maps. This example is the building block that we will use in Section **2.1** to construct a derived moduli stack of stable maps of use in Gromov-Witten theory.

Example 1.2.12. One can also use the Theorem **1.2.10** to explain the formula (12) in the Example **1.1.37**. Let $u : \mathsf{Spec}(\mathsf{k}) \to \mathsf{M}_g$ be a map classifying a curve of genus g, C, over a field k, having at worst nodal singularities. Then we can look at the derived fiber product

$$\begin{tikzcd} \Omega_C \mathsf{M}_g \arrow[r] \arrow[d] & \mathsf{Spec}(\mathsf{k}) \arrow[d, "C"] \\ \mathsf{Spec}(\mathsf{k}) \arrow[r, "C"] & \mathsf{M}_g \end{tikzcd}$$

with M_g seen as a derived stack through the inclusion $\mathsf{St}_\mathsf{k} \hookrightarrow \mathsf{dSt}_\mathsf{k}$. The derived stack of loops at C in M_g is by definition of the higher stack M_g precisely the derived stack of *automorphisms* of C

$$\Omega_C \mathsf{M}_g \simeq \mathbb{R}\,\mathrm{Aut}(C)$$

This is an *open* substack of the stack of *endomorphisms of* C, ie, the derived mapping stack $\mathbb{R}\mathsf{Map}_\mathsf{k}(C, C)$. Therefore, by the Proposition **1.1.27** the tangent complex at the identity map $\mathsf{id} : C \to C$ fits in a derived fiber product

$$\begin{tikzcd} \mathbb{T}_{\mathsf{id}, \mathbb{R}\mathsf{Map}(C,C)} \arrow[r] \arrow[d] & 0 \arrow[d, "C"] \\ 0 \arrow[r, "C"] & \mathbb{T}_{\mathsf{M}_g, C} \end{tikzcd}$$

so that

$$\mathbb{T}_{\mathsf{M}_g, C} \simeq \mathbb{T}_{\mathsf{id}, \mathbb{R}\mathsf{Map}_\mathsf{k}(C,C)}[1]$$

Finally, the Theorem **1.2.10**-(i) at the identity $f = \mathsf{id} : C \to C$, gives

$$\mathbb{T}_{\mathsf{M}_g, C} \simeq \Gamma(C, \mathbb{T}_C)[1]$$

Example 1.2.13. r The delooping argument of the Example **1.2.12** can also be used to compute the tangent complex of $\underline{\mathsf{Vect}}_n$ in the Example **1.1.38**. More generally, we can consider the derived stack $\underline{\mathsf{Perf}}$ parametrizing families of perfect complexes: if S is an affine derived scheme, then $\underline{\mathsf{Perf}}(S) = \mathsf{Perf}(S)^{\simeq} \in \mathcal{S}$ is maximal ∞-groupoid of perfect complexes on S. The derived stack $\underline{\mathsf{Perf}}$ is a union of open substacks $\underline{\mathsf{Perf}}^{[a,b]}$ parametrizing perfect complexes in Tor-amplitude $[a, b]$ (see [Lu-HAlg:Def. 7.2.4.21] or [TV07: 2.21, 2.22] for the definition of Tor-amplitude).

When $a = b = 0$ this is a derived enrichment of the classical stack of vector bundles $\underline{\mathsf{Vect}} = \cup_n \underline{\mathsf{Vect}}_n$. By [TV07], each $\underline{\mathsf{Perf}}^{[a,b]}$ is a derived Artin stack locally of finite presentation over $\mathbb{Z}$. As in the Example **1.1.38**, the tangent complex at a point $\mathsf{Spec}(R) \to \underline{\mathsf{Perf}}$ corresponding to a perfect complex $E \in \mathsf{Perf}(R)$ is given by the complex

$$\mathbb{T}_{\underline{\mathsf{Perf}},E} \simeq E \otimes E^{\vee}[1]$$

Later on, in section Chapter **3** we will consider the derived stack classifying families of perfect complexes on X, which coincides with the derived mapping stack

$$\mathbb{R}\mathsf{Map}_{\mathsf{k}}(X, \underline{\mathsf{Perf}})$$

with X a Calabi-Yau variety of dimension 3 over $\mathsf{k} = \mathbb{C}$.

We now discuss an example of mapping stack with a source given by a topological space. We need some preliminary tools:

Construction 1.2.14. The ∞-category of derived stacks being a category of presheaves of spaces, admits sheafifications of constant presheaves with values in spaces. Namely, we have a colimit-preserving functor

$$\mathcal{S} \to \mathsf{dSt}_{\mathsf{k}}$$

determined by sending the point $*$ to $\mathsf{Spec}(\mathsf{k})$ and commuting with arbitrary colimits. In particular, for any space $K \in \mathcal{S}$, we find

$$\mathsf{QCoh}(K) = \mathsf{Fun}(K, \mathsf{Mod}_{\mathsf{k}})$$

Indeed, by [Lu-SAG: 6.2.3.1], QCoh transforms colimit into limits. Since one can write $K = \mathsf{colim}_{* \to K} *$ in $\mathcal{S}$, we find

$$\mathsf{QCoh}(K) = \mathsf{lim}_{* \in K} \mathsf{QCoh}(\mathsf{Spec}(\mathsf{k})) = \mathsf{lim}_{* \in K} \mathsf{Fun}(*, \mathsf{Mod}_{\mathsf{k}}) \simeq$$

$$\simeq \mathsf{Fun}(\mathsf{colim}_{* \to K} *, \mathsf{Mod}_{\mathsf{k}}) \simeq \mathsf{Fun}(K, \mathsf{Mod}_{\mathsf{k}})$$

In this case, the pullback functor along the unique map $p : K \to *$,

$$\mathsf{QCoh}(*) = \mathsf{Mod}_{\mathsf{k}} \to \mathsf{QCoh}(K) = \mathsf{Fun}(K, \mathsf{Mod}_{\mathsf{k}})$$

corresponds to the constant diagram and therefore the pushforward p_* is given by taking the homotopy limit lim_K. It follows that

$$\mathbb{R}\Gamma(K, \mathcal{O}) = \mathsf{C}^*(K, \mathsf{k})$$

Remark 1.2.15. Under the equivalence $\mathsf{QCoh}(K) \simeq \mathsf{Fun}(K, \mathsf{Mod}_{\mathsf{k}})$ of the Construction **1.2.14**, the symmetric monoidal structure on $\mathsf{QCoh}(K)$ of the Construction **1.1.23** corresponds to the objectwise tensor product of diagrams, namely, given $E, F : K \to \mathsf{Mod}_{\mathsf{k}}$, their tensor product is the composition

$$K \xrightarrow{\Delta} K \times K \xrightarrow{E\times F} \mathsf{Mod}_{\mathsf{k}} \times \mathsf{Mod}_{\mathsf{k}} \xrightarrow{\otimes} \mathsf{Mod}_{\mathsf{k}}$$

where $\Delta : K \to K \times K$ is the diagonal inclusion.

In particular, this implies

$$\mathsf{CAlg}(\mathsf{QCoh}(K)) \simeq \mathsf{Fun}(K, \mathsf{CAlg}_{\mathsf{k}})$$

See [Lu-HAlg: 2.1.3.4.].

Construction 1.2.16. Let X be a derived scheme over a classical ring k. Another self-intersection that is preeminent in derived geometry is the self-intersection of X along its diagonal $\Delta : X \hookrightarrow X \times X$, ie, the derived fiber product in $\mathsf{dSch}_{\mathsf{k}}$

(15)
$$\begin{tikzcd} X \overset{h}{\underset{X\times X}{\times}} X \arrow[r, "p_1"] \arrow[d, "p_2"] & X \arrow[d, "\Delta"] \\ X \arrow[r, "\Delta"] & X \times X \end{tikzcd}$$

Because of the Remark **1.1.19**, the maps p_1 and p_2 are isomorphisms at the level of the classical truncations. But the derived structures on $X \overset{h}{\underset{X\times X}{\times}} X$ and X are different. The rule for the tangent complex of a fiber products (Proposition **1.1.27**) tells us that the relative tangent is given by

$$\mathbb{T}_{p_2} \simeq p_1^* \mathbb{T}_{\Delta} \simeq p_1^* \mathbb{T}_X[-1]$$

We now discuss an alternative description of $X \overset{h}{\underset{X\times X}{\times}} X$. By the Construction **1.2.14** we have $\mathsf{S}^1 \in \mathsf{dSt}_{\mathsf{k}}$ the constant sheaf with values S^1. As in the Example **A.5.13**, we have a homotopy pushout diagram in $\mathsf{dSt}_{\mathsf{k}}$

$$\begin{tikzcd} \mathsf{Spec}(\mathsf{k}) \coprod \mathsf{Spec}(\mathsf{k}) \arrow[r] \arrow[d] & \mathsf{Spec}(\mathsf{k}) \arrow[d] \\ \mathsf{Spec}(\mathsf{k}) \arrow[r] & \mathsf{S}^1 \end{tikzcd}$$

It follows from the universal property of pushouts in $\mathsf{dSt}_{\mathsf{k}}$ that for any derived scheme X over k we have a canonical equivalence of derived stacks

(16)
$$X \overset{h}{\underset{X\times X}{\times}} X \simeq \mathbb{R}\mathsf{Map}(\mathsf{Spec}(\mathsf{k}), X) \overset{h}{\underset{\mathbb{R}\mathsf{Map}(\mathsf{Spec}(\mathsf{k}),X)\times \mathbb{R}\mathsf{Map}(\mathsf{Spec}(\mathsf{k}),X)}{\times}} \mathbb{R}\mathsf{Map}(\mathsf{Spec}(\mathsf{k}), X) \simeq \mathbb{R}\mathsf{Map}_{\mathsf{k}}(\mathsf{S}^1, X)$$

In particular $\mathbb{R}\mathsf{Map}_{\mathsf{k}}(\mathsf{S}^1, X)$ is a derived scheme with truncation coinciding with $\mathsf{t}_0(X)$. We call $\mathbb{R}\mathsf{Map}_{\mathsf{k}}(\mathsf{S}^1, X)$ the *derived free loop space* of X and denote it by $\mathsf{L}X$. The terminology is inspired by free loop spaces in differential geometry but the concept is rather different: since topologically $\mathsf{L}X$ and X coincide, the loops here are infinitesimal, and visible only on functions, namely, if $X = \mathsf{Spec}(R)$ is an affine derived scheme, then the pullback (15) in $\mathsf{dAff}_{\mathsf{k}}$ become pushouts in $\mathsf{dRings}_{\mathsf{k}}$

$$
\begin{tikzcd}
X \overset{h}{\underset{X\times X}{\times}} X \simeq \mathsf{Spec}\left(R \overset{\mathbb{L}}{\underset{R \overset{\mathbb{L}}{\underset{\mathsf{k}}{\otimes}} R}{\otimes}} R) \right) \arrow[r, "p_1"] \arrow[d, "p_2"] & X = \mathsf{Spec}(R) \arrow[d, "\Delta"] \\
X = \mathsf{Spec}(R) \arrow[r, "\Delta"] & X \times X = \mathsf{Spec}(R \overset{\mathbb{L}}{\underset{\mathsf{k}}{\otimes}} R)
\end{tikzcd} \tag{17}
$$

Derived loop spaces are the basic ingredient that will allow us to treat the HKR-isomorphisms in positive and mixed characteristics in Section **2.3**. See also [BZN12; TV09; TV15; TV11] for more details.

Remark 1.2.17. The two maps p_1 and p_2 in diagram (15) are homotopic. Indeed, composing the the commutativity of (15) $\Delta \circ p_1 \sim \Delta \circ p_2$ with the first projection $\pi_1 : X \times X \to X$ we get $p_1 \simeq \pi_1 \circ \Delta \circ p_1 \sim \pi_1 \circ \Delta \circ p_2 \sim p_2$.

Remark 1.2.18. In the ∞-category of spaces $\mathcal{S}$, S^1 is the classifying space of the discrete abelian group $\mathbb{Z}$ seen as a discrete space. Since the functor $\mathcal{S} \to \mathsf{dSt}_{\mathsf{k}}$ preserves all colimits, in particular, preserves geometric realizations and the B-construction. Therefore, as a derived stack S^1 is the classifying stack of the constant stack with values in the discrete abelian group $\mathbb{Z}$:

$$\mathsf{S}^1 \simeq \mathsf{B}\mathbb{Z}$$

in $\mathsf{dSt}_{\mathsf{k}}$.

Example 1.2.19. Let us discuss the Construction **1.2.16** when $X = \mathbb{A}^1_{\mathsf{k}}$. To compute its derived loop space we use the additive group law on $\mathbb{A}^1_{\mathsf{k}}$ that allows us to conjugate the diagonal inclusion $\Delta : \mathbb{A}^1_{\mathsf{k}} \hookrightarrow \mathbb{A}^1_{\mathsf{k}} \times \mathbb{A}^1_{\mathsf{k}}$ with the inclusion of the first factor $\mathsf{id} \times 0 : \mathbb{A}^1_{\mathsf{k}} \times \mathsf{Spec}(\mathsf{k}) \hookrightarrow \mathbb{A}^1_{\mathsf{k}} \times \mathbb{A}^1_{\mathsf{k}}$ sending $x \mapsto (x, 0)$, via the isomorphism

$$\mathbb{A}^1_{\mathsf{k}} \times \mathbb{A}^1_{\mathsf{k}} \simeq \mathbb{A}^1_{\mathsf{k}} \times \mathbb{A}^1_{\mathsf{k}}$$

sending $(x, y) \mapsto (x, y - x)$. Using this isomorphism one can write the homotopy fiber product in (15) as

$$
\begin{tikzcd}
\mathbb{A}^1_{\mathsf{k}} \overset{h}{\underset{\mathbb{A}^1_{\mathsf{k}} \times \mathbb{A}^1_{\mathsf{k}}}{\times}} \mathbb{A}^1_{\mathsf{k}} \arrow[r] \arrow[d] & \mathbb{A}^1_{\mathsf{k}} \times \mathsf{Spec}(\mathsf{k}) \arrow[d, "\mathsf{id}\times 0"] \\
\mathbb{A}^1_{\mathsf{k}} \times \mathsf{Spec}(\mathsf{k}) \arrow[r, "\mathsf{id}\times 0"] & \mathbb{A}^1_{\mathsf{k}} \times \mathbb{A}^1_{\mathsf{k}}
\end{tikzcd} \tag{18}
$$

This cartesian diagram provides an equivalence of derived schemes

$$\mathsf{L}\mathbb{A}^1_{\mathsf{k}} \simeq \mathbb{A}^1_{\mathsf{k}} \overset{h}{\underset{\mathbb{A}^1_{\mathsf{k}}\times\mathbb{A}^1_{\mathsf{k}}}{\times}} \mathbb{A}^1_{\mathsf{k}} \simeq \mathbb{A}^1_{\mathsf{k}} \times \Omega_0\mathbb{A}^1_{\mathsf{k}} \simeq \mathsf{Spec}(\mathsf{k}[t]) \times \mathsf{Spec}(\mathsf{k}[\epsilon]) \simeq \mathsf{Spec}(\mathsf{k}[t,\epsilon])$$

$$\simeq \mathsf{Spec}(\mathsf{Sym}_{\mathsf{k}[t]}(\mathsf{k}[t][1]))$$

It is clear from this example that the nature of the derived loop space is very different from the loop spaces in differential geometry: topologically $\mathsf{L}\mathbb{A}^1_{\mathsf{k}}$ is $\mathbb{A}^1_{\mathsf{k}}$ but the derived ring of functions contains a new copy of $\mathsf{k}[\epsilon]$ correcting the non-transversality of the intersection, as in the Example **1.1.7**.

Remark 1.2.20. When working over a field of characteristic zero k, the computation in the Example **1.2.19** works for any algebraic group G (or more generally for any derived group scheme), yielding $\mathsf{L}G \simeq G \times \Omega_e G$ with e the identity element of G. In particular, it works for the derived loop space of $\mathsf{Spec}(\mathsf{k}[\epsilon])$ with the group structure from the Example **1.2.5**:

$$\mathsf{L}\,\mathsf{Spec}(\mathsf{k}[\epsilon]) \simeq \mathsf{Spec}(\mathsf{k}[\epsilon]) \times \left(\mathsf{Spec}(\mathsf{k}) \overset{h}{\underset{\mathsf{Spec}(\mathsf{k}[\epsilon])}{\times}} \mathsf{Spec}(\mathsf{k})\right)$$

We obtain

$$\mathsf{k} \overset{\mathbb{L}}{\underset{\mathsf{k}[\epsilon]}{\otimes}} \mathsf{k} \simeq \mathsf{Sym}_{\mathsf{k}}(0) \overset{\mathbb{L}}{\underset{\mathsf{Sym}_{\mathsf{k}}(\mathsf{k}[1])}{\otimes}} \mathsf{Sym}_{\mathsf{k}}(0) \simeq \mathsf{Sym}_{\mathsf{k}}(\mathsf{k}[2]) \simeq \mathsf{k}[u]$$

with u a generator in homological degree 2. In particular, we obtain

$$\mathsf{L}\,\mathsf{Spec}(\mathsf{k}[\epsilon]) \simeq \mathsf{Spec}(\mathsf{k}[\epsilon, u])$$

which is our first example of a derived scheme with unbounded structure sheaf.

We conclude this section with a description of the canonical action of S^1 on $\mathsf{L}X$ under rotation of loops:

Construction 1.2.21. The circle S^1 is a group object in $\mathcal{S}$ under the multiplication

$$\mathsf{S}^1 \times \mathsf{S}^1 = \mathsf{B}\mathbb{Z} \times \mathsf{B}\mathbb{Z} = \mathsf{B}(\mathbb{Z}\times\mathbb{Z}) \to \mathsf{B}\mathbb{Z} = \mathsf{S}^1 \tag{19}$$

Therefore, since the Construction **1.2.14** $\mathcal{S} \to \mathsf{dSt}_{\mathsf{k}}$ preserves cartesian products, the constant stack associated to S^1 also carries a group structure. The action of S^1 on itself via (19) induces an action

$$\mathsf{S}^1 \times \mathsf{L}X \to \mathsf{L}X \tag{20}$$

Indeed, by adjunction, a map (20) is the same as a map

$$\mathsf{S}^1 \times \mathsf{S}^1 \times \mathsf{L}X \to X \tag{21}$$

In this case, we choose the composition

$$\mathsf{S}^1 \times \mathsf{S}^1 \times \mathsf{L}X \xrightarrow{(19)} \mathsf{S}^1 \times \mathsf{L}X \xrightarrow{\mathsf{Ev}} X \tag{22}$$

Remark 1.2.22. Notice that at the level of the classical underlying schemes, the action of the Construction **1.2.21** is the identity map.

1.2.3. Linear Derived Stacks. We now review the construction of vector bundle stacks in the context of derived geometry. For the discussion in this section we work over a general (classical) commutative ring k.

Construction 1.2.23. Let X be a derived stack over k and let $E \in \mathsf{QCoh}(X)$. We consider the derived stack $\mathrm{Lin}_X(E)$ defined over X given by the functor of points sending an affine derived scheme $u : \mathsf{Spec}(A) \to X$ over X to the mapping space

$$\mathrm{Lin}_X(E)(A) := \mathsf{Map}_{\mathsf{Mod}_A}(u^*(E), A) \tag{23}$$

For E fixed, $\mathrm{Lin}_X(E)$ defines a functor of points on the affine site over X which satisfies the descent condition defining a derived stack over X. By construction, it comes with a projection $\mathrm{Lin}_X(E) \to X$ and a canonical section $s : X \to \mathrm{Lin}_X(E)$ defined on the functor of points by sending $u : \mathsf{Spec}(A) \to X$ to the zero map. We call $\mathrm{Lin}_X(E)$ the *linear derived stack* associated to E. For details see [Toë14:§3.3] and [Mon21].

Proposition 1.2.24. *[TV07:Sublemma 3.9] Let X be a derived scheme and E a perfect complex on X in Tor-amplitude $[a, b]$ with $a, b \in \mathbb{Z}$. Then $\mathrm{Lin}_X(E)$ is a derived b-geometric stack strongly of finite presentation with relative tangent complex*

$$s^*(\mathbb{T}_{\mathrm{Lin}_X(E)/X}) \simeq E^\vee$$

.

Remark 1.2.25. Let X be a derived scheme. From the formula for the tangent complex in Proposition **1.2.24** we see that $\mathrm{Lin}_X(E)$ is a smooth Artin stack if and only if $E^\vee$ is concentrated in Tor-amplitude $[0, +\infty[$ (homological conventions), ie, E is in nonpositive Tor-amplitudes. Conversely, we see that if $E^\vee$ is concentrated in Tor-amplitudes $]-\infty, 0]$ (ie, E is in nonnegative Tor-amplitudes), the functor of points of $\mathrm{Lin}(E)$ can be written as

$$\mathrm{Lin}_X(E)(A) := \mathsf{Map}_{\mathsf{Mod}_A}(u^*(E), A) \simeq \mathsf{Map}_{\mathsf{Mod}_A^{\geq 0}}(u^*(E), A)$$

because both $u^*(E)$ and A are connective. Finally, this is equivalent to

$$\simeq \mathsf{Map}_{\mathsf{dRings}_{A/.}}(\mathsf{Sym}_A^{\Delta}(u^*(E)), A)$$

where, since E is concentrated in positive Tor-amplitudes, so is $\mathsf{Sym}_{\mathscr{O}_X}^{\Delta}(E)$ (*). Notice that here we are using the *simplicial* Sym, ie, the functor $\mathsf{Sym}_A^{\Delta} : \mathsf{Mod}_A^{\geq 0} \to (\mathsf{SCR}_\mathsf{k})_{A/.}$

(*) for the t-structure of the Construction **1.1.24**.

left adjoint to the Dold-Kan functor [(*)]. This shows that $\mathrm{Lin}_X(E)$ is a relatively affine derived scheme given by the relative spectrum

$$\mathrm{Lin}_X(E) = \mathsf{Spec}_X(\mathsf{Sym}^{\Delta}_{\mathscr{O}_X}(E)) \tag{24}$$

Remark 1.2.26. The formula (24) works over any base ring k, because we use the simplicial Sym. Over a $\mathbb{Q}$-algebra k, the simplicial and the $\mathsf{E}^{\otimes}_{\infty}-\mathsf{Sym}$ coincide because the cohomology of the symmetric groups vanish in that case.

Construction 1.2.27. Every derived linear stack carries a natural action of the group scheme $\mathbb{G}_{\mathrm{m}}$ by *scaling the fibers.* At the level of the functor of points, we have

$$\mathbb{G}_{\mathrm{m}}(A) \times \mathrm{Lin}_X(E) \to \mathrm{Lin}_X(E)(A)$$

defined by the composition map

$$\mathsf{Map}_A^{\mathrm{invertible}}(A, A) \times \mathsf{Map}_A(u^*(E), A) \to \mathsf{Map}_A(u^*(E), A)$$

Remark 1.2.28. Taking into account the action of the Construction **1.2.27**, the construction Lin_X defines a functor $\mathsf{QCoh}(X)^{\mathsf{op}} \to \mathbb{G}_{\mathrm{m}} - \mathsf{dSt}_{\mathsf{k}}$ where the last is the ∞-category of derived stacks equipped with a $\mathbb{G}_{\mathrm{m}}$-action. When restricted to eventually connective objects, this functor is fully faithfull. See [Mon21:Corollary 0.3].

Definition 1.2.29. Let X be a derived stack and E a perfect complex. We define the *vector bundle stack associated to* E - $\mathbb{V}_X(E)$- as

$$\mathbb{V}_X(E) := \mathrm{Lin}_X(E^{\vee})$$

Its functor of points is determined by sending $u : \mathsf{Spec}(A) \to X$ to

$$\mathbb{V}_X(E)(A) = \mathsf{Map}_{\mathsf{Mod}_A}(A, u^*(E))$$

Remark 1.2.30. When X is a smooth affine scheme and E is a projective module of finite type, $\mathbb{V}_X(E)$ is the associated vector bundle whose sheaf of sections is E.

Example 1.2.31. When X is a derived stack locally of finite presentation over k (cf. Remark **1.1.26**) and $E = \mathbb{T}_X$ then $\mathbb{V}_X(E)$ is the derived tangent stack of X, $\mathsf{T}X$. In particular when X is a smooth scheme, we recover the tangent bundle of X.

Similarly, $\mathbb{V}_X(\mathbb{L}_X) = \mathsf{T}^*X$ is the cotangent stack of X. More generally, when $E = \mathbb{L}_X[n]$, $\mathbb{V}_X(E)$ defines the *n-shifted cotangent stack of* X.

[(*)] More precisely, the Dold-Kan functor $\mathsf{SCR}_{\mathsf{k}} \to \mathsf{Mod}^{\geq 0}_{\mathsf{k}}$ of the Remark **1.1.12** lifts to a Dold-Kan functor $\Theta_A : (\mathsf{SCR}_{\mathsf{k}})_{A/.} \to \mathsf{Mod}^{\geq 0}_A$. Since both $(\mathsf{SCR}_{\mathsf{k}})_{A/.}$ and $\mathsf{Mod}^{\geq 0}_A$ are presentable categories (see [**1**].4.4.13[]lurie-ha) and Θ_A preserves filtered colimits [Lu-SAG: 25.1.2.2], by [Lu-HTT: 5.5.2.9, 5.4.2.5, 5.3.4.4] it admits a left adjoint which we denote by Sym^{Δ}_A.

There are two cases that will be relevant to us later:

- Whenever $X = \mathsf{Spec}(R)$ is a classical smooth scheme, we will consider its (-1)-shifted cotangent bundle
$$\mathsf{T}^*[-1]X := \mathsf{Spec}(\mathsf{Sym}^{\Delta}_R(\mathbb{T}_X[1]))$$
Notice that for a derived scheme $\mathbb{T}_X$ is a priori concentrated in nonpositive Tor-amplitudes. But since X is smooth, $\mathbb{T}_X$ is in Tor-amplitude zero and therefore $\mathbb{T}_X[1]$ is in nonnegative Tor-amplitudes as in the Remark **1.2.25**;

- Whenever $X = \mathsf{Spec}(A)$ is an affine derived scheme and we consider its (-1)-shifted tangent bundle $\mathsf{T}[-1]X$. Notice that $\mathbb{L}_X$ is concentrated in nonnegative Tor-amplitudes, so we can use the Remark **1.2.25** to write
$$\mathsf{T}[-1]X := \mathsf{Spec}(\mathsf{Sym}^{\Delta}_A(\mathbb{L}_X[1]))$$

Example 1.2.32. We illustrate the paradigm of the Remark **1.2.25** with a simple example. Let $X = \mathsf{Spec}(\mathsf{k})$ with k of characteristic zero and let $E = \mathsf{k}[-1]$. Then
$$\mathbb{V}_{\mathsf{k}}(\mathsf{k}[-1]) = \mathrm{Lin}_{\mathsf{k}}(\mathsf{k}[1]) = \mathsf{Spec}(\mathsf{Sym}^{\Delta}_k(\mathsf{k}[1])) = \mathsf{Spec}(\mathsf{k}[\epsilon])$$
recovering the derived scheme of the Examples **1.1.7** and **1.1.21**. Conversely, if we consider $E = \mathsf{k}[1]$, we obtain

$$\mathbb{V}_{\mathsf{k}}(\mathsf{k}[1]) = \mathrm{Lin}_{\mathsf{k}}(\mathsf{k}[-1]) = \mathsf{Spec}(\mathsf{Sym}_k(\mathsf{k}[-1])) \simeq \mathsf{B}\mathbb{G}_{\mathsf{a}} \tag{25}$$

which is stacky but underived [(*)]. See [Toe06:Lemma 2.2.5] or [Lur11c:§4] for the last isomorphism and our discussion on affine stacks in Section **2.3** below.

Remark 1.2.33. In the Example **1.2.32** the isomorphism
$$\mathbb{V}_{\mathsf{k}}(\mathsf{k}[1]) \simeq \mathsf{B}\mathbb{G}_{\mathsf{a}}$$
makes sense over any field if we use the simplicial Sym^{Δ} instead of the $\mathsf{E}_{\infty}^{\otimes}$-version. See the Remark **1.2.26**.

Construction 1.2.34. Let $E \in \mathsf{Mod}_{\mathsf{k}}^{\leq 0}$. We describe the classical truncation of the linear derived stack $\mathrm{Lin}_{\mathsf{k}}(E)$ by evaluating at classical commutative k-algebra A
$$\mathsf{t}_0(\mathrm{Lin}_{\mathsf{k}}(E))(A) = \mathsf{Map}_{\mathsf{Mod}_A}(E \overset{\mathbb{L}}{\underset{\mathsf{k}}{\otimes}} A, A) = \mathsf{Map}_{\mathsf{Mod}_{\mathsf{k}}}(E, A)$$
Since A is classical, in particular, it is concentrated purely in degree 0 and with our assumption on E, the last mapping space becomes
$$\mathsf{t}_0(\mathrm{Lin}_{\mathsf{k}}(E))(A) = \mathsf{Map}_{\mathsf{Mod}_{\mathsf{k}}^{\leq 0}}(E, A)$$

[(*)] Here we use the fact that over a field of characteristic zero, the $\mathsf{E}_{\infty}^{\otimes} - \mathsf{Sym}$ in the formula (25), coincides with the co-simplicial $\mathsf{Sym}^{\mathrm{co}\Delta}$ (see [MRT22:Notation 1.2.10].

But now we can interpret A as a *cosimplicial commutative algebra* in degree 0. See the discussion in Reminder **2.3.42** below. In this case, using the *free cosimplicial algebra*, we find:

$$\mathsf{t}_0(\mathrm{Lin}_{\mathsf{k}}(E))(A) = \mathsf{Map}_{\mathsf{coSCR}_{\mathsf{k}}}(\mathsf{Sym}_{\mathsf{k}}^{\mathrm{co}\Delta}(E), A)$$

Example 1.2.35. Let $i : \mathbb{P}^1_{\mathbb{C}} \hookrightarrow \mathbb{P}^2_{\mathbb{C}}$ denote the inclusion of the line at infinity on $\mathbb{P}^2_{\mathbb{C}}$ with coordinates $[x : y : z]$, given by $z = 0$. We have the line bundle $\mathscr{O}_{\mathbb{P}^2_{\mathbb{C}}}(1)$ with $\mathsf{H}^0(\mathbb{P}^2_{\mathbb{C}}, \mathscr{O}_{\mathbb{P}^2_{\mathbb{C}}}(1)) = \mathbb{C}.x \oplus \mathbb{C}.y \oplus \mathbb{C}.z$ and the plane at infinity in $\mathbb{P}^2_{\mathbb{C}}$ is the zero locus of the section z, ie, we have a fiber product of schemes (which happens to be also derived)

$$\begin{tikzcd}
\mathbb{P}^1_{\mathbb{C}} \arrow[r, "i"] \arrow[d, "i"] & \mathbb{P}^2_{\mathbb{C}} \arrow[d, "z"] \\
\mathbb{P}^2_{\mathbb{C}} \arrow[r, "0"] & \mathbb{V}_{\mathbb{P}^2_{\mathbb{C}}}(\mathscr{O}_{\mathbb{P}^2_{\mathbb{C}}}(1))
\end{tikzcd}$$

Now we can use this to compute the derived self-intersection of the line at infinity in $\mathbb{P}^2_{\mathbb{C}}$. Using the fact $i^*\mathscr{O}_{\mathbb{P}^2_{\mathbb{C}}}(1) \simeq \mathscr{O}_{\mathbb{P}^1_{\mathbb{C}}}(1)$ we obtain a cube of derived pullbacks

$$\begin{tikzcd}
 & \mathbb{P}^1_{\mathbb{C}} \arrow[rr, "i"] \arrow[dd, "i"] & & \mathbb{P}^2_{\mathbb{C}} \arrow[dd, "z"] \\
\mathbb{V}_{\mathbb{P}^1_{\mathbb{C}}}(\mathscr{O}_{\mathbb{P}^1_{\mathbb{C}}}(1)[-1]) \arrow[ur] \arrow[rr] \arrow[dd] & & \mathbb{P}^1_{\mathbb{C}} \arrow[ur, "i"] \arrow[dd, "i"] & \\
 & \mathbb{P}^2_{\mathbb{C}} \arrow[rr, "0"] & & \mathbb{V}_{\mathbb{P}^2_{\mathbb{C}}}(\mathscr{O}_{\mathbb{P}^2_{\mathbb{C}}}(1)) \\
\mathbb{V}_{\mathbb{P}^2_{\mathbb{C}}}(\mathscr{O}_{\mathbb{P}^2_{\mathbb{C}}}(1)[-1]) \arrow[ur, "\pi"] \arrow[rr, "\pi"] & & \mathbb{P}^2_{\mathbb{C}} \arrow[ur, "0"] &
\end{tikzcd}$$

ie,

$$\mathbb{P}^1_{\mathbb{C}} \overset{h}{\underset{\mathbb{P}^2_{\mathbb{C}}}{\times}} \mathbb{P}^1_{\mathbb{C}} \simeq \mathbb{V}_{\mathbb{P}^1_{\mathbb{C}}}(\mathscr{O}_{\mathbb{P}^1_{\mathbb{C}}}(1)[-1])$$

Finally, notice that the classical truncation $\mathsf{t}_0(\mathbb{V}_{\mathbb{P}^1_{\mathbb{C}}}(\mathscr{O}_{\mathbb{P}^1_{\mathbb{C}}}(1)[-1])) = \mathbb{P}^1_{\mathbb{C}}$ so $\mathbb{V}_{\mathbb{P}^1_{\mathbb{C}}}(\mathscr{O}_{\mathbb{P}^1_{\mathbb{C}}}(1)[-1])$ is a non-trivial derived enrichment of $\mathbb{P}^1_{\mathbb{C}}$ with sheaf of functions corresponding to

$$\mathsf{Sym}_{\mathscr{O}_{\mathbb{P}^1_{\mathbb{C}}}}(\mathscr{O}_{\mathbb{P}^1_{\mathbb{C}}}(-1)[1]) = \bigoplus_{i \geq 0} \left(\bigwedge^i \mathscr{O}_{\mathbb{P}^1_{\mathbb{C}}}(-1) \right)[i] \simeq \mathscr{O}_{\mathbb{P}^1_{\mathbb{C}}}(-1) \oplus \mathscr{O}_{\mathbb{P}^1_{\mathbb{C}}}$$

1.2.4. Shifted Symplectic forms. In another direction, derived geometry allows us to express new forms of symplectic geometry invisible through classical algebraic geometry:

Example 1.2.36. Consider again the derived critical locus $\mathsf{dCrit}(f = \frac{1}{3}x^3)$ of the Example **1.1.32** and the explicit model for its tangent complex $\mathbb{T}_{\mathsf{dCrit}(f=\frac{1}{3}x^3)}$ given by the 2-term complex in (9). Its dual complex - the cotangent complex $\mathbb{L}_{\mathsf{dCrit}(f=\frac{1}{3}x^3)}$ - is again a $\mathsf{k}[x]/(x^2)$ two-term complex:

$$\underbrace{\mathsf{k}[x]/(x^2) \simeq \left(\mathbb{T}_{\mathbb{A}^1_{\mathsf{k}}} \underset{\mathscr{O}_{\mathbb{A}^1_{\mathsf{k}}}}{\otimes} \mathscr{O}_{\mathsf{dCrit}(f)}\right)}_{\deg\ -1} \xrightarrow{\mathrm{Hess}(f)^{\vee}=2.x} \underbrace{\mathsf{k}[x]/(x^2) \simeq \left(\mathbb{L}_{\mathbb{A}^1_{\mathsf{k}}} \underset{\mathscr{O}_{\mathbb{A}^1_{\mathsf{k}}}}{\otimes} \mathscr{O}_{\mathsf{dCrit}(f)}\right)}_{\deg\ 0} \tag{26}$$

Thanks to the symmetry of the Hessian, the complexes $\mathbb{T}_{\mathsf{dCrit}(f)}$ and $\mathbb{L}_{\mathsf{dCrit}(f)}$ differ only by a shift of -1 in homological degrees, ie

$$\mathbb{T}_{\mathsf{dCrit}(f)} \simeq \mathbb{L}_{\mathsf{dCrit}(f)}[-1] \tag{27}$$

In differential geometry, one source of identifications between the tangent and cotangent bundle are symplectic forms, ie, non-degenerate closed 2-forms. In this case we are at the presence of the same kind of structure: the equivalence (27) is part of a (-1)-*shifted symplectic structure*, ie a non-degenerate closed (-1)-shifted 2-form ω on the derived critical locus. In fact this 2-form can be checked to be exactly the d_{dR}-differential of the (-1)-shifted 1-form λ described in the Example **1.1.35**.

Of course, this structure can not be found at the level of classical algebraic varieties. Shifted symplectic forms are a discovery of Pantev-Toën-Vezzosi-Vaquié in [PTVV13] and form the building block for the applications of derived geometry in Donaldson-Thomas invariants later in Chapter **3**.

Example 1.2.37. The Example **1.2.36** can be generalized to the derived critical locus of a function f on smooth scheme U with a function f: df seen as a section of the cotangent bundle T^*U is a Lagrangian for the standard symplectic form. The derived critical locus is therefore obtained as the derived intersection of two lagrangians:

$$\begin{tikzcd} \mathsf{dCrit}(f) \arrow[r] \arrow[d] & U \arrow[d, "df"] \\ U \arrow[r, "0"] & \mathrm{T}^*U \end{tikzcd} \tag{28}$$

In general, the results of [PTVV13] show that the derived intersection of two lagrangians L_1 and L_2 on a classical symplectic smooth algebraic variety X always carries a (-1)-shifted symplectic form. This is particularly of interest when the intersection is non-transverse. See [Cal18] for an overview.

Example 1.2.38. When X is a derived Artin stack locally of finite presentation, the shifted cotangent bundle $\mathsf{T}^*X[n] := \mathbb{V}_X(\mathbb{L}_X[n])$ is n-shifted symplectic with symplectic form given by the de Rham derivative of the canonical Liouville form. See [Cal19].

Example 1.2.39. The stack BGL_n of the Example **1.1.38** carries a 2-shifted symplectic form given at the point $\mathsf{Spec}(\mathsf{k}) \to \mathsf{BGL}_n$ corresponding to the trivial vector bundle of rank n, by the trace of matrices

$$\mathfrak{gl}_n[1] \wedge \mathfrak{gl}_n[1] \simeq \mathsf{Sym}^2(\mathfrak{gl}_n)[2] \to \mathsf{k}[2]$$

sending

$$(M, N) \mapsto \frac{1}{2}\,\mathrm{tr}(A.B)$$

Example 1.2.40. The 2-form on BGL_n of the Example **1.2.39** extends to the derived stack $\underline{\mathsf{Perf}}$ of the Example **1.2.13**: at a point $\mathsf{Spec}(R) \to \underline{\mathsf{Perf}}$ (corresponding to a perfect complex $E \in \mathsf{Perf}(R)$) it is given by product of the evaluation maps

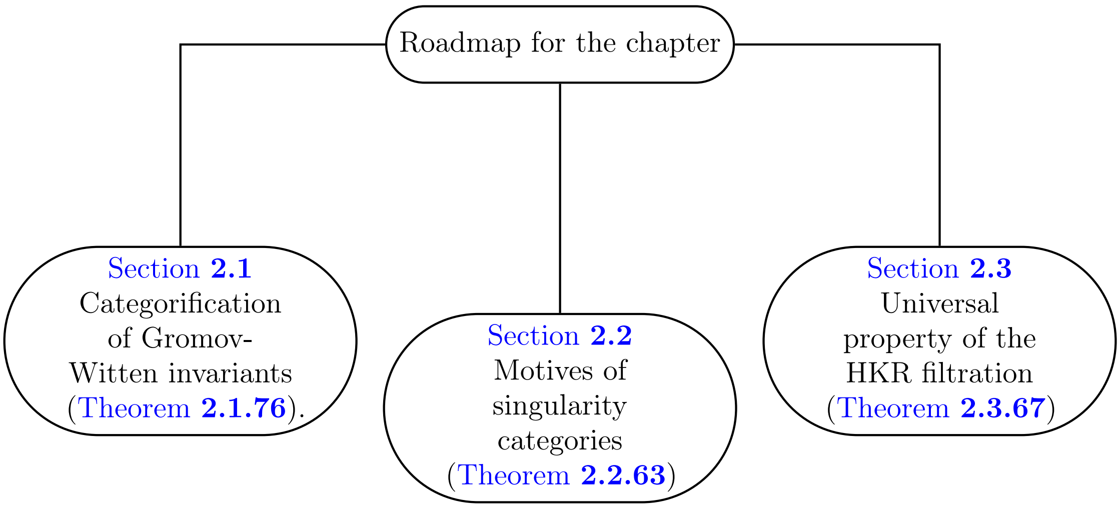
$$\mathbb{T}_{\underline{\mathsf{Perf}},E} \wedge \mathbb{T}_{\underline{\mathsf{Perf}},E} \simeq (E \otimes E^{\vee}[1]) \wedge (E \otimes E^{\vee}[1]) \to \mathscr{O}_R[2]$$

See [PTVV13:Thm 0.3]. By the AKSZ theorem [PTVV13:Thm 0.4], whenever X is a Calabi-Yau manifold of dimension d, the derived mapping stack $\mathbb{R}\mathsf{Map}_{\mathsf{k}}(X, \underline{\mathsf{Perf}})$ inherits a (2-d)-shifted symplectic form. We will return to this example in Chapter **3**.

CHAPTER 2

Survey of research works

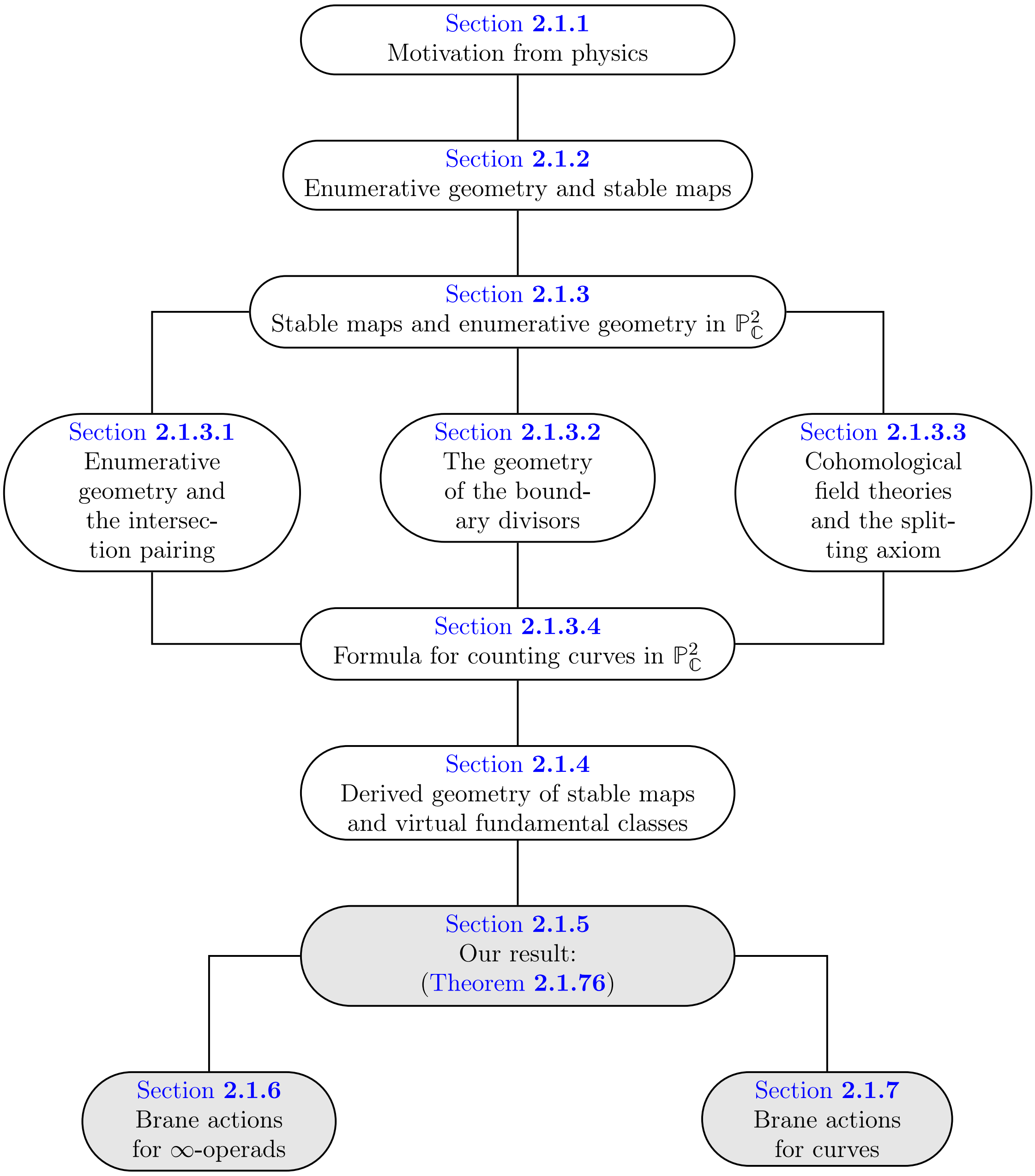
In this chapter we present a selection of three results obtained in different collaborations. Each section (Sections **2.1** to **2.3**) contains a brief introduction to the topic and an overview of the main result.



2.1. Gromov-Witten Invariants

This section contains an introduction to Gromov-Witten theory in algebraic geometry and overviews my joint work with E. Mann on the categorication of Gromov-Witten invariants [MR18] (see Theorem **2.1.76** below).

Sections **2.1.1** to **2.1.3** provide an introductory course to GW-theory in algebraic geometry aimed for advanced master students and covering standard topics in the subject. An expert reader can jump directly to Section **2.1.4**.



2.1.1. Motivation from Physics. According to general relativity, gravity is not a Newtonian force, but rather a consequence of the geometry of spacetime. In the 1920's, works of Kaluza and Klein explained that if spacetime is assumed to be five dimensional, locally of the form $\mathbb{R}^4 \times S^1$, then the curvature of the extra dimension accounts also for the electromagnetic force. See [Bou89] for a mathematical treatment. The theory of Kaluza-Klein turned out to give physically wrong predictions but the idea that the geometry of extra dimensions can account for physical forces, lasted. In [CHSW85] Candelas-Horowitz-Strominger-Witten discovered that in order for string theory to incorporate the standard model of particle physics and supersymmetry, spacetime should be locally of the form $\mathbb{R}^4 \times Y$ with Y a Calabi-Yau (CY) manifold of real dimension 6 (complex dimension 3). In these notes we will use one of the many different characterizations of CY-manifolds implied by Yau's theorem in algebraic geometry (see [Voi99:Chapter 1] for an overview):

Definition 2.1.1. Let Y be an algebraic variety over $\mathsf{k} = \mathbb{C}$. We say that Y is Calabi-Yau if its canonical bundle ω_Y is *trivializable*, ie, if there exists an isomorphism $\omega_Y \simeq \mathscr{O}_Y$.

Example 2.1.2. One of the first examples of Calabi-Yau 3-fold considered by physicists is the Fermat's quintic given by the vanishing locus

$$Y = \{x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 = 0\} \subseteq \mathbb{P}^4_{\mathbb{C}}$$

From the adjunction formula [EH16:Prop. 1.33 and §1.4.3] and the fact the canonical sheaf of $\mathbb{P}^n_{\mathbb{C}}$ is $\mathscr{O}(-n-1)$(*), for any smooth quintic Y in $\mathbb{P}^4_{\mathbb{C}}$, we have $\omega_Y \simeq \mathscr{O}(5 - 4 - 1) \simeq \mathscr{O}_Y$. The same adjunction formula also shows that the intersection of two cubics in $\mathbb{P}^5_{\mathbb{C}}$ is also Calabi-Yau.

Example 2.1.3. In complex dimension 1 the only Calabi-Yau algebraic projective varieties are elliptic curves. Indeed, for smooth projective curves we have $\omega_C \simeq \Omega^1_C$ and therefore in the Calabi-Yau case $\Omega_C \simeq \mathscr{O}_C$ one must have geometric genus (†)

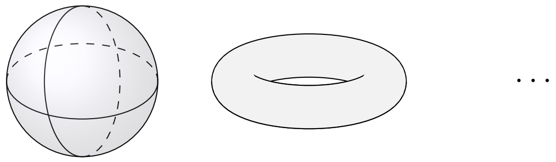
$$g = \dim_{\mathbb{C}} \mathsf{H}^0(C, \Omega^1_C) = \dim_{\mathbb{C}} \mathsf{H}^0(C, \mathscr{O}_C) = 1$$

and therefore have an elliptic curve.

In the $\mathbb{R}^4 \times Y$ model of spacetime, physical trajectories/interactions of string-like particles define 2-dimensional real surfaces inside Y with genus. Physical considerations suggest that these surfaces are actually Riemann surfaces (see the discussion in [Zwi09:§25.4]). In terms of algebraic geometry these correspond to 1-dimensional complex algebraic curves with genus

(*)This can be shown using the Euler sequence

(†)Recall that the genus of an algebraic curve over k is defined as $g := \dim_{\mathsf{k}} \mathsf{H}^1(C, \mathscr{O}_C)$. By Serre duality this agrees with $\dim_{\mathsf{k}} \mathsf{H}^0(C, \Omega^1_C)$.



The Feynmann's integral approach to mechanics [FH10] requires integration of action functionals over the space of all trajectories, which in this case means, taking integrals over the space of all algebraic curves in Y. Therefore, this is no longer a counting problem in physics, but rather one of *enumerative geometry* in Y. See [Kat06:Chapter 12].

2.1.2. Enumerative geometry via stable maps. One possibility amongst others, is to count *parametrized curves* in Y, ie, maps $f : C \to Y$ from a smooth algebraic curve C to Y, allowing for instance maps f that collapse the whole curve C to a point in Y. Posed as such, this faces several difficulties.

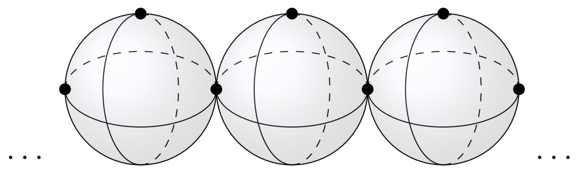
To start with, if we only allow only smooth curves, the corresponding moduli space is non-compact and therefore the counting diverges. To solve this problem one follows the strategy of Deligne-Mumford [DM69] and allow smooth curves to degenerate to singular curves with controlled nodal singularities:

Definition 2.1.4. A pre-stable curve is a connected projective curve C of genus g over k with at worst nodal singularities (*) and n marked distinct smooth points $(p_1, \cdots, p_n)$. A pre-stable curve is said to be *stable* if the group of automorphisms

$$\mathrm{Aut}(C, p_1, \cdots, p_n)$$

preserving all the points (ie $\phi(p_i) = p_i$, $\forall i$), is finite (†).

Remark 2.1.5. The finite condition on $\mathrm{Aut}(C, p_1, \cdots, p_n)$ is equivalent to the inequality $2g - 2 + n > 0$. Indeed, one can show that $(C, p_1, \cdots, p_n)$ is stable if and only if $\omega_C(\sum p_i)$ is ample (see [HM06:Ex 3.10]). When $g \geq 2$, we are authorized to have zero marked points. In the case of genus 0 the only possibility are connected trees of $\mathbb{P}^1_{\mathbb{C}}$ where each irreducible component has at least three special points (‡). Since topologically, $\mathbb{P}^1_{\mathbb{C}}$ is the two sphere S^2, so in the genus zero case, such trees can be pictured as



(*)ie, the singular points are locally modeled by the equation $xy = 0$

(†)See [HM06:Def. 2.13]

(‡)A special point is either a node or a marking.

Remark 2.1.6. Drawing curves as real surfaces gives a misleading picture of what happens at the intersection nodes. Indeed, the singularities being nodal, the underlying 2-dimensional real surfaces meet *transversally*. This can only be seen in a higher dimensional sheet. Instead, it is more convenient to simply draw curves as 1-dimensional. The picture above becomes

Notation 2.1.7. We denote by:

- $\mathsf{M}^{\mathrm{pre}}_{g,n}$ the 1-stack classifying families of *pointed pre-stable curves* and $\mathscr{C}^{\mathrm{pre}}_{g,n} \to \mathsf{M}^{\mathrm{pre}}_{g,n}$ the universal pre-stable curve.
- $\overline{\mathsf{M}}_{g,n} \subseteq \mathsf{M}^{\mathrm{pre}}_{g,n}$ the open 1-substack classifying families of *pointed stable curves* and $\overline{\mathscr{C}}_{g,n} \to \overline{\mathsf{M}}_{g,n}$ the universal stable curve [Beh97:Lemma 1]. This is the Deligne-Mumford [DM69] stacky compactification of the moduli of smooth curves. It has a coarse moduli space which is a projective variety.
- $\mathsf{M}_{g,n}$ the open substack of $\overline{\mathsf{M}}_{g,n}$ parametrizing families of smooth curves.

Example 2.1.8. We now discuss the deformation theory of pointed pre-stable curves as a simple application of derived geometry, continuing the discussion in Example **1.2.12**: by considering the homotopy fiber product taken in derived stacks(*)

$$\begin{tikzcd}
\Omega_{(C,p_1,\cdots,p_n)}\mathsf{M}^{\mathrm{pre}}_{g,n} \arrow[r] \arrow[d] & \mathsf{Spec}(\mathsf{k}) \arrow[d, "{(C,p_1,\cdots,p_n)}"] \\
\mathsf{Spec}(\mathsf{k}) \arrow[r, "{(C,p_1,\cdots,p_n)}"] & \mathsf{M}^{\mathrm{pre}}_{g,n}
\end{tikzcd}$$

we obtain a derived mapping stack of automorphisms ϕ of C, preserving the base points, ie, $\phi(p_i) = p_i$ for all i,

$$\Omega_{(C,p_1,\cdots,p_n)}\mathsf{M}^{\mathrm{pre}}_{g,n} \simeq \mathbb{R}\,\mathrm{Aut}(C, p_1, \cdots, p_n)$$

where $\mathbb{R}\,\mathrm{Aut}(C, p_1, \cdots, p_n)$ is an open substack of the mapping stack of endomorphisms $\mathbb{R}\mathsf{Map}(C, C)$ that preserve the points. But this automorphism stack is defined as a derived fiber product

$$\begin{tikzcd}
\mathbb{R}\,\mathrm{Aut}(C, p_1, \cdots, p_n) \arrow[r] \arrow[d] & \mathsf{Spec}(\mathsf{k}) \arrow[d, "{(p_1,\cdots,p_n)}"] \\
\mathbb{R}\,\mathrm{Aut}(C) \arrow[r, "\mathsf{Ev}(p_1)\times\cdots\times\mathsf{Ev}(p_n)"] & C^n
\end{tikzcd}$$

(*)with $\mathsf{M}^{\mathrm{pre}}_{g,n}$ seen as a derived stack via the inclusion $\mathsf{St}_{\mathsf{k}} \hookrightarrow \mathsf{dSt}_{\mathsf{k}}$.

establishing the fixed points, with Ev the evaluation of the automorphism at the point p_i. Therefore, we obtain a fiber sequence of tangent complexes, where using the short exact sequences[(*)]

$$0 \to \mathscr{O}_C(-p_i) \to \mathscr{O}_C \to k(p_i) \to 0$$

for each marked point p_i, we obtain that the tangent complex of $\mathsf{M}_{g,n}^{\mathrm{pre}}$ at $(C, p_1, \cdots, p_n)$ is given by the complex as in the Example **1.1.37**:

$$\mathbb{R}\Gamma(C, \mathbb{T}_C(-\sum p_i))[1] \simeq \mathbb{R}\operatorname{Hom}_{\mathscr{O}_C}(\Omega_C(\sum p_i), \mathscr{O}_C)[1]$$

We have the following results:

- [DM69:Lemma 1.3] - $\operatorname{Ext}^2_{\mathscr{O}_C}(\Omega_C(\sum p_i), \mathscr{O}_C) = 0$ for every pre-stable curve.
- [DM69:Lemma 1.4]: if moreover C is stable, $\operatorname{Ext}^0_{\mathscr{O}_C}(\Omega_C(\sum p_i), \mathscr{O}_C) = 0$

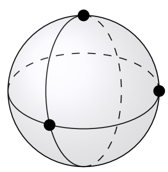
As a consequence the moduli stack $\mathsf{M}_{g,n}^{\mathrm{pre}}$ is a smooth Artin stack, and $\overline{\mathsf{M}}_{g,n}$ is a smooth Deligne-Mumford stack with tangent space at a stable curve $(C, p_1, \cdots, p_n)$ given by

$$\operatorname{Ext}^1_{\mathscr{O}_C}(\Omega_C(\sum p_i), \mathscr{O}_C)$$

By Serre duality and the Riemann-Roch formula, this coincides with $\mathsf{H}^0(C, \Omega_C^2(\sum p_i))$ of dimension

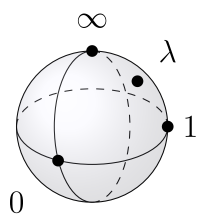
$$3g - 3 + n \tag{29}$$

Example 2.1.9. For $n = 3$, the coarse moduli spaces of $\mathsf{M}_{0,3}$ and $\overline{\mathsf{M}}_{0,3}$ coincide and contain a single point corresponding to a $\mathbb{P}^1_{\mathbb{C}}$ with three marked points:



For $n = 4$, up to automorphism, the first three points can assumed to be 0, 1 and ∞, giving us an identification between the coarse moduli space $\mathsf{M}_{0,4}$ and $\mathbb{P}^1_{\mathbb{C}} \setminus \{0, 1, \infty\}$:

$$\begin{array}{ccc} \mathsf{M}_{0,4} & \longleftrightarrow & \mathbb{P}^1_{\mathbb{C}} \setminus \{0,1,\infty\} \\ \cup & & \cup \end{array}$$

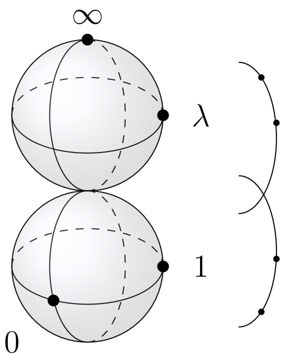


λ

(*)See for instance [Har77: II, 6.18].

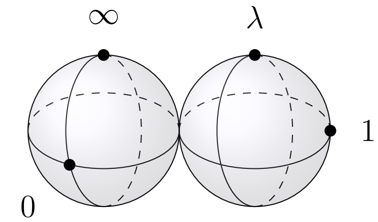
Thus, the compactification $\overline{\mathsf{M}}_{0,4}$ has coarse moduli space isomorphic to $\mathbb{P}^1_{\mathbb{C}}$ by adding the three missing points that correspond to allowing the smooth curve corresponding to a point $\lambda \in \mathbb{P}^1_{\mathbb{C}} \setminus \{0,1,\infty\}$ to degenerate to a nodal curve: as the parameter $\lambda \in \mathbb{P}^1_{\mathbb{C}} \setminus \{0,1,\infty\}$ (corresponding to the curve $\mathbb{P}^1_{\mathbb{C}}$ with the four marked points $\{0,1,\infty,\lambda\}$) gets closer for instance to ∞, we add a new curve to the moduli space $\mathsf{M}_{0,4}$ corresponding to the stable curve with two irreducible components: one irreducible component containing the two points at rest (0 and 1), and one containing the two points converging, in this case, λ and ∞:

$\lambda \rightsquigarrow \infty$

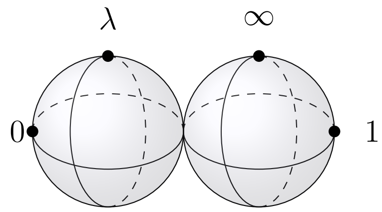


As λ gets closer to 1 (resp, 0), we compactify with the respective possibilities:

$\lambda \rightsquigarrow 1$



$\lambda \rightsquigarrow 0$



Moreover, we see that according to the our formula for dimensions in Example **2.1.8**, $\overline{\mathsf{M}}_{0,4}$ is of dimension $3 \times 0 - 3 + 4 = 1$.

Example 2.1.10. The moduli stack $\overline{\mathsf{M}}_{1,1}$ is the compactification of the moduli stack of elliptic curves.

Inspired by this compactification for curves, Kontsevich [Kon95; KM94; KM96] considers a compact moduli stack of *maps* into a smooth and proper algebraic variety Y over $\mathbb{C}$ - $\overline{\mathsf{M}}_{g,n}(Y,\beta)$.

Definition 2.1.11. A *stable map* $f : (C, p_1, ..., p_n) \to Y$ of degree $\beta \in \mathsf{H}_2(Y,\mathbb{Z})$ consists of:

- a map $f : C \to Y$ from a (at worst nodal) curve C.

- n marked points $p_1, ..., p_n \in C$ that live in the smooth locus of C;

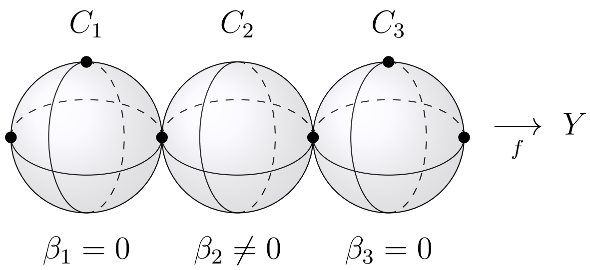
such that:

- $f_*[C] = \beta \in \mathsf{H}_2(Y, \mathbb{Z})$
- the map f is required to have only a finite number of automorphisms preserving the marked points.

Remark 2.1.12. By definition of the push forward of cycles (see [Ful98:§1.4] and [EH16:Def. 1.19]), if I denotes the set of irreducible components of C, and for each irreducible component $C_i \subseteq C$ we denote $\beta_i := f_*[C_i]$, then $\beta = \sum_{i \in I} \beta_i$.

Remark 2.1.13. Contrary to curves in $\overline{\mathsf{M}}_{g,n}$, in a stable map $f : C \to Y$ the curve C is not necessarily stable. The stability condition is equivalent to the following characterization [CK99:§7.1.1 p. 169]): denote by C_i each irreducible component of C, $f_i : C_i \to Y$ the restriction and $\beta_i = f_*[C_i]$ the degree of C_i. Then stability means that if $\beta_i = 0$ (ie, the map is constant) and C_i is of genus 0 (resp. 1), then C_i should have at least 3 special points (resp, 1)(*). See [KV07:Lemma 2.3.1] for a comparison of the two characterizations.

Example 2.1.14. The following picture illustrates a possibility for a stable map $f : C \to Y$ in genus zero with 5 marked points, where the middle irreducible component is not stable as a curve:



By allowing the curves to vary in families, we obtain:

Theorem 2.1.15 ([Kon95]). *There exists a proper Deligne-Mumford moduli stack $\overline{\mathsf{M}}_{g,n}(Y, \beta)$ parametrizing stable maps.*

Construction 2.1.16. The stack $\overline{\mathsf{M}}_{g,n}(Y, \beta)$ comes equipped with canonical evaluation maps

$$\mathsf{Ev}_i : \overline{\mathsf{M}}_{g,n}(Y, \beta) \to Y \quad i = 1, \cdots n$$

sending a stable maps $f : (C, p_1, ..., p_n) \to Y$ to $f(p_i) \in Y$.

It also comes with a stabilization map

(*)A special point is either a marked point or a nodal point.

$$\mathsf{Stab} : \overline{\mathsf{M}}_{g,n}(Y,\beta) \to \overline{\mathsf{M}}_{g,n}$$

that sends a stable map $f : (C, p_1, ..., p_n) \to X$ to the stabilization of the curve C, ie, it contracts the irreducible components of C that are not stable as curves (ie, in the sense of the Definition **2.1.4**).

Example 2.1.17. The curve obtained by applying the functor Stab to the stable map in the Example **2.1.14** is as in the points at infinity in the Example **2.1.9** with two irreducible components.

2.1.3. Stable maps and enumerative geometry in $\mathbb{P}^2_{\mathbb{C}}$. Let us now illustrate, following the example $Y = \mathbb{P}^2_{\mathbb{C}}$, how the counting of stable maps is related to the counting of rational curves and how this problem can be converted to the language of intersection theory.

2.1.3.1. ***Enumerative geometry and the intersection pairing.*** Let us forget for a second the Calabi-Yau situation and simply try to count stable maps of genus of 0 in $\mathbb{P}^2_{\mathbb{C}}$. In this case, the curve C is a finite chain of $\mathbb{P}^1_{\mathbb{C}}$'s with nodal singularities as intersections as in the Remark **2.1.5**.

Reminder 2.1.18. Recall (see [Ful98:19.1.11-(d)]) that in the case of $\mathbb{P}^2_{\mathbb{C}}$, the cycle class map induces, induces an isomorphism from Chow groups to singular homology

$$\mathsf{CH}_n(\mathbb{P}^2_{\mathbb{C}}) \xrightarrow{\sim} \mathsf{H}_{2n}(\mathbb{P}^2_{\mathbb{C}}, \mathbb{Z})$$

By the Bezout's theorem, the Chow ring $\mathsf{CH}^*(\mathbb{P}^2_{\mathbb{C}}) \simeq \mathbb{Z}[h]/([h^3])$ is the free graded algebra generated by the rational equivalence class of a line in $\mathbb{P}^2_{\mathbb{C}}$, $h := [\text{Line}] \in \mathsf{CH}^1(\mathbb{P}^2_{\mathbb{C}})$ (see [EH16:§2.1, Theorem 2.1]). Under the isomorphism $\mathsf{H}_2(\mathbb{P}^2_{\mathbb{C}}, \mathbb{Z}) \simeq \mathsf{CH}_1(\mathbb{P}^2_{\mathbb{C}}) \simeq \mathbb{Z}.[\text{Line}]$, the class $\beta \in \mathsf{H}_2(\mathbb{P}^2_{\mathbb{C}}, \mathbb{Z})$ corresponds to the integer number $d.[\text{Line}]$, given by the degree d of a codimension one subvariety.

Let us look at the moduli stack of stable maps of degree d, of genus 0 and with n marked points $\overline{\mathsf{M}}_{g=0,n}(\mathbb{P}^2_{\mathbb{C}}, \beta = d)$.

Proposition 2.1.19. *$\overline{\mathsf{M}}_{g=0,n}(\mathbb{P}^2_{\mathbb{C}}, \beta = d)$ is smooth with dimension*

$$(\dim_{\mathbb{C}} \mathbb{P}^2_{\mathbb{C}})(1-g) + \int_{\beta} c_1(T_X) + n + 3(g-1) = 3d + n - 1$$

PROOF. To see this, let us count the virtual dimension. Fix a stable map $f : C \to \mathbb{P}^2_{\mathbb{C}}$. The term $(n + 3(g-1))$ comes from infinitesimal deformations of the curve as explained in the Example **2.1.8**. The new extra term accounts for the deformations of the map $f : C \to \mathbb{P}^2_{\mathbb{C}}$: the deformation space of f is given by $\mathsf{H}^0(C, f^*\mathbb{T}_{\mathbb{P}^2_{\mathbb{C}}})$ and the obstruction space by $\mathsf{H}^1(C, f^*\mathbb{T}_{\mathbb{P}^2_{\mathbb{C}}})$. These contribute with the alternated sum

$$\mathsf{H}^0(C, f^*\mathbb{T}_{\mathbb{P}^2_{\mathbb{C}}}) - \mathsf{H}^1(C, f^*\mathbb{T}_{\mathbb{P}^2_{\mathbb{C}}}) =$$

$$\underset{\text{Hirzebruch–Riemann–Roch}}{=} \int_C c_1(f^*\mathbb{T}_{\mathbb{P}^2_{\mathbb{C}}}) + (\dim_{\mathbb{C}} \mathbb{P}^2_{\mathbb{C}}).(1-g)$$

Finally, $\mathsf{vdim}\, \overline{\mathsf{M}}_{g=0,n}(\mathbb{P}^2_{\mathbb{C}}, \beta = d)$ is given by

$$\int_C c_1(f^*\mathbb{T}_{\mathbb{P}^2_{\mathbb{C}}}) + 2(1-0) + n + 3(g-1) = \int_C c_1(f^*\mathbb{T}_{\mathbb{P}^2_{\mathbb{C}}}) + n - 1$$

By the splitting principle, we have

$$c_1(f^*\mathbb{T}_{\mathbb{P}^2_{\mathbb{C}}}) = c_1(f^*\mathsf{det}(\mathbb{T}_{\mathbb{P}^2_{\mathbb{C}}})) = c_1(f^*(\mathscr{O}(3)) = 3d$$

and finally the virtual dimension (aka expected dimension) of the moduli space is

$$3d + n - 1$$

$\square$

Remark 2.1.20. Under the current assumptions of this section, the moduli space happens to be smooth because $\mathbb{P}^2_{\mathbb{C}}$ is convex, ie, for any stable map $f : C \to \mathbb{P}^2_{\mathbb{C}}$, the cohomology groups $\mathsf{H}^1(\mathbb{P}^2_{\mathbb{C}}, f^*\mathbb{T}_{\mathbb{P}^2_{\mathbb{C}}})$ vanish. See [FP96:Theorem 2].

Construction 2.1.21. Fix $x_1, \cdots, x_n$, n points in $\mathbb{P}^2_{\mathbb{C}}$ and consider the scheme-theoretic fiber

$$\begin{tikzcd}
\bigcap_{i=1}^n \mathsf{Ev}_i^{-1}(\{x_i\}) \arrow[r] \arrow[d] & \overline{\mathsf{M}}_{0,n}(\mathbb{P}^2_{\mathbb{C}}, d) \arrow[d, "\mathsf{Ev}_1 \times \cdots \times \mathsf{Ev}_n"] \\
\{x_1\} \times \cdots \{x_n\} \arrow[r] & \mathbb{P}^2_{\mathbb{C}} \times \cdots \times \mathbb{P}^2_{\mathbb{C}}
\end{tikzcd}$$

This schematic fiber product classifies stable maps $f : (C, p_1, \cdots, p_n) \to \mathbb{P}^2_{\mathbb{C}}$ such that $f(p_i) = x_i$.

Because of equivariance under the transitive action of $\mathrm{Aut}(\mathbb{P}^2_{\mathbb{C}})$ and generic flatness ([Stacks:Section 0529]), in the case of $\mathbb{P}^2_{\mathbb{C}}$ the evaluation maps Ev_i are flat. In particular, each fiber $\mathsf{Ev}_i^{-1}(\{x_i\})$ is of the same codimension as $\{x_i\}$ in $\mathbb{P}^2_{\mathbb{C}}$, ie, 2. Therefore, the intersection $\bigcap_{i=1}^n \mathsf{Ev}_i^{-1}(\{x_i\})$ is of codimension

$$\mathrm{codim} \leq \sum_{i=1}^n 2 = 2n$$

Remark 2.1.22. The best case scenario in the Construction **2.1.21** happens when simultaneously:

- the equality is reached, ie, codim $= 2n$;
- the codimension of $\bigcap_{i=1}^{n} \mathsf{Ev}_i^{-1}(\{x_i\})$ is maximal in $\overline{\mathsf{M}}_{0,n}(\mathbb{P}^2_{\mathbb{C}}, d)$, ie, the intersection is just a collection of points,

$$\text{codim} = 3d + n - 1$$

In particular, we see that for this best case scenario to hold, the number of marked points must be

$$n = 3d - 1$$

We now establish the relation with enumerative geometry

Proposition 2.1.23. *For a generic choice of $n = 3d-1$ points $x_1, \cdots, x_{3d-1} \in \mathbb{P}^2_{\mathbb{C}}$, the schematic intersection $\bigcap_{i=1}^{n} \mathsf{Ev}_i^{-1}(\{x_i\})$ consists of a finite number of reduced points in $\overline{\mathsf{M}}_{0,n}(\mathbb{P}^2_{\mathbb{C}}, d)$, supported in the locus of stable maps $f : C \to \mathbb{P}^2_{\mathbb{C}}$ with C a smooth curve and f an immersion.*

Proof. This is a consequence of Kleiman's Bertini-type transversality theorem (see [KV07:Prop. 3.4.3] and [Kle74]). □

Proposition 2.1.24. *For a generic choice of $n = 3d-1$ points $x_1, \cdots, x_{3d-1} \in \mathbb{P}^2_{\mathbb{C}}$, each stable map $f \in \bigcap_{i=1}^{n} \mathsf{Ev}_i^{-1}(\{x_i\})$ has no repetition of the markings, ie, if $f(p_i) = x_i$, then $f^{-1}(\{x_i\}) = \{p_i\}$ with multiplicity 1.*

Proof. By the Proposition **2.1.23** we can assume $C = \mathbb{P}^1_{\mathbb{C}}$. Now, the result is obtained by showing that the locus of stable maps $\mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}}$ of degree d with repetitions of the markings is of strictly positive codimension, so that, by generic choice of the markings, it can be avoided. See [KV07:3.5.2 and 3.5.3]. □

Corollary 2.1.25. *For a generic choice of $n = 3d-1$ points $x_1, \cdots, x_{3d-1} \in \mathbb{P}^2_{\mathbb{C}}$, the finite number of points in $\bigcap_{i=1}^{n} \mathsf{Ev}_i^{-1}(\{x_i\})$ is exactly the number N_d of rational curves passing through $x_1, \cdots, x_{3d-1}$ in $\mathbb{P}^2_{\mathbb{C}}$.*

Proof. Follows directly from Proposition **2.1.23** and Proposition **2.1.24**. See [KV07:Cor. 3.5.4]. □

Finally, we discuss the approach via intersection theory and Poincaré duality, to count the number of points in the intersection $\bigcap_{i=1}^{n} \mathsf{Ev}_i^{-1}(\{x_i\})$.

Reminder 2.1.26. For this purpose, recall a more general situation from algebraic topology: if M is a compact orientable smooth manifold of dimension m, the choice of a fundamental class $[M] \in \mathsf{H}_m(M, \mathbb{Z})$ induces a non-degenerated pairing

$$\mathsf{H}^i(M,\mathbb{Z}) \otimes \mathsf{H}^{m-i}(M,\mathbb{Z}) \underset{\cup}{\to} \mathsf{H}^m(M,\mathbb{Z}) \underset{\int_{[M]}}{\to} \mathbb{Z}$$

given by

$$(\alpha,\beta) \mapsto \int_{[M]} \alpha \cup \beta$$

with $\cup$ being the cup product in cohomology. In particular,

$$\mathsf{H}^i(M,\mathbb{Z}) \simeq \mathsf{H}^{m-i}(M,\mathbb{Z})^\vee$$

At the same time, we have a linear duality given by integrating i-cochains along i-chains:

$$\mathsf{H}_i(M,\mathbb{Z}) \otimes \mathsf{H}^i(M,\mathbb{Z}) \to \mathbb{Z}$$

$$([Z],\alpha) \mapsto \int_Z \alpha$$

Combining the two dualities, we get the Poincaré duality isomorphism

$$\mathrm{PD} : \mathsf{H}_{m-i}(M,\mathbb{Z}) \simeq \mathsf{H}^{m-i}(M,\mathbb{Z})^\vee \simeq \mathsf{H}^i(M,\mathbb{Z})$$

with inverse given by the *cap* product with the fundamental class

$$\alpha \in \mathsf{H}^i(M,\mathbb{Z}) \mapsto \alpha \cap [M] \in \mathsf{H}_{m-i}(M,\mathbb{Z})$$

When Z_1 and Z_2 are compact orientable submanifolds of M of codimension, respectively, i and j, that intersect transversaly (so that the intersection is a smooth oriented submanifold of codimension $i+j$), then the Poincaré dual of the intersection $[Z_1 \cap Z_2]$ is the cup product of the Poincaré duals:

$$\mathrm{PD}([Z_1 \cap Z_2]) = \mathrm{PD}([Z_1]) \cup \mathrm{PD}([Z_2])$$

When $j = n-i$, the intersection $Z_1 \cap Z_2$ is a finite collection of points with orientation either $\epsilon = +1$ or $\epsilon = -1$. Assuming M is path-connected, the weighted counting of the number of points, $\sharp\ Z_1 \cap Z_2 := \sum_{p \in Z_1 \cap Z_2} \epsilon_p$, coincides with the image of the class $[Z_1 \cap Z_2] \in \mathsf{H}_0(M,\mathbb{Z})$ along the isomorphism $\mathsf{H}_0(M,\mathbb{Z}) \simeq \mathbb{Z}$. This number is called the *intersection number*, and it agrees with the cap pairing with the Poincaré dual classes:

$$\sharp\ (Z_1 \cap Z_2) = \int_M (\mathrm{PD}([Z_1]) \cup \mathrm{PD}([Z_2])) \tag{30}$$

Remark 2.1.27. Using the deformation to the normal cone and Gysin pullbacks (see [Ful98:§5-6, §8.4, §14.1]), it is possible to apply the Reminder **2.1.26** in algebraic geometry to the course moduli space of $M = \overline{\mathsf{M}}_{0,n}(\mathbb{P}^2_{\mathbb{C}}, d)$ using the fact that this moduli space is smooth and compact and therefore admits a fundamental class $[\overline{\mathsf{M}}_{0,n}(\mathbb{P}^2_{\mathbb{C}}, d)] \in \mathsf{H}_{3d+n-1}(\overline{\mathsf{M}}_{0,n}(\mathbb{P}^2_{\mathbb{C}}, d))$.

Corollary 2.1.28. *For each $i \geq 0$, let $\gamma_i := \mathrm{PD}([x_i])$ denote the Poincaré dual class of the homology class $[x_i] \in \mathsf{H}_0(\mathbb{P}^2_{\mathbb{C}}, \mathbb{Z})$. Following the description of the cohomology ring of $\mathbb{P}^2_{\mathbb{C}}$ in the Reminder **2.1.18**, the class γ_i, Poincaré dual of a point is the class $h^2 \in \mathsf{H}^4(\mathbb{P}^2_{\mathbb{C}}, \mathbb{Z})$. By (30) we have*

$$N_d = \sharp \left(\bigcap_{i=1}^{n} \mathsf{Ev}_i^{-1}(\{x_i\}) \right) = \int_{[\overline{\mathsf{M}}_{0,3d-1}(\mathbb{P}^2_{\mathbb{C}},d)]} \mathsf{Ev}_1^*(h^2) \cup \cdots \cup \mathsf{Ev}_{3d-1}^*(h^2) \tag{31}$$

where Ev_i^* denotes the pullback of cohomology classes along the evaluation maps.

Definition 2.1.29. The formula (31) can be generalized: for a collection of cohomology classes $\gamma_1, \cdots, \gamma_n \in \mathsf{H}^*(\mathbb{P}^2_{\mathbb{C}}, \mathbb{Q})$ such that $\sum \mathrm{codim}\gamma_i = 3d + n - 1$, one defines the Gromov-Witten invariants

$$I^d_{g,n}(\gamma_1, \cdots, \gamma_n) := \int_{[\overline{\mathsf{M}}_{0,n}(\mathbb{P}^2_{\mathbb{C}},d)]} \mathsf{Ev}_1^*(\gamma_1) \cup \cdots \cup \mathsf{Ev}_n^*(\gamma_n) \tag{32}$$

Construction 2.1.30. One can use the pushforward in homology (change of variables) along the map $\mathsf{Stab} : \overline{\mathsf{M}}_{0,n}(\mathbb{P}^2_{\mathbb{C}}, d) \to \overline{\mathsf{M}}_{0,n}$ to write the integrals (32) as

$$\int_{[\overline{\mathsf{M}}_{0,n}(\mathbb{P}^2_{\mathbb{C}},d)]} \mathsf{Ev}_1^*(\gamma_1) \cup \cdots \cup \mathsf{Ev}_n^*(\gamma_n) = \int_{\overline{\mathsf{M}}_{0,n}} \mathsf{Stab}_*(\mathsf{Ev}^*(\gamma_1) \cup \cdots \cup \mathsf{Ev}^*(\gamma_n))$$

The discussion in this section alone is not enough to determine the intersection numbers (31). There are two key ingredients that will allow us to proceed:

A) The first ingredient comes from the geometry of the boundary at infinity of the moduli space of stable maps, namely, in the form of linear relations between the different components of the divisor. We will discuss this in Section **2.1.3.2** below (see Corollary **2.1.46**).

B) The second ingredient explains how the Gromov-Witten numbers (32) can be compounded to get Gromov-Witten numbers of higher degree, more precisely, how the different $I^d_{g,n}$ can be obtained from the $I^{d_i}_{g_i,n_i}$ for decompositions $d = \sum d_i$ and $n = (\sum n_i) - k$ where k is the length of the partition of n. We will discuss this in Section **2.1.3.3**.

2.1.3.2. ***A) The geometry of the boundary divisors***. Building on the discussion in the Example **2.1.9**, we can start by describing the boundary at infinity of the inclusion $\mathsf{M}_{0,n} \subseteq \overline{\mathsf{M}}_{0,n}$ in terms of curves obtained as gluings:

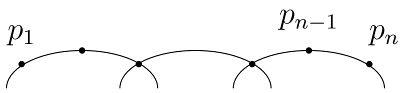
Construction 2.1.31. Given stable curves

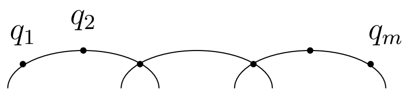
$$(C_2, p_1, \cdots, p_n) \in \overline{\mathsf{M}}_{g_1,n} \quad \text{and} \quad (C_2, q_1, \cdots, q_m) \in \overline{\mathsf{M}}_{g_2,m}$$

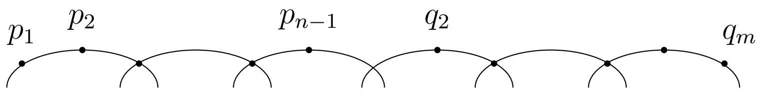
one can glue (as schemes) the last marked point of C_1 to the first marked point of C_2 and obtain this way a stable curve

$$(C_1 \coprod_{p_n=q_1}^{\text{schemes}} C_2, p_1, \cdots p_{n-1}, q_2, \cdots q_m)$$

Informally, we can picture it as

C_1:= 

and C_2:= 

$C_1 \coprod_{p_n=q_1}^{\text{schemes}} C_2$:= 

Since C_1 is of genus g_1 and C_2 is of genus g_2, the resulting gluing will be of genus $g_1 + g_2$. This operation extends to families of stable curves and defines a map of Deligne-Mumford stacks

$$m_{g_1,g_2,n,m} : \overline{\mathsf{M}}_{g_1,n} \times \overline{\mathsf{M}}_{g_2,m} \to \overline{\mathsf{M}}_{g_1+g_2,n+m-2} \tag{33}$$

which happens to be a closed immersion as soon as $g_1 \neq g_2$ or $n + m - 2 \neq 0$ [Knu83:Cor 3.9].

Construction 2.1.32. Fix $n \geq 4$ and genus zero. Then the boundary at infinity in $\overline{\mathsf{M}}_{0,n}$, ie, $\overline{\mathsf{M}}_{0,n} \setminus \mathsf{M}_{0,n}$, has several irreducible components of codimension 1 - $\mathsf{D}(A|B)$ - each one defined by a partition of the set $\{1, \cdots n\} = A \cup B$ with $\sharp A, \sharp B \geq 2$. By definition, $\mathsf{D}(A|B)$ is the image of the gluing map along the extra point

$$\overline{\mathsf{M}}_{0,(\sharp A+1)} \times \overline{\mathsf{M}}_{0,(\sharp B+1)} \hookrightarrow \overline{\mathsf{M}}_{0,n}$$

where the points in A (respectively B) appear in the copy of $\overline{\mathsf{M}}_{0,(\sharp A+1)}$ (respectively $\overline{\mathsf{M}}_{0,(\sharp B+1)}$). Each of the $\mathsf{D}(A|B)$ is a smooth divisor. These divisors intersect with normal crossings [KV07:Prop. 1.5] and the higher codimension stratum is given by the images of the gluing operations

$$\overline{\mathsf{M}}_{0,n_1} \times \overline{\mathsf{M}}_{0,n_2} \times \cdots \times \overline{\mathsf{M}}_{0,n_k} \hookrightarrow \overline{\mathsf{M}}_{0,n}$$

under partitions of n of length k. Indeed, a dimension counting using (29) shows that the image of the different consecutive gluing partitions of n with k terms, $n = n_1 + n_2 + \cdots n_k - k$ is of codimension k in $\overline{\mathsf{M}}_{0,n}$.

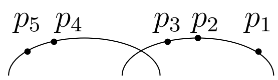
Terminology 2.1.33. We call the $\mathsf{D}(A|B)$ the boundary divisors.

Remark 2.1.34. The moduli space $\overline{\mathsf{M}}_{0,n}$ has $2^{n-1}-n-1$ boundary divisors. In order to count the required partitions of $\{1,\cdots,n\}$ we count first set-theoretic functions $f:\{1,\cdots,n\}\to\{0,1\}$. There are 2^n such functions. But since we want each subset A and B to be non-empty, we only care about surjective functions f, therefore, discounting 2^n-2 for the two surjective functions sending all values to 0 or to 1. Now, we also care that $\sharp A, \sharp B \geq 2$, therefore we must discount all functions f that send $n-1$ values to 0 and the remaining value to 1. There are n such functions, but we need to double the count to account for changing the role of 1 and 0. Therefore, we have 2^n-2-2n. Finally, the fact that the partition as a subset of $\{1,\cdots,n\}$ is given by pre-images of f and as a decomposition of subsets is invariant under the permutation of 1 and 0 in $\{0,1\}$, tells us that one must divide the counting 2^n-2-2n by 2, thus obtaining the formula in the claim.

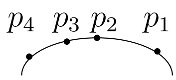
Notation 2.1.35. Let us denote by $\mathsf{D}_\infty, \mathsf{D}_0, \mathsf{D}_1$ the three divisors at infinity in $\overline{\mathsf{M}}_{0,4}$ as in Example **2.1.9**.

Construction 2.1.36. We also have a map $\overline{\mathsf{M}}_{g,n+1}\to\overline{\mathsf{M}}_{g,n}$ that forgets the last marked point and contracts the irreducible component where the last marked point lives in case removing the point breaks the stability condition (as in the stabilization map Construction **2.1.16**).

Example 2.1.37. In the case $\overline{\mathsf{M}}_{0,5}\to\overline{\mathsf{M}}_{0,4}$, the image of



is the smooth curve



Proposition 2.1.38 ([Knu83])**.** *The forgetful map $\overline{\mathsf{M}}_{g,n+1}\to\overline{\mathsf{M}}_{g,n}$ exhibits $\overline{\mathsf{M}}_{g,n+1}$ as the universal stable curve $\mathscr{C}_{g,n}$ of Notation* ***2.1.7****. In particular, it is flat.*

Proposition 2.1.39. *The inverse image of the boundary divisor $\mathsf{D}(A|B)\subseteq\overline{\mathsf{M}}_{0,n}$ under the map $\epsilon:\overline{\mathsf{M}}_{0,n+1}\to\overline{\mathsf{M}}_{0,n}$ that forgets the last point x, is the sum of boundary divisors with multiplicity 1*

$$\epsilon^*\mathsf{D}(A|B)=\mathsf{D}(A\cup\{x\}|B)+\mathsf{D}(A|B\cup\{x\})$$

PROOF. See [KV07:§1.5.10]. □

Construction 2.1.40. Let $n\geq 4$. For any subset $\{i,j,k,l\}\subseteq\{1,\cdots,n\}$ we have a forgetful map $\epsilon_{i,j,k,l}:\overline{\mathsf{M}}_{0,n}\to\overline{\mathsf{M}}_{0,4}=\overline{\mathsf{M}}_{0,\{i,j,k,l\}}$ that forget all points except i,j,k and l and stabilizes (as in Construction **2.1.16**). See [Man99:p. 93]. Under the unique automorphism sending $i\mapsto 0, j\mapsto 1, k\mapsto\infty$ and $l\mapsto\lambda$, we denote by $\mathsf{D}(ij|kl)$ the divisor corresponding to D_∞, $\mathsf{D}(ik|jl)$ the divisor corresponding to D_1 and $\mathsf{D}(jk|il)$ the image of D_0.

Corollary 2.1.41. *Since the maps $\epsilon_{\{i,j,k,l\}}$ are flat by the Proposition 2.1.38, the inverse image of the boundary divisors $\mathsf{D}(ij|kl)$, $\mathsf{D}(ik|jl)$ and $\mathsf{D}(jk|il) \subseteq \overline{\mathsf{M}}_{0,\{i,j,k,l\}}$ are divisors in $\overline{\mathsf{M}}_{0,n}$ given by the sum*

$$\epsilon^*_{i,j,k,l}(\mathsf{D}(ij|kl)) = \sum_{i,j\in A,\ \ k,l\in B} \mathsf{D}(A|B)$$
$$\epsilon^*_{i,j,k,l}(\mathsf{D}(ik|jl)) = \sum_{i,k\in A,\ \ j,l\in B} \mathsf{D}(A|B)$$
$$\epsilon^*_{i,j,k,l}(\mathsf{D}(jk|il)) = \sum_{j,k\in A,\ \ i,l\in B} \mathsf{D}(A|B)$$

Moreover, since the three divisors $\mathsf{D}(ij|kl)$, $\mathsf{D}(ik|jl)$ and $\mathsf{D}(jk|il) \subseteq \overline{\mathsf{M}}_{0,\{i,j,k,l\}} \simeq \mathbb{P}^1_{\mathbb{C}}$ are linearly equivalent, so are their pullbacks,

$$\sum_{i,j\in A,\ \ k,l\in B} \mathsf{D}(A|B) = \sum_{i,k\in A,\ \ kj,l\in B} \mathsf{D}(A|B) = \sum_{j,k\in A,\ \ i,l\in B} \mathsf{D}(A|B) \tag{34}$$

Proof. Follows by induction using the Proposition **2.1.39**. $\square$

We now discuss a similar results for the boundary of the moduli of stable maps. First we need to describe the gluing operations:

Construction 2.1.42. The gluing operations of curves of the Construction **2.1.31** can be extended to stable maps: if $f_1 : (C_1, p_1, \cdots, p_n) \to Y$ is a stable map of degree β_1 and $f_2 : (C_2, q_1, \cdots, q_m) \to Y$ is a stable map of degree β_2, such that $f_1(p_n) = f_2(q_1)$, then f_1 and f_2 defined a new map on the gluing of the two curves

$$(f : C_1 \coprod_{p_n=q_1} C_2, p_1, \cdots, p_{n-1}, q_2, \cdots, q_m) \to Y$$

defined on each irreducible component by f_1 or f_2, respectively. The result is again a stable map, since stability is determined on each irreducible component (Remark **2.1.13**). Moreover, the degree of f is the sum $\beta_1 + \beta_2$, since again, the total degree is the sum of the degrees of each irreducible components. Performed in families, we obtain a gluing map at the level of Deligne-Mumford stacks

$$\overline{\mathsf{M}}_{g_1,n_1}(Y, \beta_1) \underset{Y}{\times} \overline{\mathsf{M}}_{g_2,n_2}(Y, \beta_2) \to \overline{\mathsf{M}}_{g_1+g_2,n_1+n_2-2}(Y, \beta_1 + \beta_2) \tag{35}$$

As in Construction **2.1.31** these are closed immersions. Moreover, these can be iterated

$$\overline{\mathsf{M}}_{g_1,n_1}(Y, \beta_1) \underset{Y}{\times} \overline{\mathsf{M}}_{g_2,n_2}(Y, \beta_2) \underset{Y}{\times} \cdots \underset{Y}{\times} \overline{\mathsf{M}}_{g_k,n_k}(Y, \beta_k) \to \overline{\mathsf{M}}_{\sum g_i, \sum n_i - k}(Y, \sum \beta_i)$$

Construction 2.1.43. In genus zero, as in the Construction **2.1.31** the boundary at infinity in the coarse moduli space of $\overline{\mathsf{M}}_{0,n}(Y,\beta)$ is a divisor with normal crossings, stratified by the image of the gluing maps. More precisely, given a partition of $\{1,\cdots,n\} = A \cup B$ and a decomposition $\beta = \beta_1 + \beta_2$, we define $\mathsf{D}(A|B;\beta_1,\beta_2)$ to be the image of the closed immersion (35), ie,

$$\overline{\mathsf{M}}_{0,\sharp A+1}(Y,\beta_1) \underset{Y}{\times} \overline{\mathsf{M}}_{0,\sharp B+2}(Y,\beta_2) \simeq \mathsf{D}(A|B;\beta_1,\beta_2) \hookrightarrow \overline{\mathsf{M}}_{0,n}(Y,\beta) \tag{36}$$

The last property we will need to finalize A) is the following:

Lemma 2.1.44. *The stabilization map* $\mathsf{Stab} : \overline{\mathsf{M}}_{g,n}(Y,\beta) \to \overline{\mathsf{M}}_{g,n}$ *is flat.*

Let us now return to the enumerative geometry of curves in $\mathbb{P}^2_{\mathbb{C}}$ to conclude the computation in the Section **2.1.3.1**:

Construction 2.1.45. Let $n \geq 4$ and let $\{i,j,k,l\} \subset \{1,\cdots,n\}$. Let

$$\eta_{\{i,j,k,l\}} : \overline{\mathsf{M}}_{0,n}(\mathbb{P}^2_{\mathbb{C}},d) \to \overline{\mathsf{M}}_{0,\{i,j,k,l\}}$$

denote the composition of the stabilization map $\mathsf{Stab} : \overline{\mathsf{M}}_{0,n}(\mathbb{P}^2_{\mathbb{C}},d) \to \overline{\mathsf{M}}_{g,n}$ with the forgetful map $\epsilon_{\{i,j,k,l\}} : \overline{\mathsf{M}}_{0,n} \to \overline{\mathsf{M}}_{0,\{i,j,k,l\}}$ of the Construction **2.1.40**. Then by the Lemma **2.1.44** the inverse image of the boundary divisors $\mathsf{D}(ij|kl)$, $\mathsf{D}(ik|jl)$ and $\mathsf{D}(jk|il) \subseteq \overline{\mathsf{M}}_{0,\{i,j,k,l\}}$ are divisors in $\overline{\mathsf{M}}_{0,n}(\mathbb{P}^2_{\mathbb{C}},d)$.

Corollary 2.1.46. *The inverse image of the boundary divisors* $\mathsf{D}(ij|kl)$, $\mathsf{D}(ik|jl)$ *and* $\mathsf{D}(jk|il) \subseteq \overline{\mathsf{M}}_{0,\{i,j,k,l\}}$ *are divisors in* $\overline{\mathsf{M}}_{0,n}(\mathbb{P}^2_{\mathbb{C}},d)$ *given by the sum*

$$\eta^*_{i,j,k,l}(\mathsf{D}(ij|kl)) = \sum_{i,j\in A,\ \ k,l\in B} \mathsf{D}(A|B;d_1,d_2)$$

$$\eta^*_{i,j,k,l}(\mathsf{D}(ik|jl)) = \sum_{i,k\in A,\ \ j,l\in B} \mathsf{D}(A|B;d_1,d_2)$$

$$\eta^*_{i,j,k,l}(\mathsf{D}(jk|il)) = \sum_{j,k\in A,\ \ i,l\in B} \mathsf{D}(A|B;d_1,d_2)$$

Moreover, since the three divisors $\mathsf{D}(ij|kl)$, $\mathsf{D}(ik|jl)$ *and* $\mathsf{D}(jk|il) \subseteq \overline{\mathsf{M}}_{0,\{i,j,k,l\}} \simeq \mathbb{P}^1_{\mathbb{C}}$ *are linearly equivalent, so are their pullbacks,*

$$\eta^*_{i,j,k,l}(\mathsf{D}(ij|kl)) = \eta^*_{i,j,k,l}(\mathsf{D}(ik|jl)) = \eta^*_{i,j,k,l}(\mathsf{D}(jk|il))$$

ie,

(37)
$$\sum_{\substack{A\cup B=\{1,\cdots,n\}\\ i,j\in A,\\ k,l\in B\\ d_A+d_B=d}} \mathsf{D}(A|B;d_1,d_2) = \sum_{\substack{A\cup B=\{1,\cdots,n\}\\ i,k\in A,\\ kj,l\in B\\ d_A+d_B=d}} \mathsf{D}(A|B;d_1,d_2) = \sum_{\substack{A\cup B=\{1,\cdots,n\}\\ j,k\in A,\\ i,l\in B\\ d_A+d_B=d}} \mathsf{D}(A|B;d_1,d_2)$$

Proof. See [KV07:§2.7.5]. □

2.1.3.3. ***B) Cohomological field theories and the splitting axiom.*** We are now ready to explain the second ingredient (B) that will allow us to complete the computation in Section **2.1.3.1**. It expresses the fact that the different cohomology classes (32) satisfy a certain "splitting law" which we will now express:

Remark 2.1.47. The Definition **2.1.29** and Construction **2.1.30** of the GW numbers $I^d_{0,n}(\gamma_1,\cdots,\gamma_n)$ as a function of the cohomology classes $\gamma_1,\cdots\gamma_n$ can be reformulated as a map between cohomology groups

$$I^d_{0,n}: \mathsf{H}^*(\mathbb{P}^2_{\mathbb{C}})^{\otimes n} \to \mathsf{H}^*(\overline{\mathsf{M}}_{0,n}) \tag{38}$$

defined by the pullback-pushforward

$$(\gamma_1,\cdots,\gamma_n)\mapsto \mathsf{Stab}_*(\cup_{i=1}^n \mathsf{Ev}^*(\gamma_i))$$

along the maps

$$\begin{tikzcd} & \mathbb{R}\overline{\mathsf{M}}_{0,n}(\mathbb{P}^2_{\mathbb{C}},d) \arrow[dl, "\mathsf{Ev}_1,\cdots,\mathsf{Ev}_n"'] \arrow[dr, "\mathsf{Stab}"] & \\ (\mathbb{P}^2_{\mathbb{C}})^n & & \overline{\mathsf{M}}_{0,n} \end{tikzcd} \tag{39}$$

The operations $I^d_{0,n}$ in the (38) are indexed by the possible values for the degree $d\geq 0$.

Remark 2.1.48. Using Poincaré duality for both $\overline{\mathsf{M}}_{0,n}$ and $\mathbb{P}^2_{\mathbb{C}}$ and dualizing, the maps, the morphisms in (38) can also be formulated as maps

$$\mathsf{H}_*(\overline{\mathsf{M}}_{0,n}) \to \mathrm{Hom}_{\mathsf{k}}(\mathsf{H}^*(\mathbb{P}^2_{\mathbb{C}})^{\otimes n-1}, \mathsf{H}^*(\mathbb{P}^2_{\mathbb{C}})) \tag{40}$$

where $\mathrm{Hom}_{\mathsf{k}}$ is the vector space of classical linear maps. Concretely, it is defined by the assignment

$$\alpha\in \mathsf{H}_*(\overline{\mathsf{M}}_{0,n})\mapsto [(\gamma_1,\cdots,\gamma_{n-1})\mapsto (\mathsf{Ev}_n)_*(\mathsf{Stab}^*(\mathrm{PD}(\alpha))\cup\mathsf{Ev}_1^*(\gamma_1)\cup\cdots\cup\mathsf{Ev}^*_{n-1}(\gamma_{n-1}))]$$

Proposition 2.1.49 ([GK98; KM99]). *The family of moduli stacks* $\{\overline{\mathsf{M}}_{g,n}\}_{g\geq 0, n\geq 3}$ *forms a (modular)* operad in algebraic stacks. *The first* $n-1$ *marked points provide the inputs for operations and the last marked point provides the output. Associativity is established by the commutativity of the diagrams*

$$\begin{tikzcd} \overline{\mathsf{M}}_{g_1,n_1} \times \overline{\mathsf{M}}_{g_2,n_2} \times \overline{\mathsf{M}}_{g_3,n_3} \arrow[r, "m\times \mathrm{id}"] \arrow[d, "\mathrm{id}\times m"] & \overline{\mathsf{M}}_{g_1+g_2,n_1+n_2-2} \times \overline{\mathsf{M}}_{g_3,n_3} \arrow[d, "m"] \\ \overline{\mathsf{M}}_{g_1,n_1} \times \overline{\mathsf{M}}_{g_2+g_3,n_2+n_3-2} \arrow[r, "m"] & \overline{\mathsf{M}}_{g_1+g_2+g_3,n_1+n_2+n_3-4} \end{tikzcd}$$

By functoriality, the homology $\{\mathsf{H}_*(\overline{\mathsf{M}}_{g,n})\}_{g\geq 0, n\geq 3}$ *forms a (modular) operad in vector spaces.*

The splitting law (B) captures the compatibility of the Gromov-Witten classes along the gluing operation for curves:

Theorem 2.1.50 ([KM96; BM96]). *The collection of maps* (40) *exhibits* $\mathsf{H}^*(\mathbb{P}^2_{\mathbb{C}})$ *as an algebra over the* $\{d \in \mathbb{N}\}$*-graded operad* $\{\mathsf{H}_*(\overline{\mathsf{M}}_{0,n})\}_{n\geq 3}$. *More precisely, the maps* (40) *commute with*

$$\begin{tikzcd} \mathsf{H}_*(\overline{\mathsf{M}}_{0,n}) \otimes \mathsf{H}_*(\overline{\mathsf{M}}_{0,m}) \arrow[r] \arrow[d, "m"] & \mathrm{Hom}_{\mathsf{k}}(\mathsf{H}^*(\mathbb{P}^2_{\mathbb{C}})^{\otimes_{n-1}}, \mathsf{H}^*(\mathbb{P}^2_{\mathbb{C}})) \otimes \mathrm{Hom}_{\mathsf{k}}(\mathsf{H}^*(\mathbb{P}^2_{\mathbb{C}})^{\otimes_{m-1}}, \mathsf{H}^*(\mathbb{P}^2_{\mathbb{C}})) \arrow[d, "\circ_{n\to 1}"] \\ \mathsf{H}_*(\overline{\mathsf{M}}_{0,n+m-2}) \arrow[r] & \mathrm{Hom}_{\mathsf{k}}(\mathsf{H}^*(\mathbb{P}^2_{\mathbb{C}})^{\otimes_{n+m-2}}, \mathsf{H}^*(\mathbb{P}^2_{\mathbb{C}})) \end{tikzcd}$$

where $\circ_{n\to 1}$ *composes the output of the first* $\mathrm{Hom}_{\mathsf{k}}$ *with the first entry of the second* $\mathrm{Hom}_{\mathsf{k}}$.

Remark 2.1.51. In order to explain the proof of Theorem **2.1.50**, let us start by observing that because of Poincaré duality, and up to shifting the gradings, we can identify

$$\mathrm{Hom}_{\mathsf{k}}(\mathsf{H}^*(\mathbb{P}^2_{\mathbb{C}})^{\otimes_{n-1}}, \mathsf{H}^*(\mathbb{P}^2_{\mathbb{C}})) \simeq \mathsf{H}^*(\mathbb{P}^2_{\mathbb{C}})^{\otimes_{n-1}} \otimes \mathsf{H}^*(\mathbb{P}^2_{\mathbb{C}})$$

and under this identification, the composition map $\circ_{n\to 1}$ the tensor map with the evaluation for the duality data $e : \mathsf{H}^*(\mathbb{P}^2_{\mathbb{C}}) \otimes (\mathsf{H}^*(\mathbb{P}^2_{\mathbb{C}}))^{\vee} \to \mathsf{k}$

$$\begin{tikzcd} \mathrm{Hom}_{\mathsf{k}}(\mathsf{H}^*(\mathbb{P}^2_{\mathbb{C}})^{\otimes_{n-1}}, \mathsf{H}^*(\mathbb{P}^2_{\mathbb{C}})) \otimes \mathrm{Hom}_{\mathsf{k}}(\mathsf{H}^*(\mathbb{P}^2_{\mathbb{C}})^{\otimes_{m-1}}, \mathsf{H}^*(\mathbb{P}^2_{\mathbb{C}})) \arrow[r, "\sim"] \arrow[d, "\circ_{n\to 1}"] & \mathsf{H}^*(\mathbb{P}^2_{\mathbb{C}})^{\otimes_{n-1}} \otimes \mathsf{H}^*(\mathbb{P}^2_{\mathbb{C}}) \otimes (\mathsf{H}^*(\mathbb{P}^2_{\mathbb{C}}))^{\vee} \otimes \mathsf{H}^*(\mathbb{P}^2_{\mathbb{C}})^{\otimes_{m-2}} \otimes \mathsf{H}^*(\mathbb{P}^2_{\mathbb{C}}) \arrow[d, "\mathrm{id}\otimes e\otimes \mathrm{id}"] \\ \mathrm{Hom}_{\mathsf{k}}(\mathsf{H}^*(\mathbb{P}^2_{\mathbb{C}})^{\otimes_{n+m-2}}, \mathsf{H}^*(\mathbb{P}^2_{\mathbb{C}})) \arrow[r, "\sim"] & \mathsf{H}^*(\mathbb{P}^2_{\mathbb{C}})^{\otimes_{n+m-2}} \otimes \mathsf{H}^*(\mathbb{P}^2_{\mathbb{C}}) \end{tikzcd}$$

which under the Künneth isomorphism,

$$\mathsf{H}^*(\mathbb{P}^2_{\mathbb{C}}) \otimes \mathsf{H}^*(\mathbb{P}^2_{\mathbb{C}}) \simeq \mathsf{H}^*(\mathbb{P}^2_{\mathbb{C}} \times \mathbb{P}^2_{\mathbb{C}})$$

becomes the pullback-pushforward $q_*\Delta^*$ where $q : \mathbb{P}^2_{\mathbb{C}} \to \mathsf{Spec}(\mathsf{k})$ is the canonical projection and $\Delta : \mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}} \times \mathbb{P}^2_{\mathbb{C}}$ is the diagonal.

In terms of the Remark **2.1.51**, the content of the Theorem **2.1.50** can now interpreted via the commutativity of a certain diagram which we now present:

Construction 2.1.52. Let $A \cup B = \{1, \cdots n\}$ be a partition. Consider the commutative diagram

(41)

$$
\begin{tikzcd}
\overline{\mathsf{M}}_{0,\sharp A+1} \times \overline{\mathsf{M}}_{0,\sharp B+1} \arrow[d, equal] & \left(\coprod_{d_1 \geq 0} \overline{\mathsf{M}}_{0,\sharp A+1}(\mathbb{P}^2_{\mathbb{C}}, d_1)\right) \times \left(\coprod_{d_2 \geq 0} \overline{\mathsf{M}}_{0,\sharp B+1}(\mathbb{P}^2_{\mathbb{C}}, d_2)\right) \arrow[l, "\mathsf{Stab}_A \times \mathsf{Stab}_B"'] \arrow[r, "\mathsf{Ev}_A \times \mathsf{Ev}_B"] \arrow[dr, phantom, "\urcorner \nu"] & (\mathbb{P}^2_{\mathbb{C}})^{\sharp A+1} \times (\mathbb{P}^2_{\mathbb{C}})^{\sharp B+1} = (\mathbb{P}^2_{\mathbb{C}})^{\sharp A} \times \mathbb{P}^2_{\mathbb{C}} \times \mathbb{P}^2_{\mathbb{C}} \times (\mathbb{P}^2_{\mathbb{C}})^{\sharp B} \\
\overline{\mathsf{M}}_{0,\sharp A+1} \times \overline{\mathsf{M}}_{0,\sharp B+1} \arrow[d, equal] & \coprod_{d_1,d_2 \geq 0} \left(\overline{\mathsf{M}}_{0,\sharp A+1}(\mathbb{P}^2_{\mathbb{C}}, d_1) \underset{\mathbb{P}^2_{\mathbb{C}}}{\times} \overline{\mathsf{M}}_{0,\sharp B+1}(\mathbb{P}^2_{\mathbb{C}}, d_2)\right) \arrow[u] \arrow[r] \arrow[d, dashed, "\phi"] & (\mathbb{P}^2_{\mathbb{C}})^{\sharp A} \times \mathbb{P}^2_{\mathbb{C}} \times (\mathbb{P}^2_{\mathbb{C}})^{\sharp B} \arrow[u, "\mathsf{id}^{\sharp A} \times \Delta \times \mathsf{id}^{\sharp B}"'] \arrow[d, equal] \\
\overline{\mathsf{M}}_{0,\sharp A+1} \times \overline{\mathsf{M}}_{0,\sharp B+1} \arrow[d, "m"] \arrow[dr, phantom, "\llcorner \mu"] & \coprod_{d \geq 0} \left(\overline{\mathsf{M}}_{0,n}(\mathbb{P}^2_{\mathbb{C}}, d) \underset{\overline{\mathsf{M}}_{0,n}}{\times} (\overline{\mathsf{M}}_{0,\sharp A+1} \times \overline{\mathsf{M}}_{0,\sharp B+1})\right) \arrow[l] \arrow[d] & (\mathbb{P}^2_{\mathbb{C}})^{\sharp A} \times (\mathbb{P}^2_{\mathbb{C}}) \times (\mathbb{P}^2_{\mathbb{C}})^{\sharp B} \arrow[d, "\mathsf{id}^{\sharp A} \times * \times \mathsf{id}^{\sharp B}"] \\
\overline{\mathsf{M}}_{0,n} & \coprod_{d \geq 0} (\overline{\mathsf{M}}_{0,n}(\mathbb{P}^2_{\mathbb{C}}, d)) \arrow[l, "\mathsf{Stab}_n"'] \arrow[r, "\mathsf{Ev}"] & (\mathbb{P}^2_{\mathbb{C}})^{\sharp A} \times * \times (\mathbb{P}^2_{\mathbb{C}})^{\sharp B} \simeq (\mathbb{P}^2_{\mathbb{C}})^{\times n}
\end{tikzcd}
$$

where the squares μ and ν are cartesian and ϕ is the disjoint union of all the gluing maps with $d_1 + d_2 = d$

(42)

$$
\phi : \coprod_{d_1+d_2=d} \left(\overline{\mathsf{M}}_{0,\sharp A+1}(\mathbb{P}^2_{\mathbb{C}}, d_1) \underset{\mathbb{P}^2_{\mathbb{C}}}{\times} \overline{\mathsf{M}}_{0,\sharp B+1}(\mathbb{P}^2_{\mathbb{C}}, d_2) \right) \to \overline{\mathsf{M}}_{0,n}(\mathbb{P}^2_{\mathbb{C}}, d) \underset{\overline{\mathsf{M}}_{0,n}}{\times} (\overline{\mathsf{M}}_{0,\sharp A+1} \times \overline{\mathsf{M}}_{0,\sharp B+1})
$$

The content of the Theorem **2.1.50** can now be reformulated using the maps in the (38), as follows:

Lemma 2.1.53 (Splitting)**.** *Let $\gamma_1, \cdots, \gamma_n$ be cohomology classes in $\mathsf{H}^*(\mathbb{P}^2_{\mathbb{C}})$ and let $A \cup B = \{1, \cdots, n\}$ be a partition. Then the maps in (38) satisfy*

$$
m^*(I^d_{0,n}(\gamma_1, \cdots, \gamma_n)) = \sum_{\substack{d_1+d_2=d \\ i+j=2}} I^{d_1}_{0,\sharp A+1}((\gamma_a)_{a \in A}, h^i) \times I^{d_1}_{0,\sharp B+1}((\gamma_b)_{b \in B}, h^j)
$$

*in $\mathsf{H}^*(\overline{\mathsf{M}}_{0,\sharp A+1} \times \overline{\mathsf{M}}_{0,\sharp B+1})$. Here, h is the hyperplane class (see Reminder **2.1.18**).*

Proof. The proof uses base-change formulas along the diagram in the Construction **2.1.52**, using the Künneth decomposition for the class of the diagonal Δ in $\mathsf{H}^*(\mathbb{P}^2_{\mathbb{C}} \times \mathbb{P}^2_{\mathbb{C}})$ given by

$$
[\Delta] = \sum_{i+j=2} h^i \times h^j
$$

The final key step is the fact that the inclusion

$$
\overline{\mathsf{M}}_{0,n}(\mathbb{P}^2_{\mathbb{C}}, d) \underset{\overline{\mathsf{M}}_{0,n}}{\times} (\overline{\mathsf{M}}_{0,\sharp A+1} \times \overline{\mathsf{M}}_{0,\sharp B+1}) \hookrightarrow \overline{\mathsf{M}}_{0,n}(\mathbb{P}^2_{\mathbb{C}}, d)
$$

is a divisor with normal crossings and each morphism

$$\overline{\mathsf{M}}_{0,\sharp A+1}(\mathbb{P}^2_{\mathbb{C}}, d_1) \underset{\mathbb{P}^2_{\mathbb{C}}}{\times} \overline{\mathsf{M}}_{0,\sharp B+1}(\mathbb{P}^2_{\mathbb{C}}, d_2) \to \overline{\mathsf{M}}_{0,n}(\mathbb{P}^2_{\mathbb{C}}, d) \underset{\overline{\mathsf{M}}_{0,n}}{\times} (\overline{\mathsf{M}}_{0,\sharp A+1} \times \overline{\mathsf{M}}_{0,\sharp B+1})$$

in the disjoint union ϕ in (42), provides the first open strata. Therefore, ϕ is a finite, unramified, surjective birational morphism (see [Lee04:Proof of Prop. 11]). For more details see [FP96:Lemma 16] or [KV07: 4.3.1, 4.3.2]. See also the more recent work [PY20a:Lemma 5.13 and Lemma 5.20] and [PY22:Prop. 7.6]. □

Terminology 2.1.54. In the literature, an algebra over the operad $\{\mathsf{H}_*(\overline{\mathsf{M}}_{g,n})\}_{g\geq 0, n\geq 3}$. is known as a *cohomological field theory.*

2.1.3.4. ***Formula for counting curves of degree 2 in*** $\mathbb{P}^2_{\mathbb{C}}$. We have the following spectacular result of Kontsevich-Manin:

Theorem 2.1.55. *[KM94: 5.2.1] The numbers N_d of rational curves of degree d passing by $3d-1$ general points in the $\mathbb{P}^2_{\mathbb{C}}$ satisfy a recursive formula:*

$$N_d = \sum_{\substack{d_A+d_B=d \\ d_A\geq 1;\, d_B\geq 1}} N_{d_A}.N_{d_B}.d_A^2.d_B\left(d_B\binom{3d-4}{3d_A-2} - d_A\binom{3d-4}{3d_A-1}\right) \tag{43}$$

(*)

This theorem has originally been proved in [KM94: 5.2.1] using the associativity of quantum product via the the WDVV equations. Instead of giving a proof, in this section we reproduce the rather tedious but formative pedagogical exercise of computing the number of rational curves of degree 2 passing by 5 points in $\mathbb{P}^2_{\mathbb{C}}$,

$$N_2 = I^2_{0,5}(h^2, h^2, h^2, h^2, h^2)$$

directly from some basic axioms of GW-theory, namely, the combination of the results in Section **2.1.3.2** and Section **2.1.3.3**, together with the following properties:

Proposition 2.1.56. *In $\mathbb{P}^2_{\mathbb{C}}$, we have:*

(i) Mapping to a point: If $d=0$, we have

$$I^0_{0,n}(\gamma_1,\cdots,\gamma_n) = \begin{cases} I^0_{0,3}(\gamma_1,\gamma_2,\gamma_3) = \int(\gamma_1\cup\gamma_2\cup\gamma_3)\cap[\mathbb{P}^2_{\mathbb{C}}] & \text{if } n=3 \text{ and } \sum_{i=1}^3 \mathrm{codim}\gamma_i = 2 \\ 0 & \text{otherwise} \end{cases}$$

(ii) Two point invariants: If $n\leq 2$, the only non-zero Gromov-Witten numbers are $I^1_{0,1}(h^2,h^2)=1$, expressing the fact there is a unique line passing through two distinct points.

(*)The formula contains a link for the computation of the first numbers.

(iii) All GW-numbers containing a copy of the fundamental class $h^0 = 1$ are zero except the case $I^0_{0,3}(\gamma_1, \gamma_2, 1)$ whose value is given by $\int(\gamma_1 \cup \gamma_2 \cup 1) \cap [\mathbb{P}^2_{\mathbb{C}}]$.

(iv) Divisor equation: if $d \geq 1$ and one of the classes, say γ_n, is the hyperplane class h, then

$$I^d_{0,n}(\gamma_1, \cdots, \gamma_{n-1}, \gamma_n = h) = I^2_{0,n-1}(\gamma_1, \cdots, \gamma_{n-1}).d$$

PROOF. See [KV07: 4.2.1, 4.2.2, 4.2.3, 4.2.4]. □

Let us now use these properties to exemplify a direct computation of N_2.

Example 2.1.57. Consider the moduli of stable maps of degree 2 with $3d = 3.2 = 6$-marked points $\overline{\mathsf{M}}_{0,6}(\mathbb{P}^2_{\mathbb{C}}, 2)$ which we index as $\{q_1, q_2, p_1, p_2, p_3, p_4\}$. Let us consider four points P_1, P_2, P_3, P_4 and L_1, L_2 two lines, all in generic position in $\mathbb{P}^2_{\mathbb{C}}$. Let us name $\gamma_1, \gamma_2, \gamma_3, \gamma_4 = h^2$ and $\ell_1, \ell_2 = h$ their respective Poincaré dual classes. We consider the fiber product

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & \overline{\mathsf{M}}_{0,6}(\mathbb{P}^2_{\mathbb{C}}, 2) \arrow[d, "\mathsf{Ev}_1, \mathsf{Ev}_2, \mathsf{Ev}_3, \mathsf{Ev}_4, \mathsf{Ev}_5, \mathsf{Ev}_6"] \\ L_1 \times L_2 \times \{P_1\} \times \{P_2\} \times \{P_3\} \times \{P_4\} \arrow[r, "i"] & (\mathbb{P}^2_{\mathbb{C}})^{\times 6} \end{tikzcd}$$

ie, the collection of stable maps $f : (C, q_1, q_1, p_1, p_2, p_3, p_4) \to \mathbb{P}^2_{\mathbb{C}}$ satisfying $f(q_1) \in L_1$, $f(q_2) \in L_2$ and $f(p_i) = P_i$. By Proposition **2.1.19**, $\overline{\mathsf{M}}_{0,6}(\mathbb{P}^2_{\mathbb{C}}, 2)$ is of dimension 11. The space $(\mathbb{P}^2_{\mathbb{C}})^{\times 6}$ is of dimension 12 and therefore the embedding i is of codimension 10. As in the Construction **2.1.21**, as a consequence of flatness of the evaluation maps, Z is also of codimension 10, ie, Z is a curve in $\overline{\mathsf{M}}_{0,6}(\mathbb{P}^2_{\mathbb{C}}, 2)$. By construction its Poincaré dual is the class

$$\mathrm{PD}(Z) = \mathsf{Ev}_1^*(\ell_1) \cup \mathsf{Ev}_2^*(\ell_2) \cup \mathsf{Ev}_3^*(\gamma_1) \cup \mathsf{Ev}_4^*(\gamma_2) \cup \mathsf{Ev}_5^*(\gamma_3) \cup \mathsf{Ev}_6^*(\gamma_4)$$

We obtain a formula for computing N_2 by intersecting Z with the three boundary divisors of the Corollary **2.1.46**: setting $i = q_1$, $j = q_2$, $k = p_1$ and $l = p_2$, we have the divisor relations

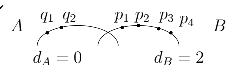
$$\int \mathrm{PD}(Z) \cap\ \eta^*_{i,j,k,l}(\mathsf{D}(ij|kl)) = \int \mathrm{PD}(Z) \cap\ \eta^*_{i,j,k,l}(\mathsf{D}(ik|jl)) = \int \mathrm{PD}(Z) \cap\ \eta^*_{i,j,k,l}(\mathsf{D}(jk|il))$$

ie,

$$\underbrace{\sum_{\substack{A \cup B = \{1, \cdots, n\} \\ i,j \in A, \\ k,l \in B \\ d_A + d_B = d}} \int \mathrm{PD}(Z) \cap \mathsf{D}(A|B; d_1, d_2)}_{\mathrm{I}} = \underbrace{\sum_{\substack{A \cup B = \{1, \cdots, n\} \\ i,k \in A, \\ j,l \in B \\ d_A + d_B = d}} \int \mathrm{PD}(Z) \cap \mathsf{D}(A|B; d_1, d_2)}_{\mathrm{II}} =$$

$$= \underbrace{\sum_{\substack{A\cup B=\{1,\cdots,n\}\\ j,k\in A,\\ i,l\in B\\ d_A+d_B=d}} \int \mathrm{PD}(Z)\cap \mathsf{D}(A|B;d_1,d_2)}_{\mathrm{III}}$$

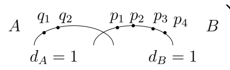
Now we use the splitting principle Lemma **2.1.53** to compute each of the classes $\int \mathrm{PD}(Z)\cap \mathsf{D}(A|B;d_1,d_2)$. There are 12 cases to compute in I:

$$\int \mathrm{PD}(Z)\cap \mathsf{D}\Bigg($$


$$\Bigg) =$$

$$= I^0_{0,3}(\ell_1,\ell_2,h^0)I^2_{0,5}(\gamma_1,\gamma_2,\gamma_3,\gamma_4,h^2) + I^0_{0,3}(\ell_1,\ell_2,h^1)I^2_{0,5}(\gamma_1,\gamma_2,\gamma_3,\gamma_4,h^1) + I^0_{0,3}(\ell_1,\ell_2,h^2)I^2_{0,5}(\gamma_1,\gamma_2,\gamma_3,\gamma_4,h^0)$$

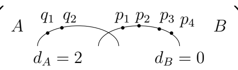
$$= \underbrace{I^0_{0,3}(h,h,1)}_{=1(\text{ by Proposition 2.1.56-(i) and (iii)})} * \underbrace{I^2_{0,5}(h^2,h^2,h^2,h^2,h^2)}_{N_2} + \underbrace{I^0_{0,3}(h,h,h)}_{=0(\text{ by Proposition 2.1.56-(i)})} * I^2_{0,5}(h^2,h^2,h^2,h^2,h) + \underbrace{I^0_{0,3}(h,h,h^2)}_{=0(\text{ by Proposition 2.1.56-(i)})} * I^2_{0,5}(h^2,h^2,h^2,h^2,1) = N_2$$

$$\int \mathrm{PD}(Z)\cap \mathsf{D}\Bigg($$


$$\Bigg) =$$

$$= I^1_{0,3}(\ell_1,\ell_2,h^0)I^1_{0,5}(\gamma_1,\gamma_2,\gamma_3,\gamma_4,h^2) + I^1_{0,3}(\ell_1,\ell_2,h^1)I^1_{0,5}(\gamma_1,\gamma_2,\gamma_3,\gamma_4,h^1) + I^1_{0,3}(\ell_1,\ell_2,h^2)I^1_{0,5}(\gamma_1,\gamma_2,\gamma_3,\gamma_4,h^0)$$

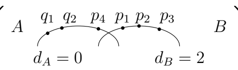
$$= \underbrace{I^1_{0,3}(h,h,1)}_{=0(\text{ by Proposition 2.1.56-(iii) })} * I^1_{0,5}(h^2,h^2,h^2,h^2,h^2) + \underbrace{I^1_{0,3}(h,h,h)}_{=0(\text{ by Proposition 2.1.56-(iv) and (ii)})} * I^1_{0,5}(h^2,h^2,h^2,h^2,h) + I^1_{0,3}(h,h,h^2) * \underbrace{I^1_{0,5}(h^2,h^2,h^2,h^2,1)}_{=0(\text{ by Proposition 2.1.56-(iii)})} = 0$$

$$\int \mathrm{PD}(Z)\cap \mathsf{D}\Bigg($$


$$\Bigg) =$$

$$= I^2_{0,3}(\ell_1,\ell_2,h^0)I^0_{0,5}(\gamma_1,\gamma_2,\gamma_3,\gamma_4,h^2) + I^2_{0,3}(\ell_1,\ell_2,h^1)I^0_{0,5}(\gamma_1,\gamma_2,\gamma_3,\gamma_4,h^1) + I^2_{0,3}(\ell_1,\ell_2,h^2)I^0_{0,5}(\gamma_1,\gamma_2,\gamma_3,\gamma_4,h^0)$$

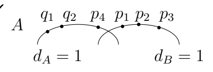
$$= I^2_{0,3}(h,h,1) \underbrace{* I^0_{0,5}(h^2,h^2,h^2,h^2,h^2)}_{=0(\text{ by Proposition 2.1.56-(i) })} + I^2_{0,3}(h,h,h) \underbrace{* I^0_{0,5}(h^2,h^2,h^2,h^2,h)}_{=0(\text{ by Proposition 2.1.56-(i)})} + I^2_{0,3}(h,h,h^2) * \underbrace{I^0_{0,5}(h^2,h^2,h^2,h^2,1)}_{=0(\text{ by Proposition 2.1.56-(i)})} = 0$$

$$\int \mathrm{PD}(Z)\cap \mathsf{D}\Bigg($$


$$\Bigg) =$$

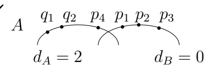
$$= I^0_{0,4}(\ell_1,\ell_2,\gamma_4,h^0)I^2_{0,4}(\gamma_1,\gamma_2,\gamma_3,h^2) + I^0_{0,4}(\ell_1,\ell_2,\gamma_4,h^1)I^2_{0,4}(\gamma_1,\gamma_2,\gamma_3,h^1) + I^0_{0,4}(\ell_1,\ell_2,\gamma_4,h^2)I^2_{0,4}(\gamma_1,\gamma_2,\gamma_3,h^0)$$

$$= \underbrace{I^0_{0,4}(h,h,h^2,1)}_{=0(\text{ by Proposition 2.1.56-(i) })} * I^2_{0,4}(h^2,h^2,h^2,h^2) + \underbrace{I^0_{0,4}(h,h,h^2,h)}_{=0(\text{ by Proposition 2.1.56-(i)})} * I^2_{0,4}(h^2,h^2,h^2,h) + \underbrace{I^0_{0,4}(h,h,h^2,h^2)}_{=0(\text{ by Proposition 2.1.56-(i)})} * I^2_{0,4}(h^2,h^2,h^2,1) = 0$$

$\int \mathrm{PD}(Z) \cap \mathsf{D}\Bigg($  $\Bigg) =$

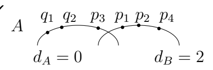
$$= I^1_{0,4}(\ell_1,\ell_2,\gamma_4,h^0)I^1_{0,4}(\gamma_1,\gamma_2,\gamma_3,h^2) + I^1_{0,4}(\ell_1,\ell_2,\gamma_4,h^1)I^1_{0,4}(\gamma_1,\gamma_2,\gamma_3,h^1) + I^1_{0,4}(\ell_1,\ell_2,\gamma_4,h^2)I^1_{0,4}(\gamma_1,\gamma_2,\gamma_3,h^0)$$

$$= \underbrace{I^1_{0,4}(h,h,h^2,1)}_{=0(\text{ by Proposition } \mathbf{2.1.56}\text{-(iii) })} * I^1_{0,4}(h^2,h^2,h^2,h^2) + \underbrace{I^1_{0,4}(h,h,h^2,h)}_{=0(\text{ by Proposition } \mathbf{2.1.56}\text{-(iii) and (ii)})} * I^1_{0,4}(h^2,h^2,h^2,h) + I^1_{0,4}(h,h,h^2,h^2) * \underbrace{I^1_{0,4}(h^2,h^2,h^2,1)}_{=0(\text{ by Proposition } \mathbf{2.1.56}\text{-(iii)})} = 0$$

$\int \mathrm{PD}(Z) \cap \mathsf{D}\Bigg($  $\Bigg) =$

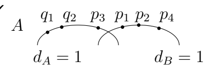
$$= I^2_{0,4}(\ell_1,\ell_2,\gamma_4,h^0)I^0_{0,4}(\gamma_1,\gamma_2,\gamma_3,h^2) + I^2_{0,4}(\ell_1,\ell_2,\gamma_4,h^1)I^0_{0,4}(\gamma_1,\gamma_2,\gamma_3,h^1) + I^2_{0,4}(\ell_1,\ell_2,\gamma_4,h^2)I^0_{0,4}(\gamma_1,\gamma_2,\gamma_3,h^0)$$

$$= I^2_{0,4}(h,h,h^2,1) * \underbrace{I^0_{0,4}(h^2,h^2,h^2,h^2)}_{=0(\text{ by Proposition } \mathbf{2.1.56}\text{-(i) })} + I^2_{0,4}(h,h,h^2,h) * \underbrace{I^0_{0,4}(h^2,h^2,h^2,h)}_{=0(\text{ by Proposition } \mathbf{2.1.56}\text{-(i)})} + I^2_{0,4}(h,h,h^2,h^2) * \underbrace{I^0_{0,4}(h^2,h^2,h^2,1)}_{=0(\text{ by Proposition } \mathbf{2.1.56}\text{-(i)})} = 0$$

$\int \mathrm{PD}(Z) \cap \mathsf{D}\Bigg($  $\Bigg) =$

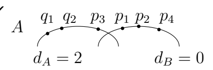
$$= I^0_{0,4}(\ell_1,\ell_2,\gamma_3,h^0)I^2_{0,4}(\gamma_1,\gamma_2,\gamma_4,h^2) + I^0_{0,4}(\ell_1,\ell_2,\gamma_3,h^1)I^2_{0,4}(\gamma_1,\gamma_2,\gamma_4,h^1) + I^0_{0,4}(\ell_1,\ell_2,\gamma_3,h^2)I^2_{0,4}(\gamma_1,\gamma_2,\gamma_4,h^0)$$

$$= \underbrace{I^0_{0,4}(h,h,h^2,1)}_{=0(\text{ by Proposition } \mathbf{2.1.56}\text{-(i) })} * I^2_{0,4}(h^2,h^2,h^2,h^2) + \underbrace{I^0_{0,4}(h,h,h^2,h)}_{=0(\text{ by Proposition } \mathbf{2.1.56}\text{-(i)})} * I^2_{0,4}(h^2,h^2,h^2,h) + \underbrace{I^0_{0,4}(h,h,h^2,h^2)}_{=0(\text{ by Proposition } \mathbf{2.1.56}\text{-(i)})} * I^2_{0,4}(h^2,h^2,h^2,1) = 0$$

$\int \mathrm{PD}(Z) \cap \mathsf{D}\Bigg($  $\Bigg) =$

$$= I^1_{0,4}(\ell_1,\ell_2,\gamma_3,h^0)I^1_{0,4}(\gamma_1,\gamma_2,\gamma_4,h^2) + I^1_{0,4}(\ell_1,\ell_2,\gamma_3,h^1)I^1_{0,4}(\gamma_1,\gamma_2,\gamma_4,h^1) + I^1_{0,4}(\ell_1,\ell_2,\gamma_3,h^2)I^1_{0,4}(\gamma_1,\gamma_2,\gamma_4,h^0)$$

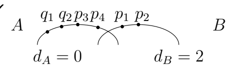
$$= \underbrace{I^1_{0,4}(h,h,h^2,1)}_{=0(\text{ by Proposition } \mathbf{2.1.56}\text{-(iii) })} * I^1_{0,4}(h^2,h^2,h^2,h^2) + \underbrace{I^1_{0,4}(h,h,h^2,h)}_{=0(\text{ by Proposition } \mathbf{2.1.56}\text{-(iii) and (ii)})} * I^1_{0,4}(h^2,h^2,h^2,h) + I^1_{0,4}(h,h,h^2,h^2) * \underbrace{I^1_{0,4}(h^2,h^2,h^2,1)}_{=0(\text{ by Proposition } \mathbf{2.1.56}\text{-(iii)})} = 0$$

$\int \mathrm{PD}(Z) \cap \mathsf{D}\Bigg($  $\Bigg) =$

$$= I^2_{0,4}(\ell_1,\ell_2,\gamma_3,h^0)I^0_{0,4}(\gamma_1,\gamma_2,\gamma_4,h^2) + I^2_{0,4}(\ell_1,\ell_2,\gamma_3,h^1)I^0_{0,4}(\gamma_1,\gamma_2,\gamma_4,h^1) + I^2_{0,4}(\ell_1,\ell_2,\gamma_3,h^2)I^0_{0,4}(\gamma_1,\gamma_2,\gamma_4,h^0)$$

$$= I^2_{0,4}(h,h,h^2,1) * \underbrace{I^0_{0,4}(h^2,h^2,h^2,h^2)}_{=0(\text{ by Proposition } \mathbf{2.1.56}\text{-(i) })} + I^2_{0,4}(h,h,h^2,h) * \underbrace{I^0_{0,4}(h^2,h^2,h^2,h)}_{=0(\text{ by Proposition } \mathbf{2.1.56}\text{-(i)})} + I^2_{0,4}(h,h,h^2,h^2) * \underbrace{I^0_{0,4}(h^2,h^2,h^2,1)}_{=0(\text{ by Proposition } \mathbf{2.1.56}\text{-(i)})} = 0$$

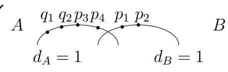
$$\int \mathrm{PD}(Z) \cap \mathsf{D}\left(\quad \right) =$$



$$= I^0_{0,5}(\ell_1,\ell_2,\gamma_3,\gamma_4,h^0)I^2_{0,3}(\gamma_1,\gamma_2,h^2) + I^0_{0,5}(\ell_1,\ell_2,\gamma_3,\gamma_4,h^1)I^2_{0,3}(\gamma_1,\gamma_2,h^1) + I^0_{0,5}(\ell_1,\ell_2,\gamma_3,\gamma_4,h^2)I^2_{0,3}(\gamma_1,\gamma_2,h^0)$$

$$= I^2_{0,5}(h,h,h^2,h^2,1) * \underbrace{I^0_{0,3}(h^2,h^2,h^2)}_{=0(\text{ by Proposition } \mathbf{2.1.56}\text{-(i) })} + I^2_{0,5}(h,h,h^2,h^2,h) * \underbrace{I^0_{0,3}(h^2,h^2,h)}_{=0(\text{ by Proposition } \mathbf{2.1.56}\text{-(i)})} + I^2_{0,5}(h,h,h^2,h^2,h^2) * \underbrace{I^0_{0,3}(h^2,h^2,1)}_{=0(\text{ by Proposition } \mathbf{2.1.56}\text{-(i)})} = 0$$

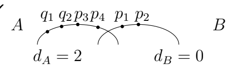
$$\int \mathrm{PD}(Z) \cap \mathsf{D}\left(\quad \right) =$$



$$= I^1_{0,5}(\ell_1,\ell_2,\gamma_3,\gamma_4,h^0)I^1_{0,3}(\gamma_1,\gamma_2,h^2) + I^1_{0,5}(\ell_1,\ell_2,\gamma_3,\gamma_4,h^1)I^1_{0,3}(\gamma_1,\gamma_2,h^1) + I^1_{0,5}(\ell_1,\ell_2,\gamma_3,\gamma_4,h^2)I^1_{0,3}(\gamma_1,\gamma_2,h^0)$$

$$= \underbrace{I^1_{0,5}(h,h,h^2,h^2,1)}_{=0(\text{ by Proposition } \mathbf{2.1.56}\text{-(iii) })} * I^1_{0,3}(h^2,h^2,h^2) + \underbrace{I^1_{0,5}(h,h,h^2,h^2,h)}_{=1(\text{ by Proposition } \mathbf{2.1.56}\text{-(iii) and (ii)})} * \underbrace{I^1_{0,3}(h^2,h^2,h)}_{=1(\text{ by Proposition } \mathbf{2.1.56}\text{-(iii) and (ii)})} + \underbrace{I^1_{0,5}(h,h,h^2,h^2,h^2)}_{=0(\text{ by Proposition } \mathbf{2.1.56}\text{-(iv) and (i)})} * I^1_{0,3}(h^2,h^2,1) = 1$$

$$\int \mathrm{PD}(Z) \cap \mathsf{D}\left(\quad \right) =$$



$$= I^2_{0,5}(\ell_1,\ell_2,\gamma_3,\gamma_4,h^0)I^0_{0,3}(\gamma_1,\gamma_2,h^2) + I^2_{0,5}(\ell_1,\ell_2,\gamma_3,\gamma_4,h^1)I^0_{0,3}(\gamma_1,\gamma_2,h^1) + I^2_{0,5}(\ell_1,\ell_2,\gamma_3,\gamma_4,h^2)I^0_{0,3}(\gamma_1,\gamma_2,h^0)$$

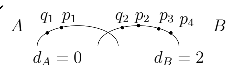
$$= I^2_{0,5}(h,h,h^2,h^2,1) * \underbrace{I^0_{0,3}(h^2,h^2,h^2)}_{=0(\text{ by Proposition } \mathbf{2.1.56}\text{-(i) })} + I^2_{0,5}(h,h,h^2,h^2,h) * \underbrace{I^0_{0,3}(h^2,h^2,h)}_{=0(\text{ by Proposition } \mathbf{2.1.56}\text{-(iv) and (i)})} + I^2_{0,5}(h,h,h^2,h^2,h^2) * \underbrace{I^0_{0,3}(h^2,h^2,1)}_{=0(\text{ by Proposition } \mathbf{2.1.56}\text{-(i)})} = 0$$

In conclusion, we obtain

$$\text{side I } = N_2 + 1$$

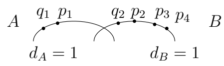
We now compute the 12 cases of side II:

$$\int \mathrm{PD}(Z) \cap \mathsf{D}\left(\quad \right) =$$



$$= I^0_{0,3}(\ell_1,\gamma_1,h^0)I^2_{0,5}(\ell_2,\gamma_2,\gamma_3,\gamma_4,h^2) + I^0_{0,3}(\ell_1,\gamma_1,h^1)I^2_{0,5}(\ell_2,\gamma_2,\gamma_3,\gamma_4,h^1) + I^0_{0,3}(\ell_1,\gamma_1,h^2)I^2_{0,5}(\ell_2,\gamma_2,\gamma_3,\gamma_4,h^0)$$

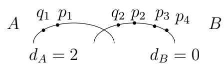
$$= \underbrace{I^0_{0,3}(h,h^2,1)}_{=1(\text{ by Proposition } \mathbf{2.1.56}\text{-(i) and (iii)})} * I^2_{0,5}(h,h^2,h^2,h^2,h^2) + \underbrace{I^0_{0,3}(h,h^2,h)}_{=0(\text{ by Proposition } \mathbf{2.1.56}\text{-(i)})} * I^2_{0,5}(h,h^2,h^2,h^2,h) + \underbrace{I^0_{0,3}(h,h^2,h^2)}_{=0(\text{ by Proposition } \mathbf{2.1.56}\text{-(i)})} * I^2_{0,5}(hitsrniningal,h^2,h^2,h^2,1) = 0$$

$$\int \mathrm{PD}(Z)\cap \mathsf{D}\Bigg($$


$$\Bigg) =$$

$$= I^1_{0,3}(\ell_1,\gamma_1,h^0)I^1_{0,5}(\ell_2,\gamma_2,\gamma_3,\gamma_4,h^2) + I^1_{0,3}(\ell_1,\gamma_1,h^1)I^1_{0,5}(\ell_2,\gamma_2,\gamma_3,\gamma_4,h^1) + I^1_{0,3}(\ell_1,\gamma_1,h^2)I^1_{0,5}(\ell_2,\gamma_2,\gamma_3,\gamma_4,h^0)$$

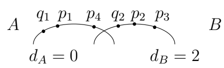
$$= \underbrace{I^1_{0,3}(h,h^2,1)}_{=0(\text{ by Proposition } \mathbf{2.1.56}\text{-(iii) })} *I^1_{0,5}(h,h^2,h^2,h^2,h^2) + \underbrace{I^1_{0,3}(h,h^2,h)}_{=0(\text{ by Proposition } \mathbf{2.1.56}\text{-(iii) and (ii)})} *I^1_{0,5}(h,h^2,h^2,h^2,h) + \underbrace{I^1_{0,3}(h,h^2,h^2)}_{=0(\text{ by Proposition } \mathbf{2.1.56}\text{-(iii)})} *I^1_{0,5}(h,h^2,h^2,h^2,1) = 0$$

$$\int \mathrm{PD}(Z)\cap \mathsf{D}\Bigg($$


$$\Bigg) =$$

$$= I^2_{0,3}(\ell_1,\gamma_1,h^0)I^0_{0,5}(\ell_2,\gamma_2,\gamma_3,\gamma_4,h^2) + I^2_{0,3}(\ell_1,\gamma_1,h^1)I^0_{0,5}(\ell_2,\gamma_2,\gamma_3,\gamma_4,h^1) + I^2_{0,3}(\ell_1,\gamma_1,h^2)I^0_{0,5}(\ell_2,\gamma_2,\gamma_3,\gamma_4,h^0)$$

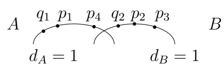
$$= I^2_{0,3}(h,h^2,1) \underbrace{*I^0_{0,5}(h,h^2,h^2,h^2,h^2)}_{=0(\text{ by Proposition } \mathbf{2.1.56}\text{-(i) })} + I^2_{0,3}(h,h^2,h) \underbrace{*I^0_{0,5}(h,h^2,h^2,h^2,h)}_{=0(\text{ by Proposition } \mathbf{2.1.56}\text{-(i)})} + I^2_{0,3}(h,h^2,h^2) * \underbrace{I^0_{0,5}(h,h^2,h^2,h^2,1)}_{=0(\text{ by Proposition } \mathbf{2.1.56}\text{-(iii)})} = 0$$

$$\int \mathrm{PD}(Z)\cap \mathsf{D}\Bigg($$


$$\Bigg) =$$

$$= I^0_{0,4}(\ell_1,\gamma_1,\gamma_4,h^0)I^2_{0,4}(\ell_2,\gamma_2,\gamma_3,h^2) + I^0_{0,4}(\ell_1,\gamma_1,\gamma_4,h^1)I^2_{0,4}(\ell_2,\gamma_2,\gamma_3,h^1) + I^0_{0,4}(\ell_1,\gamma_1,\gamma_4,h^2)I^2_{0,4}(\ell_2,\gamma_2,\gamma_3,h^0)$$

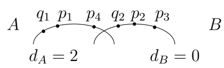
$$= \underbrace{I^0_{0,4}(h,h^2,h^2,1)}_{=0(\text{ by Proposition } \mathbf{2.1.56}\text{-(i) })} *I^2_{0,4}(h,h^2,h^2,h^2) + \underbrace{I^0_{0,4}(h,h^2,h^2,h)}_{=0(\text{ by Proposition } \mathbf{2.1.56}\text{-(i)})} *I^2_{0,4}(h,h^2,h^2,h) + \underbrace{I^0_{0,4}(h,h^2,h^2,h^2)}_{=0(\text{ by Proposition } \mathbf{2.1.56}\text{-(i)})} *I^2_{0,4}(h,h^2,h^2,1) = 0$$

$$\int \mathrm{PD}(Z)\cap \mathsf{D}\Bigg($$


$$\Bigg) =$$

$$= I^1_{0,4}(\ell_1,\gamma_1,\gamma_4,h^0)I^1_{0,4}(\ell_2,\gamma_2,\gamma_3,h^2) + I^1_{0,4}(\ell_1,\gamma_1,\gamma_4,h^1)I^1_{0,4}(\ell_2,\gamma_2,\gamma_3,h^1) + I^1_{0,4}(\ell_1,\gamma_1,\gamma_4,h^2)I^1_{0,4}(\ell_2,\gamma_2,\gamma_3,h^0)$$

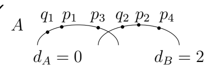
$$= \underbrace{I^1_{0,4}(h,h^2,h^2,1)}_{=0(\text{ by Proposition } \mathbf{2.1.56}\text{-(iii) })} *I^1_{0,4}(h,h^2,h^2,h^2) + \underbrace{I^1_{0,4}(h,h^2,h^2,h)}_{=1(\text{ by Proposition } \mathbf{2.1.56}\text{-(iv) and (ii)})} * \underbrace{I^1_{0,4}(h,h^2,h^2,h)}_{=1(\text{ by Proposition } \mathbf{2.1.56}\text{-(iv) and (ii)})} +I^1_{0,4}(h,h^2,h^2,h^2) * \underbrace{I^1_{0,4}(h,h^2,h^2,1)}_{=0(\text{ by Proposition } \mathbf{2.1.56}\text{-(iii)})} = 1$$

$$\int \mathrm{PD}(Z)\cap \mathsf{D}\Bigg($$


$$\Bigg) =$$

$$= I^2_{0,4}(\ell_1,\gamma_1,\gamma_4,h^0)I^0_{0,4}(\ell_2,\gamma_2,\gamma_3,h^2) + I^2_{0,4}(\ell_1,\gamma_1,\gamma_4,h^1)I^0_{0,4}(\ell_2,\gamma_2,\gamma_3,h^1) + I^2_{0,4}(\ell_1,\gamma_1,\gamma_4,h^2)I^0_{0,4}(\ell_2,\gamma_2,\gamma_3,h^0)$$

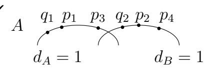
$$= \underbrace{I^2_{0,4}(h,h^2,h^2,1)}_{=0(\text{ by Proposition } \mathbf{2.1.56}\text{-(iii) })} ast I^0_{0,4}(h,h^2,h^2,h^2), \ +I^2_{0,4}(h,h^2,h^2,h) * \underbrace{I^0_{0,4}(h,h^2,h^2,h)}_{=0(\text{ by Proposition } \mathbf{2.1.56}\text{-(iv) and (i)})} +I^2_{0,4}(h,h^2,h^2,h^2) * \underbrace{I^0_{0,4}(h,h^2,h^2,1)}_{=0(\text{ by Proposition } \mathbf{2.1.56}\text{-(iii)})} = 0$$

$$\int \mathrm{PD}(Z)\cap \mathsf{D}\Bigg(\quad$$ 

$$\Bigg) =$$

$$= I^0_{0,4}(\ell_1,\gamma_1,\gamma_3,h^0)I^2_{0,4}(\ell_2,\gamma_2,\gamma_4,h^2) + I^0_{0,4}(\ell_1,\gamma_1,\gamma_3,h^1)I^2_{0,4}(\ell_2,\gamma_2,\gamma_4,h^1) + I^0_{0,4}(\ell_1,\gamma_1,\gamma_3,h^2)I^2_{0,4}(\ell_2,\gamma_2,\gamma_4,h^0)$$

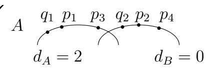
$$= \underbrace{I^0_{0,4}(h,h^2,h^2,1)}_{=0(\text{ by Proposition 2.1.56-(i) })} *I^2_{0,4}(h,h^2,h^2,h^2) + \underbrace{I^0_{0,4}(h,h^2,h^2,h)}_{=0(\text{ by Proposition 2.1.56-(i)})} *I^2_{0,4}(h,h^2,h^2,h) + \underbrace{I^0_{0,4}(h,h^2,h^2,h^2)}_{=0(\text{ by Proposition 2.1.56-(i)})} *I^2_{0,4}(h,h^2,h^2,1) = 0$$

$$\int \mathrm{PD}(Z)\cap \mathsf{D}\Bigg(\quad$$ 

$$\Bigg) =$$

$$= I^1_{0,4}(\ell_1,\gamma_1,\gamma_3,h^0)I^1_{0,4}(\ell_2,\gamma_2,\gamma_4,h^2) + I^1_{0,4}(\ell_1,\gamma_1,\gamma_3,h^1)I^1_{0,4}(\ell_2,\gamma_2,\gamma_4,h^1) + I^1_{0,4}(\ell_1,\gamma_1,\gamma_3,h^2)I^1_{0,4}(\ell_2,\gamma_2,\gamma_4,h^0)$$

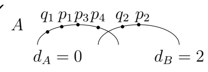
$$= \underbrace{I^1_{0,4}(h,h^2,h^2,1)}_{=0(\text{ by Proposition 2.1.56-(iii) })} *I^1_{0,4}(h,h^2,h^2,h^2) + \underbrace{I^1_{0,4}(h,h^2,h^2,h)}_{=1(\text{ by Proposition 2.1.56-(iv) and (ii)})} * \underbrace{I^1_{0,4}(h,h^2,h^2,h)}_{=1(\text{ by Proposition 2.1.56-(iv) and (ii)})} +I^1_{0,4}(h,h^2,h^2,h^2)* \underbrace{I^1_{0,4}(h,h^2,h^2,1)}_{=0(\text{ by Proposition 2.1.56-(iii)})} = 1$$

$$\int \mathrm{PD}(Z)\cap \mathsf{D}\Bigg(\quad$$ 

$$\Bigg) =$$

$$= I^2_{0,4}(\ell_1,\gamma_1,\gamma_3,h^0)I^0_{0,4}(\ell_2,\gamma_2,\gamma_4,h^2) + I^2_{0,4}(\ell_1,\gamma_1,\gamma_3,h^1)I^0_{0,4}(\ell_2,\gamma_2,\gamma_4,h^1) + I^2_{0,4}(\ell_1,\gamma_1,\gamma_3,h^2)I^0_{0,4}(\ell_2,\gamma_2,\gamma_4,h^0)$$

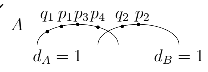
$$= \underbrace{I^2_{0,4}(h,h^2,h^2,1)}_{=0(\text{ by Proposition 2.1.56-(iii) })} ast I^0_{0,4}(h,h^2,h^2,h^2), +I^2_{0,4}(h,h^2,h^2,h)* \underbrace{I^0_{0,4}(h,h^2,h^2,h)}_{=0(\text{ by Proposition 2.1.56-(iv) and (i)})} +I^2_{0,4}(h,h^2,h^2,h^2)* \underbrace{I^0_{0,4}(h,h^2,h^2,1)}_{=0(\text{ by Proposition 2.1.56-(iii)})} = 0$$

$$\int \mathrm{PD}(Z)\cap \mathsf{D}\Bigg(\quad$$ 

$$\Bigg) =$$

$$= I^0_{0,5}(\ell_1,\gamma_1,\gamma_3,\gamma_4,h^0)I^2_{0,3}(\ell_2,\gamma_2,h^2) + I^0_{0,5}(\ell_1,\gamma_1,\gamma_3,\gamma_4,h^1)I^2_{0,3}(\ell_2,\gamma_2,h^1) + I^0_{0,5}(\ell_1,\gamma_1,\gamma_3,\gamma_4,h^2)I^2_{0,3}(\ell_2,\gamma_2,h^0)$$

$$= I^2_{0,5}(h,h^2,h^2,h^2,1)* \underbrace{I^0_{0,3}(h,h^2,h^2)}_{=0(\text{ by Proposition 2.1.56-(i) })} +I^2_{0,5}(h,h^2,h^2,h^2,h)* \underbrace{I^0_{0,3}(h,h^2,h)}_{=0(\text{ by Proposition 2.1.56-(i)})} +I^2_{0,5}(h,h^2,h^2,h^2,h^2)* \underbrace{I^0_{0,3}(h,h^2,1)}_{=0(\text{ by Proposition 2.1.56-(i)})} = 0$$

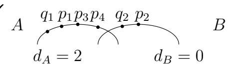
$$\int \mathrm{PD}(Z)\cap \mathsf{D}\Bigg(\quad$$ 

$$\Bigg) =$$

$$= I^1_{0,5}(\ell_1,\gamma_1,\gamma_3,\gamma_4,h^0)I^1_{0,3}(\ell_2,\gamma_2,h^2) + I^1_{0,5}(\ell_1,\gamma_1,\gamma_3,\gamma_4,h^1)I^1_{0,3}(\ell_2,\gamma_2,h^1) + I^1_{0,5}(\ell_1,\gamma_1,\gamma_3,\gamma_4,h^2)I^1_{0,3}(\ell_2,\gamma_2,h^0)$$

$$= \underbrace{I^1_{0,5}(h,h^2,h^2,h^2,1)}_{=0(\text{ by Proposition 2.1.56-(iii) })} *I^1_{0,3}(h,h^2,h^2) + I^1_{0,5}(h,h^2,h^2,h^2,h)* \underbrace{I^1_{0,3}(h,h^2,h)}_{=0(\text{ by Proposition 2.1.56-(iv) and (ii)))}} +I^1_{0,5}(h,h^2,h^2,h^2,h^2)* \underbrace{I^1_{0,3}(h,h^2,1)}_{=0(\text{ by Proposition 2.1.56-(iv) and (ii)})} = 0$$

$$\int \mathrm{PD}(Z) \cap \mathsf{D}\left(\right) =$$



$$= I^2_{0,5}(\ell_1, \gamma_1, \gamma_3, \gamma_4, h^0) I^0_{0,3}(\ell_2, \gamma_2, h^2) + I^2_{0,5}(\ell_1, \gamma_1, \gamma_3, \gamma_4, h^1) I^0_{0,3}(\ell_2, \gamma_2, h^1) + I^2_{0,5}(\ell_1, \gamma_1, \gamma_3, \gamma_4, h^2) I^0_{0,3}(\ell_2, \gamma_2, h^0)$$

$$= \underbrace{I^2_{0,5}(h, h^2, h^2, h^2, 1)}_{=0(\text{ by Proposition } \mathbf{2.1.56}\text{-(iii) })} * I^0_{0,3}(h, h^2, h^2) + I^2_{0,5}(h, h^2, h^2, h^2, h) * \underbrace{I^0_{0,3}(h, h^2, h)}_{=0(\text{ by Proposition } \mathbf{2.1.56}\text{-(i)))}} + I^2_{0,5}(h, h^2, h^2, h^2, h^2) * \underbrace{I^0_{0,3}(h, h^2, 1)}_{=0(\text{ by Proposition } \mathbf{2.1.56}\text{-(i))}} = 0$$

In conclusion, we obtain

$$\text{side II } = 1 + 1$$

Thus, from the divisor relations we obtain the equation

$$N_2 + 1 = 1 + 1$$

so that $N_2 = 1$

Remark 2.1.58. The strategy used in the Example **2.1.57** replaced in $\overline{\mathsf{M}}_{0,3d}(\mathbb{P}^2_{\mathbb{C}}, d)$, gives an alternative proof of Kontsevich's formula (43) in the general case. See [KV07; FP96].

This finishes our introductory course in Gromov-Witten theory.

2.1.4. Derived geometry of stable maps and virtual fundamental classes. Now that we have discussed the counting mechanics in the case of $Y = \mathbb{P}^2_{\mathbb{C}}$ (Example **2.1.57**), we can discuss what goes wrong in the general situation when Y is not *convex* (see Remark **2.1.20**): for general Y, the moduli stacks $\overline{\mathsf{M}}_{g,n}(Y, \beta)$ are not smooth and do not have a well-defined fundamental class. Therefore, the integrals (31) are not defined and the discussion of Section **2.1.3.1** does not apply.

The first solution to this problem appeared in the works of Beherend-Fantechi [BF97; Beh97] where a correction is obtained via the introduction of a *virtual fundamental class* $[\overline{\mathsf{M}}_{g,n}(Y, \beta)]^{vir}_{BF} \in \mathsf{H}_{\text{exp.dim}}(\overline{\mathsf{M}}_{g,n}(Y, \beta)]^{vir})$ against which we can integrate and get the correct intersection numbers.

Definition 2.1.59. Let $\beta \in \mathsf{H}_2(Y; \mathbb{Q})$ and $\gamma_1, \cdots, \gamma_n \in \mathsf{H}^*(Y, \mathbb{Q})$. We set

$$I^\beta_{g,n}(\gamma_1, \cdots, \gamma_n) := \int_{[\overline{\mathsf{M}}_{g,n}(Y,\beta)]^{vir}} \cup^n_{i=1} \mathsf{Ev}^*(\gamma_i)$$

The construction of $[\overline{\mathsf{M}}_{g,n}(Y,\beta)]^{vir}_{BF}$ relies on the fact that deformations and obstructions for stable maps are concentrated in homological degrees 0 and 1 as seen above. More precisely, the central ingredient of Behrend-Fantechi is the notion of a *perfect obstruction theory*:

Definition 2.1.60. [BF97: 5.1] Let X be a (classical) Deligne-Mumford stack. A *perfect obstruction theory* on X is a perfect complex $E \in \mathsf{Perf}(X)$ concentrated in Tor-amplitudes $[1,0]$ (homological notations), together with a map

$$\phi : E \to \mathbb{L}_X$$

which is an isomorphism on H^0 and surjective on H^1.

Finding an intrinsic definition of a perfect obstruction theory has been one of the triggers for the development of derived geometry: the works of Toën-Vezzosi [HAG-II] and Lurie [Lur04] provide a natural framework for Behrend-Fantechi's formalism in terms of quasi-smooth derived schemes (Definition **1.1.30**):

Lemma 2.1.61. *Let X be a quasi-smooth derived scheme (or Deligne-Mumford stack) and let $i : \mathsf{t}_0 X \hookrightarrow X$ be the closed immersion of its classical truncation. Then the induced map given by functoriality of cotangent complexes in $\mathsf{QCoh}(X)$*

$$E := i^*\mathbb{L}_X \to \mathbb{L}_{\mathsf{t}_0 X}$$

is a perfect obstruction theory.

Proof. This is a consequence of Lurie's connectivity estimates [Lu-HAlg: 7.4.3.12] as explained in [STV15:Proposition 1.2]. $\square$

As a consequence to Lemma **2.1.61**, all we need in order to give a canonical sense to Behrend-Fantechi's perfect obstruction theories on the moduli space of space maps, is to construct a quasi-smooth derived enrichment, whose cotangent complex coincides with the obstruction theory of stable maps. This is where we use the general mechanism to produce non-trivial derived stacks out of classical ones, reviewed in Section **1.2.2**: derived mapping stacks

Construction 2.1.62. Let Y be a smooth projective algebraic variety. We define the derived moduli stack of pre-stable maps as a derived mapping stack relative to $\mathsf{M}^{\mathrm{pre}}_{g,n}$:

$$\mathbb{R}\mathsf{Map}_{i(\mathsf{M}^{\mathrm{pre}}_{g,n})}(i(\mathscr{C}^{\mathrm{pre}}_{g,n}), i(Y \times \mathsf{M}^{\mathrm{pre}}_{g,n}))$$

where $i : \mathsf{St}_\mathsf{k} \subseteq \mathsf{dSt}_\mathsf{k}$ is the inclusion of higher stacks in derived stacks. Since the universal curve is flat over $\mathsf{M}^{\mathrm{pre}}_{g,n}$, the truncation of this derived stack is the classical higher stack of maps

$$\mathsf{t}_0\mathbb{R}\mathsf{Map}_{i(\mathsf{M}^{\mathrm{pre}}_{g,n})}(i(\mathscr{C}^{\mathrm{pre}}_{g,n}), i(Y \times \mathsf{M}^{\mathrm{pre}}_{g,n})) \simeq \mathsf{Map}_{\mathsf{M}^{\mathrm{pre}}_{g,n}}(\mathscr{C}^{\mathrm{pre}}_{g,n}, Y \times \mathsf{M}^{\mathrm{pre}}_{g,n})$$

See Remark **1.2.9**. Finally, Kontsevich's moduli stack of stable maps is an open substack in the classical truncation

$$\overline{\mathsf{M}}_{g,n}(Y,\beta) \subseteq \mathsf{t}_0\mathbb{R}\mathsf{Map}_{\mathsf{M}^{\mathrm{pre}}_{g,n}}(\mathscr{C}^{\mathrm{pre}}_{g,n}, Y \times \mathsf{M}^{\mathrm{pre}}_{g,n})$$

given by those maps that satisfy the stability condition of the Definition **2.1.11**. The universal property of the cotangent complex allows us to equip an open substack of the truncation of a derived stack with a unique derived structure (see [HAG-II: 2.2.2.9]). We obtain this way a canonical derived Deligne-Mumford stack of stable maps

$$\mathbb{R}\overline{\mathsf{M}}_{g,n}(Y,\beta)$$

naturally defined over $\mathsf{M}^{\mathrm{pre}}_{g,n}$.

Proposition 2.1.63. *The derived stack $\mathbb{R}\overline{\mathsf{M}}_{g,n}(Y,\beta)$ is a quasi-smooth derived stack over k with tangent complex at point f given by the homotopy cofiber*

$$\mathbb{T}_{\mathbb{R}\overline{\mathsf{M}}_{g,n}(Y,\beta),f} := \mathbb{R}\Gamma(C, \mathrm{cofiber}(\mathbb{T}_C(-\sum p_i) \to f^*\mathbb{T}_Y))$$

Proof. Being an open substack of a derived mapping stack, the cotangent complex of $\mathbb{R}\overline{\mathsf{M}}_{g,n}(Y,\beta)$ can be computed by the formula in Theorem **1.2.10**-(i) for derived mapping stacks relativity to the base $S = \mathsf{M}^{\mathrm{pre}}_{g,n}$. Notice now that the fiber of $\mathbb{R}\overline{\mathsf{M}}_{g,n}(Y,\beta) \to \mathsf{M}^{\mathrm{pre}}_{g,n}$ at a curve $(C, p_1, \cdots, p_n)$ is precisely the derived mapping stack $\mathbb{R}\mathsf{Map}_\beta((C, p_1, \cdots, p_n), Y)$ over k classifying stable maps of degree β with source C. The tangent complex of $\mathbb{R}\mathsf{Map}_\beta((C, p_1, \cdots, p_n), Y)$ relatively to k at a point f is (by the same Theorem **1.2.10**-(i)) given by $\mathbb{R}\Gamma(C, f^*\mathbb{T}_Y)$ which, since C is a nodal curve, is concentrated in Tor-amplitudes 0 and -1 homological. Therefore, $\mathbb{R}\overline{\mathsf{M}}_{g,n}(Y,\beta) \to \mathsf{M}^{\mathrm{pre}}_{g,n}$ is relatively quasi-smooth. But since $\mathsf{M}^{\mathrm{pre}}_{g,n}$ is a smooth Artin stack, it follows that $\mathbb{R}\overline{\mathsf{M}}_{g,n}(Y,\beta)$ is also quasi-smooth over k.

Finally, combined with the discussion in the Example **2.1.8**, we see that if $f : (C, p_1, \cdots, p_n) \to Y$ is a stable map of degree β, the tangent complex of $\mathbb{R}\overline{\mathsf{M}}_{g,n}(Y,\beta)$ relatively to the base field k at f is given by the homotopy cofiber

$$\mathbb{T}_{\mathbb{R}\overline{\mathsf{M}}_{g,n}(Y,\beta),f} := \mathbb{R}\Gamma(C, \mathrm{cofiber}(\mathbb{T}_C(-\sum p_i) \to f^*\mathbb{T}_Y))$$

□

Remark 2.1.64. Formalizing the quasi-smoothness of the stack of stable maps first envisioned in [Kon95] has been one of the original motivations for the development of derived geometry.

Corollary 2.1.65. *The perfect obstruction theory of Behrend-Fantechi re-appears naturally from the procedure in Lemma **2.1.61**.*

For more details about the discussion in this section see [STV15:§2.2].

Remark 2.1.66. A computation similar to the one of the Proposition **2.1.19** shows that the expected dimension (ie, Euler characteristic of the tangent complex) of $\mathbb{R}\overline{\mathsf{M}}_{g,n}(Y,\beta)$ is given by

$$\chi(\mathbb{T}_{\mathbb{R}\overline{\mathsf{M}}_{g,n}(Y,\beta),f}) = \int_\beta c_1(\mathbb{T}_Y) + (1-g)\dim Y + 3g - 3 + n$$

We will *not* explain here how to use the perfect obstruction theory of the Corollary **2.1.65** to construct $[\overline{\mathsf{M}}_{g,n}(Y,\beta)]^{vir}_{BF}$. Instead, we will describe a different construction, initially proposed by Kontsevich, that also uses the quasi-smoothness of the derived structure but in a different way. The compatibility of the two methods was first obtained by Kapranov-Fontanine [CFK09] using the Grothendieck-Riemann-Roch theorem. To explain it, we start with the following lemma that allows us to define virtual fundamental classes in G-theory:

Reminder 2.1.67. Let X be a derived stack. Recall that we have a stable k-linear ∞-category $\mathsf{Coh}^{\mathsf{b}}(X)$ given as a full subcategory of $\mathsf{QCoh}(X)$ (cf. Construction **1.1.23**) spanned by those objects that are cohomologically bounded and whose cohomology $\mathsf{H}^i(E)$ are coherent modules over $\mathsf{H}^0(\mathscr{O}_X)$. When X is a classical scheme, the homotopy category of $\mathsf{Coh}^{\mathsf{b}}(X)$ recovers the derived 1-category of bounded coherent sheaves. Recall also that, by definition, the G spectrum of X is the K-spectrum of $\mathsf{Coh}^{\mathsf{b}}(X)$, ie, $\mathsf{G}(X) := \mathsf{K}(\mathsf{Coh}^{\mathsf{b}}(X))$.

Lemma 2.1.68. *Let X be a quasi-smooth derived Artin stack over $\mathbb{C}$. Then, the derived structure sheaf $\mathscr{O}_X$ is bounded coherent, ie, is an object in $\mathsf{Coh}^{\mathsf{b}}(X)$.*

Proof. See [Toe12:Sublemma 2.3] for the result for quasi-smooth derived schemes and [Kha22:Corollary 6.1] for the general result for quasi-smooth derived Artin stacks. ☐

We also have the following lemma which is a consequence of devissage:

Lemma 2.1.69. *[Kha22:Corollary 3.4] Let X be a noetherian derived Artin stack and let $i : \mathsf{t}_0X \to X$ be the inclusion of the classical truncation. Then the direct image map*

$$i_* : \mathsf{G}(\mathsf{t}_0X) \to \mathsf{G}(X)$$

is an equivalence of G-spectra. In particular, at the level of G_0 we have

$$i_*^{-1}[\mathrm{E}] = \sum_i (-1)^i\, [\mathsf{H}^i(\mathrm{E})]$$

Construction 2.1.70 (G-theoretic virtual fundamental class)**.** Let X be a quasi-smooth derived Artin stack over $\mathbb{C}$. The G-theoretic fundamental class of X, denoted $[X]_{\mathsf{G}}$, is defined as

$$[X]_{\mathsf{G}} = [\mathscr{O}_X] \in \mathsf{G}_0(X)$$

The G-theoretic *virtual* fundamental class of X, denoted $[X]^{vir}_{\mathsf{G}}$, is the class in $\mathsf{G}_0(\mathsf{t}_0 X)$ defined by

$$[X]^{vir}_{\mathsf{G}} := i_*^{-1}[\mathscr{O}_X] = \sum_i (-1)^i \, [\pi_i(\mathscr{O}_X)] \in \mathsf{G}_0(\mathsf{t}_0 X)$$

This construction was first suggested by Kontsevich in [Kon95: 1.4.2]. For more details see [Toë14:page 192] or [Kha22:Construction 6.2].

The comparison of the G-theoretic class of Construction **2.1.70** with Beherend-Fantechi's virtual fundamental class happens via the Grothendieck-Riemann-Roch transformation and was first established in the works of Kapranov-Fontanine [CFK09] in the case of a quasi-smooth derived scheme admitting an embedding in a smooth ambient variety. Recently, Khan extended this comparison to quasi-smooth derived Artin stacks answering [Toë09:Question 4.7] in general:

Theorem 2.1.71. *Let X be a quasi-smooth Artin stack over $\mathbb{C}$ with classical truncation $i : \mathsf{t}_0 X \to X$. Then $\mathsf{t}_0 X$ admits a functorial virtual fundamental class à la Behrend-Fantechi $[X]^{vir}_{BF} \in \mathsf{CH}_*(\mathsf{t}_0 X)$ and we have*

$$\tau_{\mathsf{t}_0 X}([X]^{vir}_{\mathsf{G}}) = \mathsf{Todd}(i^* \mathbb{L}_X) \cap [X]^{vir}_{BF} \tag{44}$$

where $\tau_{\mathsf{t}_0 X} : \mathsf{G}_0(\mathsf{t}_0 X)_{\mathbb{Q}} \to \mathsf{CH}_*(\mathsf{t}_0 X)_{\mathbb{Q}}$ is the homological Chern character for the classical truncation $\mathsf{t}_0 X$ induced by the Chern character via Borel-Moore homology (generalizing the construction of [Ful98:§18]).

Proof. See [Kha19:§3.1] for the construction of the fundamental class à la Behrend-Fantechi using the deformation to the derived normal stack of [KR18] and [Kha19:§3.3] for the comparison with Behrend-Fantechi in the case X is Deligne-Mumford.

See [Kha19:§3.5] and [Kha22:Theorem 6.12 and Remark 6.13] for the formula (44) generalizing the result of Kapranov-Fontanine [CFK09:Theorem 4.2.3]. □

Example 2.1.72. Let Y be a smooth variety over $\mathbb{C}$ and let E be a vector bundle of rank r. The total space of E is the vector bundle stack of the Definition **1.2.29**, $\mathbb{V}_Y(\mathrm{E}) = \mathsf{Spec}_Y(\mathsf{Sym}_{\mathscr{O}_Y}(\mathrm{E}^\vee))$. Consider the derived self-intersection of the zero section

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[d, "j"] & Y \arrow[d, "0"] \\ Y \arrow[r, "0"] & \mathbb{V}_Y(\mathrm{E}) \end{tikzcd}$$

It is clear that the truncation $\mathsf{t}_0(X)$ is Y and so the inclusion of the classical truncation $i : Y = \mathsf{t}_0(X) \to X$ provides a right inverse to $j : X \to Y$, ie, $j \circ i = \mathsf{id}_Y$. Because i_* induces an isomorphism on G-theory (Lemma **2.1.69**) and since G is functorial, we find that $i_*^{-1} = j_*$. The G-theoretic virtual fundamental class of X of the

Construction **2.1.70** therefore coincides with the class $[j_*\mathscr{O}_X] \in \mathsf{G}_0(Y) = \mathsf{G}(\mathsf{t}_0 X)$. But from the computation of the derived intersection we have an equivalence in $\mathsf{Coh}^{\mathsf{b}}(Y)$

$$j_*\mathscr{O}_X \simeq \mathsf{Sym}_{\mathscr{O}_Y}(\mathrm{E}^\vee[1]) = \bigoplus_{i\geq 0}\left(\bigwedge^i \mathrm{E}^\vee\right)[i]$$

and so

$$[X]^{vir}_{\mathsf{G}} = \sum (-1)^i [\bigwedge^i \mathrm{E}^\vee] \in \mathsf{G}_0(Y)$$

Now, since $Y = \mathsf{t}_0 X$ is smooth, the homological Chern character is

$$\tau_Y(-) = \mathsf{Ch}(-) \cap (\mathsf{Todd}(\mathbb{T}_Y) \cap [Y]) = (\mathsf{Ch}(-) \cup \mathsf{Todd}(\mathbb{T}_Y)) \cap [Y]$$

where Ch is the standard Chern character. Combined with (44), we obtain

$$[X]^{vir}_{BF} = \tau_Y([X]^{vir}_{\mathsf{G}}) \cap \mathsf{Todd}(i^*\mathbb{T}_X)^{-1} = (\mathsf{Ch}([X]^{vir}_{\mathsf{G}}) \cup \mathsf{Todd}(\mathbb{T}_Y) \cup \mathsf{Todd}(i^*\mathbb{T}_X)^{-1}) \cap [Y]$$

$$= (\mathsf{Ch}(\sum (-1)^i [\bigwedge^i \mathrm{E}^\vee]) \cup \mathsf{Todd}(\mathbb{T}_Y) \cup \mathsf{Todd}(i^*\mathbb{T}_X)^{-1}) \cap [Y]$$

Now, by the [FP96:Example 3.2.5], we have

$$\mathsf{Ch}(\sum (-1)^i [\bigwedge^i \mathrm{E}^\vee]) = c_r(\mathrm{E}) \cup \mathsf{Todd}(\mathrm{E})^{-1}$$

where $c_r(\mathrm{E})$ is the top Chern class of E. Finally, using the cofiber sequence for tangent complexes for the map $j : X \to Y$, we find

$$\mathbb{T}_j \to \mathbb{T}_X \to j^*\mathbb{T}_Y$$

and using the derived pullback, we find

$$\mathbb{T}_j \simeq j^*\mathbb{T}_0 \simeq j^*\mathrm{E}[-1]$$

and therefore,

$$\mathrm{E}[-1] = i^*j^*\mathrm{E}[-1] \to i^*\mathbb{T}_X \to i^*j^*\mathbb{T}_Y = \mathbb{T}_Y$$

which implies

$$\mathsf{Todd}(i^*\mathbb{T}_X) = \mathsf{Todd}(\mathbb{T}_Y) \cup \mathsf{Todd}(\mathrm{E})^{-1}$$

Finally, we find

$$[X]^{vir}_{BF} = (c_r(\mathrm{E}) \cup \mathsf{Todd}(E)^{-1} \cup \mathsf{Todd}(\mathbb{T}_Y) \cup \mathsf{Todd}(\mathbb{T}_Y)^{-1} \cup \mathsf{Todd}(\mathrm{E})) \cap [Y] = c_r(\mathrm{E}) \cap [Y]$$

ie, in this case the virtual fundamental class is the top Chern class of the vector bundle.

Example 2.1.73. In [PY22] the authors use Theorem **2.1.71** to defined Gromov-Witten numbers in non-archimedean geometry in motivic Borel-Moore homology.

2.1.5. Our result: Categorification of cohomological field theories.

We now explain the results of [MR18] obtained in collaboration with E. Mann. Our main result is a generalization of the notion of cohomological field theory of the Theorem **2.1.50** and the splitting principle Lemma **2.1.53**. We exhibit a mechanism that allows us to lift the splitting axiom from singular cohomology/Chow rings, to a more vast class of cohomology theories satisfying the conditions in the Theorem **2.1.71**. In fact, even more generally, we show how to lift the action to the level of derived categories of coherent sheaves.

Our work was motivated by preceding work of Givental-Lee [Lee04; Giv00] who constructed a virtual class in G-theory using the deformation to the normal cone of Behrend-Fantechi [BF97] and showed that a form of splitting axiom holds. As a corollary of our main theorem we recover their results (see Theorem **2.1.78** below).

To explain our result we need first to introduce a key construction in category theory, namely, the category of spans:

Definition 2.1.74. Let C be an $(\infty, 1)$-category with fiber products. The $(\infty, 1)$-category of spans in C has objects given the objects of C, a morphism from X to Y is given by a diagram

$$\begin{tikzcd} & Z \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}$$

in C and compositions are given by homotopy fiber products. Moreover, the cartesian product in C, induces a symmetric monoidal structure $\times$ in $\mathsf{Spans}(\mathsf{C})$ where every object is dualizable . See [Hau18; DK19; GR17] for details.

Remark 2.1.75. Let X be an object in $\mathsf{Spans}(\mathsf{C})$. Notice that an endomorphism of X in $\mathsf{Spans}(\mathsf{C})$ corresponds to the data of a morphism $Z \to X \times X$ in C. More generally, a map $X^n \to X$ in $\mathsf{Spans}(\mathsf{C})$ corresponds to a morphism $Z \to X^{n+1}$ in C.

Our main technical result is the following:

Theorem 2.1.76. *(Mann-R. [MR18:Theorem 1.1.2]) Let Y be a smooth and proper algebraic variety over $\mathsf{k} = \mathbb{C}$. Then the derived stacks $\mathbb{R}\overline{\mathsf{M}}_{0,n}(Y, \beta)$ endow Y with a* lax *action of the operad of stable curves $\overline{\mathsf{M}}_{0,n}$ in the category of spans in derived stacks*

$$\overline{\mathsf{M}}_{0,n} \to \mathsf{Map}_{\mathsf{Spans}}(Y^n, Y)$$

with corresponding multiplication given by the spans

$$
\begin{tikzcd}
 & \mathbb{R}\overline{\mathsf{M}}_{0,n}(Y,\beta) \arrow[dl, "\mathsf{Ev}_1,\cdots\mathsf{Ev}_{n-1},\mathsf{Stab}"'] \arrow[dr, "\mathsf{Ev}_n"] & \\
Y^{n-1}\times\overline{\mathsf{M}}_{0,n} & & Y
\end{tikzcd} \tag{45}
$$

Remark 2.1.77. The terminology *lax* means that associativity does not hold strictly, but only up to a natural transformation. See the Construction **2.1.110** below.

Our main result establishes a notion of *categorified cohomological field theory*, generalizing Theorem **2.1.50**. It is obtained by applying derived categories $\mathsf{Coh}^{\mathsf{b}}$ to the Theorem **2.1.76**:

Theorem 2.1.78. *[MR18:Theorem 1.1.1] Let Y be a smooth and proper algebraic variety over $\mathsf{k} = \mathbb{C}$. Then, the derived category of bounded coherent sheaves $\mathsf{Coh}^{\mathsf{b}}(Y)$ is a lax-algebra over the operad of categories $\{\mathsf{Coh}^{\mathsf{b}}(\overline{\mathsf{M}}_{0,n})\}_{n\geq 3}$, with multiplication functor*

$$\mathsf{Coh}^{\mathsf{b}}(Y)^{\otimes_{n-1}}\otimes\mathsf{Coh}^{\mathsf{b}}(\overline{\mathsf{M}}_{0,n})\longrightarrow\mathsf{Coh}^{\mathsf{b}}(Y)$$

given by the pullback-pushforward along the diagrams (45).

We also showed the following:

Theorem 2.1.79. *[MR18:Corollary 4.3.3] The categorical action of the Theorem **2.1.78** descends to the G-theory of Y and we recover this way the K-theoretic Gromov-Witten invariants of Givental-Lee [Lee04; Giv00][(*)].*

Remark 2.1.80. More generally than what is stated in the Theorem **2.1.79**, if $\mathscr{E}$ is a cohomology theory satisfying the axioms of the Theorem **2.1.71**, then $\mathscr{E}(Y)$ acquires a lax action of the operad $\{\mathscr{E}(\overline{\mathsf{M}}_{0,n})\}_n$. We did not write extended version in our original paper because at the time the Theorem **2.1.71** was not available.

Remark 2.1.81. The reason Theorem **2.1.76** is restricted to genus zero is because when the paper was written, the theory of modular ∞-operads was not yet available. Nowadays, the formalism of modular operads is available [HRY20] but the construction of the brane action is still missing.

Remark 2.1.82. The novelty of Theorems **2.1.78** and **2.1.79** in respect to [KM96; BM96; Lee04; Giv00] is that the formalism and functoriality of virtual classes is contained in the derived stacks $\mathbb{R}\overline{\mathsf{M}}_{g,n}(Y,\beta)$ and in the definition of pullbacks and

(*)See ??

pushforwards for maps of derived stacks. The novelty of Theorem **2.1.79** with respect to [Lee04] is the fact that the action lifts to derived categories.

Remark 2.1.83. The difficulty in proving Theorem **2.1.78** is all in the Theorem **2.1.76**. The strategy to prove Theorem **2.1.76** uses the idea of *brane actions* discover by Toën for ∞-operads [Lu-HAlg; Toë13]. In particular, the proof also works in the context of the homotopy theory of topological spaces to show that any loop space $\mathsf{Map}(\mathsf{S}^1, X)$ is automatically an algebra over the operad of little disks $\mathsf{E}_2^{\otimes}$ in the category of co-spans of spaces. Recently this result has been extended to include the operad of framed little disks [Pou23]. See the Remark **2.1.92** below.

A survey of our results appeared in [MR21]. More recently Theorem **2.1.76** served as model to obtain recursive formulas for Gromov-Witten invariants in non-archimedian geometry [PY20a; PY22].

2.1.6. Brane Actions. In order to explain the idea behind the proof of the Theorem **2.1.76** we will first discuss a simpler version of the same underlying mechanism that goes by the name of *brane action*, and was first discovered by B. Toën in [Toë13]. We assume the reader is familiar with ∞-operads [Lu-HAlg].

Definition 2.1.84. Let C be an ∞-category with homotopy pushouts. Then the ∞-category of co-spans in C is the category of spans on the opposite of C, ie, $\mathsf{coSpans}(C) := \mathsf{Spans}(\mathsf{C}^{\mathsf{op}})$. Objects are those of C and a morphism from X to Y is a a pair of morphisms $X \to Z \leftarrow Y$ in C. Compositions are defined by taking homotopy pushouts.

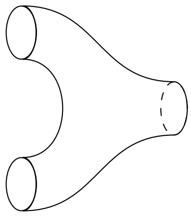
Construction 2.1.85. Let X be a topological space and consider the free loop space $\mathsf{Map}(\mathsf{S}^1, X)$ (this time, and contrary to the discussion in Construction **1.2.16**, we mean the topological space of continuous maps from S^1 to X). This space carries a circle action, obtained by pre-composition with the group structure on S^1. But there is something else going on: Sullivan-Chas [CS99] constructed a loop product on the homology of the $\mathsf{Map}(\mathsf{S}^1, X)$ which, following [CJ02], can be modeled at the level of singular chains via a pullback-pushforward along the following diagram

$$\begin{tikzcd} & \mathsf{Map}(\mathsf{S}^1 \vee \mathsf{S}^1, X) \arrow[dl] \arrow[dr] & \\ \mathsf{Map}(\mathsf{S}^1, X) \times \mathsf{Map}(\mathsf{S}^1, X) & & \mathsf{Map}(\mathsf{S}^1, X) \end{tikzcd} \tag{46}$$

induced by composition with

$$\mathsf{S}^1 \coprod \mathsf{S}^1 \to 8 = \mathsf{S}^1 \vee \mathsf{S}^1 \leftarrow \mathsf{S}^1 \tag{47}$$

Here the first map is the canonical inclusion of the two circles and the second is map given by going around the figure 8. Up to homotopy, one can understand the diagram (47) as the pair of pants



where the first map gives the inclusion of the two circles on the left and the second map is the inclusion of the circle on the right. This map can also be seen as a morphism

$$\mathsf{S}^1 \coprod \mathsf{S}^1 \dashrightarrow \mathsf{S}^1 \tag{48}$$

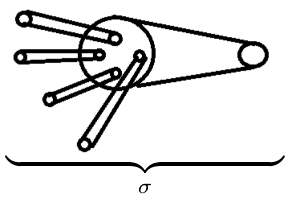
in the ∞-category of *co-spans* in spaces - $\mathsf{coSpans}(\mathcal{S})$ (cf. Definition **2.1.84**).

In this case, the algebra structure on the homology of the free loop space is a consequence of a simpler incarnation of the mechanism behind the Theorem **2.1.76**:

Proposition 2.1.86. *The homotopy type S^1, seen as an object in the symmetric monoidal ∞-category $\mathsf{coSpans}(\mathcal{S})^{\amalg}$ carries the structure of a $\mathsf{E}_2^{\otimes}$-algebra with multiplication given by the operation (47). More generally, given a configuration of n little disks $\sigma \in \mathsf{E}_2^{\otimes}(n)$ we have an induced multiplication given by the n^{th}-pants associated to σ:*

$$\coprod_n \mathsf{S}^1 \to \vee_n \mathsf{S}^1 \leftarrow \mathsf{S}^1$$

FIGURE 1. n-pants given by σ



To recover (46) we apply the functor $\mathsf{Map}(-, X)$ to (47) to obtain a *span* [(*)] in the ∞-category of spaces:

Construction 2.1.87. Let $X \in \mathcal{S}$. The functor $\mathsf{Map}(-, X) : \mathcal{S}^{\mathsf{op}} \to \mathcal{S}$ sends limits to limits and therefore, extends to a symmetric monoidal functor

$$\mathsf{coSpans}(\mathcal{S})^{\amalg} \to \mathsf{Spans}(\mathcal{S})^{\times}$$

(*) aka correspondance.

Corollary 2.1.88 (of Proposition **2.1.86**). *The space* $\mathsf{Map}(\mathsf{S}^1, X)$ *acquires the structure of an* $\mathsf{E}_2^{\otimes}$*-algebra in* $\mathsf{Spans}(\mathscr{S})^{\times}$ *with multiplication given by the span in* (46).

Remark 2.1.89. Before unraveling the mechanism behind the Proposition **2.1.86**, let us remark that this result has analogues for higher dimensional spheres S^n, namely, by a result of Ginot-Tradler-Zeinalian [GTZ12], the sphere S^n, seen as an object in the ∞-category of cobordisms in spaces, carries a structure of $\mathsf{E}_{n+1}^{\otimes}$-algebra with multiplication given by co-spans

$$\mathsf{S}^n \coprod \mathsf{S}^n \to \vee \mathsf{S}^n \leftarrow \mathsf{S}^n \tag{49}$$

As in the Construction **2.1.87**, $\mathsf{Map}(\mathsf{S}^n, X)$, seen as an object in the ∞-category of spans in spaces, carries a structure of $\mathsf{E}_{n+1}^{\otimes}$-algebra.

The following remark unravels some of the mystery:

Remark 2.1.90. The circle S^1 is weakly equivalent to the space $\mathsf{E}_2^{\otimes}(2)$ of binary operations in the little disks operad. Similarly, S^n is weakly equivalent to $\mathsf{E}_{n+1}^{\otimes}(2)$.

To really explain what is going on, we need a definition:

Definition 2.1.91. [J.Lurie] Let $\mathscr{O}$ be a monochromatic ∞-operad with $\mathscr{O}(0) \simeq \mathscr{O}(1) \simeq *$. Let $\sigma \in \mathscr{O}(n)$ be a n-ary operation.

- The space of extensions of σ - $\mathrm{Ext}(\sigma)$ - is the homotopy fiber product

  $$\mathrm{Ext}(\sigma) := \{\sigma\} \underset{\mathscr{O}(n)}{\times} \mathscr{O}(n+1)$$

  where the map $\mathscr{O}(n+1) \to \mathscr{O}(n)$ forgets the last entry.

- We say that $\mathscr{O}$ is coherent if for each pair of composable operations σ, τ, the natural square

  $$\begin{tikzcd} \mathrm{Ext}(\mathsf{id}) \arrow[r] \arrow[d] & \mathrm{Ext}(\sigma) \arrow[d] \\ \mathrm{Ext}(\tau) \arrow[r] & \mathrm{Ext}(\sigma \circ \tau) \end{tikzcd}$$

  is homotopy-cocartesian.

Remark 2.1.92. The space of extensions defined in Definition **2.1.91** is what Lurie calles in [Lu-HAlg:Notation 5.1.1.8] the space of *strict extensions*. In fact, in [Lu-HAlg:Def. 3.3.1.4] Lurie gives another definition of $\mathrm{Ext}(\sigma)$ for operads $\mathscr{O}$ without assuming the space of unitary operations $\mathscr{O}(1)$ to be contractible. These two definitions are claimed to be equivalent (without proof) under the map in [Lu-HAlg:Construction 5.1.1.9, Remark 5.1.1.10].

In our proof of the Theorem **2.1.94** in [MR18] below, we used a *third* definition of $\mathrm{Ext}(\sigma)$ in the case the space unitary operations $\mathcal{O}(1)$ is contractible. But while the comparison with the two definitions above seems obvious, we did not provide details (see the proof of the Theorem **2.1.94** below). More recently, Pourcelot [Pou23:Theorem 4.1] showed that the three definitions agree even for ∞-operads with non-contractible space of unitary operations, such as the operad of framed little disks.

Example 2.1.93. ([Lu-HAlg], [Toë13]) The ∞-operads $\mathsf{E}_k^{\otimes}$ are coherent. Indeed, given $\sigma \in \mathsf{E}_k^{\otimes}(n)$, the space of extensions is computed as an homotopy pullback

$$\begin{tikzcd}
\mathrm{Ext}(\sigma) \simeq \vee_n \mathsf{S}^{k-1} \arrow[r] \arrow[d] & \mathsf{E}_k^{\otimes}(n+1) \arrow[d, "\text{Forget Last entry}"] \\
* \arrow[r, "\sigma"] & \mathsf{E}_k^{\otimes}(n)
\end{tikzcd}$$

and the diagram in the coherence condition becomes the pushout

$$\begin{tikzcd}
\mathrm{Ext}(\mathsf{id}) = \mathsf{S}^{k-1} \arrow[r] \arrow[d] & \mathrm{Ext}(\sigma) = \vee_n \mathsf{S}^{k-1} \arrow[d] \\
\mathrm{Ext}(\tau) = \vee_m \mathsf{S}^{k-1} \arrow[r] & \mathrm{Ext}(\sigma \circ \tau) = \vee_{n+m-1} \mathsf{S}^{k-1}
\end{tikzcd}$$

The following theorem is due to Toën using rectification methods for ∞-operads. In [MR18] we provided a purely ∞-categorical proof which we sketch here:

Theorem 2.1.94 (Toën)**.** *Let $\mathcal{O}$ be a coherent ∞-operad (in the ∞-category of spaces). Then, $\mathcal{O}(2) = \mathrm{Ext}(\mathsf{id})$, seen as an object in the ∞-category of co-spans in spaces, carries an action of $\mathcal{O}$ with multiplication given by the*

(50) $$\sigma \in \mathcal{O}(n) \mapsto \coprod_n \mathrm{Ext}(\mathsf{id}) \to \mathrm{Ext}(\sigma) \leftarrow \mathrm{Ext}(\mathsf{id})$$

Before sketching the proof let us provide a few remarks:

Remark 2.1.95. Let C be an ∞-category with finite products and $\mathcal{O}$ an ∞-operad. An $\mathcal{O}$-algebra X in $\mathsf{Spans}(\mathsf{C})$ is the same as a compatible family of maps $\mathcal{O}(n) \mapsto \mathsf{Map}_{\mathsf{Spans}(\mathsf{C})}(X^n, X)$ assembling to a map of ∞-operads

$$\mathcal{O}^{\otimes} \to \mathsf{Spans}(\mathsf{C})^{\otimes}$$

This is the same as a symmetric monoidal functor

(51) $$\mathsf{Env}(\mathcal{O})^{\otimes} \to \mathsf{Spans}(\mathsf{C})^{\otimes}$$

from the symmetric monoidal envelope of $\mathscr{O}$.

We now use the fact that the construction $\mathsf{C} \mapsto \mathsf{Spans}(\mathsf{C})$ admits a left adjoint $\mathsf{D} \mapsto \mathsf{Tw}(\mathsf{D})$ given by the ∞-category of twisted arrows.

Definition 2.1.96. Let D be a ∞-category. We consider a new ∞-category $\mathsf{Tw}(\mathsf{D})$ where objets are morphisms in D and morphisms between $u : X \to Y$ and $v : A \to B$ are given by commutative diagrams

$$\begin{tikzcd} X \arrow[r] \arrow[d, "u"] & A \arrow[d, "v"] \\ Y & B \arrow[l] \end{tikzcd}$$

See [Lu-HAlg:§5.2.1] and [Bar13; HHLN20] for details.

The following proposition appeared first in [Ras14:§20] and has now been proved in [HHLN20:Thm 2.18]:

Proposition 2.1.97. *Let C be an ∞-category with finite limits and D an ∞-category. Then there is a canonical equivalence between the space of ∞-functors $F : \mathsf{D} \to \mathsf{Spans}(\mathsf{C})$ and the space of ∞-functors $\tilde{F} : \mathsf{Tw}(\mathsf{D}) \to C$ such that for any composable pair of morphisms in D, $f : x \to y$ and $g : y \to z$, the canonical morphism*

$$\tilde{F}(g \circ f) \to \tilde{F}(f) \underset{\tilde{F}(\mathsf{id}_y)}{\times} \tilde{F}(g)$$

is an equivalence in C.

Remark 2.1.98. Following the Proposition **2.1.97**, the data of a symmetric monoidal functor (51) is equivalent to the data of a functor

$$\mathsf{Tw}(\mathsf{Env}(\mathscr{O}))^{\otimes} \to \mathsf{C}^{\times}$$

In the case C=$\mathscr{S}^{\mathsf{op}}$, where we obtain co-spans, this corresponds to a ∞-functor

$$F : \mathsf{Tw}(\mathsf{Env}(\mathscr{O}))^{\otimes} \to (\mathscr{S}^{\mathsf{op}})^{\amalg}$$

which via the Grothendieck construction, this becomes equivalent to the data of a cartesian fibration

$$\mathscr{B}\mathscr{O} \to \mathsf{Tw}(\mathsf{Env}(\mathscr{O}))^{\otimes} \tag{52}$$

Proof of Theorem **2.1.94**. In conclusion, to prove the theorem all we need is to exhibit the cartesian fibration (52) that captures the formula (50). For this purpose we start by considering the source map

$$\mathsf{Fun}(\Delta^1, \mathsf{Tw}(\mathsf{Env}(\mathscr{O}))^{\otimes}) \to \mathsf{Tw}(\mathsf{Env}(\mathscr{O}))^{\otimes}$$

which is a cartesian fibration under compositions of morphisms (see [Lu-HTT: 2.4.7.5, 2.4.7.11]). We summarize here the proof in [MR18:Theorem 2.1.7]:

consider a non-full subcategory $\mathscr{B}\mathscr{O} \subseteq \mathsf{Fun}(\Delta^1, \mathsf{Tw}(\mathsf{Env}(\mathscr{O}))^{\otimes})$ whose:

(i) objects are those twisted morphisms

$$\sigma := (\langle n\rangle, \sigma_1 : X_1 \to Y_1, ..., \sigma_n : X_n \to Y_n) \xrightarrow{f} \delta := (\langle 1\rangle, \delta : U \to V)$$

over the unique active map $\langle n\rangle \to \langle 1\rangle$ such that the corresponding twisted arrow

$$\begin{tikzcd} \bigoplus_{i\in\langle n\rangle^\circ} X_i \arrow[r, "x"] \arrow[d, "\oplus_{i\in\langle n\rangle^\circ}\sigma_i"] & U \arrow[d, "\delta"] \\ \bigoplus_{i\in\langle n\rangle^\circ} Y_i & V \arrow[l, "y"] \end{tikzcd}$$

satisfies the following two conditions:

(a) the active map $x : \bigoplus_{i\in\langle n\rangle^\circ} X_i \to U$ is semi-inert in $\mathscr{O}^{\otimes}$ and is defined over one of the maps

$$\langle m\rangle := p(\bigoplus_{i\in\langle n\rangle^\circ} X_i) \to \langle m+1\rangle$$

corresponding to an inclusion that misses a single $\langle m+1\rangle$;

(b) the map y is an equivalence.

(ii) A morphism in $\mathsf{Fun}(\Delta[1], \mathsf{Tw}(\mathsf{Env}(\mathscr{O}))^{\otimes})$ over a morphism

$$\lambda := (\langle\gamma\rangle, \lambda_1 : A_1 \to B_1, ..., \lambda_\gamma : A_\gamma \to B_\gamma) \xrightarrow{g} \sigma$$

in $\mathsf{Tw}(\mathsf{Env}(\mathscr{O}))^{\otimes}$ is a commutative square

$$\begin{tikzcd} (\langle 1\rangle, \omega : W \to Z) \arrow[r, "h"] & \delta \\ \lambda \arrow[u, "t"] \arrow[r, "g"] & \sigma \arrow[u, "f"'] \end{tikzcd}$$

over

$$\begin{tikzcd} \langle 1\rangle \arrow[r, "id"] & \langle 1\rangle \\ \langle\gamma\rangle \arrow[u, "p(t)"] \arrow[r, "g"] & \langle n\rangle \arrow[u, "p(f)"'] \end{tikzcd}$$

such that

(a) both t and f satisfy the conditions of item 1);

(b) in the induced diagram

$$\begin{tikzcd} & & W \arrow[rr, "h"] & & U \\ \bigoplus_{\alpha\in\langle\gamma\rangle^\circ} A_\alpha \simeq \bigoplus_{i\in\langle n\rangle^\circ}\bigoplus_{j\in g^{-1}\{i\}} A_j \arrow[urr] \arrow[rrrr, "g"] & & & & \bigoplus_{i\in\langle n\rangle^\circ} X_i \arrow[u] \end{tikzcd}$$

the map h sends the unique element $p(W) - p(\bigoplus_{\alpha\in\langle\gamma\rangle^+} A_\alpha)$ to the missing element in $\langle m+1\rangle$.

The fibers of the composite map

$$\pi : \mathscr{B}\mathscr{O} \subseteq \mathsf{Fun}(\Delta[1], \mathsf{Tw}(\mathsf{Env}(\mathscr{O}))^{\otimes}) \to \mathsf{Tw}(\mathsf{Env}(\mathscr{O}))^{\otimes}$$

over an object $\sigma := (\langle n\rangle, \sigma_1 : X_1 \to Y_1, ..., \sigma_n : X_n \to Y_n)$ provide a third model for the space of extensions $\mathrm{Ext}(\bigoplus_{i\in\langle\rangle^\circ}\sigma_i) \simeq \coprod_{i\in\langle n\rangle^\circ}\mathrm{Ext}(\sigma_i)$ of the Remark **2.1.92**.

To conclude one shows that π remains a cartesian fibration. In our case relies on the contractibility of $\mathscr{O}(1)$ and the definition of semi-inert morphisms. For general coherent operads where $\mathscr{O}(1)$ is not necessarily contractible, this has been generalized in [Pou23:Theorem 2.17].

□

2.1.7. Brane actions and GW-theory. We now explain how to deduce Theorem **2.1.76** from Theorem **2.1.94**.

Remark 2.1.99. To extend the formalism of brane actions from ∞-operads with values in $\mathscr{S}$ to ∞-operads in derived stacks we use the Theorem **2.1.94** objectwise on functors of points.

Remark 2.1.100. Contrary to the case of stable curves (cf Proposition **2.1.38**) the forgetful map $\mathsf{M}^{\mathrm{pre}}_{g,n+1} \to \mathsf{M}^{\mathrm{pre}}_{g,n}$ is not the universal curve $\mathscr{C}^{\mathrm{pre}}_{g,n} \to \mathsf{M}^{\mathrm{pre}}_{g,n}$. The issue is that, for pre-stable curves, forgetting the last point does not force contraction of the non-stable components. See [BZFN10:Remark 2.1.7]. Therefore, when we consider the operad in stacks given by $\mathscr{O}(n) := \mathsf{M}^{\mathrm{pre}}_{0,n+1}$, the derived stack of extensions of a given curve $\sigma := (C, p_1, \cdots, p_n) \in \mathsf{M}^{\mathrm{pre}}_{g,n} = \mathscr{O}(n)$ given by the fiber $\mathrm{Ext}(\sigma) := \{\sigma\} \underset{\mathsf{M}^{\mathrm{pre}}_{g,n}}{\times} \mathsf{M}^{\mathrm{pre}}_{g,n+1}$ is not the curve C itself. In particular, when we map to a algebraic variety Y as in the Construction **2.1.87** we won't get the derived stack of maps $\mathbb{R}\mathsf{Map}(C, Y)$.

To correct the problem in the Remark **2.1.100**, we consider a slighly modified version of the stack of pre-stable curves defined by Costello:

Construction 2.1.101. Let $\mathrm{NE}(Y) \subseteq \mathsf{H}_2(Y, \mathbb{Z})$ denote subset given by the classes $f_*[C]$ for every map $f : C \to X$ with C a curve. We consider the moduli stack

$\mathfrak{M}_{g,n,\beta}$ parametrizing pre-stable curves with a formal index $\beta_i \in \mathrm{NE}(Y)$ attached to each irreducible component, such that:

- $\beta = \sum_i \beta_i$;
- if $\beta_i = 0$ then the irreducible component C_i is stable.

Example 2.1.102. By definition, the moduli stack $\mathfrak{M}_{0,3,0}$ is the compactification $\overline{\mathsf{M}}_{0,3} = \mathsf{Spec}(\mathbb{C})$.

Proposition 2.1.103. *(Costello [Cos06]) The stacks $\mathfrak{M}_{g,n,\beta}$ are smooth Artin stacks. Moreover, the maps*

$$\mathfrak{M}_{g,n+1,\beta} \to \mathfrak{M}_{g,n,\beta} \tag{53}$$

obtained by forgetting the last marking and stabilizing the curve, exhibit $\mathfrak{M}_{g,n+1,\beta}$ as the universal curve. In particular, it is flat over $\mathfrak{M}_{g,n,\beta}$.

Construction 2.1.104. The derived stack of stable maps $\mathbb{R}\overline{\mathsf{M}}_{g,n}(Y,\beta)$ of the Construction **2.1.62** can also be obtained using the stacks $\mathfrak{M}_{g,n,\beta}$ instead of $\mathsf{M}^{\mathrm{pre}}_{g,n}$. We consider the relative derived mapping stack

$$\mathbb{R}\mathsf{Map}_{\mathfrak{M}_{g,n,\beta}}(\mathfrak{M}_{g,n+1,\beta}, Y \times \mathfrak{M}_{g,n,\beta})$$

Since $\mathfrak{M}_{g,n+1,\beta}$ is the universal curve (Proposition **2.1.103**) this is also a stack of maps from a curve to $f : C \to Y$. The derived moduli stack $\mathbb{R}\overline{\mathsf{M}}_{g,n}(Y,\beta)$ lives as a connected component of $\mathbb{R}\mathsf{Map}_{\mathfrak{M}_{g,n,\beta}}(\mathfrak{M}_{g,n+1,\beta}, Y \times \mathfrak{M}_{g,n,\beta})$ classifying maps $f : C \to X$ with $f_*[C] = \beta$. See [MR18:§3.2].

Construction 2.1.105. [MR18:Prop. 3.1.4] We consider the following ($\mathrm{NE}(Y)$-graded) operad $\mathscr{O}$ in 1-stacks:

$$\mathscr{O}_{n,\beta} := \begin{cases} \emptyset & \text{if } n = 0,1 \text{ and } \beta \neq 0 \\ \mathsf{Spec}(\mathbb{C}) & \text{if } n = 0,1 \text{ and } \beta = 0 \\ \mathfrak{M}_{0,n+1,\beta} & \text{otherwise} \end{cases}$$

with operations given by the gluing of curves. The concatenation with the unique unary and nullary operations are given by (53).

Remark 2.1.106. Let $c : \mathsf{Spec}(\mathbb{C}) \to \mathscr{O}_{n-1,\beta} = \mathfrak{M}_{0,n,\beta}$ be a point classifying curve $(C, p_1, \cdots, p_n)$. The *stack of extensions* $\mathrm{Ext}(c)$ (defined objectwise on the functor of points), is given by the pullback in derived stacks

$$\begin{tikzcd} \mathrm{Ext}(c) \arrow[r] \arrow[d] & \mathscr{O}_{n,\beta} = \mathfrak{M}_{0,n+1,\beta} \arrow[d] \\ \mathsf{Spec}(\mathbb{C}) \arrow[r] & \mathscr{O}_{n-1,\beta} = \mathfrak{M}_{0,n,\beta} \end{tikzcd}$$

Since the maps $\mathfrak{M}_{g,n+1,\beta} \to \mathfrak{M}_{g,n,\beta}$ are flat (cf. Proposition **2.1.103**), we obtain an isomorphism

$$\mathrm{Ext}(c) \simeq C$$

When $\mathsf{id} : \mathsf{Spec}(\mathbb{C}) \to \mathcal{O}(1,0)$ classifies the unique unary operation, we obtain

$$\begin{tikzcd}
\mathrm{Ext}(\mathsf{id}) \arrow[r] \arrow[d] & \mathcal{O}_{2,0} = \mathfrak{M}_{0,3,0} = \mathsf{Spec}(\mathbb{C}) \arrow[d] \\
\mathsf{Spec}(\mathbb{C}) \arrow[r] & \mathcal{O}_{1,0} = \mathsf{Spec}(\mathbb{C})
\end{tikzcd}$$

and therefore, an isomorphism

$$\mathrm{Ext}(\mathsf{id}) \simeq \mathsf{Spec}(\mathbb{C})$$

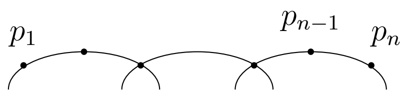
Finally, for every $c : \mathsf{Spec}(\mathbb{C}) \to \mathcal{O}_{n-1,\beta} = \mathfrak{M}_{0,n,\beta}$, the co-span analogous to the one in (50)

$$\coprod_{n-1} \mathrm{Ext}(\mathsf{id}) \to \mathrm{Ext}(c) \leftarrow \mathrm{Ext}(\mathsf{id}) \tag{54}$$

presents the curve as a co-span between the marked points:

$$\coprod_{n-1} \mathsf{Spec}(\mathbb{C}) \underbrace{\to}_{p_1,\cdots,p_{n-1}} C \underbrace{\leftarrow}_{p_n} \mathsf{Spec}(\mathbb{C}) \tag{55}$$

The figure Figure **1** is therefore replaced by the picture

$\{p_1\} \coprod \cdots \coprod \{p_{n-1}\} \hookrightarrow$  $\hookleftarrow \{p_n\}$

Remark 2.1.107. The operad $\mathcal{O}$ of the Construction **2.1.105** satisfies the assumptions $\mathcal{O}(0) = \mathcal{O}(1) = *$ of the Definition **2.1.91** but does not satisfy the coherency condition: indeed, while the composition operation is obtained by gluing two curves C_1 and C_2 by taking the pushout in the category of schemes $C_1 \coprod_{p_n=q_1}^{\text{schemes}} C_2$, the coherency condition is about the pushout in derived stacks $C_1 \coprod_{p_n=q_1}^{\mathsf{dSt_k}} C_2$. However, this is not an issue: the canonical comparison map

$$C_1 \coprod_{p_n=q_1}^{\mathsf{dSt_k}} C_2 \to C_1 \coprod_{p_n=q_1}^{\text{schemes}} C_2$$

becomes an isomorphism after composition with the $\mathbb{R}\mathsf{Map}(-, Y)$-functor, when Y is a derived Artin stack [(*)]. This is more than sufficient to our needs, since we apply to Y a smooth and proper algebraic variety. See [MR18:§3.1.2].

We can now reformulate the main technical result of our paper from which we deduce the Theorem **2.1.76**:

Proposition 2.1.108. *[MR18:Prop. 3.1.8 and 3.2.1] Let Y be a smooth and proper algebraic variety over $\mathsf{k} = \mathbb{C}$. Then, the formalism of brane actions for the (*$\mathrm{NE}(Y)$*-graded) operad in stacks $\mathscr{O}$ of the Construction **2.1.105** endows Y with an action of $\mathscr{O}$ in the ∞-category of co-spans, with multiplication given by the co-span*

(56)
$$\begin{tikzcd}
 & \mathbb{R}\overline{\mathsf{M}}_{0,n+1}(Y,\beta) \arrow[dl, "\mathsf{Ev}_1,\cdots\mathsf{Ev}_n,\text{forget}"'] \arrow[dr, "\mathsf{Ev}_{n+1}"] & \\
Y^n \times \mathscr{O}_{n,\beta} & & Y
\end{tikzcd}$$

Remark 2.1.109. The associativity of the action is a consequence of the commutativity of a diagram similar to (41) replacing the moduli of stable curves $\overline{\mathsf{M}}_{0,n}$ by the stacks $\mathfrak{M}_{0,n,\beta}$. Namely for every decomposition $\beta = \beta_1 + \beta_2$ we have a commutative diagram of derived stacks

(57)
$$\begin{tikzcd}
\mathfrak{M}_{0,n,\beta_1} \times \mathfrak{M}_{0,m,\beta_2} \arrow[d, equal] & \mathbb{R}\overline{\mathsf{M}}_{0,n}(Y,\beta_1) \times \mathbb{R}\overline{\mathsf{M}}_{0,m}(Y,\beta_2) \arrow[l] \arrow[r] & (Y)^n \times (Y)^m = (Y)^{n-1} \times Y \times Y \times (Y)^{m-1} \\
\mathfrak{M}_{0,n,\beta_1} \times \mathfrak{M}_{0,m,\beta_2} \arrow[d, equal] & \mathbb{R}\overline{\mathsf{M}}_{0,n}(Y,\beta_1) \underset{Y}{\times} \mathbb{R}\overline{\mathsf{M}}_{0,m}(Y,\beta_2) \arrow[u] \arrow[r] \arrow[d, dashed, "\phi"] \arrow[ur, phantom, "\urcorner\nu"] & (Y)^{n-1} \times Y \times (Y)^{m-1} \arrow[u, "\mathsf{id}^{n-1}\times\Delta\times\mathsf{id}^{m-1}"'] \arrow[d, equal] \\
\coprod_{\beta_1+\beta_2=\beta} \mathfrak{M}_{0,n\beta_1} \times \mathfrak{M}_{0,m,\beta_2} \arrow[d, "m"] \arrow[dr, phantom, "\llcorner\mu"] & \mathbb{R}\overline{\mathsf{M}}_{0,n}(Y,\beta) \underset{\mathfrak{M}_{0,n+m-2,\beta}}{\times} (\mathfrak{M}_{0,n,\beta_1} \times \mathfrak{M}_{0,m,\beta_2}) \arrow[l] \arrow[d] & (Y)^{n-1} \times (Y) \times (Y)^{m-1} \arrow[d, "\mathsf{id}^{n-1}\times*\times\mathsf{id}^{m-1}"] \\
\mathfrak{M}_{0,n+m-2,\beta} & \mathbb{R}\overline{\mathsf{M}}_{0,n+m-2}(Y,\beta) \arrow[l] \arrow[r] & (Y)^{n-1} \times * \times (Y)^{m-1} \simeq (Y)^{\times_{n+m-2}}
\end{tikzcd}$$

where the two squares μ and ν are cartesian and this time, contrary to the case in the (41), the canonical map ϕ is an equivalence of derived stacks.

Construction 2.1.110. The stabilization maps $\mathfrak{M}_{0,n,\beta} \to \overline{\mathsf{M}}_{0,n}$ define a map of operads from $\mathscr{O}$ to the Construction **2.1.105** to the operad of stable curves of the Proposition **2.1.49** that forgets the $\mathrm{NE}(Y)$-grading:

$$\mathscr{O} \to \{\overline{\mathsf{M}}_{0,n}\}_n$$

In the ∞-category of spans in derived stacks, this can also be seen as a *lax* map of operads (without gradings) in the reverse direction

[(*)]This follows from Tannakian duality. Combine [BHL15:Thm 1.4] and [Lur11b:Thm 7.1] to obtain the criterion of [Bha16]. More generally, following loc.cit and [HR17:p. 4] we could also take Y to be a perfect stack in the sense of [BZFN10].

(58) $$\{\overline{\mathsf{M}}_{0,n}\}_n \dashrightarrow \mathscr{O}$$

given by the span

$$\begin{tikzcd} & \coprod_\beta \mathfrak{M}_{0,n,\beta} \arrow[dl] \arrow[dr] & \\ \overline{\mathsf{M}}_{0,n} & & \coprod_\beta \mathfrak{M}_{0,n,\beta} \end{tikzcd}$$

Notice that this map is not strongly compatible with the operations on both sides: the obstruction comes from the failure of the canonical map ϕ

$$\phi : \coprod_{\beta_1+\beta_2=\beta} \mathfrak{M}_{0,n,\beta_1} \times \mathfrak{M}_{0,m,\beta_2} \to \mathfrak{M}_{0,n+m-2,\beta_1+\beta_2} \underset{\overline{\mathsf{M}}_{0,n+m-2}}{\times} (\overline{\mathsf{M}}_{0,n} \times \overline{\mathsf{M}}_{0,m})$$

to be an equivalence. In fact, the r.h.s is a divisor with normal crossings inside $\mathfrak{M}_{0,n+m-2,\beta_1+\beta_2}$ and the l.h.s corresponds to the inclusions of open strata of each component. Nevertheless, ϕ defines a coherent relation between the two operads and defines a lax structure on Equation (58)

PROOF OF THE THEOREM **2.1.76**:
We *pullback* the action of the Proposition **2.1.108** along the *lax* map of operads (58) of the Construction **2.1.110**. □

This concludes the discussion in this section.

2.2. Motives and Categories of Singularities

This section overviews the results of [BRTV18], a joint work with A.Blanc, B. Toën and G. Vezzosi on motives of singularity categories (see Theorem **2.2.63** below).

A video lecture covering the materials in this section is available here

Our main result concerns the comparison of two invariants of singularities of a function f on a smooth variety U. When U is defined over $\mathbb{C}$ and f has isolated singularities, an example of such invariant is the *Milnor number* - $\mu(f)$ - defined as the dimension of the underlying vector space of the Jacobian algebra:

$$\mu(f) := \dim_{\mathbb{C}} \mathcal{O}_U/(\partial_1 f, \cdots \partial_n f) \tag{59}$$

The results of [Mil68] show that $\mu(f)$ is an invariant of the singularity. Our result compares two more refined invariants:

- at one side a cohomological invariant: the cohomology of vanishing cycles;
- at the other side, a categorical invariant - the category of singularities of Eisenbud-Orlov, and its interpretation in terms of Matrix factorizations $\mathrm{MF}(U, f)$

We will describe and give examples of both invariants below and explain the comparison theorem. Here's a roadmap for this section:

Section **2.2.1**
Motivation from physics: Mirror symmetry

Section **2.2.2**
Categories of Singularities

Section **2.2.3**
Categories of singularities and derived geometry

Section **2.2.4**
Vanishing Cycles

Section **2.2.5**
Our results
(Theorem **2.2.63**)

2.2.1. Motivations from physics: Mirror Symmetry. Physicists [LVW89; GP90; HV00] noticed a remarkable ambiguity in string theory, namely, in the choice of the CY 3-fold Y used to compactify the extra dimensions of space-time discussed in Section **2.1.1**. More precisely, it was noticed that the physical theories do not change when Y is replaced by another CY 3-fold Y' under the condition that the Hodge numbers of Y - $h^{p,q}(Y)$(*) - are related to the Hodge numbers of Y' under the formula

$$h^{p,q}(Y) = h^{3-p,q}(Y')$$

In particular, such pairs Y and Y' will have Euler characteristics of opposite sign. See [Voi99] for an overview of the mathematics underlying mirror symmetry and its physical origins.

Remark 2.2.1. A much simpler form of mirror symmetry (T-duality) happens for certain physical theories on a circle, where it has been remarked that a circle of radius r and one of radius $\frac{1}{r}$ are indistinguishable [LSY15:page 219].

Example 2.2.2. The first examples of mirror symmetry also appeared in [GP90], such as the mirror to the Fermat quintic in the Example **2.1.2** with Hodge diamond given by

$$Y: \quad \begin{matrix} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 1 & & 101 & & 101 & & 1 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{matrix} \qquad\qquad Y^{\mathsf{Mirror}}: \quad \begin{matrix} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 101 & & 0 & \\ 1 & & 1 & & 1 & & 1 \\ & 0 & & 101 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{matrix}$$

Its mirror Y^{Mirror} has been explicitly computed using resolution of singularities of an orbifold quotient. See [HKKPTVVZ03: 6.5.1]. One of the first striking applications of mirror symmetry in mathematics is the computation by Candelas et al [COGP91] of the Gromov-Witten numbers of the Fermat quintic Y (cf. Section **2.1**), entirely in terms of its mirror Y^{Mirror}.

Kontsevich [Kon94] suggested an explanation for the general phenomena of mirror symmetry, namely, as a duality between symplectic (A side) and complex manifolds (B side), in the form of an equivalence of derived categories pairing Fukaya categories Fuk on the A-side with derived categories of coherent sheaves $\mathsf{Coh}^{\mathsf{b}}$ on the B-side. This proposal, known as *homological mirror symmetry*, triggered a new interest and results in derived categories. For instance, see [PZ98] for the full proof of this conjecture in the case of elliptic curves.

(*)ie, $h^{p,q}(Y) := \dim_{\mathsf{k}} \mathsf{H}^q(Y, \Omega^p_Y)$.

At the same time, discoveries in physics [GVW89; VW89; Mar90] indicated that mirror symmetry still makes sense away from the Calabi-Yau situation (for instance, in the Fano case), not as a duality between varieties, but instead, between pairs (U, f) consisting of U a variety and f a function - so called Landau-Ginzburg (LG) pairs. In the case of Fano manifolds, this has been made precise by a theorem of Auroux [Aur07]. Kontsevich's homological version of mirror symmetry in this case, requires a new kind of derived categories - so-called derived categories of singularities $\mathsf{D}_{\mathsf{Sing}}$ [Orl04; Sei11]. These categories have a mathematical interest of their own as invariants of singularities. For instance, key computations by Efimov, Dycheroff and Preygel show that the Hochschild type invariants of $\mathsf{D}_{\mathsf{Sing}}$ are related to the vanishing cycles of f - see Proposition **2.2.30** and Proposition **2.2.42** below. When $\mathsf{k} = \mathbb{C}$, the study of $\mathsf{D}_{\mathsf{Sing}}$ has been related to derived geometry, namely, it has been shown in [Pre11] that derived geometry induces in $\mathsf{D}_{\mathsf{Sing}}$ a 2-*periodic dg-structure* (Section **2.2.3** below). One of the motivations for our work [BRTV18] was to extend such results to positive and mixed characteristics with arithmetic applications in mind (see the Application **2.2.67** below).

The following terminology will accompany us in the next sections:

Definition 2.2.3. A LG-pair (or Landau-Ginzburg) pair over a base scheme S is a pair (U, f) consisting of a S-scheme $U \to S$ together with a function $f : U \to \mathbb{A}^1_S$.

2.2.2. Categories of Singularities. Let us now turn our attention away from physics and describe in detail the categories of singularities. These categories appeared long before the motivations from string physics, namely, in the study of hypersurface singularities by Eisenbud in [Eis80]. By a famous theorem of Serre [Ser65b:IV, Theorem 9], a classical affine scheme $Y = \mathsf{Spec}(A)$ is regular if and only if it is of finite global projective dimension, ie, every finitely generated A-module M admits a finite resolution by finitely generated projective A-modules

$$0 \to P_n \to P_{n-1} \to \cdot \to P_2 \to P_1 \to P_0 \to M$$

which stops after a finite number of steps equal to the dimension $n - 1 = \dim(Y)$. (*). In general, if Y is not smooth, any finitely generated A-module M admits a resolution by projective A-modules but there is no reason for it to stop. The main discovery of Eisenbud concerns the case where Y is a singular hypersurface given as the zero locus of a function f:

Proposition 2.2.4. *[Eis80:Theorem 6.1] Let A be a regular local ring, $f \in A$ and set $R = A/(f)$. Then every minimal R-projective resolution of a finitely generated R-module M becomes 2-periodic after* $\dim A + 1$ *steps*

$$... \to F \to Q \to F \to Q \to P_n \to ... \to P_2 \to P_1 \to P_0 \to M$$

(*)And in this case, all the different notions of dimension agree (Krull, depth, Tor, global)

PROOF. We want to show that a minimal resolution of M by projective R-modules behaves 2-periodically after a certain point. Notice that a (classical)R-module is the same as an A-module annihilated by f. The algorithm to construct a minimal resolution of M goes as follows. As M is finitely generated over R, we can produce a map

$$P_0 := R^{m_0} \to M$$

for m_0 = number of generators of M that we fix. The map is determined by the choice of generators. Then we can take the kernel of this map to obtain a new module M_1 that fits in an exact sequence

$$0 \to M_1 \to P_0 \to M \to 0$$

As R is Noetherian, M_1 is finitely generated and one can continue to apply this strategy to M_1 to obtain an exact sequence

$$0 \to M_2 \to P_1 := R^{m_1} \to M_1 \to 0$$

and by induction we get

$$0 \to M_m \to P_{m-1} \to \cdots \to P_1 \to P_0 \to M$$

with M_m finitely generated over R. Here's the key lemma discovered by Eisenbud, which we will not prove here:

- For $d \geq 1 + \mathrm{depth}(R)$, M_d is *maximal Cohen-Macaulay module*, ie

$$\mathrm{depth}(M_d) = \dim(R)$$

We now combine this with the following facts:

(i) The Auslander-Buschsbaum formula

$$\mathrm{pd}_A(M_d) = \mathrm{depth}(A) - \mathrm{depth}(M_d)$$

(ii) The fact that A is assume to be regular, by Serre's theorem, implies

$$\mathrm{depth}(A) = \dim(A)$$

(iii) As R is a ring of functions of an hypersurface,

$$\dim(R) = \dim(A) - 1$$

Combining these facts, one obtains

$$\mathrm{pd}_A(M_d) \underset{\text{Auslender-Bauschlaum}}{=} \mathrm{depth}(A) - \mathrm{depth}(M_d) \underset{A \text{ regular}}{=} \dim(A) - \mathrm{depth}(M_d) \underset{Eisenbud}{=}$$

$$= \dim(A) - \dim(R) \underset{hypersurface}{=} \dim(A) - (\dim(A) - 1) = 1$$

This says that when we see M_d as an A-module, it admits a minimal projective resolution of length 1

$$0 \longrightarrow E_1 \xrightarrow{\delta_0} E_0 \to M_d$$

where E_1 and E_0 are finitely generated projective A-modules. Moreover, since M_d is an R-module, seen as a A-module we have $f.M_d = 0$. In particular, one must have a null-homotopy

(60)
$$\begin{tikzcd} 0 \arrow[r] & E_1 \arrow[r, "\delta_0"] \arrow[d, ".f"'] & E_0 \arrow[r] \arrow[d, ".f"] \arrow[dl, "\delta_1"'] & M_d \\ 0 \arrow[r] & E_1 \arrow[r, "\delta_0"] & E_0 \arrow[r] & M_d \end{tikzcd}$$

satisfying

$$\delta_0\delta_1 = .f = \delta_1\delta_0$$

Finally by reducing mod f we obtain a 2-periodic resolution of M_d

$$\cdots \to E_1/f \to E_0/f \to E_1/f \to E_0/f \to M_d$$

In conclusion we obtain a resolution of M as an R-module

$$\cdots \to E_1/f \to E_0/f \to E_1/f \to E_0/f \to P_{d-1} \to P_d \to \cdots \to P_0 \to M$$

See also [Dyc11:§2.1] for more details.

$\square$

The category of matrix factorization $\mathrm{MF}(U, f)$ is designed to capture the 2-periodic part of the resolutions appearing in Proposition **2.2.4**:

Construction 2.2.5. ([Dou01], [Orl04:§3.1]) Let (U, f) be an LG-pair (Definition **2.2.3**) over $\mathsf{k} = \mathbb{C}$ with U an affine smooth scheme over k. A matrix factorisation on the pair (U, f) is the data of two vector bundles E_0, E_1 on U together with differentials

$$\delta_0 : E_0 \to E_1 \quad , \quad \delta_1 : E_1 \to E_0$$

satisfying

$$\delta_0\delta_1 = .f = \delta_1\delta_0$$

The collection of matrix factorisations on the pair (U, f) forms a $\mathbb{Z}/2$-dg-graded category $\mathrm{MF}(U, f)$: if $P = (E_0, E_1, \delta_0, \delta_1)$ and $Q = (Q_0, Q_1, \lambda_0, \lambda_1)$ are matrix factorisations then we have a 2-periodic chain complex of morphisms $\underline{\mathrm{HOM}}(P, Q)$ defined by

(61) $$\mathrm{Hom}(E_0, Q_1) \oplus \mathrm{Hom}(E_1, Q_0) \to \mathrm{Hom}(E_0, Q_0) \oplus \mathrm{Hom}(E_1, Q_1)$$

and

(62) $$\mathrm{Hom}(E_0, Q_1) \oplus \mathrm{Hom}(E_1, Q_0) \leftarrow \mathrm{Hom}(E_0, Q_0) \oplus \mathrm{Hom}(E_1, Q_1)$$

with differentials

$$d_{\to}(v) = \lambda \circ v - v \circ \delta \quad \text{and} \quad d_{\leftarrow}(v) = \delta \circ v + v \circ \lambda$$

We will denote by $\mathrm{hMF}(U, f)$ the homotopy 1-category of matrix factorisations where objects are matrix factorisations and morphisms are given by extracting H^0 of the complexes $\underline{\mathrm{HOM}}(P, Q)$, ie, homotopy classes of naive maps. The category $\mathrm{hMF}(X, f)$ admits a natural structure of triangulated category with shift functor given by

$$(E_0, E_1, \delta_0, \delta_1)[1] := (E_1, E_0, -\delta_0, -\delta_1)$$

Remark 2.2.6. The definition of $\mathrm{MF}(U, f)$ and $\mathrm{hMF}(U, f)$ has to be slightly modified when U is non-affine. See [BW12:Example 2.14] and [Orl04; Orl12; LP13; PV11]. Under the assumption that U has enough vector bundles, this extension to the non-affine case in loc. cit. correspond to the left Kan extension of the Construction **2.2.5** from affine to non-affine LG-models. See [LP13: 2.11], [Efi12:§5] and [BW12:§3].

Example 2.2.7. By definition, $\mathrm{hMF}(\mathsf{Spec}(\mathbb{C}), 0)$ is the homotopy category of 2-periodic complexes.

Example 2.2.8. We describe the steps to compute $\mathrm{hMF}(\mathbb{A}^2_{\mathsf{k}}, x^2 + y^2)$ with $\mathsf{k} = \mathbb{C}$. The hypersurface we are interested in is $U_0 = \{(x, y) \in \mathbb{A}^2_{\mathsf{k}} : x^2 + y^2 = 0\}$. Under the change of coordinates $z = x + iy, w = x - iy$ we find $U_0 = \{(z, w) \in \mathbb{C}^2 : z.w = 0\}$ and an isomorphism of categories $\mathrm{hMF}(\mathbb{A}^2_{\mathsf{k}}, x^2 + y^2) \simeq \mathrm{hMF}(\mathbb{A}^2_{\mathsf{k}}, z.w)$. As an object of this category, we can consider the following matrix factorization

$$\delta_0 = (-*z) : E_0 := \mathsf{k}[z, w] \to E_1 := \mathsf{k}[z, w];\ \delta_1 := (-*w) : E_1 = \mathsf{k}[z, w] \to E_0 = \mathsf{k}[z, w]$$

with

$$\delta_0\delta_1 = - * z * w = \delta_1\delta_0$$

We denote this matrix factorization by G. One can show that G is a compact generator of the category $\mathrm{hMF}(\mathbb{A}^2_{\mathsf{k}}, z.w)$. This a result of Dyckerhoff [Dyc11:Theorem 4.1]. Using this fact, the category $\mathrm{hMF}(\mathbb{A}^2_{\mathsf{k}}, z.w)$ becomes equivalent to the category of dg-modules over the 2-periodic dg-algebra $\underline{\mathrm{HOM}}(G, G)$ which one can explicitly compute: In both even and odd degrees one has $\mathsf{k}[z, w] \oplus \mathsf{k}[z, w]$. The differential even$\to$ odd is given by the matrix

$$\begin{pmatrix} z & -z \\ -w & w \end{pmatrix}$$

and odd$\to$ even is given by

$$\begin{pmatrix} w & z \\ w & z \end{pmatrix}$$

and one checks that $d^2 = 0$. Now one can explicitely compute

$$\mathsf{H}^0(\underline{\mathrm{HOM}}(G, G)) \simeq \mathsf{k}[x, y]/(x, y) \simeq \mathsf{k}$$

and

$$\mathsf{H}^1(\underline{\mathrm{HOM}}(G, G)) \simeq 0$$

In other words, as a $\mathbb{Z}$-graded complex, we get

$$\underline{\mathrm{HOM}}(G,G) \simeq \bigoplus_{i\in\mathbb{Z}} \mathsf{k}[2i]$$

and the dg-algebra algebra structure on endomorphisms of G corresponds to the free dg-algebra $\mathsf{k}[u,u^{-1}]$ with u a generator in homological degree -2. As a corollary (cf. [Lu-HAlg: 7.1.2.1]) we get an equivalence of categories between $\mathrm{hMF}(\mathbb{A}^2_{\mathsf{k}}, x^2+y^2)$ and the homotopy category of 2-periodic complexes, or in a more suggestive form following the Example **2.2.7**, an equivalence of 1-categories

$$\mathrm{hMF}(\mathbb{A}^2_{\mathsf{k}}, x^2+y^2) \simeq \mathrm{hMF}(\mathsf{Spec}(\mathbb{C}), 0)$$

Let us now describe Orlov's approach. We started the discussion in this section by mentioning Serre's characterization of regularity in terms of finite global projective dimension. The same result can be expressed by comparing the derived categories of coherent sheaves and perfect complexes (ie, locally given by finite resolutions by finitely generated projective modules).

Notation 2.2.9. Let Y be a derived scheme. We denote by:

- $\mathsf{D}^{\mathsf{b}}_{\mathsf{Coh}}(Y)$ = the derived category of bounded coherent sheaves (cf. **??**).
- $\mathsf{D}_{\mathsf{Perf}}(Y)$ = derived category of perfect complexes [(*)].

Remark 2.2.10. In general, there is no inclusion $\mathsf{D}_{\mathsf{Perf}}(Y) \subseteq \mathsf{D}^{\mathsf{b}}_{\mathsf{Coh}}(Y)$. This is only the case when Y is eventually coconnective, such as when $Y = U_0$ is the zero locus of a regular function f on a smooth scheme U (since in this case U_0 is quasi-smooth). See [GR17:Chapter 1] and [Toe12:Sublemma 2.3].

Definition 2.2.11. Let (U,f) be a LG-pair (Definition **2.2.3**) over k with U smooth and f flat. Let U_0 be the classical schematic zero locus of f. Orlov's category of singularities of U_0 - $\mathsf{D}_{\mathsf{Sing}}(U_0)$ - is defined as the Verdier quotient

$$\mathsf{D}_{\mathsf{Perf}}(U_0) \subseteq \mathsf{D}^{\mathsf{b}}_{\mathsf{Coh}}(U_0) \to \mathsf{D}_{\mathsf{Sing}}(U_0) := \mathsf{D}^{\mathsf{b}}_{\mathsf{Coh}}(U_0)/\mathsf{D}_{\mathsf{Perf}}(U_0)$$

Remark 2.2.12. $\mathsf{D}_{\mathsf{Sing}}(U_0)$ captures exactly those coherent sheaves on U_0 which do not admit projective resolutions of finite amplitude. It follows from Serre's theorem that U_0 is smooth if and only if the inclusion $\mathsf{D}_{\mathsf{Perf}}(U_0) \subseteq \mathsf{D}^{\mathsf{b}}_{\mathsf{Coh}}(U_0)$ is essentially surjective, if and only if $\mathsf{D}_{\mathsf{Sing}}(X_0)$ vanishes.

Recall that the category of matrix factorizations $\mathrm{MF}(U,f)$ in the Construction **2.2.5** was defined to capture the 2-periodic part of the resolution that starts after a finite resolution by vector bundles:

[(*)] This is the homotopy category of the k-linear stable ∞-category $\mathsf{Perf}(Y)$ of the Construction **1.1.23**.

$$\underbrace{\ldots \to F \to Q \to F \to Q}_{\in \mathrm{MF}(U,f)} \to \underbrace{P_n \to \ldots \to P_2 \to P_1 \to P_0}_{\in D_{\mathsf{Perf}}(U_0)} \to \underbrace{M}_{D^{\mathsf{b}}_{\mathsf{Coh}}(U_0)}$$

This picture suggests that both $\mathrm{MF}(U, f)$ and $\mathsf{D}_{\mathsf{Sing}}(U_0)$ should contain the same information, since both result from discounting from $\mathsf{D}^{\mathsf{b}}_{\mathsf{Coh}}(U_0)$ the contribution of finite resolutions by projective modules, ie, $\mathsf{D}_{\mathsf{Perf}}(U_0)$. This is indeed the case:

Theorem 2.2.13. *([Orl04:Theorem 3.9]) Let U be a smooth affine scheme over $\mathsf{k} = \mathbb{C}$, with $f : U \to \mathbb{A}^1_{\mathsf{k}}$ a flat map. Assume that the only critical value of f is 0. Then the assignment*

$$\mathrm{hMF}(U, f) \to \mathsf{D}_{\mathsf{Sing}}(U_0)$$

defined by sending

$$(E_1, E_0, \delta_0, \delta_1) \mapsto \operatorname{coker} \delta_0$$

is an equivalence of triangulated categories.

Proof. We combine Eisenbud's Proposition 2.2.4 with Orlov's observation that every object in $\mathsf{D}_{\mathsf{Sing}}(U_0)$ is equivalent to the image along the quotient map $\mathsf{D}^{\mathsf{b}}_{\mathsf{Coh}}(U_0) \to \mathsf{D}_{\mathsf{Sing}}(U_0)$ of a complex concentrated in one degree. $\square$

Remark 2.2.14. When f admits more than one critical value, one must replace $\mathrm{hMF}(U, f)$ by the direct sum over the critical values $\bigoplus_{\lambda \in \text{ Critical Values}} \mathrm{hMF}(U, f - \lambda)$.

Remark 2.2.15. For applications, it is more convenient to replace the triangulated categories $\mathrm{hMF}(U, f)$ and $\mathsf{D}_{\mathsf{Sing}}(U_0)$ by their respective idempotent completions (aka Karoubi envelope), $\mathrm{hMF}(U, f)^{\mathsf{idem}}$ and $\mathsf{D}_{\mathsf{Sing}}(U_0)^{\mathsf{idem}}$:

- The category $\mathrm{hMF}(U, f)^{\mathsf{idem}}$ can be explicited described as

$$\mathrm{hMF}(U, f)^{\mathsf{idem}} \simeq \mathrm{hMF}(\widehat{U}, \mathrm{TE}(f)) \tag{63}$$

  where $\widehat{U}$ is the formal completion of the critical locus Z of f inside U and $\mathrm{TE}(f)$ is the Taylor expansion of f along the critical locus. See [Dyc11:Theorem 5.7].

- A similar statement holds of $\mathsf{D}_{\mathsf{Sing}}(U_0)^{\mathsf{idem}}$. For a start it is easy to see that it only depends on the formal completion of Z along the *zero locus* U_0: if we denote by $\mathsf{D}_{\mathsf{Perf}_Z}(U_0)$ and $\mathsf{D}_{\mathsf{Coh}^{\mathsf{b}}_Z}(U_0)$ the categories with support along Z, we have the vertical open/closed localization exact sequences of idempotent complete triangulated categories

(64)
$$\begin{tikzcd}
\mathsf{D}_{\mathsf{Perf}_Z}(U_0) \arrow[r] \arrow[d] & \mathsf{D}_{\mathsf{Coh}^b_Z}(U_0) \arrow[r] \arrow[d] & \mathsf{D}_{\mathsf{Coh}^b_Z}(U_0)/\mathsf{D}_{\mathsf{Perf}_Z}(U_0) =: \mathsf{D}_{\mathsf{Sing}_Z}(U_0)^{\mathsf{idem}} \arrow[d, "\sim"] \\
\mathsf{D}_{\mathsf{Perf}}(U_0) \arrow[r] \arrow[d, "\text{restriction}"] & \mathsf{D}_{\mathsf{Coh}^b}(U_0) \arrow[r] \arrow[d, "\text{restriction}"] & \mathsf{D}_{\mathsf{Sing}}(U_0)^{\mathsf{idem}} \arrow[d] \\
\mathsf{D}_{\mathsf{Perf}}(U_0 \setminus Z) \arrow[r, "\sim"] & \mathsf{D}_{\mathsf{Coh}^b}(U_0 \setminus Z) \arrow[r] & 0
\end{tikzcd}$$

where the equivalence at the bottom horizontal line follows from Serre's theorem, since $U_0 \setminus Z$ is smooth over k (by definition of Z) and therefore also regular. For more details see [Orl11].

Notice however that the claim in (i) is slightly stronger, namely (63) only depends on the formal completion of Z along U (not U_0), equipped with the Taylor expansion of f. To reach a similar claim for singularity categories we will need the extra piece of information encoded in the 2-periodic structure discussed in Section **2.2.3** below. See the Construction **2.2.25**

2.2.3. Categories of Singularities and derived geometry. The equivalence of the Theorem **2.2.13** only captures the 1-categorical and triangulated structures. But we saw in Construction **2.2.5** that $\mathrm{hMF}(U, f)$ is, by definition, the homotopy category of a $\mathbb{Z}/2\mathbb{Z}$-graded dg-category. Also, the categories $\mathsf{D}_{\mathsf{Perf}}(U_0)$, $\mathsf{D}^{\mathsf{b}}_{\mathsf{Coh}}(U_0)$ and $\mathsf{D}_{\mathsf{Sing}}(X_0)^{\mathsf{idem}}$ have canonical dg-enhancements: the first two have dg-enhancements given by the dg-categories of perfect complexes $\mathsf{Perf}(U_0)$ and bounded coherent complexes $\mathsf{Coh}^{\mathsf{b}}(U_0)$. For the third, we define:

Definition 2.2.16. Let (U, f) be a LG-pair (Definition **2.2.3**) over a base ring k with U smooth over k. We define a dg-category $\mathsf{Sing}(U_0)$ as the quotient in the ∞-category of small îdempotent complete dg-categories $\mathsf{dgcat}^{\mathsf{idem}}_{\infty,\mathsf{k}}$(*),

$$\mathsf{Perf}(U_0) \to \mathsf{Coh}^{\mathsf{b}}(U_0) \to \mathsf{Sing}(U_0) \tag{65}$$

Remark 2.2.17. By construction, the homotopy category $\mathsf{hSing}(U_0)$ is already idempotent complete, and we have $\mathsf{hSing}(U_0) \simeq \mathsf{D}_{\mathsf{Sing}}(U_0)^{\mathsf{idem}}$. See [Dri].

In particular, one can ask if the equivalence of Theorem **2.2.13** can be lifted to an equivalence of the dg-enhancements. This is partially the matter of our joint work with A.Blanc-Vezzosi-Toën. In fact we show that Orlov's equivalence lifts to

(*)Recall that $\mathsf{dgcat}^{\mathsf{idem}}_{\infty,\mathsf{k}}$ is a full subcategory of $\mathsf{dgCat}_{\mathsf{k}}$ spanned by those small dg-categories that are idempotent complete. Moreover, this inclusion admits a left adjoint given by the idempotent completion. See [Toë11].

an equivalence of idempotent complete 2-periodic dg-categories, functorially in the pair (U, f) (see the Theorem **2.2.45** below). In order to explain this, one must first find the 2-periodic dg-enhancement on $\mathsf{Sing}(U_0)$. This is where derived geometry becomes crucial via the discussion in Section **1.2.1**. The discovery that the 2-periodic structure is due to derived geometry is due to Preygel in [Pre11]:

Construction 2.2.18. Let U be a smooth scheme over a ring k and let $f : U \to \mathbb{A}^1_{\mathsf{k}}$ be a flat map. Via the Example **1.2.5**, and since f is assumed to be flat - so that U_0 coincides with the derived zero locus of f - the derived fiber product $\Omega_0\mathbb{A}^1_{\mathsf{k}}$ acts on U_0 via the derived loop action

$$a : \Omega_0\mathbb{A}^1_{\mathsf{k}} \times U_0 \to U_0$$

Notice that, as explained in the Remark **1.2.6**, this action is the identity on the classical truncations. Now, since $\Omega_0\mathbb{A}^1_{\mathsf{k}}$ is a $\mathsf{E}^{\otimes}_{\infty}$- group,

$$\boxplus : \Omega_0\mathbb{A}^1_{\mathsf{k}} \times \Omega_0\mathbb{A}^1_{\mathsf{k}} \to \Omega_0\mathbb{A}^1_{\mathsf{k}}$$

the ∞-category $\mathsf{Coh}^{\mathsf{b}}(\Omega_0\mathbb{A}^1_{\mathsf{k}})$ becomes a symmetric monoidal category via the convolution product induced by the pushforward along the group structure.

$$\boxplus_* : \mathsf{Coh}^{\mathsf{b}}(\Omega_0\mathbb{A}^1_{\mathsf{k}}) \otimes \mathsf{Coh}^{\mathsf{b}}(\Omega_0\mathbb{A}^1_{\mathsf{k}}) \simeq \mathsf{Coh}^{\mathsf{b}}(\Omega_0\mathbb{A}^1_{\mathsf{k}} \times \Omega_0\mathbb{A}^1_{\mathsf{k}}) \to \mathsf{Coh}^{\mathsf{b}}(\Omega_0\mathbb{A}^1_{\mathsf{k}})$$

Moreover, the action of $\Omega_0\mathbb{A}^1_{\mathsf{k}}$ on U_0 induces an action at the level of dg-categories induced by pushforward along a:

$$a_* \circ \boxplus_* : \mathsf{Coh}^{\mathsf{b}}(\Omega_0\mathbb{A}^1_{\mathsf{k}}) \otimes \mathsf{Coh}^{\mathsf{b}}(U_0) \to \mathsf{Coh}^{\mathsf{b}}(U_0) \tag{66}$$

Remark 2.2.19. In the Construction **2.2.18** we assumed f to be flat so that U_0 is underived. In fact, the construction works the same way if we drop the flatness assumption and replace U_0 by the derived zero fiber.

To understand the categorical action in the Construction **2.2.18** we need first to understand the symmetric monoidal dg-category $\mathsf{Coh}^{\mathsf{b}}(\Omega_0\mathbb{A}^1_{\mathsf{k}})^{\boxplus}$.

Notation 2.2.20. Let $\mathsf{k}[u]$ denote the commutative differential graded algebra with a generator in homological degree -2. We denote by $\mathsf{Mod}(\mathsf{k}[u])^{\otimes}$ the k-linear ∞-category of $\mathsf{k}[u]$-modules, ie, chain complexes E together with a map of degree $u : E \to E[-2]$ corresponding to the action of the element u. We denote by $\mathsf{Perf}(\mathsf{k}[u])$ the k-linear ∞-category of perfect $\mathsf{k}[u]$-modules.

Lemma 2.2.21. *Let* k *be a regular ring. Then, there is a canonical equivalence of symmetric monoidal dg-categories*

$$\mathsf{Coh}^{\mathsf{b}}(\Omega_0\mathbb{A}^1_{\mathsf{k}})^{\boxplus} \simeq \mathsf{Perf}(\mathsf{k}[u])^{\otimes}$$

PROOF. This is because k seen as a coherent sheaf over $\mathsf{k}[\epsilon] = \mathscr{O}_{\Omega_0\mathbb{A}^1_{\mathsf{k}}}$ is a compact generator of the category $\mathsf{Coh}^{\mathsf{b}}(\Omega_0\mathbb{A}^1_{\mathsf{k}}) = \mathsf{Coh}^{\mathsf{b}}(\mathsf{Spec}(\mathsf{k}[\epsilon]))$. An explicit computation gives an equivalence of associative dg-algebras

$$\mathbb{R}\,\mathrm{Hom}_{\mathsf{k}[\epsilon]}(\mathsf{k},\mathsf{k}) \simeq \mathsf{k}[u]$$

See [Pre11:Prop. 3.1.4] for the result over $\mathsf{k} = \mathbb{C}$ and [BRTV18:Lemma 2.39] for the result over a general basis. $\square$

The following result unravels the mystery:

Proposition 2.2.22. *([Pre11:Lemma 3.1.9] over $\mathsf{k} = \mathbb{C}$ and [BRTV18:2.43 and 2.45] over a general basis.) Via the comparison in Lemma **2.2.21**,*

(i) Let (U, f) be an LG-pair with U regular. The exact sequence

$$\mathsf{Perf}(\Omega_0\mathbb{A}^1_{\mathsf{k}}) \subseteq \mathsf{Coh}^{\mathsf{b}}(\Omega_0\mathbb{A}^1_{\mathsf{k}}) \to \mathsf{Sing}(\Omega_0\mathbb{A}^1_{\mathsf{k}})$$

identifies with the exact-sequence

$$\mathsf{Perf}(\mathsf{k}[u])^{u-\mathrm{torsion}} \subseteq \mathsf{Perf}(\mathsf{k}[u]) \to \mathsf{Perf}(\mathsf{k}[u, u^{-1}])$$

where $\mathsf{Perf}(\mathsf{k}[u])^{u-\mathrm{torsion}}$ denotes the full subcategory of $\mathsf{Perf}(\mathsf{k}[u])$ where u acts nilpotently and $\mathsf{Perf}(\mathsf{k}[u, u^{-1}])$ is the dg-category of 2-periodic perfect complexes over k.[(*)]

(ii) the exact sequence

$$\mathsf{Perf}(U_0) \subseteq \mathsf{Coh}^{\mathsf{b}}(U_0) \to \mathsf{Sing}(U_0)$$

identifies with

$$\mathsf{Coh}^{\mathsf{b}}(U_0)^{u-\mathrm{torsion}} \subseteq \mathsf{Coh}^{\mathsf{b}}(U_0) \to \mathsf{Coh}^{\mathsf{b}}(U_0) \otimes_{\mathsf{k}[u]} \mathsf{k}[u, u^{-1}] =: \mathrm{Sing}^{2-\mathsf{per}}(U, f)$$

*where $\mathsf{Coh}^{\mathsf{b}}(U_0)^{u-\mathrm{torsion}}$ denotes the full subcategory spanned by objects where u acts nilpotently via a_* and the equivalence of Lemma **2.2.21**. In particular, we have*

$$\mathsf{Sing}(U_0) \simeq \mathrm{Sing}^{2-\mathsf{per}}(U, f)$$

and therefore the category of singularities acquires a canonical 2-periodic structure induced by the derived action.

Remark 2.2.23. This result shows that the 2-periodicity in $\mathsf{Sing}(U_0)$ was hidden in the action of the derived group stack $\Omega_0\mathbb{A}^1_{\mathbb{C}}$ of the Example **1.2.5**.

(*) At the level of presentable categories, $\mathsf{Mod}_{\mathsf{k}[u,u^{-1}]}$ is a presentable localization of $\mathsf{Mod}_{\mathsf{k}[u]}$. In particular, the forgetful map $\mathsf{Mod}_{\mathsf{k}[u,u^{-1}]} \to \mathsf{Mod}_{\mathsf{k}[u]}$ is fully faithful with image consisting of those $\mathsf{k}[u]$-modules E with $u : E \to E[-2]$ an equivalence.

Remark 2.2.24. The dg-category $\mathsf{Perf}(\mathsf{k}[u,u^{-1}])$ is a symmetric monoidal ∞-category because $\mathsf{k}[u,u^{-1}]$ is an $\mathsf{E}_\infty^{\otimes}$-algebra in Mod_k. Therefore, $\mathsf{Perf}(\mathsf{k}[u,u^{-1}])$ can be interpreted as an object in $\mathsf{CAlg}(\mathsf{dgCat}_\mathsf{k}^{\mathsf{idem}})$. The ∞-category of module-objects

$$\mathsf{Mod}_{\mathsf{Perf}(\mathsf{k}[u,u^{-1}])}(\mathsf{dgCat}_\mathsf{k}^{\mathsf{idem}})$$

is equivalent to the ∞-category of 2-periodic dg-categories $\mathsf{dgCat}_\mathsf{k}^{\mathsf{idem},2-\mathsf{per}}$. See [BRTV18:Notation 2.9 and Remark 2.12] for details.

Construction 2.2.25. Let $\mathsf{k}=\mathbb{C}$. We saw in the Remark **2.2.15**-(ii) that the homotopy category $\mathsf{D}_{\mathsf{Sing}}(U_0)^{\mathsf{idem}}$ only depends on the formal completion of the critical locus along U_0. Now we can show something slightly better.

First, we can show that the same statement as in Remark **2.2.15**-(ii) is true when enhanced with the 2-periodic structure on the dg-category $\mathrm{Sing}^{2-\mathsf{per}}(U,f)$ of the Proposition **2.2.22**. Indeed, this follows from the same diagram (64) and the observation that the $\mathsf{k}[u]$-linearity on $\mathsf{Coh}^\mathsf{b}(U_0)$ actually is also defined on $\mathsf{Coh}^\mathsf{b}_Z(U_0)$ and the inclusion

$$\mathsf{Coh}^\mathsf{b}_Z(U_0)\to\mathsf{Coh}^\mathsf{b}(U_0)$$

is $\mathsf{k}[u]$-equivariant. See [Pre11:Prop. 4.1.6]. But in fact we can show something better, namely that $\mathrm{Sing}^{2-\mathsf{per}}(U,f)$ only depends on the formal completion $\widehat{U}$ of U (not U_0) along the critical locus $Z=\mathsf{crit}(f)$ of f, together with the Taylor expansion of f.

Let (U,f) be an LG-pair and let Z be the (underived) critical locus of f. Assume f only has a critical value at $0^{(*)}$. Then, the map $f:U\to\mathbb{A}^1_\mathsf{k}$ induces a map at the level of formal completions

$$\mathrm{TE}(f):\widehat{U}^Z\to\widehat{\mathbb{A}^1}$$

given by the Taylor expansion of f. Now, using the exponential map

$$\mathsf{exp}:\widehat{\mathbb{G}_\mathsf{a}}\to\widehat{\mathbb{G}_\mathrm{m}}$$

we can instead consider the *formal* LG-pair $(\widehat{U}^Z),\mathrm{TE}(f))$. Following [Pre11:Lemma 6.1.1], multiplication by $\mathsf{exp}(\mathrm{TE}(f))$ determines a natural equivalence of the identity functor $\mathsf{id}_{\mathsf{Coh}^\mathsf{b}(\widehat{U}^Z)}$, ie, a map of groups

$$\mathbb{Z}\to\mathrm{Aut}(\mathsf{id}_{\mathsf{Coh}^\mathsf{b}(\widehat{U}^Z)})$$

which is the same as a functor

$$\mathsf{S}^1=\mathsf{B}\mathbb{Z}=\mathsf{S}^1\to\mathsf{Fun}^\mathsf{k}(\mathsf{Coh}^\mathsf{b}(\widehat{U}^Z),\mathsf{Coh}^\mathsf{b}(\widehat{U}^Z))$$

$^{(*)}$Otherwise we consider the sum of the different singularity categories at each critical value.

ie, a k-linear action of the topological circle S^1 on the dg-category $\mathsf{Coh}^{\mathsf{b}}_Z(U) = \mathsf{Coh}^{\mathsf{b}}(\widehat{U}^Z)$. In particular extracting fixed points for this action yields a category $\mathsf{Coh}^{\mathsf{b}}(\widehat{U}^Z)^{\mathsf{hS}^1}$ canonically equipped with a $\mathsf{k}^{\mathsf{hS}^1} = \mathsf{C}^*(\mathsf{BS}^1, \mathsf{k}) = \mathsf{k}[u]$-linear structure. By the [Pre11:Proposition. 7.1.4] we have a $\mathsf{k}[u]$-linear equivalence of $\mathsf{k}[u]$-linear categories

$$\mathsf{Coh}^{\mathsf{b}}(\widehat{U}^Z)^{\mathsf{hS}^1} \simeq \mathsf{Coh}^{\mathsf{b}}(\widehat{U_0}^Z)$$

where the r.h.s is the formal completion along the zero locus, equipped with the $\mathsf{k}[u]$-linear structure of the Proposition **2.2.22**. In particular, this implies that $\mathrm{Sing}^{2-\mathsf{per}}(U, f)$ only depends on the formal completion $\widehat{U}^Z$ and the S^1-action corresponding to the function $\mathsf{exp}(\mathrm{TE}(f))$.

Related Works 2.2.26. In [Pip22b], Pippi generalized the construction of categories of singularities and matrix factorizations to Landau-Ginzburg pairs $(U, f_1, \cdots, f_n)$ with several functions extending previous works of Orlov [Orl06] and Burke-Walker [BW15].

Before concluding this section, we state a computational result for categories of matrix factorizations:

Proposition 2.2.27. *([Pre11:Theorem 4.1.3]) Let (U, f) and (V, g) be* LG*-pairs over $\mathsf{k} = \mathbb{C}$ with U and V smooth. Then the exterior product of coherent sheaves induces an equivalence of 2-periodic small idempotent complete dg-categories*

$$\bigoplus_{\lambda \in \mathit{Crit}(-f) \cap \mathit{Crit}(g)} \mathrm{Sing}^{2-\mathsf{per}}(U, f+\lambda) \underset{\mathsf{k}[u,u^{-1}]}{\otimes} \mathrm{Sing}^{2-\mathsf{per}}(V, g-\lambda) \simeq \mathrm{Sing}^{2-\mathsf{per}}(U \times V, f-g)$$

Example 2.2.28. As an instance of the Proposition **2.2.27** we have over $\mathsf{k} = \mathbb{C}$ (or more generally over any ring with a square root of -1)

$$\mathrm{Sing}^{2-\mathsf{per}}(\mathbb{A}^1_{\mathsf{k}}, x^2) \underset{\mathsf{k}[u,u^{-1}]}{\otimes} \mathrm{Sing}^{2-\mathsf{per}}(\mathbb{A}^1_{\mathsf{k}}, y^2) \simeq \mathrm{Sing}^{2-\mathsf{per}}(\mathbb{A}^2, x^2 + y^2)$$

Remark 2.2.29. Combining the Proposition **2.2.27** with the computation of the Example **2.2.8**, we obtain

$$\mathrm{Sing}^{2-\mathsf{per}}(\mathbb{A}^2, x^2 + y^2) \simeq \mathrm{Sing}^{2-\mathsf{per}}(\mathsf{Spec}(\mathsf{k}), 0) \simeq \mathsf{Perf}(\mathsf{k}[u, u^{-1}])$$

inducing a stability behavior of singularities under addition of two extra dimensions and a non-degenerated quadratic form.

$$\mathrm{Sing}^{2-\mathsf{per}}(\mathsf{Spec}(\mathbb{C}), 0) \simeq \mathrm{Sing}^{2-\mathsf{per}}(\mathbb{A}^2_{\mathbb{C}}, x^2+y^2) \simeq ... \simeq \mathrm{Sing}^{2-\mathsf{per}}(\mathbb{A}^{2n}_{\mathbb{C}}, x_1^2+y_1^2+...+x_n^2+y_n^2)$$

In this categorical case this is known as Knörrer periodicity. See [Pre11:Remark 4.1.5].

To conclude this section, we summarize the computations of different Hochschild invariants[(*)] of the categories $\mathrm{Sing}^{2-\mathsf{per}}(U,f)$, combining independent results of Preygel, Efimov, Dyckerhoff and Li-Pomerleano and inspired by initial insights from string theory [KR04]:

Proposition 2.2.30. *([Pre11:Thm 8.2.6], [Dyc11:Cor. 6.4], [Efi12:Theorem 1.3] and [**Lin2013**])*

*(i) The Hochschild cohomology of $\mathrm{Sing}^{2-\mathsf{per}}(U,f)$ is given by the 2-periodized derived ring of functions on the derived critical locus of f (cf.Remark **1.1.33**):*

$$\mathsf{HH}^{\bullet}_{2-\mathsf{per}}(\mathrm{Sing}^{2-\mathsf{per}}(U,f)) \simeq \mathscr{O}_{\mathsf{dCrit}(U,f)}[u,u^{-1}]$$

In particular, when $X=\mathsf{Spec}(R)$ is an affine smooth scheme and f has isolated singularities, we recover the Jacobian algebra concentrated in even degree

$$\mathsf{HH}^{*}_{2-\mathsf{per}}(\mathrm{Sing}^{2-\mathsf{per}}(U,f)) = R/(\partial_1 f,\cdots,\partial_n f)$$

(ii) The complex of Hochschild homology of $\mathrm{Sing}^{2-\mathsf{per}}(U,f)$ is given by the 2-periodized graded module of differential forms with differential given by wedge with df:

$$\mathsf{HH}^{2-\mathsf{per}}_{\bullet}(\mathrm{Sing}^{2-\mathsf{per}}(U,f)) \simeq \mathbb{R}\Gamma(\Omega^{\bullet}_U[u,u^{-1}],u.(-df\wedge -))$$

(iii) The complex of cyclic homology is given by the 2-periodized twisted de Rham complex

$$\mathsf{HC}^{2-\mathsf{per}}_{\bullet}(\mathrm{Sing}^{2-\mathsf{per}}(U,f)) \simeq \mathbb{R}\Gamma(\Omega^{\bullet}_U[u,u^{-1}][[v]],v.\mathsf{d}_{\mathsf{dR}}+u(-df\wedge -))$$

Proof. (Summary) Since the identity functor is a continuous functor, the Hochschild homology of a dg-category T coincides with that of its completion under colimits $\widehat{T}$ [Toë07:Cor. 8.2]. This implies that we can replace the small idempotent dg-category $\mathrm{Sing}^{2-\mathsf{per}}(U,f)$ by the big presentable dg-category

$$\mathrm{MF}^{2-\mathsf{per},\infty}_{\mathsf{dg}}(U,f) := \mathsf{Ind}(\mathrm{Sing}^{2-\mathsf{per}}(U,f))$$

its Ind-completion and replace dg-functors by colimit preserving k-linear dg-functors. But in this case, [Pre11:Theorem 4.2.3] gives us an explicit description for continuous functors between MF categories:[(†)]

(*)Recall that the Hochschild cohomology of a dg-category $T\in\mathsf{dgCat}^{\mathsf{idem}}_{\mathsf{k}}$ is given by the complex of automorphisms of its identity dg-functor - see [Toë07:§8.1]. We will discuss Hochschild homology type invariants in more detail in Section **2.3**.

(†)Here $\mathsf{Fun}^{\mathsf{L}}$ denotes the full-subcategory spanned by functors preserving colimits.

$$\mathsf{Fun}^{\mathsf{L}}_{2-\mathsf{per}}(\mathrm{MF}^{2-\mathsf{per},\infty}_{\mathsf{dg}}(U,f),\mathrm{MF}^{2-\mathsf{per},\infty}_{\mathsf{dg}}(V,g))\simeq \mathrm{MF}^{2-\mathsf{per},\infty}_{\mathsf{dg}}(U\times V,-f+g)$$

identifying the identity functor with the diagonal bimodule Δ. The equivalences (i), (ii) and (iii) in the proposition are a consequence of the computation of the automorphisms of Δ combined with the HKR isomorphism (see [Pre11:Prop. 8.2.4] and Section **2.3** below).

□

Further Reading 2.2.31. See also [aT13; Seg13] for similar results.

We now recall a construction of [KKP08] and [Shk14] capturing the periodic cyclic homology of a 2-periodic dg-category:

Construction 2.2.32. Let $T\in \mathsf{dgCat}^{\mathsf{idem},2-\mathsf{per}}_{\mathsf{k}}$. Then its *periodic cyclic homology* $\mathsf{HP}_{2-\mathsf{per}}(T)$ can be viewed as a vector bundle over the formal punctured disk $\mathsf{Spf}(\mathsf{k}((v)))$ with parameter v of degree 2, with a natural Gauss-Manin connection ∇^{GM}_v. See [Shk14].

Remark 2.2.33. The formal connection in the Construction **2.2.32** has a categorical origin. See [Iwa21; Iwa22].

Proposition 2.2.34. *(See [Efi12:Thm 1.3]) We have an isomorphism of bundles with connection*

$$(\mathsf{HP}_{2-\mathsf{per}}(\mathrm{Sing}^{2-\mathsf{per}}(U,f)),\nabla^{GM}_u)\simeq(\mathbb{R}\Gamma(\Omega^{\bullet}_U[u,u^{-1}],\mathsf{d}_{\mathsf{dR}}+(-df\wedge -)),\nabla_u:=\tfrac{d}{du}+\tfrac{\Gamma}{u}+\tfrac{f}{u^2})$$

with $\Gamma_{|\Omega^p}=\frac{-p}{2}.\mathsf{id}$.

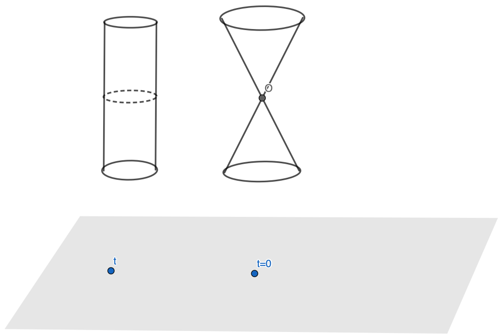
2.2.4. Vanishing Cycles.

We now turn our attention to a second invariant of singularities, namely, vanishing cycles. We start with a simple example:

Example 2.2.35. Consider the complex affine plane $U=\mathbb{C}^2$ with the function $f:\mathbb{C}^2\to\mathbb{C}$ given by $f(x,y)=x^2+y^2$. The only critical point is the origin $(0,0)\in\mathbb{A}^2_{\mathbb{C}}$ where $df=2xdx+2ydy$ vanishes. When the parameter $t\in\mathbb{C}$ varies, the fibers $U_t:=f^{-1}(\{t\})$ for $t\neq 0$ are all topologically isomorphic to a cylinder. At $t=0$, the fiber $U_0:=f^{-1}(\{0\})$ is topologically isomorphic to a double cone. The picture illustrates this:[(*)]

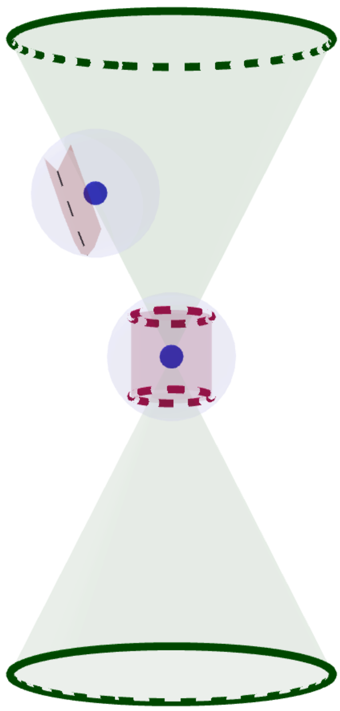
[(*)]Since we are working over $\mathsf{k}=\mathbb{C}$, we can use a change of coordinates of the Example **2.2.8** to rewrite $f(x,y)=zw$.

FIGURE 2. Family $x^2 + y^2 = t$



As $t \to 0$, the dotted circle collapses to the origin. For each point $x \in U_0$ we consider the closed ball $\mathsf{B}(x, \epsilon)$ of radius $\epsilon > 0$ around x in $\mathbb{C}^2$. We take $|t|$ small enough and consider the intersection $F_{x,\epsilon} := U_t \cap \mathsf{B}(x, \epsilon)$. The following picture illustrates $F_{x,\epsilon}$ when $x = 0$ and $x \neq 0$.

FIGURE 3. Milnor Fibers



We see that for $x \neq 0$, $F_{x,\epsilon}$ is contractible but when $x = 0$, $F_{x,\epsilon}$ is topologically a cylinder and therefore has the homotopy type of a circle. In order to capture how this information varies with the point x, we consider the chain complex of singular cochains $\mathsf{C}^*(F_{x,\epsilon}, \mathbb{Z})$ and the reduced version $\tilde{\mathsf{C}}^*(F_{x,\epsilon}, \mathbb{Z})$, as a function of $x \in U_0$. We find the class of the dotted circle in Figure **2** in the unique non-zero class $\mathsf{H}^1(F_{x,\epsilon}, \mathbb{Z})$.

Construction 2.2.36. The topological construction of the Example **2.2.35** can be generalized to any polynomial function $f : U \to \mathbb{A}^1_{\mathbb{C}}$ on a smooth $\mathbb{C}$-scheme U. The space $F_{x,\epsilon}$ is called the *Milnor fiber* at $x \in U_0$, the cohomology $\mathsf{C}^*(F_{x,\epsilon}, \mathbb{Z})$ is called the *nearby cohomology* at x and the reduced version $\tilde{\mathsf{C}}^*(F_{x,\epsilon}, \mathbb{Z})$ is called the *vanishing cohomology* at x. For more details see [Mil68].

Lemma 2.2.37. *[Mil68]Assume f has an isolated singularity at x. Then, the complex of singular cochains of $\mathsf{C}^*(F_{x,\epsilon}, \mathbb{Z})$ is concentrated in degrees 0 and $n-1$, where n is the complex dimension of U. More precisely*

$$\mathsf{H}^i(F_{x,\epsilon}, \mathbb{Z}) \simeq \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z}^\mu & i = n-1 \\ 0 & i \neq 0, n-1 \end{cases}$$

where μ coincides with the Milnor number *of f at the origin (cf. (59)).*

Remark 2.2.38. In the case of the Example **2.2.35**, $F_{x=0,\epsilon}$ is diffeomorphic to the cylinder $\mathsf{T}^*\mathsf{S}^1$ and therefore $\mu = 1$.

Proposition 2.2.39. *Let (U, f) be as in the Construction **2.2.36** and assume that f is proper. The assignments $x \mapsto \mathsf{C}^*(F_{x,\epsilon}, \mathbb{Z})$ and $x \mapsto \tilde{\mathsf{C}}^*(F_{x,\epsilon}, \mathbb{Z})$ are the stalks of chain-complexed valued sheaves for the complex analytic topology on U_0, respectively, Ψ_f and $\mathscr{V}_f$. Moreover:*

(i) these fit in a short exact sequence of sheaves

$$\mathbb{Z}_{U_0} \to \Psi_f \to \mathscr{V}_f \tag{67}$$

where $\mathbb{Z}_{U_0}$ is the constant sheaf.

(ii) $\mathbb{R}\Gamma(U_0, \Psi_f) \simeq \mathsf{C}^*(U_t, \mathbb{Z})$;

(iii) U_0 is smooth if and only if $\mathbb{R}\Gamma(U_0, \mathscr{V}_f) = 0$;

PROOF. See [Dim04:§4.2]. □

We now discuss the monodromy action. The following general construction illustrates part of the mechanism:

Construction 2.2.40. We place ourselves in the ∞-category of spaces $\mathcal{S}$ and consider a commutative diagram

(68)
$$\begin{tikzcd} Y \arrow[r] \arrow[d] & X \arrow[d] \\ \mathsf{S}^1 \arrow[r] & * \end{tikzcd}$$

which we assume, is *not* homotopy cartesian. By the universal property of homotopy pullbacks, this is the same as a commutative diagram

(69)
$$\begin{tikzcd} Y \arrow[rr] \arrow[dr] & & X \times \mathsf{S}^1 \arrow[dl] \\ & \mathsf{S}^1 & \end{tikzcd}$$

Now, since S^1 is the classifying space of the discrete group $\mathbb{Z}$ (see Remark **1.2.18**) we can apply the reverse mechanism of the Example **1.2.4** ,with $G = \mathbb{Z}$, both to $Y \to \mathsf{S}^1$ and $X \times \mathsf{S}^1 \to \mathsf{S}^1$ to produce monodromy actions of $\mathbb{Z} = \Omega_1\mathsf{S}^1$ on the homotopy fibers

$$\begin{tikzcd} \tilde{Y} \arrow[r] \arrow[d] & Y \arrow[d] \\ * \arrow[r] & \mathsf{S}^1 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X \arrow[r] \arrow[d] & X \times \mathsf{S}^1 \arrow[d] \\ * \arrow[r] & \mathsf{S}^1 \end{tikzcd}$$

To describe an explicit model for this homotopy pullback, we can replace the inclusion of the point $1 : * \to \mathsf{S}^1$ by the weak-equivalent universal cover given by the exponential map $\mathsf{exp} : \mathbb{R} \to \mathsf{S}^1$.

By construction, X comes equipped with the trivial $\mathbb{Z}$ action. Moreover, the map $Y \to X \times \mathsf{S}^1$ induces a $\mathbb{Z}$-equivariant *specialization* map

(70)
$$\mathsf{sp} : \tilde{Y} \to X$$

which, by the Example **1.2.4**, after passing to homotopy quotients, recovers the morphism in (69):

$$\tilde{Y}/\mathbb{Z} \simeq Y \to X/\mathbb{Z} \simeq X \times \mathsf{S}^1$$

Now, passing to singular cochains[(*)] we obtain a map in $\mathsf{CAlg}(\mathsf{Mod}_{\mathbb{Z}})$

$$\mathsf{C}^*(X, \mathbb{Z}) \xrightarrow{\mathsf{sp}^*} \mathsf{C}^*(\tilde{Y}, \mathbb{Z})$$

compatible with the $\mathbb{Z}$-actions. Let us denote by $\mathrm{fib}(\mathsf{sp}^*)$ the homotopy fiber of the underlying morphism of modules

[(*)]Recall that the ∞-categorical version of singular cochains sends homotopy colimits of spaces to homotopy limits of commutative differential graded algebras.

$$\mathrm{fib}(\mathsf{sp}^*) \to \mathsf{C}^*(X, \mathbb{Z}) \to \mathsf{C}^*(\tilde{Y}, \mathbb{Z})$$

The monodromy actions extends naturally to $\mathrm{fib}(\mathsf{sp}^*)$. Taking homotopy fixed points, we obtain equivalences in $\mathsf{CAlg}(\mathsf{Mod}_{\mathbb{Z}})$

$$\mathsf{C}^*(*, \mathbb{Z})^{\mathsf{h}\mathbb{Z}} \simeq \mathsf{C}^*(\mathsf{S}^1, \mathbb{Z}) \tag{71}$$

$$\mathsf{C}^*(\tilde{Y}, \mathbb{Z})^{\mathsf{h}\mathbb{Z}} \simeq \mathsf{C}^*(Y, \mathbb{Z}), \qquad \mathsf{C}^*(X, \mathbb{Z})^{\mathsf{h}\mathbb{Z}} \underset{\text{Künneth}}{\simeq} \mathsf{C}^*(X, \mathbb{Z}) \underset{\mathbb{Z}}{\otimes} \mathsf{C}^*(\mathsf{S}^1, \mathbb{Z}) \tag{72}$$

and two presentations for the same map of algebras $(\mathsf{sp}^*)^{\mathsf{h}\mathbb{Z}}$:

$$\begin{tikzcd} \mathsf{C}^*(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathsf{C}^*(\mathsf{S}^1, \mathbb{Z}) \arrow[r] \arrow[d, "\sim"] & \mathsf{C}^*(Y, \mathbb{Z}) \arrow[d, "\sim"] \\ \mathsf{C}^*(X, \mathbb{Z})^{\mathsf{h}\mathbb{Z}} \arrow[r, "(\mathsf{sp}^*)^{\mathsf{h}\mathbb{Z}}"] & \mathsf{C}^*(\tilde{Y}, \mathbb{Z})^{\mathsf{h}\mathbb{Z}} \end{tikzcd} \tag{73}$$

with fiber given by

$$\mathrm{fib}(\mathsf{sp}^*)^{\mathsf{h}\mathbb{Z}} \to \mathsf{C}^*(X, \mathbb{Z}) \underset{\mathbb{Z}}{\otimes} \mathsf{C}^*(\mathsf{S}^1, \mathbb{Z}) \to \mathsf{C}^*(Y, \mathbb{Z}) \tag{74}$$

(since taking homotopy fixed points commutes with homotopy fibers). In a sense, $\mathrm{fib}(\mathsf{sp}^*)^{\mathsf{h}\mathbb{Z}}$ measures the cohomological failure of the commutative square (68) to be homotopy cartesian.

We now discuss a sheaf-theoretic version of the Construction **2.2.40**:

Construction 2.2.41. The formalism of vanishing cycles of [Mr0b; Mr0a; KS90] provides a sheaf-theoretic analogue for the Construction **2.2.40**. Let $f : U \to \mathbb{C}$ be a proper map and consider the restriction of f to a small closed disk D centered at 0 in $\mathbb{C}$. Denote by $\mathsf{D}^* = \mathsf{D} \setminus \{0\}$ the punctured disk and consider the strict cartesian diagrams of topological spaces

$$\begin{tikzcd} U_{|\mathsf{D}^*} \arrow[r, "j"] \arrow[d, "f_{|\mathsf{D}^*}"] & U_{|\mathsf{D}} \arrow[d, "f_{|\mathsf{D}}"] & U_0 \arrow[l, "i"'] \arrow[d] \\ \mathsf{D}^* \arrow[r, hook] & \mathsf{D} & \{0\} \arrow[l] \end{tikzcd} \tag{75}$$

Consider the universal cover $\widetilde{\mathsf{D}^*} \to \mathsf{D}^*$ and take the (strict) fiber product

$$
\begin{tikzcd}
\tilde{U} \arrow[r, "\pi"] \arrow[d] & U_{|\mathsf{D}^*} \arrow[r, "j"] \arrow[d, "f_{|\mathsf{D}^*}"] & U_{|\mathsf{D}} \arrow[d, "f_{|\mathsf{D}}"] & U_0 \arrow[l, "i"'] \arrow[d] \\
\widetilde{\mathsf{D}^*} \arrow[r] & \mathsf{D}^* \arrow[r, hook] & \mathsf{D} & \{0\} \arrow[l]
\end{tikzcd}
\tag{76}
$$

This last fiber product is in fact a homotopy fiber product because the universal cover is a Serre fibration. Since the disk D is contractible and D^* is homotopy equivalent to S^1 via the inclusion of the boundary $\mathsf{S}^1 = \partial\mathsf{D} \hookrightarrow \mathsf{D}^*$, the diagram (76) is essentially homotopically of the form (68) discussed in the Construction **2.2.40**.

We now consider $\mathbb{Z}_{\tilde{U}}$ the constant sheaf on $\tilde{U}$ and denote by $j_*\pi_*(\mathbb{Z}_{\tilde{U}})$ its derived pushforward to $U_{|\mathsf{D}}$(*). The unit of the adjunction for pullbacks/pushforwards on derived categories of sheaves, gives us a unit map

$$\mathbb{Z}_{U_{|\mathsf{D}}} \to j_*\pi_*(\mathbb{Z}_{\tilde{U}})$$

which we can then pullback to a map of sheaves on U_0 to obtain a specialization map

$$\mathsf{sp} : \mathbb{Z}_{U_0} \to i^*j_*\pi_*(\mathbb{Z}_{\tilde{U}}) \tag{77}$$

By definition, the sheaf of nearby cycles Ψ_f of the Proposition **2.2.39** is precisely

$$\Psi_f := i^*j_*\pi_*(\mathbb{Z}_{\tilde{U}})$$

and the map (77) is the map $\mathbb{Z}_{U_0} \to \Psi_f$ of (67). The fact that $\mathbb{R}\Gamma(U_0, \Psi_f) \simeq \mathsf{C}^*(\tilde{U}, \mathbb{Z})$ is a consequence of proper base change.

The group of deck transformations of $\tilde{\mathsf{D}}^*$ over D^* is $\mathbb{Z}$, isomorphic to the fundamental group of D^*. By construction, it acts on the sheaf $i^*j_*\pi_*\mathbb{Z}_{\tilde{U}}$ providing the analogue of the monodromy action of the Construction **2.2.40**. Moreover, $\mathbb{Z}$ acts trivially on $\mathbb{Z}_{U_0}$ and the specialization map (77) is $\mathbb{Z}$-equivariant. Passing to homotopy fixed points at the level of sheaves we obtain

$$\mathsf{sp}^{\mathsf{h}\mathbb{Z}} : (\mathbb{Z}_{U_0})^{\mathsf{h}\mathbb{Z}} \to (i^*j_*\pi_*(\mathbb{Z}_{\tilde{U}}))^{\mathsf{h}\mathbb{Z}} \simeq i^*j_*\mathbb{Z}_{U_{|\mathsf{D}^*}} \tag{78}$$

Finally, since taking homotopy fixed points is a right adjoint, it commutes with limits and in particular, with taking derived global sections. We obtain a map of algebras

$$\mathsf{sp}^{\mathsf{h}\mathbb{Z}} : \mathsf{C}^*(U_0, \mathbb{Z}) \underset{\mathbb{Z}}{\otimes} \mathsf{C}^*(\mathsf{S}^1, \mathbb{Z}) \to \mathsf{C}^*(U_{|\mathsf{D}^*}, \mathbb{Z}) \tag{79}$$

(*)Here we are using the formalism of derived sheaves on manifolds, and the six operations, as developed in [KS90]. See [RS18:Remark 4.2] for an ∞-categorical construction of the derived category of sheaves in the topological setting.

as in (73). By definition of the sheaf of vanishing cycles $\mathscr{V}_f$ as a cofiber of (77), we obtain a fiber sequence analogous to (80):

$$\mathbb{H}(U_0, \mathscr{V}_f)^{\mathsf{h}\mathbb{Z}}[-1] \to \mathsf{C}^*(U_0, \mathbb{Z}) \underset{\mathbb{Z}}{\otimes} \mathsf{C}^*(\mathsf{S}^1, \mathbb{Z}) \to \mathsf{C}^*(U_{|\mathsf{D}^*}, \mathbb{Z}) \tag{80}$$

Finally, we discuss the relation between vanishing cycles and the categories of singularities of Section **2.2.3**. Historically, a first step was Kashiwara's computation of vanishing cycles in terms of $\mathscr{D}$-modules via the Riemann-Hilbert correspondence [Kas83]. A further pioneering work was Kapranov's [Kap91] identifying $\mathscr{D}$-modules with modules over the de Rham algebra. The final step was the comparison between vanishing cycle cohomology and twisted de Rham cohomology by Sabbah and Saito in [SS14; Sab10] establishing proofs for conjectures of Kontsevich-Soibelman in [KS11].

Proposition 2.2.42. *(See [Efi12:Thm 1.2] and [Sab10]) Let U be a smooth quasi-projective variety over $\mathsf{k} = \mathbb{C}$ and $f : U \to \mathbb{A}^1_{\mathsf{k}}$ whose only critical value is 0. Then we have an isomorphism of bundles with connection over a punctured disk with formal parameter v of degree 2*

$$(\mathbb{R}\Gamma(\Omega^\bullet_U((v)), v.\mathsf{d}_{\mathsf{dR}} + (-df \wedge -)), \nabla_v := \frac{d}{dv} + \frac{f}{v^2}) \simeq \mathrm{RH}^{-1}(\mathscr{V}_f, T)$$

where RH^{-1} is the inverse to the Riemann-Hilbert equivalence between constructible sheaves and holonomic $\mathscr{D}$-modules on U_0 and T is the monodromy operator on vanishing cycles.

The combination the Proposition **2.2.42** and Proposition **2.2.34** establish a precise link between the theory of matrix factorizations and the formalism of vanishing cycles:

Corollary 2.2.43. *(See [Efi12:Thm 1.1])*

$$(\mathsf{HP}_{2-\mathsf{per}}(\mathrm{Sing}^{2-\mathsf{per}}(U, f)), \nabla_u^{GM}) \simeq \mathrm{RH}^{-1}(\mathscr{V}_f, T(-1))$$

*where on the l.h.s we use the connection of the Construction **2.2.32**.*

More recent works of Lunts and Schnürer's [LS17:Theorem 1.2] combined with those of [IS13], show that this connection between the two theories can be expressed as an equivalence of classes in a certain Grothendieck group of motives.

In Section **2.2.5** below we overview the results of joint work with Blanc, Toën and Vezzosi (Theorem **2.2.63**) that provides a motivic lift of Efimov's comparison and in a sense, a categorification of Lunts and Schnürer's results

2.2.5. Our result: Comparing motives of singularity categories and vanishing cycles. Our first result in [BRTV18] shows that Orlov's equivalence in the Theorem **2.2.13** can indeed be lifted to the 2-periodic dg-enhancements:

Remark 2.2.44. When U is not regular, and in order to preserve the match with matrix factorizations, Burke-Walker [BW12] and Efimov-Positselski [EP15] observe that the definition of $\mathsf{Sing}(U_0)$ in the Definition **2.2.16** needs to be slightly modified to discard the singularities of U. More precisely, instead of $\mathsf{Coh}^{\mathsf{b}}(U_0)$ we consider the full subcategory $\mathsf{Coh}^{\mathsf{b}}(U_0)_{\mathsf{Perf}(U)}$ of those bounded coherent sheaves that are perfect on U after pushforward along $U_0 \hookrightarrow U$. In this case we define the *relative category of singularities* $\mathsf{Sing}(U_0/U)$ to be the quotient $\mathsf{Coh}^{\mathsf{b}}(U_0)_{\mathsf{Perf}(U)}/\mathsf{Perf}(U_0)$ in $\mathsf{dgcat}^{\mathsf{idem}}_{\infty,\mathsf{k}}$. When U is regular, we have $\mathsf{Sing}(U_0/U) = \mathsf{Sing}(U_0)$. See [BRTV18:Definition 2.23 and Prop. 2.25].

Theorem 2.2.45. *Let A be commutative ring. Then:*

(i) The two constructions of matrix factorizations MF *of the Construction **2.2.5** and* $\mathrm{Sing}^{\mathsf{2-per}}$ *of the Proposition **2.2.22** are defined functorially for* LG_A*-pairs and satisfy the Thom-Sebastiani theorem. In particular, both are given by lax-monoidal functors*

$$\mathrm{MF}, \mathrm{Sing}^{\mathsf{2-per}} : \mathrm{LG}_A \to \mathsf{dgCat}_A^{\mathsf{idem,2-per}}$$

where LG_A *denotes the category of* LG*-pairs over A;*

*(ii) An adjoint to Orlov's functor of the Theorem **2.2.13** can be lifted to a lax monoidal natural transformation*

$$\mathrm{Orl}^{-1} : \mathrm{Sing}^{\mathsf{2-per}} \to \mathrm{MF}$$

(iii) Orl^{-1} *is an equivalence whenever f is a non-zero divisor, U/A is separated and every coherent $\mathcal{O}_X$-module is a quotient of a vector bundle (eg, U is regular).*

Proof. The fact that MF extends to a general base ring is [BRTV18:2.11 and 2.12]. The construction for $\mathrm{Sing}^{\mathsf{2-per}}$ is [BRTV18:2.43 and 2.45]. For (ii) and (iii) see [BRTV18: 2.49]. □

Remark 2.2.46. It has been proved in [Pip22b] that one can drop the hypothesis assuming f is a non-zero divisor.

We now turn to the main purpose of [BRTV18]. In the Example **2.2.35**, Construction **2.2.40** and Construction **2.2.41** we used topological and sheaf-theoretic methods to define Milnor fibers and the sheaves of nearby and vanishing cycles. Following [Mr0b:Exposé XIII, XIV, XV] there is a purely algebraic description using the étale topology and the derived categories of ℓ-adic sheaves of [Eke90; BBD82]. In fact, Ayoub [Ayo07a; Ayo07b; Ayo14; Ayo10] has shown that the construction of nearby and vanishing cycles can be performed at the level of motives, before taking the Betti or de Rham realizations.

In [BRTV18] we show that the comparison of Corollary **2.2.43** can be lifted to the level of ℓ-adic sheaves of vanishing cycles using a new way of assigning a motive to the category of singularities. This mechanism relies on the motivic stable homotopy theory of schemes SH of Morel-Voevodsky [MV99; Voe98] and on a heavy use of the formalism of six operations. For economical reasons we had to compromise and leave the formal details out of this survey. We have instead tried to provide a shortcut to our main Theorem **2.2.63** below:

Construction/Theorem 2.2.47 (Stable Motivic Homotopy Theory)**.** Let S be a base scheme and Sm_S the category of smooth schemes over S. It is a symmetric monoidal category for the cartesian product. By definition, $\mathsf{SH}_S^{\otimes}$ is a presentable stable symmetric monoidal ∞-category together with a symmetric monoidal ∞-functor

$$\Sigma_+^{\infty} : \mathrm{Sm}_S^{\times} \to \mathsf{SH}_S^{\otimes}$$

universal with respect to the following properties (see [Rob15:Cor. 1.2]) :

(i) The image of an elementary Nisnevich square in Sm_S is a pushout square in SH_S.

(ii) (Homotopy invariance) The natural projection $\mathbb{A}^1_S \longrightarrow S$ is sent to an equivalence.

(iii) (Stability) Let $S \longrightarrow \mathbb{P}^1_S$ be the point at infinity and consider its image in SH_S. The cofiber of this map in SH_S, denoted as $(\mathbb{P}^1_S, \infty)$, is $\otimes$-invertible.

The primary purpose of the category SH is to serve as a container for all cohomology theories of algebraic varieties, such as algebraic K-theory[(*)]

Construction/Theorem 2.2.48. In SH_S there exists an object KGL_S[(†)] which represents homotopy-invariant algebraic K-theory, ie, for any S-smooth scheme X

$$\pi_n \, \mathsf{Map}_{\mathsf{SH}_S}(\Sigma_+^{\infty} X, \mathsf{KGL}_S) \simeq \mathsf{KH}_n(X)$$

where KH is the homotopy invariant K-theory spectrum [Wei89; Cis13]. When S is regular, this coincides with algebraic K-theory. It is beyond the scope of this thesis

(*)Recall that the connective algebraic K-theory of a scheme X is, by definition, the Waldhausen K-theory spectrum of the dg-category of perfect complexes $\mathsf{Perf}(X)$. See [Sch11] for an introduction to the subject and [TT90; Wal85] for details.

(†)In [BRTV18] we used the notation $\mathsf{B}\mathbb{U}$ instead of KGL.

to review this material. Instead, we redirect the reader to the beautiful survey [AE17] and to the original sources [MV99; Voe98].

Remark 2.2.49. The object KGL of the Construction/Theorem **2.2.48** reflects the projective bundle theorem in a form of Bott-periodicity $\mathsf{KGL}(-1)[-2] \simeq \mathsf{KGL}$ where $(1)[2] := (\mathbb{P}^1, \infty) \in \mathsf{SH}_S$.

In our context, algebraic K-theory provides the mechanism to assign a motive to every dg-category:

Construction 2.2.50. Let now $S = \mathsf{Spec}(A)$ be a classical affine scheme. For each A-dg-category T we consider the object $\mathscr{M}_A(T) \in \mathsf{SH}_A$ which represents homotopy invariant K-theory twisted by T, ie, defined by the universal property:

$$\mathsf{Map}_{\mathsf{SH}_A}(\Sigma^\infty_+ X, \mathscr{M}_A(T)) \simeq \mathrm{KH}(\mathrm{Perf}(X) \underset{A}{\otimes} T)$$

where on the r.h.s we have the homotopy invariant K-theory spectrum of the tensor product of dg-categories $\mathsf{Perf}(X) \otimes_A T$. Notice in particular that when $T = \mathsf{Perf}(A)$, $\mathscr{M}(\mathsf{Perf}(A)) = \mathsf{KGL}_A$. The assignment $T \mapsto \mathscr{M}_A(T)$ can be made ∞-functorial and symmetric lax monoidal

$$\mathscr{M}_A : \mathsf{dgCat}_A^{\mathsf{idem},\otimes} \to \mathsf{SH}_A^{\otimes}$$

Moreover,

(i) Because of localization for non-connective K-theory, $\mathscr{M}_A$ sends exact sequences of idempotent complete dg-categories[(*)] to cofiber/fiber sequences in SH_A.

(ii) Because $\mathscr{M}_A$ is lax monoidal and $\mathscr{M}_A(\mathsf{Perf}(A)) = \mathsf{KGL}_A$, the functor $\mathscr{M}_A$ factors through KGL_A-modules

$$\mathscr{M}_A : \mathsf{dgCat}_A^{\mathsf{idem}} \to \mathsf{Mod}_{\mathsf{KGL}_A}(\mathsf{SH}_A) \tag{81}$$

Remark 2.2.51. Over $A = \mathbb{C}$, composing the functor (81) with the topological realization $\mathsf{SH}_{\mathbb{C}} \to \mathsf{Sp}$, recovers Blanc's topological K-theory of dg-categories [Bla15].

We now want to apply Construction **2.2.50** to 2-periodic dg-categories:

Construction 2.2.52. Since by definition (see the Remark **2.2.24**) we have

$$\mathsf{dgCat}_A^{\mathsf{idem},2-\mathsf{per}} = \mathsf{Mod}_{\mathsf{Perf}(A[u,u^{-1}])}(\mathsf{dgCat}_A^{\mathsf{idem}})$$

(*) ie, cofiber sequences in $\mathsf{dgCat}^{\mathsf{idem}}$

and the functor $\mathscr{M}_A$ of (81) is symmetric lax monoidal, it induces a functor at the level of $\mathsf{Perf}(A[u, u^{-1}])$-modules

$$\mathscr{M}_A^{2-\mathsf{per}} : \mathsf{dgCat}_A^{\mathsf{idem},2-\mathsf{per}} \to \mathsf{Mod}_{\mathscr{M}_A(\mathsf{Perf}(A[u,u^{-1}]))}(\mathsf{Mod}_{\mathsf{KGL}_A}(\mathsf{SH}_A)) \tag{82}$$

Finally, we use Construction **2.2.52** to extract ℓ-adic sheaves:

Construction 2.2.53. We consider the symmetric monoidal $\mathbb{Q}_\ell$-adic realization

$$\mathscr{R}_\ell : \mathsf{SH}_S \to \mathsf{Sh}_{\mathbb{Q}_\ell}(S)$$

where ℓ is a prime invertible in $S = \mathsf{Spec}(A)$ and $\mathsf{Sh}_{\mathbb{Q}_\ell}(S)$ is the ∞-category of $\mathbb{Q}_\ell$-adic sheaves, whose homotopy category recovers the derived category of [Eke90; BBD82]. See [CD16] for the construction of the realization and [BRTV18:§3.7] for the ∞-categorical construction.

Notation 2.2.54. Let $S = \mathsf{Spec}(A)$ be a classical affine scheme.The ℓ-adic realization of dg-categories, denoted $\mathscr{R}_A^{\mathsf{dg}}$, is lax monoidal composition

$$\mathsf{dgcat}_{\infty,S}^{\mathsf{idem}} \xrightarrow{\mathscr{M}_A} \mathsf{SH}_S \xrightarrow{\mathscr{R}_\ell} \mathsf{Sh}_{\mathbb{Q}_\ell}(S)$$

Definition 2.2.55. Consider the free commutative algebra object in $\mathsf{Sh}_{\mathbb{Q}_\ell}(S)$ generated by the Tate twist $\mathbb{Q}_{\ell,S}[\beta] = \mathsf{Sym}_{\mathbb{Q}_\ell}(\mathbb{Q}_{\ell,A}(1)[2])$ and define $\mathbb{Q}_{\ell,S}[\beta, \beta^{-1}]$ to be the commutative algebra object obtained by inverting the generator $\beta = (1)[2]$:

$$\mathbb{Q}_{\ell,S}[\beta, \beta^{-1}] := \mathsf{Sym}_{\mathbb{Q}_\ell}(\mathbb{Q}_{\ell,A}(1)[2])[\mathbb{Q}_{\ell,A}(1)[2]^{-1}] = \bigoplus_{i \in \mathbb{Z}} \mathbb{Q}_{\ell,A}(i)[2i]$$

Lemma 2.2.56. *The Chern character* Ch *induces an equivalence in* $\mathsf{Sh}_{\mathbb{Q}_\ell}(S)$

$$\mathscr{R}_A^{\mathsf{dg}}(\mathsf{Perf}(A)) = \mathscr{R}_\ell(\mathsf{KGL}_A) \xrightarrow[\sim]{\mathsf{Ch}} \mathbb{Q}_{\ell,S}[\beta, \beta^{-1}]$$

Proof. This is the result of a computation of Riou [Rio10] using the γ-filtration. See also [CD19: 14.2.17, 16.1.7] and our [BRTV18: 3.35]. The element β corresponds to the inverse of the algebraic Bott element. See [BRTV18:§3.1.2]. □

We can now explain our main theorem comparing the ℓ-adic realization of the categories of singularities and the ℓ-adic sheaf of vanishing cycles. First we describe a variation of the context in which the result applies:

Context 2.2.57. Fix a diagram of schemes

$$\eta \xrightarrow{j_\eta} S \xleftarrow{i_\sigma} \sigma \tag{83}$$

with S an excellent strictly local henselian trait, namely, the spectrum of an excellent henselian discrete valuation ring A, with uniformizer π, generic point $\eta = \mathsf{Spec}(K)$ and closed point $\sigma = \mathsf{Spec}(A/m)$ with $m = (\pi)$ the maximal ideal. Assume that $\mathsf{k} := A/m$ is a algebraically closed perfect field of characteristic $p \geq 0$. Moreover, fix a separable closure $\bar{K}$ of K (inside a fixed algebraic closure), and set $\bar{\eta} := \mathsf{Spec}\ \bar{K}$ and $j_{\bar{\eta}} : \bar{\eta} \to S$.

In parallel, we also fix $p : U \to S$ a proper flat scheme over S with U regular and consider the LG-pair (U, f) where f is defined as the composite

$$f := (U \xrightarrow{p} S \xrightarrow{\pi} \mathbb{A}^1_S), \tag{84}$$

π being our fixed uniformizer.

In this case we have cartesian diagrams

$$\begin{tikzcd}
U_\eta \arrow[r, hook, "j"] \arrow[d, "p_\eta"] \arrow[dr, phantom, "\lrcorner"] & U \arrow[d, "p"] \arrow[dr, phantom, "\llcorner"] & U_\sigma = U_0 \arrow[l, "i"'] \arrow[d, "p_\sigma"] \\
\eta \arrow[r, hook, "j_\eta"'] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & S \arrow[d, "\pi"] \arrow[dr, phantom, "\llcorner"] & \sigma \arrow[l, "i_\sigma"] \arrow[d] \\
\mathbb{G}_{m,S} \arrow[r, hook, "j_0"'] \arrow[dr, "q"'] & \mathbb{A}^1_S \arrow[d] & S \arrow[l, hook', "i_0"] \arrow[dl, equal] \\
& S &
\end{tikzcd} \tag{85}$$

which, since p is flat, are also derived fiber products.

Example 2.2.58. The Context **2.2.57** is reminiscent of the topological case discussed in the Construction **2.2.41** replacing the affine line by a formal disk around 0: the scheme S plays the role of a formal disk; σ of the center of the disk; η of the punctured disk and $\bar{\eta}$ the universal cover; the Galois group $\mathsf{Gal}(\bar{\eta}/\eta)$ plays the role of the discrete groupe $\mathbb{Z}$. Examples of the Context **2.2.57** include the cases where:

- $A = \mathbb{C}[[t]]$ with $m = (t)$ and $K = \mathbb{C}((t))$;
- $A = \overline{\mathbb{F}_p}[[t]]$ with $m = (t)$ and $K = \overline{\mathbb{F}_p}((t))$;
- A the strict henselization of $\mathbb{Z}_p$ along $\mathbb{F}_p \subseteq \overline{\mathbb{F}_p}$.

Definition 2.2.59. Assume the Context **2.2.57**. Using the six operations for ℓ-adic sheaves, we define the ℓ-adic hyper-cohomology of the punctured disk η by

$$\mathbb{H}_{\mathbb{Q}_\ell}(\eta) := (i_\sigma)^*(j_\eta)_*\mathbb{Q}_{\ell,\eta}$$

in $\mathsf{Sh}_{\mathbb{Q}_\ell}(\sigma)$.

Remark 2.2.60. As expected from the cohomology of a circle, at the level of the underlying object we have

$$\mathbb{H}_{\mathbb{Q}_\ell}(\eta) \simeq \mathbb{Q}_{\ell,\sigma} \oplus \mathbb{Q}_{\ell,\sigma}(-1)[-1]$$

See [BRTV18:Lemma 4.16].

Remark 2.2.61. By lax functoriality of $(i_\sigma)^*$ and $(j_\eta)_*$, $\mathbb{H}_{\mathbb{Q}_\ell}(\eta)$ is an algebra object. In [BRTV18:Lemma 4.34] we show that $\mathbb{H}_{\mathbb{Q}_\ell}(\eta)$ satisfies a formula similar to (71), namely, as homotopy fixed points for the trivial action of $\mathsf{Gal}(\bar{\eta}/\eta)$, on the unit object.

$$\mathbb{H}_{\mathbb{Q}_\ell}(\eta) \simeq (\mathbb{Q}_{\ell,\sigma})^{\mathsf{hGal}(\bar{\eta}/\eta)}$$

Remark 2.2.62. The cohomology of the punctured disk η of the Definition **2.2.59** is the ℓ-adic replacement for the role of the cohomology of the circle S^1 in the Construction **2.2.40** and Construction **2.2.41**.

We can now state our main theorem: first it identifies the motivic realization of the dg-category of 2-periodic complexes with the β-periodized cohomology of the punctured disk, the identification being compatible with the algebra structures on each side. Second it compares their respective modules: namely, the motive of matrix factorization categories with the β-periodized inertia-invariant vanishing cycles:

Theorem 2.2.63. *[BRTV18:Prop. 4.27 and Corollary 4.43] In the Context **2.2.57** we have canonical equivalences of algebra objects in $\mathsf{Sh}_{\mathbb{Q}_\ell}(\sigma)$*

$$\mathscr{R}^{\mathsf{dg}}(\mathsf{Perf}(A[u,u^{-1}]))_{|\sigma} \simeq \mathbb{H}_{\mathbb{Q}_\ell}(\eta)[\beta,\beta^{-1}]$$

and a compatible equivalence of their respective modules

$$\mathscr{R}^{\mathsf{dg}}_A(\mathrm{Sing}^{2-\mathsf{per}}(U,f))_{|\sigma} \simeq \mathbb{H}_{\text{ét}}(U_0,\mathscr{V}_f[-1])^{\mathsf{hGal}(\bar{\eta}/\eta)}[\beta,\beta^{-1}]$$

Proof of the Theorem **2.2.63**: (Sketch) By construction, as in the Construction **2.2.40** and Construction **2.2.40**, β-periodized homotopy fixed points on vanishing cycles $\mathbb{H}_{\text{ét}}(U_0,\mathscr{V}_f[\beta,\beta^{-1})^{\mathsf{hGal}(\bar{\eta}/\eta)}[-1]$ is the fiber in $\mathsf{Sh}_{\mathbb{Q}_\ell}(\sigma)$ of the map

$$\mathbb{H}_{\mathbb{Q}_\ell}(U_0)[\beta,\beta^{-1}] \underset{\mathbb{Q}_{\ell,\sigma}[\beta,\beta^{-1}]}{\otimes} \mathbb{H}_{\mathbb{Q}_\ell}(\eta)[\beta,\beta^{-1}] \to \mathbb{H}_{\mathbb{Q}_\ell}(U_\eta)[\beta,\beta^{-1}]$$

constructed using the ℓ-adic formalism of vanishing cycles (β-periodized) analogous to (80). To compare with matrix factorizations we use the two key exact sequences of dg-categories: the first is the sequence (65) in the Definition **2.2.16** defining Sing. The second is the localization sequence for derived categories of bounded coherent sheaves for the open-closed pair (U,U_0,U_η), already used in the middle vertical column in (64). Combining the two we find a diagram of idempotent complete A-dg-categories

$$\begin{tikzcd}
\mathsf{Perf}(U_0) \arrow[r, hook] & \mathsf{Coh}^{\mathsf{b}}(U_0) \arrow[r] \arrow[d, "\text{pushforward}"] & \mathrm{Sing}^{2-\mathsf{per}}(U,f) \\
& \mathsf{Coh}^{\mathsf{b}}_{U_0}(U) \arrow[d, hook] & \\
& \mathsf{Coh}^{\mathsf{b}}(U) \underset{U \text{ regular}}{=} \mathsf{Perf}(U) \arrow[d, "\text{restriction}"] & \\
& \mathsf{Coh}^{\mathsf{b}}(U_\eta) \underset{U_\eta \text{ regular}}{=} \mathsf{Perf}(U_\eta) &
\end{tikzcd} \tag{86}$$

which, after motivic realization, turns into a cofiber/fiber sequence (Construction **2.2.50**-(i)). Now the key steps are:

- identifying $\mathscr{R}^{dg}_S(\mathsf{Perf}(U_0))_{|\sigma} = \mathbb{H}_{\mathbb{Q}_\ell}(U_0)[\beta, \beta^{-1}]$ [BRTV18: 3.13][(*)];
- identifying $\mathscr{R}^{dg}_S(\mathsf{Perf}(U))_{|\sigma} = \mathbb{H}_{\mathbb{Q}_\ell}(U_0)[\beta, \beta^{-1}]$ [BRTV18:Lemma 3.25 and Prop. 4.10];
- identifying $\mathscr{R}^{dg}_S(\mathsf{Perf}(U_\eta))_{|\sigma} = \mathbb{H}_{\mathbb{Q}_\ell}(U_\eta)[\beta, \beta^{-1}]$ [BRTV18:Lemma 3.13];
- Showing that the induced map

$$\mathscr{R}^{dg}_S(\mathsf{Perf}(U_0))_{|\sigma} \to \mathscr{R}^{dg}_S(\mathsf{Perf}(U))_{|\sigma}$$

  is null-homotopic [BRTV18:Lemma 3.26] [(†)].In particular, its cofiber is given by

$$\mathbb{H}_{\mathbb{Q}_\ell}(U_0)[\beta, \beta^{-1}] \oplus \mathbb{H}_{\mathbb{Q}_\ell}(U_0)[\beta, \beta^{-1}][1] \simeq \mathbb{H}_{\mathbb{Q}_\ell}(U_0)[\beta, \beta^{-1}] \underset{\mathbb{Q}_{\ell,\sigma}}{\otimes} \mathbb{H}_{\mathbb{Q}_\ell}(\eta)$$

  where the last equivalence is obtained by combining Remark **2.2.60** with β-periodicity.

(*) We write $|_\sigma$ for the pullback $i^*_\sigma : \mathsf{Sh}_{\mathbb{Q}_\ell}(S) \to \mathsf{Sh}_{\mathbb{Q}_\ell}(\sigma)$.

(†) This is a consequence of the fact that the class of the Koszul algebra $\mathsf{K}(A,0)$ is zero in K-theory of A because of the cofiber sequence

$$\begin{tikzcd}
A \arrow[r, "0"] \arrow[d] & A \arrow[d] \\
0 \arrow[r] & \mathsf{K}(A,0)
\end{tikzcd}$$

Finally, completing the triangles, we obtain a fiber sequence

$$\mathscr{R}_A^{\mathsf{dg}}(\mathrm{Sing}^{\mathsf{2-per}}(U,f))_{|\sigma} \to \mathbb{H}_{\mathbb{Q}_\ell}(U_0)[\beta,\beta^{-1}] \underset{\mathbb{Q}_{\ell,\sigma}}{\otimes} \mathbb{H}_{\mathbb{Q}_\ell}(\eta) \to \mathbb{H}_{\mathbb{Q}_\ell}(U_\eta)[\beta,\beta^{-1}]$$

The main part of the work is then the identification of this sequence with the one coming from the theory of vanishing cycles.

□

Remark 2.2.64. In fact in [BRTV18:Corollary 4.43] we prove a more general version of Theorem **2.2.63** showing that the assumption that S is strictly local is not necessary. In particular the conclusion of the theorem holds in the cases $A = \mathbb{F}_p[[t]]$, $K = \mathbb{F}_p((t))$ and $A = \mathbb{Z}_p$, $K = \mathbb{Q}_p$. However, in these cases we have to replace invariants under the Galois group by invariants under the inertia part. See [BRTV18:§4.1].

Remark 2.2.65. The assumption that U is regular is crucial to our result since otherwise, as explained in the Remark **2.2.44**, we have to modify the definition of Sing and therefore of the first row cofiber sequence in (86).

Application 2.2.66. In [Pip22a] Pippi extended the results of Orlov and Burke-Walker associating categories of matrix factorizations of Sections **2.2.2** and **2.2.3** to Landau-Ginzburg pairs (U, s) where s is a section of a vector bundle not necessarily the trivial bundle, and proved a version of Theorem **2.2.63**.

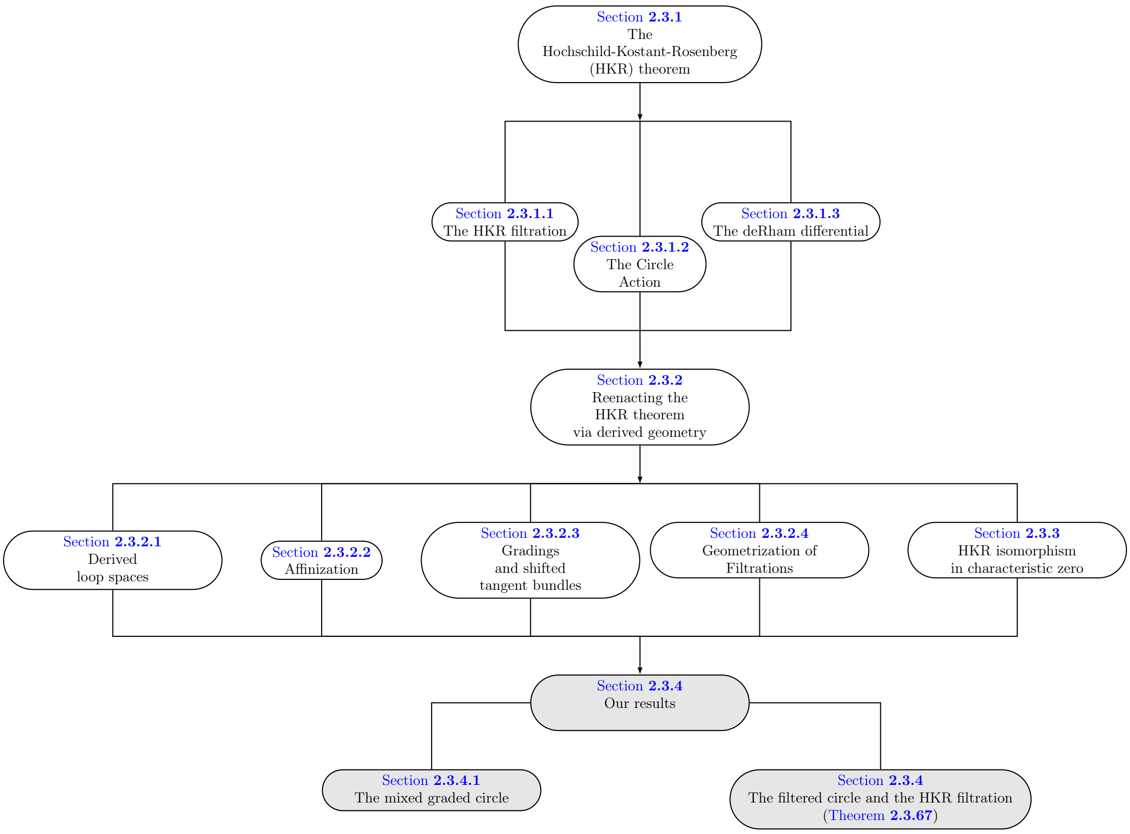
Application 2.2.67. In [TV17; TV22], Toën-Vezzosi proposed a new approach to the Bloch's conductor conjecture using Theorem **2.2.63** as a starting input. Part of this conjecture has now been proved by Beraldo-Pippi [BP22; BP23].

2.3. The filtered circle and the universal property of the HKR filtration

This section overviews the results of [MRT22], a joint work with T. Moulinos and B. Toën, on the Hochshild-Kostant-Rosenberg theorem in positive and mixed characteristic (see Theorem **2.3.67** below).

Advertisement 2.3.1. The contents of this section have a less geometrical flavor than the results discussed in Sections **2.1** and **2.2** before and Chapter **3** ahead, and seem quite far from enumerative geometry. However, one of the motivations for our Theorem **2.3.67** was to lead, in future works, to the construction of Donaldson-Thomas enumerative invariants in positive characteristic via the methods of shifted symplectic geometry discussed in Chapter **3**. See the Application **2.3.85** below.

Here's a roadmap for this section:



2.3.1. Hochschild-Kostant-Rosenberg (HKR) theorem.

In this section we fix k a (classical) commutative ring. The Hochschild-Kostant-Rosenberg (HKR) theorem [HKR62] establishes Hochschild homology classes as algebraic analogues of differential forms:

Reminder 2.3.2. Let R be a (classical) commutative k-algebra. The Hochschild homology groups $\mathsf{HH}_i(R/\mathsf{k})$ of R relativitely to k are defined by

$$\mathsf{HH}_i(R/\mathsf{k}) := \mathrm{Tor}^i_{R\otimes_{\mathsf{k}} R}(R, R)$$

Theorem 2.3.3 (HKR isomorphism)**.** *Let $X = \mathsf{Spec}(R)$ be a smooth affine scheme over k. Then there is an isomorphism identifying the Hochschild homology groups of the algebra R with the modules of i-differential forms of X - $\Omega^i_{X/\mathsf{k}}$.*

$$\mathsf{HH}_i(R/\mathsf{k}) \simeq \bigoplus_{i=1}^{n} \Omega^i_{X/\mathsf{k}} \tag{87}$$

PROOF. [HKR62]. See also [Wei94:Theorem 9.4.7]. □

2.3.1.1. ***The HKR filtration.*** The goal of this first section is to explain a reformulation of the Theorem **2.3.3** - see Proposition **2.3.15** below. We start by discussing the l.h.s of (87):

Remark 2.3.4. The groups $\mathrm{HH}_i(R/\mathsf{k})$ are defined for every commutative k-algebra R as the homology groups of the derived tensor product of k-algebras

$$\mathsf{HH}(R/\mathsf{k}) := R \underset{R \overset{\mathbb{L}}{\underset{\mathsf{k}}{\otimes}} R}{\overset{\mathbb{L}}{\otimes}} R \tag{88}$$

where R is seen as a $R \overset{\mathbb{L}}{\underset{\mathsf{k}}{\otimes}} R$-algebra using the multiplication map $R \overset{\mathbb{L}}{\underset{\mathsf{k}}{\otimes}} R \to R$. In particular, this shows that $\mathsf{HH}(R/\mathsf{k})$ carries the structure of derived ring (cf. Remark **1.1.12**).

Remark 2.3.5. Derived geometry [TV11; BZN12] offers a different perspective of the Remark **2.3.4**: through the equivalences in the Construction **1.2.16**, the derived tensor product $R \underset{R \overset{\mathbb{L}}{\underset{\mathsf{k}}{\otimes}} R}{\overset{\mathbb{L}}{\otimes}} R$ is the derived ring of functions $\mathscr{O}_{\mathsf{L}X}$ on the loop derived scheme $\mathsf{L}X$. More precisely, by definition, $\mathscr{O}_{\mathsf{L}X}$ is a derived ring and its underlying $\mathsf{E}_\infty^{\otimes}$-ring is $\Theta(\mathscr{O}_{\mathsf{L}X}) = \mathsf{HH}(R/\mathsf{k})$. See the Remark **1.1.12** for notations.

Example 2.3.6. The Remark **2.3.5** combined with the Example **1.2.19** give us the computation of HH for $\mathsf{k}[t]$:

$$\mathsf{HH}(\mathsf{k}[t]/\mathsf{k}) \simeq \mathscr{O}_{\mathsf{L}\mathbb{A}^1_{\mathsf{k}}} \simeq \mathsf{k}[t,\epsilon] \simeq \mathsf{k}[t] \oplus \mathsf{k}[t][1]$$

with zero differential. Notice that this is compatible with the HKR isomorphism since $\Omega^0_{\mathsf{k}[t]/\mathsf{k}} = \mathsf{k}[t]$ and $\Omega^1_{\mathsf{k}[t]/\mathsf{k}} = \mathsf{k}[t].dt$.

The same strategy allows us to compute $\mathsf{HH}(\mathsf{k}[x,y]/\mathsf{k})$. Indeed, we have

$$\mathsf{L}\mathbb{A}^2_{\mathsf{k}} = \mathbb{R}\mathsf{Map}_{\mathsf{k}}(\mathsf{S}^1, \mathbb{A}^2_{\mathsf{k}}) \simeq \mathbb{R}\mathsf{Map}_{\mathsf{k}}(\mathsf{S}^1, \mathbb{A}^1_{\mathsf{k}}) \times \mathbb{R}\mathsf{Map}_{\mathsf{k}}(\mathsf{S}^1, \mathbb{A}^1_{\mathsf{k}}) = \mathsf{L}\mathbb{A}^1_{\mathsf{k}} \times \mathsf{L}\mathbb{A}^1_{\mathsf{k}}$$

Therefore,

$$\mathsf{HH}(\mathsf{k}[x,y]/\mathsf{k})) = \mathscr{O}_{\mathsf{L}\mathbb{A}^2_{\mathsf{k}}} \simeq \mathscr{O}_{\mathsf{L}\mathbb{A}^1_{\mathsf{k}}} \overset{\mathbb{L}}{\underset{\mathsf{k}}{\otimes}} \mathscr{O}_{\mathsf{L}\mathbb{A}^1_{\mathsf{k}}} \simeq \mathsf{HH}(\mathsf{k}[x]/\mathsf{k}) \overset{\mathbb{L}}{\underset{\mathsf{k}}{\otimes}} \mathsf{HH}(\mathsf{k}[y]/\mathsf{k}) \simeq \mathsf{k}[x,\epsilon_1] \overset{\mathbb{L}}{\underset{\mathsf{k}}{\otimes}} \mathsf{k}[y,\epsilon_2] \simeq$$

$$\simeq \mathsf{k}[x,y] \oplus (\mathsf{k}[x,y]dx \oplus \mathsf{k}[x,y]dy)[1] \oplus (\mathsf{k}[x,y]dx\wedge dy)[2]$$

More generally, the same strategy computes HH of polynomial rings.

Construction 2.3.7. Assume R is classical and flat over k. In this case there is a more down-to-earth description of Hochschild homology. Namely, the derived tensor product (88) can be obtained as a *un-normalized Dold-Kan* construction of the simplicial object $(R,R)_\bullet$ in discrete k-modules given by

$$\cdots \; R\otimes_{\mathsf{k}} R^{\otimes 3} \;\overset{\partial_3,\,\partial_2,\,\partial_1,\,\partial_0}{\rightrightarrows\rightrightarrows}\; R\otimes_{\mathsf{k}} R\otimes_{\mathsf{k}} R \;\overset{\partial_2,\,\partial_1,\,\partial_0}{\rightrightarrows\rightarrow}\; R\otimes_{\mathsf{k}} R \;\underset{\partial_0}{\overset{\partial_1}{\rightrightarrows}}\; R \tag{89}$$

with

- $\partial_0(r_0\otimes r_1\otimes\cdots r_n) = r_0.r_1\otimes r_2\otimes\cdots r_n$
- $\partial_i(r_0\otimes r_1\otimes\cdots r_n) = r_0\otimes r_2\otimes\cdots\otimes r_i.r_{i+1}\otimes\cdots\otimes r_n$
- $\partial_n(r_0\otimes r_1\otimes\cdots r_n) = r_n.r_0\otimes r_1\otimes\cdots r_{n-1}$;

By [Lu-HAlg:§1.2.3], the Dold-Kan complex $|(R,R)_\bullet|$ is quasi-isomorphic to the geometric realization of the simplicial object (89) : $\Delta^{\mathsf{op}}\to\mathsf{Mod}_{\mathsf{k}}$ seen as a diagram in the ∞-category $\mathsf{Mod}_{\mathsf{k}}$. We can now check that

$$\mathsf{HH}(R/\mathsf{k}) \simeq |(R,R)_\bullet| \tag{90}$$

Indeed, the r.h.s is the homotopy colimit of the simplicial object $(R,R)_\bullet$. For the l.h.s one considers the bar-resolution of R as an $R\otimes_{\mathsf{k}} R^{\mathsf{op}}$-module given by the augmented simplicial object

$$
\begin{tikzcd}
\cdots\; R^{\otimes 5} \arrow[r, shift left=6, "\partial_3"] \arrow[r, shift left=2, "\partial_2"] \arrow[r, shift right=2, "\partial_1"] \arrow[r, shift right=6, "\partial_0"'] & R^{\otimes 4} \arrow[l, shift left=4, "\epsilon_2"'] \arrow[l, "\epsilon_1"'] \arrow[l, shift right=4, "\epsilon_0"'] \arrow[r, shift left=4, "\partial_2"] \arrow[r, "\partial_1"] \arrow[r, shift right=4, "\partial_0"'] & R^{\otimes 3} \arrow[l, shift left=2, "\epsilon_1"'] \arrow[l, shift right=2, "\epsilon_0"'] \arrow[r, shift left=2, "\partial_1"] \arrow[r, shift right=2, "\partial_0"'] & R^{\otimes 2} \arrow[l, "\epsilon_0"'] \arrow[r] & R
\end{tikzcd}
$$

which exhibits

$$\mathsf{colim}_{[n]\in\Delta}\, R^{n+2} \simeq R$$

Finally we have

$$l.h.s = R \underset{R\otimes_{\mathsf{k}} R}{\otimes} R \simeq R \underset{R\otimes_{\mathsf{k}} R}{\otimes} (\mathrm{colim}_{[n]\in\Delta}\, R^{n+2}) \simeq \mathsf{colim}_{[n]\in\Delta} R \underset{R\otimes R}{\otimes} (R^{n+2}) \simeq \mathsf{colim}_{[n]\in\Delta} R^{n+1}$$

In terms of the explicit model (90), the HKR-isomorphism (87) is given by the anti-symmetrization maps

$$\epsilon_n : \Omega^i_{R/\mathsf{k}} \to \mathsf{HH}_i(R/\mathsf{k}) \qquad r_0.dr_1 \wedge \cdots \wedge dr_n \mapsto \sum_{\sigma\in\Sigma_n} (-1)^{\mathsf{sign}(\sigma)} [r_0 \otimes r_{\sigma(1)} \otimes \cdots \otimes r_{\sigma(n)}]$$

Let us now discuss the r.h.s of (87):

Remark 2.3.8. The r.h.s of the HKR isomorphism (87) is given by the homology groups of the chain complex

$$\bigoplus_{i=1}^{n} \Omega^i_{X/\mathsf{k}}[i] \tag{91}$$

endowed with zero differentials. In other words, by the direct sum in $\mathsf{Mod}_{\mathsf{k}}$ of the shifted projective modules $\Omega^i_{X/\mathsf{k}}[i]$.

Furthermore, the direct sum decomposition (91) can be interpreted as a *grading*. Recall what this means:

Construction 2.3.9. Let $\mathbb{Z}^{\mathrm{disc}}$ be the category with objects given by the integer numbers and only with identity morphisms. A graded object in an ∞-category $\mathscr{C}$ is a functor $\mathsf{N}(\mathbb{Z}^{\mathrm{disc}})^{\mathsf{op}} \to \mathscr{C}$. We write $\mathscr{C}^{\mathsf{gr}}$ for the ∞-category of graded objects $\mathsf{Fun}(\mathsf{N}(\mathbb{Z})^{\mathsf{op}}, \mathscr{C})$. By construction, we have

$$\mathscr{C}^{\mathsf{gr}} = \prod_{n\in\mathbb{Z}} \mathscr{C}$$

We call the projection onto the n^{th}-coordinate

$$\mathscr{C}^{\mathsf{gr}} = \prod_{n\in\mathbb{Z}} \mathscr{C} \to \mathscr{C}$$

the graded piece of weight n. The direct sum provides a functor $\bigoplus : \mathscr{C}^{\mathsf{gr}} \to \mathscr{C}$. Moreover, if $\mathscr{C}$ is a symmetric monoidal category, $\mathscr{C}^{\mathsf{gr}}$ inherits a symmetric monoidal structure under Day convolution. See [MRT22:§2.2] for an overview.

Example 2.3.10. We interpret (91) as a graded object $\Omega^*_{X/\mathsf{k}}$ in Mod_k with graded piece of weight i given by $\Omega^i_X[i]$ if $i \in \{0, \cdots, n\}$ and zero otherwise.

The reason why we want to interpret (91) as a graded object is made clear by the following reformulation of the HKR Theorem **2.3.3**:

Proposition 2.3.11. *Let R be a smooth k-algebra. Then the chain complex $\mathsf{HH}(R/\mathsf{k}) \in \mathsf{Mod}_\mathsf{k}$ carries a natural* filtration *whose associated graded piece of weight i is $\Omega^i_{R/\mathsf{k}}[i]$.*

In order to explain this result, let us first explain what we mean by *filtration*:

Construction 2.3.12. Consider $(\mathbb{Z}, \leq)$ the poset of integers numbers and let $\mathbb{Z}^{\leq}$ be the 1-category associated to this poset [(*)]. Let $\mathscr{C}$ be an ∞-category. A *filtered object* in $\mathscr{C}$ is a functor $\mathsf{N}(\mathbb{Z}^{\leq})^{\mathsf{op}} \to \mathscr{C}$. We denote by $\mathsf{Fil}(\mathscr{C})$ the ∞-category of filtered objects in $\mathscr{C}$, ie, $\mathsf{Fun}(\mathsf{N}(\mathbb{Z}^{\leq})^{\mathsf{op}}, \mathscr{C})$. Informally, we picture a filtered object X in $\mathscr{C}$ as a diagram

$$\cdots \to X_{n+1} \to X_n \to X_{n-1} \to \cdots \to X_2 \to X_1 \to X_0 \to X_{-1} \to X_{-2} \to \cdots$$

We define:

- The n^{th}-graded piece of a filtered object X is the cofiber in $\mathscr{C}$

  $$\mathsf{gr}^n(X) := X_n/X_{n+1}$$

  The construction sending a filtered object X to the collection of its graded pieces $(\mathsf{gr}^n(X))_{n\in\mathbb{Z}}$ defines a functor

  $$\mathsf{gr} : \mathsf{Fil}(\mathscr{C}) \to \mathscr{C}^{\mathsf{gr}}$$

  called the *associated graded object.*

- Given a filtered object $X : \mathsf{N}(\mathbb{Z}^{\leq})^{\mathsf{op}} \to \mathscr{C}$ its image under the colimit functor $\mathsf{colim} : \mathsf{Fun}(\mathsf{N}(\mathbb{Z}^{\leq})^{\mathsf{op}}, \mathscr{C}) \to \mathscr{C}$ is called the *underlying object* of X.

For more details see [MRT22:§2.2].

[(*)] Recall that if $(P, \leq)$ is a poset, the category $P^{\leq}$ has objects given by the elements in P and a unique morphism $x \to y$ if and only if $x \leq y$. Compositions are defined using the transitivity of the partial order.

Let us now explain the proof of the Proposition **2.3.11**.

Construction 2.3.13. Recall that $\mathsf{Mod}_{\mathsf{k}}$ admits a t-structure (cf. Construction **1.1.24**). In particular, for every object $E \in \mathsf{Mod}_{\mathsf{k}}$ the successive truncations

$$\cdots \to \tau_{\geq 2} E \to \tau_{\geq 1} E \to \tau_{\geq 0} E \to \tau_{\geq -1} E \to \cdots$$

can be interpret as a *filtered object* $\tau_{\geq} E \in \mathsf{Fil}(\mathsf{Mod}_{\mathsf{k}})$. We call this the Whitehead tower filtration of E. The assignment $E \mapsto \tau_{\geq} E$ provides a functor

$$\tau_{\geq} : \mathsf{Mod}_{\mathsf{k}} \to \mathsf{Fil}(\mathsf{Mod}_{\mathsf{k}}) \tag{92}$$

Finally, assume that E is connected (ie, $\tau_{\geq 0} E \simeq E$). In this case, by definition of a t-structure, the graded piece of weight i is the homology of E shifted to degree i, $\mathsf{H}_i(E)[i]$.

Proof of the Proposition **2.3.11**: Consider the Whitehead filtration on $\mathsf{HH}(R/\mathsf{k})$. By the Theorem **2.3.3** and the Construction **2.3.13** its graded pieces of weight i are precisely $\Omega^i_{R/\mathsf{k}}[i]$. □

To conclude this section we explain how to extend the filtration of the Proposition **2.3.11** to derived rings:

Construction 2.3.14. Both sides of the HKR isomorphism make sense for derived rings over k:

- The definition of HH remains unchanged via the derived tensor product formula (88);
- The r.h.s (91) still makes sense if we replace the projective module of Kähler forms $\Omega_{R/\mathsf{k}}$ by the cotangent complex $\mathbb{L}_{R/\mathsf{k}}$ and the wedge powers $\Omega^i[i]$ by the derived wedge powers $(\Lambda^i \mathbb{L}_{R/\mathsf{k}})[i]$;

These define ∞-functors

$$\mathsf{dRings}_{\mathsf{k}} \to \mathsf{Mod}_{\mathsf{k}} \qquad A \mapsto \mathsf{HH}(A/\mathsf{k}) \tag{93}$$

$$\mathsf{dRings}_{\mathsf{k}} \to \mathsf{Mod}_{\mathsf{k}} \qquad A \mapsto (\Lambda^i \mathbb{L}_{A/\mathsf{k}})[i] \tag{94}$$

We have the following result:

Proposition 2.3.15. *Let A be a derived ring over k. Then the chain complex $\mathsf{HH}(A/\mathsf{k}) \in \mathsf{Mod}_{\mathsf{k}}$ carries a natural* filtration *whose associated graded piece of weight i is $(\Lambda^i \mathbb{L}^i_{A/\mathsf{k}})[i]$.*

PROOF. A proof is available in [NS18: IV. 4.1]. We reproduce here the key arguments. The basic fact is that both constructions (93) and (94) preserve sifted colimits. In particular, since $\mathsf{dRings}_{\mathsf{k}}$ is the sifted completion of the 1-category of polynomial rings $\mathsf{Poly}_{\mathsf{k}}$ (cf. Remark **1.1.12**), this means that both functors (93) and (94) are obtained as left Kan extensions from their restrictions to polynomial algebras. See for instance the [BMS19:Example 2.2] for the cotangent complex and [BMS19:Remark 2.3] for HH for direct proofs. But on polynomial algebras, we know that HH carries the Whitehead tower filtration (Proposition **2.3.11**) given by the composition

$$\mathsf{Poly}_{\mathsf{k}} \xrightarrow{\mathsf{HH}} \mathsf{Mod}_{\mathsf{k}} \xrightarrow{(92)} \mathsf{Fil}(\mathsf{Mod}_{\mathsf{k}}) \tag{95}$$

Notice that by the connectivity of HH implied by the proof of the Proposition **2.3.11**, the composition

$$\mathsf{Poly}_{\mathsf{k}} \xrightarrow{\mathsf{HH}} \mathsf{Mod}_{\mathsf{k}} \xrightarrow{(92)} \mathsf{Fil}(\mathsf{Mod}_{\mathsf{k}}) \xrightarrow{\mathsf{colim}} \mathsf{Mod}_{\mathsf{k}} \tag{96}$$

is again HH. Therefore, we can consider the left Kan extension of (95) from polynomial rings to all derived rings

$$\begin{tikzcd} \mathsf{Poly}_{\mathsf{k}} \arrow[r, "\mathsf{HH}"] \arrow[d] & \mathsf{Mod}_{\mathsf{k}} \arrow[r, "(92)"] & \mathsf{Fil}(\mathsf{Mod}_{\mathsf{k}}) \\ \mathsf{dRings}_{\mathsf{k}} \arrow[urr, dashed, "\mathsf{HH}^{\mathsf{Fil}}"'] & & \end{tikzcd} \tag{97}$$

Since the functor $\mathsf{colim} : \mathsf{Fil}(\mathsf{Mod}_{\mathsf{k}}) \to \mathsf{Mod}_{\mathsf{k}}$ preserves all colimits, it follows that the composition

$$\begin{tikzcd} \mathsf{Poly}_{\mathsf{k}} \arrow[r, "\mathsf{HH}"] \arrow[d] & \mathsf{Mod}_{\mathsf{k}} \arrow[r, "(92)"] & \mathsf{Fil}(\mathsf{Mod}_{\mathsf{k}}) \arrow[r, "\mathsf{colim}"] & \mathsf{Mod}_{\mathsf{k}} \\ \mathsf{dRings}_{\mathsf{k}} \arrow[urr, dashed] \arrow[urrr, dashed] & & & \end{tikzcd} \tag{98}$$

gives back the functor (93). The functor $\mathsf{HH}^{\mathsf{Fil}}$ obtained in (97) therefore endows HH of derived rings with a filtration. It remains to show the claim about its graded pieces. For this purpose we consider the composition of (95) with the associated-graded functor

$$\mathsf{Poly}_{\mathsf{k}} \xrightarrow{\mathsf{HH}} \mathsf{Mod}_{\mathsf{k}} \xrightarrow{(92)} \mathsf{Fil}(\mathsf{Mod}_{\mathsf{k}}) \xrightarrow{\mathsf{gr}} \mathsf{Mod}^{\mathsf{gr}}_{\mathsf{k}} \tag{99}$$

sending a polynomial ring R to the graded object $((\Lambda^i \Omega^1_{R/\mathsf{k}})[i])_{i\in\mathbb{Z}}$. Since gr commutes with all colimits (see [Lurpt:§3.2] or as a consequence of the Lemma **2.3.53** below) and (94) is the left Kan extension of its restriction to polynomial rings, we find that the graded pieces of $\mathsf{HH}^{\mathsf{Fil}}(A)$ on a derived ring, are given by the $(\Lambda^i \mathbb{L}_{A/\mathsf{k}})[i]$. □

When k is a $\mathbb{Q}$-algebra, the conclusion of the Proposition **2.3.15** simplifies:

Construction 2.3.16. Let $(E_n)_{n\in\mathbb{Z}} \in \mathsf{Mod}^{\mathsf{gr}}_{\mathsf{k}}$ be graded object. We assign to it a filtered object defined as follows

$$\cdots \hookrightarrow \bigoplus_{n\geq 2} E_n \hookrightarrow \bigoplus_{n\geq 1} E_n \hookrightarrow \bigoplus_{n\geq 0} E_n \hookrightarrow \bigoplus_{n\geq -1} E_n \hookrightarrow \bigoplus_{n\geq -2} E_n \hookrightarrow \cdots$$

This procedure defines a functor

$$\mathrm{triv} : \mathsf{Mod}^{\mathsf{gr}}_{\mathsf{k}} \to \mathsf{Fil}(\mathsf{Mod}_{\mathsf{k}})$$

Terminology 2.3.17. We say that a filtered object $X \in \mathsf{Fil}(\mathsf{Mod}_{\mathsf{k}})$ admits a *splitting* if there exists an equivalence of filtered objects

$$X \simeq \mathrm{triv}(\mathsf{gr}(X))$$

Remark 2.3.18. Notice that a splitting on a filtered object $X \in \mathsf{Fil}(\mathsf{Mod}_{\mathsf{k}})$ induce retractions of the maps $X_n \to X_{n-1}$.

Proposition 2.3.19. *Let* k *be a* $\mathbb{Q}$*-algebra and* $A \in \mathsf{dRings}_{\mathsf{k}}$. *Then* $\mathsf{HH}^{\mathsf{Fil}}(A)$ *admits a splitting given by the anti-symmetrization maps. In particular, we have an equivalence of objects in* $\mathsf{Mod}_{\mathsf{k}}$

$$\mathsf{HH}(A/\mathsf{k}) \simeq \bigoplus_{i=1}^{n} (\Lambda^i \mathbb{L}_{A/\mathsf{k}})[i] \tag{100}$$

Proof. See [Yek02:Theorem 4.8] and [Lod92:Prop. 1.3.16, Remark 3.2.3, Prop. 5.4.6]. See also [Bar68] and [Qui70: 8.6] □

Warning 2.3.20. In positive characteristic we know that under certain hypothesis the filtration $\mathsf{HH}^{\mathsf{Fil}}$ also splits. See for instance [AV20] and [Yek02]. However, this is not the general case: see [ABM21] for counter-examples.

The main purpose of our main result (Theorem **2.3.67** below) is to clarify the universal property of the filtered object $\mathsf{HH}^{\mathsf{Fil}}(A/\mathsf{k})$ with respect to the derived ring A, independently of the characteristic of k. In order to explain this we need to bring into play three aspects that the formulation of the Proposition **2.3.15** has ignored:

- The structure of derived ring on $\mathsf{HH}(A/\mathsf{k})$ (cf. Remark **2.3.4**);
- The action of the topological circle S^1 on $\mathsf{HH}(A/\mathsf{k})$.
- A key ingredient that hasn't been mentioned yet, namely, the de Rham differential d_{dR} on the r.h.s of (100) mapping $\Lambda^i\mathbb{L}_{A/\mathsf{k}} \to \Lambda^{i+1}\mathbb{L}_{A/\mathsf{k}}$.

2.3.1.2. ***The circle action.***

Construction 2.3.21. Since in the ∞-category $\mathsf{CAlg}_{\mathsf{k}}$ derived tensor products are pushouts (Remark **1.1.11**), we find

$$\mathsf{HH}(R/k) := R \coprod^{\mathsf{h}}_{R\coprod^{\mathsf{h}}_{\mathsf{k}}R} R \simeq R \overset{\mathbb{L}}{\underset{\mathsf{k}}{\otimes}} (* \coprod^{\mathsf{h}}_{*\coprod *} *) \simeq R \overset{\mathbb{L}}{\underset{\mathsf{k}}{\otimes}} \mathsf{S}^1 \tag{101}$$

where the middle equivalence uses the fact the ∞-category $\mathsf{CAlg}_{\mathsf{k}}$ is *tensored* over the ∞-category of spaces $\mathscr{S}$ [(*)]. Therefore, the action of S^1 on itself (cf (19)) induces an action of S^1 on HH

$$\mathsf{S}^1 \overset{\mathbb{L}}{\underset{\mathsf{k}}{\otimes}} \mathsf{HH}(R/\mathsf{k}) = \mathsf{S}^1 \overset{\mathbb{L}}{\underset{\mathsf{k}}{\otimes}} \left(\mathsf{S}^1 \overset{\mathbb{L}}{\underset{\mathsf{k}}{\otimes}} R\right) \simeq (\mathsf{S}^1 \times \mathsf{S}^1) \overset{\mathbb{L}}{\underset{\mathsf{k}}{\otimes}} R \underset{(19)}{\to} \mathsf{S}^1 \overset{\mathbb{L}}{\underset{\mathsf{k}}{\otimes}} R = \mathsf{HH}(R/\mathsf{k}) \tag{102}$$

Remark 2.3.22. Notice that by definition of the tensor action of $\mathscr{S}$ on $\mathsf{CAlg}_{\mathsf{k}}$, the action (102) corresponds to a map of monoid objects in $\mathscr{S}$

$$\mathsf{S}^1 \to \mathrm{End}_{\mathsf{CAlg}_{\mathsf{k}}}(\mathsf{HH}(R/\mathsf{k}))$$

which is the same as a functor $\mathsf{BS}^1 \to \mathsf{CAlg}_{\mathsf{k}}$ sending the unique object $\bullet$ in BS^1 to $\mathsf{HH}(R/\mathsf{k})$.

Notation 2.3.23. We write $\mathsf{S}^1 - \mathsf{CAlg}_{\mathsf{k}}$ for the ∞-category of objects in $\mathsf{CAlg}_{\mathsf{k}}$ endowed with an action of the circle as in the Remark **2.3.4**, ie

$$\mathsf{S}^1 - \mathsf{CAlg}_{\mathsf{k}} := \mathsf{Fun}(\mathsf{BS}^1, \mathsf{CAlg}_{\mathsf{k}})$$

Similarly, we write $\mathsf{S}^1 - \mathsf{Mod}_{\mathsf{k}}$ for the linear version

---

[(*)]By definition, the tensor structure over $\mathscr{S}$ means that for every $K \in \mathscr{S}$ and $A, B \in \mathsf{CAlg}_{\mathsf{k}}$, we have

$$\mathsf{Map}_{\mathscr{S}}(K, \mathsf{Map}_{\mathsf{CAlg}_{\mathsf{k}}}(A, B)) \simeq \mathsf{Map}_{\mathsf{CAlg}_{\mathsf{k}}}(A \overset{\mathbb{L}}{\underset{\mathsf{k}}{\otimes}} K, B)$$

In particular, it follows from this universal property that if K_1 and K_2 are objects in $\mathscr{S}$, we have $(K_1 \times K_2) \overset{\mathbb{L}}{\underset{\mathsf{k}}{\otimes}} A \simeq K_1 \overset{\mathbb{L}}{\underset{\mathsf{k}}{\otimes}} K_2 \overset{\mathbb{L}}{\underset{\mathsf{k}}{\otimes}} A$.

$$\mathsf{S}^1 - \mathsf{Mod}_\mathsf{k} := \mathsf{Fun}(\mathsf{BS}^1, \mathsf{Mod}_\mathsf{k}) \underset{\textit{Construction } \mathbf{1.2.14}}{=} \mathsf{QCoh}(\mathsf{BS}^1)$$

Remark 2.3.24. Following the Construction **1.2.14** and the Remark **1.2.15**, $\mathsf{S}^1 - \mathsf{Mod}_\mathsf{k}$ inherits the symmetric monoidal structure of Mod_k and we have

$$\mathsf{S}^1 - \mathsf{CAlg}_\mathsf{k} = \mathsf{CAlg}(\mathsf{S}^1 - \mathsf{Mod}_\mathsf{k})$$

Remark 2.3.25. The abelian group structure on S^1 induces a structure of $\mathsf{E}_\infty^\otimes$-algebra on the complex of singular chains $\mathsf{C}_*(\mathsf{S}^1, \mathsf{k})$. The diagonal $\mathsf{S}^1 \to \mathsf{S}^1 \times \mathsf{S}^1$ makes $\mathsf{C}_*(\mathsf{S}^1, \mathsf{k})$ a commutative Hopf algebra in $\mathsf{E}_\infty^\otimes$-algebras. In particular, the ∞-category $\mathsf{Mod}_{\mathsf{C}_*(\mathsf{S}^1,\mathsf{k})}(\mathsf{Mod}_\mathsf{k})$ inherits a symmetric monoidal structure. The underlying object functor induces a symmetric monoidal equivalence

$$\mathsf{QCoh}(\mathsf{BS}^1)^\otimes = \mathsf{Fun}(\mathsf{BS}^1, \mathsf{Mod}_\mathsf{k})^\otimes \simeq \mathsf{Mod}_{\mathsf{C}_*(\mathsf{S}^1,\mathsf{k})}(\mathsf{Mod}_\mathsf{k})^{\otimes_{\mathrm{Hopf}}}$$

See [Arp20:§2].

Remark 2.3.26. Alternatively, we can describe $\mathsf{QCoh}(\mathsf{BS}^1)^\otimes$ in terms of $\mathsf{C}^*(\mathsf{S}^1, \mathsf{k})$-comodules, where $\mathsf{C}^*(\mathsf{S}^1, \mathsf{k})$ is equipped with the Hopf algebra structure induced from the group structure on BS^1. This induces an equivalence of symmetric monoidal ∞-categories

$$\mathsf{QCoh}(\mathsf{BS}^1) \simeq \mathsf{C}^*(\mathsf{S}^1, \mathsf{k}) - \mathsf{CoMod}_\mathsf{k}^\otimes$$

where on comodules the symmetric monoidal structure is induced by the natural multiplicative structure on singular cochains. See [BZN12: 3.10].

Construction 2.3.27. To obtain the circle action in the explicit model of the Construction **2.3.7**, one remarks that the simplicial object (89) carries extra symmetries given by the action of the cyclic groups C_{n+1} on the level n of the simplicial object. Namely, the cyclic permutation τ_n acts by

$$\tau_n(r_0 \otimes r_1 \otimes ... \otimes r_n) = (-1)^n r_n \otimes r_0 \otimes ... \otimes r_{n-1}$$

with the following compatibilities:

- $\tau^n \delta_i = \delta_{i-1}\tau_{n-1}$ for $1 \leq i \leq n$;
- $\tau_n^{n+1} = \mathsf{id}$
- $\tau_n \delta_0 = \delta_n$
- $\tau^n \epsilon_i = \epsilon_{i-1}\tau_{n-1}$ for $1 \leq i \leq n$;
- $\tau_n \epsilon_0 = \epsilon_n \tau_{n+1}^2$;

The cyclic category Λ, introduced by Connes in [Con83], is defined by adding to Δ the cyclic permutations $\tau_n : [n] \to [n]$ and imposing the relations above. See also [Lod92:Chapter 6]. The observation that (89) possesses the extra cyclic symmetries means that (89) : $\Delta^{\mathsf{op}} \to \mathsf{Mod}_{\mathsf{k}}$ lifts to a functor $\Lambda^{\mathsf{op}} \to \mathsf{Mod}_{\mathsf{k}}$.

Finally, the circle action of the Remark **2.3.4** is a consequence of a computation due to Connes [Con83:Theoreme 10] showing that the classifying space of the category Λ (obtained by weakly-inverting all morphisms) is the classifying space of the circle:

$$\mathsf{N}(\Lambda)[W_{\text{all}}^{-1}] \simeq \mathsf{BS}^1 = \mathsf{K}(\mathbb{Z}, 2)$$

Indeed, we have a commutative diagram

$$\begin{tikzcd}
\mathsf{Fun}(\mathsf{N}(\Lambda^{\mathsf{op}}), \mathsf{Mod}_{\mathsf{k}}) \arrow[rr, "\text{forget}"] \arrow[d, "\mathsf{LKE}"] & & \mathsf{Fun}(\mathsf{N}(\Delta)^{\mathsf{op}}, \mathsf{Mod}_{\mathsf{k}}) \arrow[d, "\mathsf{colim}"] \\
\mathsf{Fun}(\mathsf{BS}^1, \mathsf{Mod}_{\mathsf{k}}) = \mathsf{Fun}(\mathsf{N}(\Lambda^{\mathsf{op}})[W_{\text{all}}^{-1}], \mathsf{Mod}_{\mathsf{k}}) \arrow[rr, "\text{forget}"] & & \mathsf{Fun}(\mathsf{N}(\Delta^{\mathsf{op}})[W_{\text{all}}^{-1}], \mathsf{Mod}_{\mathsf{k}}) \simeq \mathsf{Mod}_{\mathsf{k}}
\end{tikzcd}$$

which uses the fact that the classifying space of the category $\mathsf{N}(\Delta)$ is contractible [Qui73:Corollary I p.8]. This diagram implies that the geometric realization $|(R, R)_\bullet|$ admits a circle action. Finally, it remains to compare the circle actions on both sides of the (90), namely, the one constructed here and the one of the Construction **2.3.21**. This is done in [Hoy15:after Remark 1.4].

Finally, we formulate the universal property of Hochschild homology:

Proposition 2.3.28. *The assignment* $R \mapsto [\mathsf{S}^1 \circlearrowright \mathsf{HH}(R/\mathsf{k})]$ *provides a left adjoint to the forgetful functor* $\mathsf{S}^1 - \mathsf{CAlg}_{\mathsf{k}} \to \mathsf{CAlg}_{\mathsf{k}}$.

Proof. Indeed, this follows from a general argument: if G is a group object in the ∞-category $\mathscr{S}$ and $\bullet \in \mathsf{B}G$ denotes the unique point then, for any ∞-category $\mathscr{C}$ tensored over $\mathscr{S}$, one can show by computation that the left Kan extension along the inclusion $\bullet : * \to \mathsf{B}G$

$$\mathsf{LKE} : \mathscr{C} \to \mathsf{Fun}(\mathsf{B}G, \mathscr{C})$$

sends an object $C \in \mathscr{C}$ to the functor $\mathsf{B}G \to \mathscr{C}$ sending $\bullet$ to the tensor $G \otimes C$ and $G = \mathrm{Aut}_\bullet(\mathsf{B}G)$ to the action of G on $G \otimes C$ as in the Construction **2.3.21**. □

Panorama 2.3.29. In recent years, Hochschild homology, more precisely its *topological version*(*) together with its circle action and cyclotomic structure [NS18], found spectacular applications in both arithmetic geometry [BMS19] and algebraic topology, such as the redshift conjecture [HW22]. See [KN; Mat22; Hes23] for surveys. As we shall explain below, our discussion here is closer in spirit to the stacky interpretation of the construction of *prismatic cohomology* as introduced by Drinfeld [Dri18; Dri20; Dri21a; Dri21b] and developed by Lurie-Bhatt [BL22]. We will discuss this again below.

(*)ie, when the base ring $\mathsf{k} = \mathbb{S}$ is the sphere spectrum.

An explicit description of the circle action of the Construction **2.3.21** on the underlying chain complex HH can be given in terms of the notion of *mixed modules* [Bur86; Kas87]:

Remark 2.3.30. Let $\mathsf{H}_*(\mathsf{S}^1, \mathsf{k})$ be the strictly associative dg-algebra enconding the homology of the circle with coefficients in k. As a dg-algebra over k, this is the split square zero algebra with a generator ϵ in (homological) degree 1, therefore verifying $\epsilon^2 = 0$, ie

$$\mathsf{H}_*(\mathsf{S}^1, \mathsf{k}) = \mathsf{k}[\epsilon]/\epsilon^2, \quad |\epsilon| = 1$$

Lemma 2.3.31. *Fix γ a generator of $\mathsf{H}_1(\mathsf{S}^1, \mathsf{k})$. Then the assignment $\epsilon \mapsto \gamma$ determines an equivalence of $\mathsf{E}_1^{\otimes}$-algebras*

$$\mathsf{H}_*(\mathsf{S}^1, \mathsf{k}) \simeq \mathsf{C}_*(\mathsf{S}^1, \mathsf{k})$$

In other words, the $\mathsf{E}_\infty^{\otimes}$-algebra $\mathsf{C}_*(\mathsf{S}^1, \mathsf{k})$ is formal as a $\mathsf{E}_1^{\otimes}$-algebra, independently of the characteristic of k.

Construction 2.3.32. The Lemma **2.3.31** implies in particular that we have an equivalence left module ∞-categories

$$\mathsf{Mod}_{\mathsf{C}_*(\mathsf{S}^1,\mathsf{k})}(\mathsf{Mod}_\mathsf{k}) \simeq \mathsf{LMod}_{\mathsf{C}_*(\mathsf{S}^1,\mathsf{k})}(\mathsf{Mod}_\mathsf{k}) \simeq \mathsf{LMod}_{\mathsf{H}_*(\mathsf{S}^1,\mathsf{k})}(\mathsf{Mod}_\mathsf{k})$$

which is not compatible with the symmetric monoidal structures. In any case, this allow us to describe a circle action on $M \in \mathsf{Mod}_\mathsf{k}$ as the action of an operator $\epsilon : M[1] \to M$ verifying $\epsilon^2 \sim 0$. In the case where $M = \mathsf{HH}$ this is the Connes operator B. See [Hoy15:Theorem 2.3]. for an explicit description in terms of the cyclic objects. In particular, this induces maps

$$B : \mathsf{HH}_n(R/\mathsf{k}) \to \mathsf{HH}_{n+1}(R/\mathsf{k})$$

To conclude this section, we discuss the compatibility between the circle action of the Construction **2.3.21** and the HKR filtration $\mathsf{HH}^{\mathsf{Fil}}$ of the Proposition **2.3.15**:

Construction 2.3.33. The t-structure in $\mathsf{QCoh}(\mathsf{BS}^1)$ of the Construction **1.1.24**, corresponds to the objectwise t-structure on $\mathsf{Fun}(\mathsf{BS}^1, \mathsf{Mod}_\mathsf{k})$ under the equivalence in Construction **1.2.14**. In particular, the Whithead tower

$$\tau_\geq : \mathsf{QCoh}(\mathsf{BS}^1) \to \mathsf{Fil}(\mathsf{QCoh}(\mathsf{BS}^1))$$

is homotopic to the composition with the Whitehead tower functor on Mod_k

$$\mathsf{QCoh}(\mathsf{BS}^1) = \mathsf{Fun}(\mathsf{BS}^1, \mathsf{Mod}_\mathsf{k}) \xrightarrow{\tau_\geq \circ -} \mathsf{Fun}(\mathsf{BS}^1, \mathsf{Fil}(\mathsf{Mod}_\mathsf{k})) \simeq \mathsf{Fil}(\mathsf{QCoh}(\mathsf{BS}^1))$$

In other words, since each $\tau_{\geq n}$ is a functor and $\tau_{\geq n} \to \mathsf{id}$ is a natural transformation, each level of the Whitehead tower of an object $E \in \mathsf{QCoh}(\mathsf{BS}^1)$ carries a circle action and the transition morphisms are compatible with it.

It follows that the functor $\mathsf{HH}^{\mathsf{Fil}}$ admits a canonical lifting

$$\begin{tikzcd} & \mathsf{Fil}(\mathsf{QCoh}(\mathsf{BS}^1)) \arrow[d, "\mathsf{forget}"] \\ \mathsf{dRings}_{\mathsf{k}} \arrow[ur, dashed] \arrow[r, "\mathsf{HH}^{\mathsf{Fil}}"] & \mathsf{Fil}(\mathsf{Mod}_{\mathsf{k}}) \end{tikzcd}$$

Problem 2.3.34. Consider the filtered object $\mathsf{HH}^{\mathsf{Fil}}$ with the levelwise circle action of the Construction **2.3.33**. By definition, the graded pieces $\mathsf{gr}^i(\mathsf{HH}^{\mathsf{Fil}})$ live in the heart of the t-structure on $\mathsf{QCoh}(\mathsf{BS}^1) = \mathsf{Fun}(\mathsf{BS}^1, \mathsf{Mod}_{\mathsf{k}})$, ie, in the ∞-category

$$\mathsf{Fun}(\mathsf{BS}^1, \mathsf{Mod}_{\mathsf{k}}^{\heartsuit}) \simeq \mathsf{Fun}(\mathsf{h}(\mathsf{BS}^1), \mathsf{Mod}_{\mathsf{k}}^{\heartsuit}) \simeq \mathsf{Fun}(*, \mathsf{Mod}_{\mathsf{k}}^{\heartsuit}) = \mathsf{Mod}_{\mathsf{k}}^{\heartsuit}$$

In particular, each $\mathsf{gr}^i(\mathsf{HH}^{\mathsf{Fil}})$ is concentrated in a single homological degree and the induced circle action is trivial.

2.3.1.3. ***The deRham differential.***

Notation 2.3.35. Let R be a smooth polynomial k-algebra. We denote by d_{dR} de Rham differential given by k-linear maps

$$\mathsf{d}_{\mathsf{dR}} : \Omega^i_{R/\mathsf{k}} \to \Omega^{i+1}_{R/\mathsf{k}}$$

with $\mathsf{d}_{\mathsf{dR}}^2 = 0$.

The de Rham differential is related to the circle action of Section **2.3.1.2**, more precisely, to the B-operator of the Construction **2.3.32**:

Proposition 2.3.36. *Let R be a smooth k-algebra. Then the HKR-isomorphism (87) matches the B-operator with the d_{dR}-differential, ie, the diagrams*

(103)
$$\begin{tikzcd} \mathsf{HH}_i(R/\mathsf{k}) \arrow[r, "B"] \arrow[d, "\sim"] & \mathsf{HH}_{i+1}(R/\mathsf{k}) \arrow[d, "\sim"] \\ \Omega^i_{R/\mathsf{k}} \arrow[r, "\mathsf{d}_{\mathsf{dR}}"] & \Omega^{i+1}_{R/\mathsf{k}} \end{tikzcd}$$

commute.

PROOF. See [Lod92:Prop. 2.3.3]. □

This finally brings us to the central problem that motivated our work. The Proposition **2.3.36** expresses the operator B - a consequence of the circle action - as the deRham differential on forms. However, we saw in the Problem **2.3.34** that the naive circle action on the graded pieces is trivial. This problem has been identified in [Ant19]. So the content of the Proposition **2.3.36** cannot be interpreted as a consequence of the circle action, at least, directly. Furthermore, the Proposition **2.3.36** does not integrate the multiplicative structure on HH, the HKR-filtration and how the $\Omega^i_{X/\mathsf{k}}$ appear as associated graded pieces. This is the starting point of our work in [MRT22] under the motto that the *circle itself needs to be considered as a filtered group in a non-trivial fashion* in order to bring together these elements in a coherent way. See our main result Theorem **2.3.67** below.

A first step, is a re-interpretation of the de Rham differential as an *extra structure* on the *graded* module $\Omega^*_{R/\mathsf{k}}$ of the Example **2.3.10**, namely, as a structure of *mixed graded module*:

Construction 2.3.37. Consider $\mathsf{k}[\epsilon]_{\mathsf{gr}}$ the $\mathbb{Z}$-graded strictly associative dg-algebra freely generator by an element ϵ in homological degree 1 and weight 1, and *strictly* verifying $\epsilon^2 = 0$. As an object, this lives in the 1-category of strictly associative algebra objects in the 1-category of $\mathbb{Z}$-graded objects in the 1-category of chain complexes - $\mathrm{Ch}(\mathsf{k})^{\mathsf{gr}}$- with respect to the Day convolution. By definition of $\mathsf{k}[\epsilon]_{\mathsf{gr}}$ a structure of strict left $\mathsf{k}[\epsilon]_{\mathsf{gr}}$-module in $\mathrm{Ch}(\mathsf{k})^{\mathsf{gr}}$ consists of a $\mathbb{Z}$-graded chain complex $M \in \mathrm{Ch}(\mathsf{k})^{\mathsf{gr}}$, together with an action of an operator

$$\epsilon : M(1)[1] \to M$$

.
satisfying the strict relation $\epsilon \circ \epsilon = 0$ and where $M(1)[1]$ is graded chain complex obtained from M positioning the graded piece of weight i in weight $i+1$ but with a homological shift of 1, ie if $M = (M_i)_{i\in\mathbb{Z}}$, $M(1)[1] = (M_{i+1}[1])_{i\in\mathbb{Z}}$.

The $\mathbb{Z}$-graded associative dg-algebra $\mathsf{k}[\epsilon]_{\mathsf{gr}}$ carries a strictly commutative graded Hopf structure

$$\mathsf{k}[\epsilon]_{\mathsf{gr}} \to \mathsf{k}[\epsilon]_{\mathsf{gr}} \underset{\mathsf{gr}}{\otimes} \mathsf{k}[\epsilon]_{\mathsf{gr}}$$

determined by

$$\epsilon \mapsto \epsilon \otimes 1 + 1 \otimes \epsilon$$

This makes the 1-category of strict left graded dg-modules $\mathrm{LMod}^{\mathsf{strict}}_{\mathsf{k}[\epsilon]_{\mathsf{gr}}}(\mathrm{Ch}(\mathsf{k})^{\mathsf{gr}})$ a symmetric monoidal 1-category. We define the symmetric monoidal ∞-category of mixed graded modules over k - $\mathsf{Mod}_{\mathsf{k}}^{\epsilon-\mathsf{gr},\otimes}$ - as its underlying symmetric monoidal ∞-category obtained by inverting weight-wise quasi-isomorphisms.

Remark 2.3.38. Forgetting the left module structure over $\mathsf{k}[\epsilon]_{\mathsf{gr}}$ induces an ∞-functor (after passing to Dwyer-Kan localizations)

$$\mathsf{Mod}_{\mathsf{k}}^{\epsilon-\mathsf{gr}} \to \mathsf{Mod}_{\mathsf{k}}^{\mathsf{gr}}$$

Furthermore, after composition with the functor $\bigoplus : \mathsf{Mod}_{\mathsf{k}}^{\mathsf{gr}} \to \mathsf{Mod}_{\mathsf{k}}$[(*)], we find a direct sum decomposition in $\mathsf{Mod}_{\mathsf{k}}$

$$\bigoplus(\mathsf{k}[\epsilon]_{\mathsf{gr}}) \simeq \mathsf{k} \oplus \mathsf{k}[1]$$

which exhibits $\mathsf{k}[\epsilon]_{\mathsf{gr}}$ as a *graded* upgrade of the associative algebra $\mathsf{k}[\epsilon]/\epsilon^2$ in the Remark **2.3.30**.

Definition 2.3.39. A *mixed graded algebra* over k is a commutative algebra object in the symmetric monoidal ∞-category $\mathsf{Mod}_{\mathsf{k}}^{\epsilon-\mathsf{gr},\otimes}$

Proposition 2.3.40. *Let R be a polynomial algebra over k. Then the de Rham differential induces on the $\mathbb{Z}$-graded module $\Omega^*_{R/\mathsf{k}}$ of the Example **2.3.10** a structure of mixed graded module.*

Proof. In the polynomial case, all this can be seen at the level of the strict 1-category of $\mathbb{Z}$-graded chain complexes as in the Construction **2.3.37**, where the check is tautological. □

2.3.2. Reenacting the HKR theorem via derived geometry.

2.3.2.1. ***Derived Loop Spaces.*** In the Remark **2.3.5** we saw that Hochschild homology admits an interpretation via derived geometry, namely, as the derived ring of functions of the derived loop space $\mathsf{L}X$. We now re-interpret the circle action of the Section **2.3.1.2** in this language:

Construction 2.3.41. Using the Remark **2.3.5** we see that the circle action of the Remark **2.3.4** is a consequence of the circle action on $\mathsf{L}X$ of the Construction **1.2.21**: indeed, we have a pullback square of derived stacks

$$\begin{tikzcd} \mathsf{L}X \arrow[r, "p"] \arrow[d, "q"] & \mathsf{L}X/\mathsf{S}^1 \arrow[d, "a"] \\ \mathsf{Spec}(\mathsf{k}) \arrow[r, "e"] & \mathsf{BS}^1 \end{tikzcd}$$

and since $\mathsf{L}X$ is an affine derived stack (and therefore of finite cohomological dimension over k), we have a base change property for this cartesian square:

$$\begin{tikzcd} \mathsf{QCoh}(\mathsf{L}X) \arrow[d, "q_*"] & \mathsf{QCoh}(\mathsf{L}X/\mathsf{S}^1) = \mathsf{QCoh}(\mathsf{L}X)^{\mathsf{hS}^1} \arrow[l, "p^*"'] \arrow[d, "a_*"] \\ \mathsf{Mod}_{\mathsf{k}} & \mathsf{QCoh}(\mathsf{BS}^1) = \mathsf{Fun}(\mathsf{BS}^1, \mathsf{Mod}_{\mathsf{k}}) \arrow[l, "e^*"'] \end{tikzcd}$$

[(†)] which implies that

(*) Construction **2.3.9**

(†) See Construction **1.2.14**

$$e^*a_*\mathscr{O}_{\mathsf{L}X/\mathsf{S}^1} \simeq p_*q^*\mathscr{O}_{\mathsf{L}X/\mathsf{S}^1} \simeq \mathbb{R}\Gamma(\mathsf{L}X, \mathscr{O}_{\mathsf{L}X}) := \mathsf{HH}(X/\mathsf{k})$$

and shows that $\mathsf{HH}(X/\mathsf{k})$ is in the image of e^*. Since all functors are lax monoidal, we obtain a circle action compatible with the commutative algebra structure. This circle action is canonically equivalent to the one of the (102): indeed, the structure sheaf of $\mathscr{O}_{\mathsf{L}X/\mathsf{S}^1}$ seen as an object in the equivariant derived category $\mathsf{QCoh}(\mathsf{L}X)^{\mathsf{hS}^1}$ is an equivariant lift of $\mathscr{O}_{\mathsf{L}X}$ given by the circle action on $\mathsf{L}X$ that corresponds to (102) via

$$\mathsf{S}^1 \to \mathsf{Map}_{\mathsf{dAff}_{\mathsf{k}}^{\mathsf{op}}}(\mathsf{L}X, \mathsf{L}X) \simeq \mathsf{Map}_{\mathsf{dRings}_{\mathsf{k}}}(\mathsf{HH}(A), \mathsf{HH}(A))$$

2.3.2.2. ***Affinization.*** Let k be a commutative ring. We recall here the results of [Toe06].

Reminder 2.3.42. Cosimplicial commutative algebras over a ring k form a model category [Toe06:Thm 2.1.2] and we denote by $\mathsf{coSCR}_{\mathsf{k}}$ its underlying ∞-category. As for simplicial rings, we also have a *co-normalized Dold-Kan* construction given by a functor $\Theta^{cc} : \mathsf{coSCR}_{\mathsf{k}} \to \mathsf{Mod}_{\mathsf{k}}^{\leq 0}$ [K$\check{9}$3] that is conservative and commutes with derived tensor products. Furthermore, Θ^{cc} admits a left adjoint with we will denote as $\mathsf{Sym}^{\mathrm{co}\Delta}$. See our discussion in [MRT22:§3.1].

Construction 2.3.43. We denote by $\mathsf{Spec}^{\mathrm{co}\Delta} : \mathsf{coSCR}_{\mathsf{k}}^{\mathsf{op}} \to \mathsf{St}_{\mathsf{k}}$ the ∞-functor sending an object $\mathrm{A} \in \mathsf{coSCRk}$ to the (higher) stack which sends a classical commutative ring B to the mapping space $\mathsf{Map}_{\mathsf{coSCRk}}(\mathrm{A}, \mathrm{B})$.

Review 2.3.44. By [Toe06: 2.2.3] the ∞-functor $\mathsf{Spec}^{\mathrm{co}\Delta} : \mathsf{coSCRk}^{\mathsf{op}} \to \mathsf{St}_{\mathsf{k}}$ is fully faithful and admits a left adjoint $\mathsf{C}^*_{\Delta}(-, \mathscr{O})$ that enhances the standard $\mathsf{E}_{\infty}^{\otimes}$-algebra structure of cohomology of global sections $\mathsf{C}^*(-, \mathscr{O})$ with a structure of cosimplicial commutative algebra, namely, it provides a lifting in $\widehat{\mathsf{Cat}_{\infty}}$

$$\begin{tikzcd} & \mathsf{coSCRk} \arrow[d, "\theta"] \\ \mathsf{St}_{\mathsf{k}} \arrow[ur, "\mathsf{C}^*_{\Delta}(-,\mathscr{O})"] \arrow[r, "\mathsf{C}^*(-,\mathscr{O})"'] & \mathsf{CAlg}_{\mathsf{k}} \end{tikzcd}$$

We say that $X \in \mathsf{St}_{\mathsf{k}}$ is *affine* if it lives in the essential image of $\mathsf{Spec}^{\mathrm{co}\Delta}$. More generally, given $X \in \mathsf{St}_{\mathsf{k}}$, we define its affinization as the stack $\mathsf{Spec}^{\mathrm{co}\Delta}(\mathsf{C}^*_{\Delta}(X, \mathscr{O}))$ [Toe06: 2.3.2].

In particular, when K is topological space seen as a constant *higher* stack (via the version of the Construction **1.2.14** for higher stacks), we denote its affinization by

$$\mathsf{Aff}(K) := \mathsf{Spec}^{\mathrm{co}\Delta}(\mathsf{C}^*_{\Delta}(K, \mathsf{k})) \tag{104}$$

2.3.2.3. ***Gradings and shifted tangent bundles.*** Derived geometry also allows us to re-interpret the r.h.s of (100) geometrically:

Construction 2.3.45. Let $A \in \mathsf{dRings}_{\mathsf{k}}$ be a derived ring over k and let $X = \mathsf{Spec}(A)$. Then we can form the (-1)-*shifted tangent bundle* of X using the Example **1.2.31** and the Remark **1.2.26**,

$$\mathsf{T}[-1]X := \mathsf{Spec}(\mathsf{Sym}^{\Delta}_{A}(\mathbb{L}_{A/\mathsf{k}}[1]))$$

where $\mathbb{L}_{A/\mathsf{k}}$ is the relative cotangent complex over k (see Construction **1.1.25**). By construction, this is again an affine derived scheme, defined over X.

Remark 2.3.46. The derived scheme $\mathsf{T}[-1]X$ is defined independently of the characteristic of k, precisely because we use the simplicial Sym^{Δ} as discussed in the Remark **1.2.26**.

Example 2.3.47. When $A = R$ is a classical smooth k-algebra, the cotangent complex $\mathbb{L}_{A/\mathsf{k}}$ coincides with the projective module of Kähler differentials $\Omega^1_{R/\mathsf{k}}$ (cf. Proposition **1.1.27**). In this case, we find

$$\mathscr{O}_{\mathsf{T}[-1]X} = \mathsf{Sym}^{\Delta}_{R}(\mathbb{L}_{R/\mathsf{k}}[1]) \simeq \bigoplus_{i=1}^{n} \Omega^i_{R/\mathsf{k}}[i] \tag{105}$$

This discussion shows that both sides of the Theorem **2.3.3** admit a geometric counterpart: the derived loop space $\mathsf{L}X$ and the shifted tangent bundle $\mathsf{T}[-1]X$. This is independent of the characteristic of the base ring k.

The grading observed in the Example **2.3.10** also has a geometric counterpart given by the scaling action of $\mathbb{G}_{\mathrm{m}}$ on derived linear stacks (cf. Construction **1.2.27**). The key fact is contained in the following result:

Lemma 2.3.48. *Let k be a commutative ring. Then, the comonad induced by pullback along the canonical atlas $e : \mathsf{Spec}(\mathsf{k}) \to \mathsf{B}\mathbb{G}_{\mathrm{m}\,\mathsf{k}}$ induces a symmetric monoidal equivalence of ∞-categories making the diagram commute*

$$\begin{tikzcd} \mathsf{Mod}^{\mathsf{gr}}_{\mathsf{k}} \arrow[dr, "\bigoplus"'] & & \mathsf{QCoh}(\mathsf{B}\mathbb{G}_{\mathrm{m}\,\mathsf{k}}) \arrow[ll, "\sim"'] \arrow[dl, "e^*"] \\ & \mathsf{Mod}_{\mathsf{k}} & \end{tikzcd}$$

where on graded objects we use the symmetric monoidal structure induced by Day convolution.

Proof. See [Spi10] over the ring $\mathbb{Z}$ and [Mou19] over the sphere spectrum. $\square$

Definition 2.3.49. We defined a *graded derived stack* to be a derived stack over $\mathsf{B}\mathbb{G}_{\mathrm{m}}$.

Remark 2.3.50. To give a graded derived stack $Z \to \mathsf{B}\mathbb{G}_{\mathrm{m}\,\mathsf{k}}$ is equivalent to give a derived stack Y with an action of the group stack $\mathbb{G}_{\mathrm{m}\,\mathsf{k}}$. Indeed, the argument of the Construction **1.2.2** applied to the ∞-topos of derived stacks tells us that the derived fiber product Y

(106)
$$\begin{tikzcd} Y \arrow[r] \arrow[d] & Z \arrow[d] \\ \mathsf{Spec}(\mathsf{k}) \arrow[r, "e"] & \mathsf{B}\mathbb{G}_{\mathrm{m}\,\mathsf{k}} \end{tikzcd}$$

carries a monodromy action of $\Omega\mathsf{B}\mathbb{G}_{\mathrm{m}} = \mathbb{G}_{\mathrm{m}}$ and that we can recover Z as the derived quotient of Y with respect to this action

$$Z = Y/\mathbb{G}_{\mathrm{m}}$$

Again, this is a consequence of [Lu-HTT:Proposition 6.2.3.15]. This procedure defines an equivalence of ∞-categories

$$\mathbb{G}_{\mathrm{m}} - \mathsf{dSt}_{\mathsf{k}} \simeq (\mathsf{dSt}_{\mathsf{k}})_{./\mathsf{B}\mathbb{G}_{\mathrm{m}}}$$

We call Y the underlying $\mathbb{G}_{\mathrm{m}}$-stack of Z.

Remark 2.3.51. The Lemma **2.3.48** implies that for any map of derived stacks $p : Z \to \mathsf{B}\mathbb{G}_{\mathrm{m}\,\mathsf{k}}$ relatively affine (or more generally satisfying the sufficient conditions for base change [MRT22: 2.2.14]), the pushforward $p_*(\mathscr{O}_Z)$ is canonically a graded object. Indeed, the base change formula associated to cartesian square (106)

$$\begin{tikzcd} \mathsf{QCoh}(Y) \arrow[d, "\mathbb{R}\Gamma"] & \mathsf{QCoh}(Z) \arrow[l, "\text{pullback}"'] \arrow[d, "p_*"] \\ \mathsf{Mod}_{\mathsf{k}} & \mathsf{QCoh}(\mathsf{B}\mathbb{G}_{\mathrm{m}}) = \mathsf{Mod}^{\mathsf{gr}}_{\mathsf{k}} \arrow[l, "e^*=\bigoplus"'] \end{tikzcd}$$

implies that $\mathbb{R}\Gamma(Y, \mathscr{O}_Y)$ has a natural grading.

Construction 2.3.52. Let X be an affine derived scheme over k. The shifted tangent stack $\mathsf{T}[-1]X$ carries the $\mathbb{G}_{\mathrm{m}}$-action of the Construction **1.2.27**. The associated graded stack is the homotopy quotient $\mathsf{T}[-1]X\,/\mathbb{G}_{\mathrm{m}} \to \mathsf{B}\mathbb{G}_{\mathrm{m}}$. It fits in a cartesian square

(107)
$$\begin{tikzcd} \mathsf{T}[-1]X \arrow[r] \arrow[d] & \mathsf{T}[-1]X\,/\mathbb{G}_{\mathrm{m}} \arrow[d] \\ \mathsf{Spec}(\mathsf{k}) \arrow[r, "e"] & \mathsf{B}\mathbb{G}_{\mathrm{m}\,\mathsf{k}} \end{tikzcd}$$

and the same base change argument used in the Remark **2.3.5** combined with the equivalence of the Lemma **2.3.48** endows $\mathscr{O}_{\mathsf{T}[-1]X}$ with a canonical grading.

2.3.2.4. ***Geometrization of Filtrations***. The following result is originally due to C. Simpson:

Lemma 2.3.53. *[Simpson] Let* k *be a commutative ring. Then the Rees bundle construction induces a symmetric monoidal equivalence of* ∞*-categories*

$$\mathsf{Fil}(\mathsf{Mod}_{\mathsf{k}}) \simeq \mathsf{QCoh}([\mathbb{A}^1_{\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}])$$

where on filtered objects we use the symmetric monoidal structure induced by Day convolution and on the r.h.s we use the quotient stack of the Example ***1.1.36****. Moreover, under this equivalence:*

- *the pullback along the open immersion* $1 : \mathsf{Spec}(\mathsf{k}) = \mathbb{G}_{\mathrm{m}\,\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}} \to [\mathbb{A}^1_{\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}]$

$$\mathsf{QCoh}([\mathbb{A}^1_{\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}]) \to \mathsf{QCoh}(\mathsf{Spec}(\mathsf{k}))$$

  corresponds to the colimit functor

$$\mathsf{Fil}(\mathsf{Mod}_{\mathsf{k}})) \to \mathsf{Mod}_{\mathsf{k}}$$

- *the pullback along the closed immersion* $0 : \mathsf{B}\mathbb{G}_{\mathrm{m}\,\mathsf{k}} \to [\mathbb{A}^1_{\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}]$

$$\mathsf{QCoh}([\mathbb{A}^1_{\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}]) \to \mathsf{QCoh}(\mathsf{Spec}(\mathsf{k}))$$

  corresponds to the associated graded

$$\mathsf{gr} : \mathsf{Fil}(\mathsf{Mod}_{\mathsf{k}}) \to \mathsf{Mod}_{\mathsf{k}}^{\mathsf{gr}}$$

PROOF. See [Mou19] for the result over the sphere spectrum. □

In light of the Lemma **2.3.53**, we define

Definition 2.3.54. A *filtered derived stack* over k is a derived stack Z equipped with a map $Z \to [\mathbb{A}^1_{\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}]$. Its associated graded stack Z^{gr} and associated underlying stack Z^{u} are defined by the cartesian diagrams of derived stacks

(108)
$$\begin{tikzcd} Z^{\mathsf{gr}} \arrow[r] \arrow[d] & Z \arrow[d] & Z^{\mathrm{u}} \arrow[l] \arrow[d] \\ \mathsf{B}\mathbb{G}_{\mathrm{m}\,\mathsf{k}} \arrow[r, "0"] & {[\mathbb{A}^1_{\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}]} & \mathsf{Spec}(\mathsf{k}) \arrow[l, "1"'] \end{tikzcd}$$

Remark 2.3.55. The Definition **2.3.54** is motivated by the following observation, similar to the Remark **2.3.51**: if $p : Z \to [\mathbb{A}^1_{\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}]$ is a filtered derived stack either relatively affine or more generally of finite cohomological dimension (see the discussion in [MRT22: 2.2.14]) then the derived pushforward $p_*(\mathscr{O}_Z)$ is a filtered object in $\mathsf{Mod}_{\mathsf{k}}$. Indeed, using base change for the right cartesian square in (108)

$$\begin{tikzcd}
\mathsf{QCoh}(Z^{\mathrm{u}}) \arrow[d, "\mathbb{R}\Gamma"] & \mathsf{QCoh}(Z) \arrow[l, "\text{pullback}"'] \arrow[d, "p_*"] \\
\mathsf{Mod}_{\mathsf{k}} & \mathsf{QCoh}([\mathbb{A}^1_{\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}]) = \mathsf{Fil}(\mathsf{Mod}_{\mathsf{k}}) \arrow[l, "1^*=\mathsf{colim}"']
\end{tikzcd}$$

Construction 2.3.56. The structure map $\mathbb{A}^1_{\mathsf{k}} \to \mathsf{Spec}(\mathsf{k})$ is canonically equivariant with respect to the $\mathbb{G}_{\mathrm{m}\,\mathsf{k}}$-action on $\mathbb{A}^1_{\mathsf{k}}$ of the Example **1.1.36**. In particular it induces a map

$$[\mathbb{A}^1_{\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}] \to \mathsf{B}\mathbb{G}_{\mathrm{m}\,\mathsf{k}} \tag{109}$$

Let now Y be a stack endowed with a $\mathbb{G}_{\mathrm{m}}$-action and take $Z = [Y/\mathbb{G}_{\mathrm{m}}] \to \mathsf{B}\mathbb{G}_{\mathrm{m}\,\mathsf{k}}$ the associated graded stack as in the Remark **2.3.50**. We define the associated *split* filtered stack $Z^{\mathsf{split}} \to [\mathbb{A}^1_{\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}]$ to be the pullback

$$\begin{tikzcd}
Z^{\mathsf{split}} \arrow[r] \arrow[d] & Z \arrow[d] \\
{[\mathbb{A}^1_{\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}]} \arrow[r] & \mathsf{B}\mathbb{G}_{\mathrm{m}\,\mathsf{k}}
\end{tikzcd}$$

By construction, it is equivalent to the quotient stack $[Y \times \mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]$ where we let $\mathbb{G}_{\mathrm{m}}$ act on the product coordinate-wise. The associated graded stack $(Z^{\mathsf{split}})^{\mathsf{gr}}$ is canonically equivalent to Z because the map (109) is a right inverse to the inclusion $0 : \mathsf{B}\mathbb{G}_{\mathrm{m}\,\mathsf{k}} \to [\mathbb{A}^1_{\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}]$.

The underlying stack $(Z^{\mathsf{split}})^{\mathrm{u}}$ is obtained as a fiber product

$$\begin{tikzcd}
(Z^{\mathsf{split}})^{\mathrm{u}} \arrow[r] \arrow[d] & Z^{\mathsf{split}} \arrow[r] \arrow[d] & Z \arrow[d] \\
\mathsf{Spec}(\mathsf{k}) = \mathbb{G}_{\mathrm{m}\,\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}} \arrow[r] & {[\mathbb{A}^1_{\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}]} \arrow[r] & \mathsf{B}\mathbb{G}_{\mathrm{m}\,\mathsf{k}}
\end{tikzcd}$$

but since the composition at the bottom $\mathsf{Spec}(\mathsf{k}) \to \mathsf{B}\mathbb{G}_{\mathrm{m}\,\mathsf{k}}$ coincides with the canonical atlas, we see that $(Z^{\mathsf{split}})^{\mathrm{u}}$ coincides with the underlying $\mathbb{G}_{\mathrm{m}\,\mathsf{k}}$-stack Y. In particular, we see that the underlying $\mathbb{G}_{\mathrm{m}}$-stack of the filtration is equivalent to the underlying stack of the associated graded. This provides a geometrization of the Terminology **2.3.17**.

2.3.3. HKR isomorphism in characteristic zero.

The first formulation of the HKR isomorphism in terms of derived geometry is due to Ben-Zvi and Nadler in [BZN12]. We can summarize their main theorem as follows:

Theorem 2.3.57. *Let k be a $\mathbb{Q}$-algebra, let $\mathsf{B}\mathbb{G}_{\mathsf{a}\,\mathsf{k}}$ denote the classifying stack of the additive group and let $X = \mathsf{Spec}(A)$ be an affine derived scheme over k. Then:*

(i) The composition with the affinization map $\mathsf{S}^1 \to \mathsf{Aff}(\mathsf{S}^1)$ *of the* (104) *induces an equivalence of derived mapping stacks*

$$\mathbb{R}\mathsf{Map}_{\mathsf{k}}(\mathsf{Aff}(\mathsf{S}^1), X) \simeq \mathbb{R}\mathsf{Map}_{\mathsf{k}}(\mathsf{S}^1, X) =: \mathsf{L}X$$

(ii) The formality of the cohomology $\mathsf{C}^*_{\Delta}(\mathsf{B}\mathbb{G}_{\mathsf{a}\,\mathsf{k}}, \mathscr{O}) \simeq \mathsf{Sym}_{\mathsf{k}}(\mathsf{k}[-1])$ *in characteristic zero, induces an equivalence of derived mapping stacks*

$$\mathbb{R}\mathsf{Map}_{\mathsf{k}}(\mathsf{B}\mathbb{G}_{\mathsf{a}\,\mathsf{k}}, X) \simeq \mathsf{T}[-1]X$$

(iii) The unit of the group $\mathbb{Z} \to \mathbb{G}_{\mathsf{a}\,\mathsf{k}}$ *induces an isomorphism of abelian group stacks*

$$\mathsf{Aff}(\mathsf{S}^1) \simeq \mathsf{B}\mathbb{G}_{\mathsf{a}}$$

(iv) Combining (i),(ii) and (iii), we obtain an equivalence of derived schemes

$$\mathsf{L}X \simeq \mathsf{T}[-1]X \tag{110}$$

matching the circle action on the l.h.s with the $\mathsf{B}\mathbb{G}_{\mathsf{a}}$ *-action on the r.h.s.. In particular, after passing to the derived rings of functions we recover a HKR-isomorphism as in* ***Proposition 2.3.19*** *enhanched with the structures of derived rings, and matching the circle action with the de Rham differential.*

Remark 2.3.58. Both $\mathsf{L}X$ and $\mathsf{T}[-1]X$ carry an extra structure of group scheme over X. Respectively, the first under composition of loops and the second under linear sums of tangent vectors. The HKR isomorphism (110) is not a map of group schemes over X with respect to these group structures. The obstruction is given by the Todd class of X: more precisely, each group structure produces a canonical trivialization of the respective canonical bundle

$$\alpha : \mathscr{O}_{\mathsf{L}X} \simeq \omega_{\mathsf{L}X} \qquad \beta : \mathscr{O}_{\mathsf{T}[-1]X} \simeq \omega_{\mathsf{T}[-1]X}$$

and the difference of trivializations

$$\mathscr{O}_{\mathsf{T}[-1]X} \underset{\beta}{\simeq} \omega_{\mathsf{T}[-1]X} \simeq \mathrm{HKR}^*(\omega_{\mathsf{L}X}) \underset{\mathrm{HKR}^*(\alpha^{-1})}{\simeq} \mathrm{HKR}^*(\mathscr{O}_{\mathsf{L}X}) = \mathscr{O}_{\mathsf{T}[-1]X} \in \mathbb{R}\Gamma(\mathsf{T}[-1]X, \mathscr{O}_{\mathsf{T}[-1]X}) = \bigoplus_{i=0}^{n} \mathbb{R}\Gamma(X, \Omega^i_X)$$

is the Todd class. See [Pri20; Mou22].

2.3.4. Our results. We now discuss our main result, namely, a universal property of the HKR-filtration as a universal filtered derived ring endowed with an action of the filtered circle. But before that, we describe our construction of a mixed graded circle:

2.3.4.1. ***The geometry behind the de Rham differential - the mixed graded circle.***

Construction 2.3.59. Consider $\mathsf{k}[\eta]$ the trivial square zero extension on the complex $\mathsf{k}\oplus\mathsf{k}[-1]$ as an object in the ∞-category of cosimplicial graded commutative algebras $\mathsf{coSCR}^{\mathsf{gr}}_{\mathsf{k}}$ with the copy of $\mathsf{k}[-1]$ in weight -1 and consider its associated affine ($\mathbb{G}_{\mathrm{m}}$-)stack $\mathsf{Spec}^{\mathrm{co}\Delta}(\mathsf{k}[\eta])$.

Warning 2.3.60. The weight 1 used in [MRT22:Construction 3.4.2] for the direct summand $\mathbb{Z}_{(p)}[-1]$ is incorrect and should instead be weight -1. This does not affect the computations that follow in the paper but is important to match the weight 1 in $\mathsf{k}[\epsilon]_{\mathsf{gr}}$.

Our first result is the following:

Proposition 2.3.61. *Let k be a $\mathbb{Z}_{(p)}$ algebras and let $X=\mathsf{Spec}(A)$ be an affine derived scheme over k. Then we have an equivalence of derived $\mathbb{G}_{\mathrm{m}}$-stacks*

$$\mathbb{R}\mathsf{Map}(\mathsf{Spec}^{\mathrm{co}\Delta}(\mathsf{k}[\eta]),X)\simeq\mathsf{T}[-1]X$$

*where the l.h.s is equipped with the $\mathbb{G}_{\mathrm{m}}$-action induced by $\mathsf{Spec}^{\mathrm{co}\Delta}(\mathsf{k}[\eta])$. and the r.h.s is equipped with the $\mathbb{G}_{\mathrm{m}}$-action of the Construction **2.3.52**.*

The novelty comes from the following result:

Proposition 2.3.62. *[MRT22:Cor 3.4.18] The ($\mathbb{G}_{\mathrm{m}}$-)stack $\mathsf{Spec}^{\mathrm{co}\Delta}(\mathsf{k}[\eta])$ admits a unique $\mathsf{E}^{\otimes}_{\infty}$-group structure, compatible with the grading and the base point induced by the augmentation $\mathsf{k}[\eta]\to\mathsf{k}$.*

Definition 2.3.63. We define the *mixed graded circle* as the quotient stack

$$\mathsf{S}^1_{\epsilon-\mathsf{gr}}:=[\mathsf{Spec}^{\mathrm{co}\Delta}(\mathsf{k}[\eta])/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}]\to\mathsf{B}\mathbb{G}_{\mathrm{m}}$$

The relevance of the mixed circle, is that even over $\mathsf{k}=\mathbb{Z}_{(p)}$, it recovers the notion of mixed graded modules with their symmetric monoidal structure:

Proposition 2.3.64. *[MRT22:4.2.3-(ii)] Let k be a $\mathbb{Z}_{(p)}$-algebra. Then we have a symmetric monoidal equivalence*

$$\mathsf{QCoh}(\mathsf{B}_{\mathsf{B}\mathbb{G}_{\mathrm{m}}}(\mathsf{S}^1_{\epsilon-\mathsf{gr}}))^{\otimes}\simeq\mathsf{Mod}^{\epsilon-\mathsf{gr},\otimes}_{\mathsf{k}}$$

Remark 2.3.65. When k is a $\mathbb{Q}$-algebra, the canonical map $\mathsf{Spec}^{\mathrm{co}\Delta}(\mathsf{k}[\eta])\to\mathsf{B}\mathbb{G}_{\mathsf{a}\,\mathsf{k}}$ classifying the element η is an equivalence of group stacks. In particular, it induces an equivalence of classifying stacks over $\mathsf{B}\mathbb{G}_{\mathrm{m}}$

$$\mathsf{B}_{\mathsf{B}\mathbb{G}_{\mathrm{m}}}(\mathsf{S}^1_{\epsilon-\mathsf{gr}})\simeq[\mathsf{B}(\mathsf{B}\mathbb{G}_{\mathsf{a}})/\mathbb{G}_{\mathrm{m}}]=\mathsf{B}(\mathbb{G}_{\mathrm{m}}\ltimes\mathsf{B}\mathbb{G}_{\mathsf{a}})$$

presenting representations of $\mathbb{G}_{\mathrm{m}} \ltimes \mathsf{B}\mathbb{G}_{\mathsf{a}}$ as a model for mixed graded complexes. This explains the role of the copy of $\mathsf{B}\mathbb{G}_{\mathsf{a}}$ in the Theorem **2.3.57** and why it does not work away from characteristic zero.

Construction 2.3.66. It follows from the Proposition **2.3.61** and the Proposition **2.3.62** that the shifted tangent stack $\mathsf{T}[-1]X$ not only comes equipped with a $\mathbb{G}_{\mathrm{m}}$-action, but also, with an extra action of the mixed graded circle

$$\mathsf{Spec}^{\mathrm{co}\Delta}(\mathsf{k}[\eta]) \times \mathbb{R}\mathsf{Map}(\mathsf{Spec}^{\mathrm{co}\Delta}(\mathsf{k}[\eta]), X) \to \mathbb{R}\mathsf{Map}(\mathsf{Spec}^{\mathrm{co}\Delta}(\mathsf{k}[\eta]), X)$$

corresponding, via the the evaluation map, to the composition with the group structure

$$\mathsf{Spec}^{\mathrm{co}\Delta}(\mathsf{k}[\eta]) \times \mathsf{Spec}^{\mathrm{co}\Delta}(\mathsf{k}[\eta]) \times \mathbb{R}\mathsf{Map}(\mathsf{Spec}^{\mathrm{co}\Delta}(\mathsf{k}[\eta]), X) \to \mathsf{Spec}^{\mathrm{co}\Delta}(\mathsf{k}[\eta]) \times \mathbb{R}\mathsf{Map}(\mathsf{Spec}^{\mathrm{co}\Delta}(\mathsf{k}[\eta]), X) \to X$$

which descends to an action of the mixed graded circle

$$\mathsf{S}^1_{\epsilon-\mathsf{gr}} \times [(\mathsf{T}[-1]X)/\mathbb{G}_{\mathrm{m}}] \to [(\mathsf{T}[-1]X)/\mathbb{G}_{\mathrm{m}}]$$

This action enhances the conclusion of the Construction **2.3.52**, namely, as a consequence of the symmetric monoidal equivalence Proposition **2.3.64** and base-change with respect to the cartesian diagram

$$\begin{tikzcd}
{[(\mathsf{T}[-1]X)/\mathbb{G}_{\mathrm{m}}]} \arrow[r] \arrow[d] & {[(\mathsf{T}[-1]X)/\mathbb{G}_{\mathrm{m}}]/\mathsf{S}^1_{\epsilon-\mathsf{gr}}} \arrow[d] \\
\mathsf{B}\mathbb{G}_{\mathrm{m}} \arrow[r] & \mathsf{B}_{\mathsf{B}\mathbb{G}_{\mathrm{m}}} \mathsf{S}^1_{\epsilon-\mathsf{gr}}
\end{tikzcd}$$

the structure sheaf $\mathscr{O}_{\mathsf{T}[-1]X}$ comes canonically equipped with a structure of mixed graded algebra.

2.3.4.2. *The filtered circle and the HKR filtration.*

We now state our main result:

Theorem 2.3.67. *[MRT22] Fix a prime p and let $k = \mathbb{Z}_{(p)}$ be the local ring at p. Then, there exists a filtered abelian group stack*

$$\mathsf{S}^1_{\mathsf{Fil}} \to [\mathbb{A}^1_{\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}]$$

implementing a filtration on (the affinization of) the topological circle S^1, with associated graded abelian group stack given by the mixed graded circle $\mathsf{S}^1_{\epsilon-\mathsf{gr}}$ of the Definition 2.3.63.

Moreover, for any affine derived scheme $X = \mathsf{Spec}\, A$:

(i) the filtered complex of global functions on the derived mapping stack

$$\mathsf{L}_{\mathsf{Fil}}\, X := \mathbb{R}\mathsf{Map}_{[\mathbb{A}^1_{\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}]}(\mathsf{S}^1_{\mathsf{Fil}}, X \times [\mathbb{A}^1_{\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}])$$

recovers the HKR filtration $\mathsf{HH}^{\mathsf{Fil}}(A)$ of the Proposition 2.3.15, but enhanced with a compatible action of the filtered circle $\mathsf{S}^1_{\mathsf{Fil}}$ which is also compatible with the filtered multiplicative structure.

(ii) its associated graded is $\bigoplus_{i=0}^{n}(\Lambda^i \mathbb{L}_{A/\mathsf{k}})[i]$ with its graded multiplicative structure, enhanced with a compatible action of the graded circle *$(\mathsf{S}^1_{\mathsf{Fil}})^{\mathsf{gr}}$ that recovers the de Rham differential.*

(iii) its underlying complex is $\mathsf{HH}(A)$ and the induced action of $(\mathsf{S}^1_{\mathsf{Fil}})^{\mathsf{u}}$ recovers the circle action of Section 2.3.1.2 and the compatible algebra structure.

Finally, the filtered derived ring $\mathsf{HH}^{\mathsf{Fil}}(A)$ equipped with the action of the filtered circle $\mathsf{S}^1_{\mathsf{Fil}}$ is universal with respect to A-filtered derived rings with $\mathsf{S}^1_{\mathsf{Fil}}$-action.

Terminology 2.3.68. We call the filtered stack $\mathsf{L}_{\mathsf{Fil}}\, X$ the filtered loop stack of X.

Construction 2.3.69. Let $X = \mathsf{Spec}(A)$ be an affine derived scheme over k. Since the filtration on $\mathsf{HH}^{\mathsf{Fil}}(A)$ is compatible with the $\mathsf{S}^1_{\mathsf{Fil}}$-action, taking fixed points $\mathsf{HH}^{\mathsf{Fil}}(A)^{\mathsf{h}\mathsf{S}^1_{\mathsf{Fil}}}$ gives a new filtered complex.

We also proved the following results:

Proposition 2.3.70. *Let k be a $\mathbb{Z}_{(p)}$-algebra. Then:*

(i) The associated graded pieces of weight i of $\mathsf{HH}^{\mathsf{Fil}}(A)^{\mathsf{h}\mathsf{S}^1_{\mathsf{Fil}}}$ are the truncated complete derived deRham complex $\mathbb{L}\mathsf{DR}(X/\mathsf{k})^{\geq i}$ of [Bha12];

(ii) *Whenever X is quasi-smooth, the underlying object of* $\mathsf{HH}^{\mathsf{Fil}}(A)^{\mathsf{hS}^1_{\mathsf{Fil}}}$ *is the negative cyclic homology* $\mathsf{HC}^-(X/\mathsf{k})$*;*

(iii) $\mathsf{HH}^{\mathsf{Fil}}(A)^{\mathsf{hS}^1_{\mathsf{Fil}}}$ *recovers the filtration constructed by Antieau in [Ant19] for discrete rings and in [BMS19] for p-adic rings.*

Remark 2.3.71. When k is a $\mathbb{Q}$-algebra the filtration on $\mathsf{HC}^-(X/\mathsf{k})$ recovers the one of [Lod92: 5.1.12].

Before addressing other applications, let us quickly discuss the proof of the Theorem **2.3.67**. It goes in two main steps:

Step 1) The construction of the filtered circle $\mathsf{S}^1_{\mathsf{Fil}}$;

Step 2) Showing that the construction in Step 1) satisfies the statements in the theorem.

In this survey we will merely address Step 1). Readers who want to follow Step 2) should refer back to our work.

The construction of $\mathsf{S}^1_{\mathsf{Fil}}$ is inspired by the results in characteristic zero (cf. Section **2.3.3**) where the classifying stack $\mathsf{B}\mathbb{G}_{\mathsf{a}\,\mathsf{k}}$ had the miraculous property of being simultaneously equivalent to the affinization of the topological circle S^1 and to the underlying stack of the mixed graded circle $\mathsf{S}^1_{\epsilon-\mathsf{gr}}$. To extend this result, independently of the characteristic of k, we use the group scheme of big Witt vectors W as the universal extension of $\mathbb{G}_{\mathsf{a}}$ equipped with a Frobenius endomorphism.

Recollection 2.3.72. We recall the definition and main properties of big Witt vectors following [Hes15; KN]. For any (classical) commutative ring R, the collection of formal power series in one variable of the form $1+tR[[t]]$ are invertible with respect to multiplication. In particular, this forms an abelian group. We define the abelian group scheme of big Witt vectors as the functor of points given by

$$(\mathsf{W}, +) : \mathsf{CRings}_{\mathsf{k}} \to \mathsf{AbGrp} \qquad R \mapsto ((1 + t.R[[t]]), \times)$$

The underlying scheme of W is an infinite product of copies of $\mathbb{A}^1_{\mathsf{k}}$ given by extracting the list of coefficients of the corresponding formal power series.

In fact W carries an extra multiplication but to construct it we need a choice of *Ghost coordinates*: for every commutative ring R we consider the map given by the derivative of the logarithm

$$-\frac{d}{dt}\log : (1 + t.R[[t]])^{\times} \to R[[t]], \qquad f \mapsto -\frac{f'}{f}$$

Notice that this map transforms products of formal power series into sums. Therefore, it defines a map of abelian group schemes

$$\mathsf{Ghost} : (\mathsf{W}, +) \to (\prod_{i=1}^{\infty} \mathbb{G}_{\mathsf{a}\,\mathsf{k}}, +)$$

where on the r.h.s we have the coordinate-wise additive law. It is a theorem that there exists a unique ring structure $(\mathsf{W}, +, *)$ that lifts via Ghost the coordinate-wise ring structure on the product of ring schemes $(\prod_{i=1}^{\infty} \mathbb{G}_{\mathsf{a}\,\mathsf{k}}, +, *)$. A key property of the Ghost map is that after base change to $\mathbb{Q}$ it becomes an isomorphism of ring stacks

$$\mathsf{Ghost} \otimes \mathbb{Q} : \mathsf{W} \otimes \mathbb{Q} \simeq \prod_{i=1}^{\infty} \mathbb{G}_{\mathsf{a}} \otimes \mathbb{Q} \tag{111}$$

Furthermore, W comes equipped with:

- Frobenius endomorphisms

  $$\mathrm{Frob}_n : \mathsf{W} \to \mathsf{W}, \quad \forall n \in \mathbb{N}$$

  determined by the shift maps on Ghost coordinates

  $$\mathrm{Shift}_n : \prod_{i=1}^{\infty} \mathbb{G}_{\mathsf{a}} \to \prod_{i=1}^{\infty} \mathbb{G}_{\mathsf{a}} \qquad (\omega_i) \mapsto (\omega_{ni})$$

  and in particular, verifying the relations $\mathrm{Frob}_n \circ \mathrm{Frob}_m = \mathrm{Frob}_{nm}$;
- An action of the multiplicative monoid $(\mathbb{A}^1_, *)$.

Finally, we need the quotient of W of p-typical Witt vectors W_p. It is defined as follows: let $S_p := \{1, p, p^2, \cdots\} \subseteq \mathbb{N}$ and let $W^{S_p}(R) \subseteq W(R)$ denote the subset spanned by those formal power series that can be written as products

$$\prod_{i=1}^{\infty}(1 - \lambda_n t^n) \qquad \text{for } \lambda_n = 0 \text{ if } n \in S_p$$

In fact $\mathsf{W}^{S_p} \subseteq \mathsf{W}$ forms a subgroup scheme and we define the group of p-typical Witt vectors as the quotient abelian group scheme

$$\mathsf{W}_p := \mathsf{W}/\mathsf{W}^{S_p}$$

By construction, the Ghost coordinates descent to W_p as a map of abelian groups

$$\begin{tikzcd} \mathsf{W} \arrow[r, "\mathsf{Ghost}"] \arrow[d] & \prod_{i \in \mathbb{N}} \mathbb{G}_{\mathsf{a}} \arrow[d, "\pi_{S_p}"] \\ \mathsf{W}_p \arrow[r, dashed] & \prod_{i \in S_p} \mathbb{G}_{\mathsf{a}} \end{tikzcd}$$

and so does the Frobenius endomorphism

$$\mathrm{Frob}_p : \mathsf{W}_p \to \mathsf{W}_p$$

which in this case in terms of Ghost coordinates becomes the shift by one to the left

$$\mathrm{Shift}_p = \mathrm{Shift} : \prod_{i \in S_p}^{\infty} \mathbb{G}_{\mathsf{a}} \to \prod_{i \in S_p}^{\infty} \mathbb{G}_{\mathsf{a}} \quad (\omega_1, \omega_p, \omega_{p^2}, \omega_{p^3}, \cdots) \mapsto (\omega_p, \omega_{p^2}, \omega_{p^3}, \cdots)$$

and the monoid action $\mathbb{A}^1 \times \mathsf{W}_p \to \mathsf{W}_p$.

Remark 2.3.73. As we shall see in Proposition **2.3.80** below, the isomorphism (111) is, secretly, the key ingredient for the splitting of the HKR filtration in characteristic zero.

Remark 2.3.74. The construction of the ring structure $*$ on W via Ghost coordinates is quite obscure. A theorem of Almkvist [Alm78] sheds some light on the situation: for every ring R we consider the category $\mathrm{End}(\mathsf{Proj}_R^{\mathsf{ft}})$ whose objects are endomorphisms $f : M \to M$ of M a projective module of finite type over R and morphisms are $(M, f) \to (N, g)$ are morphisms in $\mathsf{Proj}_R^{\mathsf{ft}}$ compatible with the endomorphisms. Given such a pair (M, f), we use the *characteristic polynomial* $P_f(t) := \sum_{i \geq 0} \mathsf{Tr}(\Lambda^i f) t^i$ to define the map

$$\mathrm{End}(\mathsf{Proj}_R^{\mathsf{ft}}) \to \mathsf{W}(R) \quad (M, f) \mapsto \frac{1}{P_f(t)}$$

It follows from the definition of the characteristic polynomial that this construction sends the composition of endomorphism to the product of formal power series and therefore defines a map of monoids

$$(\mathrm{End}(\mathsf{Proj}_R^{\mathsf{ft}}), \circ) \to (\mathsf{W}(R), +)$$

But on endomorphisms we also have the tensor product operation $\otimes$. A key observation by Almkvist is that $\otimes$ is sent to the multiplication of Witt vectors $*$ as defined abstractly via Ghost coordinates in the Recollection **2.3.72**.

$$(\mathrm{End}(\mathsf{Proj}_R^{\mathsf{ft}}), \circ, \otimes) \to (\mathsf{W}(R), +, *)$$

We now explain the construction of the filtered circle

Construction 2.3.75. Consider the family $\mathsf{W}_p \times \mathbb{A}^1 \to \mathbb{A}^1$ as a trivial abelian group stack relativiely to $\mathbb{A}^1$ and the morphism of $\mathbb{A}^1$-relative abelian group stacks given by

$$\begin{tikzcd} \mathsf{W}_p \times \mathbb{A}^1 \arrow[rr, "\mathscr{G}_p"] \arrow[dr] & & \mathsf{W}_p \times \mathbb{A}^1 \arrow[dl] \\ & \mathbb{A}^1 & \end{tikzcd} \qquad (f, a) \mapsto (\mathrm{Frob}_p(f) - [a^{p-1}].f, a)$$

Let $\mathrm{Ker}\,\mathscr{G}_p$ be the $\mathbb{A}^1$-relative abelian group stack given by the kernel. Notice that:

- The fiber of $\mathrm{Ker}\,\mathscr{G}_p$ at $a = 1$ is the k-group scheme parametrizing fixed points of the Frobenius endomorphism

$$(\mathrm{Ker}\,\mathscr{G}_p)_{|1} = \mathsf{Fix}$$

- The fiber of $\mathrm{Ker}\,\mathscr{G}_p$ at $a = 0$ is the k-group scheme given by the kernel of the Frobenius endomorphism

$$(\mathrm{Ker}\,\mathscr{G}_p)_{|0} = \mathsf{Ker}$$

Moreover, one can check that the action of $\mathbb{G}_{\mathrm{m}}$ on $W_p \times \mathbb{A}^1$ restricts to $\mathrm{Ker}\,\mathscr{G}_p$.

Definition 2.3.76. We define the *filtered circle* $\mathsf{S}^1_{\mathsf{Fil}}$ as the relative classifying stack

$$\mathsf{S}^1_{\mathsf{Fil}} := \mathsf{B}_{[\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]}([\mathrm{Ker}\,\mathscr{G}_p/\mathbb{G}_{\mathrm{m}}]) \to [\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]$$

Since $\mathrm{Ker}\,\mathscr{G}_p$ is abelian, $\mathsf{S}^1_{\mathsf{Fil}}$ inherits a structure of filtered abelian group stack.

Remark 2.3.77. The filtered circle $\mathsf{S}^1_{\mathsf{Fil}}$ establishes a filtration on the abelian group stack BFix with associated graded stack given by the abelian graded group stack $[(\mathsf{BKer})/\mathbb{G}_{\mathrm{m}}]$.

The following result summarizes two essential ingredients in proving the Theorem **2.3.67**:

Proposition 2.3.78. *Let* k *be a* $\mathbb{Z}_{(p)}$*-algebra. Then:*

(i) The unit of the group $\mathbb{Z} \to \mathsf{Fix}$ *induces an equivalence of group stacks*

$$\mathsf{Aff}(\mathsf{S}^1) \simeq \mathsf{BFix} = (\mathsf{S}^1_{\mathsf{Fil}})^{\mathrm{u}}$$

which induces an equivalence of symmetric monoidal categories

$$\mathsf{QCoh}(\mathsf{BS}^1)^{\otimes} \simeq \mathsf{QCoh}(\mathsf{B}^2\mathsf{Fix})^{\otimes}$$

(ii) The choice of the generator $\eta \in \mathsf{H}^1(\mathsf{BKer}, \mathscr{O})$ *corresponding to the projection along the first Ghost coordinate* [(*)] *induces an isomorphism of graded group stacks*

$$(\mathsf{S}^1_{\mathsf{Fil}})^{\mathsf{gr}} = [(\mathsf{BKer})/\mathbb{G}_{\mathrm{m}}] \simeq \mathsf{S}^1_{\epsilon-\mathsf{gr}}$$

which induces an equivalence of symmetric monoidal categories

$$\mathsf{QCoh}([(\mathsf{BKer}/\mathbb{G}_{\mathrm{m}})])^{\otimes} \simeq \mathsf{QCoh}(\mathsf{S}^1_{\epsilon-\mathsf{gr}}) \underset{\textit{Proposition 2.3.64}}{\simeq} \mathsf{Mod}_{\mathsf{k}}^{\epsilon-\mathsf{gr},\otimes}$$

To conclude let us explain how to recover the HKR isomorphism in characteristic zero:

Construction 2.3.79. We consider the split filtered stack

$$[(\mathsf{B}\mathbb{G}_{\mathsf{a}\,\mathsf{k}})/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}]^{\mathsf{split}} \to [\mathbb{A}^1_{\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}]$$

provided by the Construction **2.3.56**. By construction this is a filtered group stack with group structure induced from the group structure on $\mathsf{B}\mathbb{G}_{\mathsf{a}\,\mathsf{k}}$.

Proposition 2.3.80. *Let* k *be a* $\mathbb{Q}$*-algebra. Then the Ghost coordinate on Witt vectors induces an isomorphism of filtered group stacks*

$$(\mathsf{S}^1_{\mathsf{Fil}})_{\mathsf{k}} \simeq [(\mathsf{B}\mathbb{G}_{\mathsf{a}\,\mathsf{k}})/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}]^{\mathsf{split}} \tag{112}$$

which induces the splitting of the HKR filtration in the Proposition ***2.3.19*** *and Theorem* ***2.3.57****.*

PROOF. Consider the map

$$\Psi : \mathbb{G}_{\mathsf{a}\,\mathsf{k}} \times \mathbb{A}^1_{\mathsf{k}} \to \left(\prod_{i=0}^{\infty} \mathbb{G}_{\mathsf{a}\,\mathsf{k}}\right) \times \mathbb{A}^1_{\mathsf{k}}$$

sending

$$(x,a) \to [(x, a^{p-1}.x, a^{p^2-1}x, a^{p^3-1}x, \cdots), a]$$

This defines a map of group schemes over $\mathbb{A}^1$ with respect to the additive group laws, and is injective on the functor of points and with image consisting precisely of the group scheme $\mathrm{Ker}(\Sigma_p)$ where Σ_p is the group map relative to $\mathbb{A}^1$

$$\Sigma_p : \left(\prod_{i\in S_p} \mathbb{G}_{\mathsf{a}}\right) \times \mathbb{A}^1 \to \left(\prod_{i\in S_p} \mathbb{G}_{\mathsf{a}}\right) \times \mathbb{A}^1$$

$$((x_i)_{i\in S_p}, a) \mapsto (\mathrm{Shift}((x_i)_{i\in S_p}) - a^{p-1}(x_i)_{i\in S_p}, a)$$

[(*)] ie, the composition $\mathsf{Ker} \subseteq \mathsf{W}_p \to \prod_{i\in\mathbb{Z}} \mathbb{G}_{\mathsf{a}} \to \mathbb{G}_{\mathsf{a}}$.

Therefore, we have an isomorphism of group schemes

$$\mathbb{G}_{\mathsf{a}\,\mathsf{k}} \times \mathbb{A}^1_{\mathsf{k}} \xrightarrow[\sim]{\Psi} \operatorname{Ker} \Sigma_p$$

Moreover, one can check that this isomorphism is compatible with the $\mathbb{G}_{\mathrm{m}}$-action on both sides. Therefore, Ψ provides a splitting for the filtered group scheme $\operatorname{Ker}(S_p)$ independently of the characteristic of k. With rational coefficients, the Ghost map provides another $\mathbb{G}_{\mathrm{m}}$-equivariant(*) isomorphism

$$(\mathbb{G}_{\mathsf{a}\,\mathsf{k}} \times \mathbb{A}^1_{\mathsf{k}}) \otimes \mathbb{Q} \xrightarrow[\sim]{\Psi} \operatorname{Ker}(\Sigma_p) \otimes \mathbb{Q} \xleftarrow[\sim]{\mathsf{Ghost}\otimes\mathbb{Q}} \operatorname{Ker} \mathscr{G}_{p^\infty} \otimes \mathbb{Q}$$

and therefore an isomorphism of filtered abelian group stacks

$$\begin{tikzcd}
{[(\mathbb{G}_{\mathsf{a}\,\mathsf{k}} \times \mathbb{A}^1_{\mathsf{k}})/\mathbb{G}_{\mathrm{m}}] \otimes \mathbb{Q}} \arrow[r, "\Psi", "\sim"'] \arrow[dr] & {[\operatorname{Ker}(\Sigma_p)/\mathbb{G}_{\mathrm{m}}] \otimes \mathbb{Q}} \arrow[d] & {[\operatorname{Ker} \mathscr{G}_p/\mathbb{G}_{\mathrm{m}}] \otimes \mathbb{Q}} \arrow[l, "\mathsf{Ghost}\otimes\mathbb{Q}"', "\sim"] \arrow[dl] \\
& {[\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]} &
\end{tikzcd}$$

Notice also that by construction, the filtered stack $[(\mathbb{G}_{\mathsf{a}\,\mathsf{k}} \times \mathbb{A}^1_{\mathsf{k}})/\mathbb{G}_{\mathrm{m}}]$ is the split filtered stack associated to the graded stack $\mathbb{G}_{\mathsf{a}}/\mathbb{G}_{\mathrm{m}}$ as in the construction Construction **2.3.56**:

$$[\mathbb{G}_{\mathsf{a}}/\mathbb{G}_{\mathrm{m}}]^{\mathsf{split}} \otimes \mathbb{Q} \simeq [\big(\mathbb{G}_{\mathsf{a}\,\mathsf{k}} \times \mathbb{A}^1_{\mathsf{k}}\big)/\mathbb{G}_{\mathrm{m}}] \otimes \mathbb{Q}$$

Passing to classifying spaces B on the composition

$$[\mathbb{G}_{\mathsf{a}}/\mathbb{G}_{\mathrm{m}}]^{\mathsf{split}} \otimes \mathbb{Q} \simeq [\operatorname{Ker} \mathscr{G}_p/\mathbb{G}_{\mathrm{m}}] \otimes \mathbb{Q}$$

we obtain the splitting in (112).

□

Remark 2.3.81. The abelian group stack $\mathsf{B}\operatorname{Ker}\mathscr{G}_p \to \mathbb{A}^1$ has another characterization via Cartier duality [Car62]. Let $\widehat{\mathbb{G}_{\mathrm{m}}}$ be the formal multiplicative group, obtained by formal completion of $\mathbb{G}_{\mathrm{m}}$ at 1. The deformation to the normal bundle [Mou21:Theorem 1.5] at 1 produces a family of formal groups over $\mathbb{A}^1$

$$\operatorname{Def}_{\widehat{\mathbb{G}_{\mathrm{m}}}} \to \mathbb{A}^1$$

whose fibers at any point $\lambda \neq 0$ are isomorphic to $\widehat{\mathbb{G}_{\mathrm{m}}}$ and the fiber at 0 is the additive formal group $\widehat{\mathbb{G}_{\mathsf{a}}}$. In terms of filtrations this corresponds to the I-adic filtration on $\mathscr{O}(\widehat{\mathbb{G}_{\mathrm{m}\,\mathsf{k}}}) = \mathsf{k}[[t]]$. The results of [SS01] (see the discussion in [MRT22: 6.3.3]) show that the Cartier dual of $\widehat{\mathbb{G}_{\mathrm{m}}}$ is Fix, ie

$$\mathsf{Map}^{\mathsf{Gr}}(\widehat{\mathbb{G}_{\mathrm{m}}}, \widehat{\mathbb{G}_{\mathrm{m}}}) \simeq \mathsf{Fix}$$

the Cartier dual of $\widehat{\mathbb{G}_{\mathsf{a}}}$ is Ker

(*)Since the $\mathbb{G}_{\mathrm{m}}$-action on W_p is activated through Ghost coordinates.

$$\mathsf{Map}^{\mathsf{Gr}}(\widehat{\mathbb{G}_{\mathsf{a}}}, \widehat{\mathbb{G}_{\mathrm{m}}}) \simeq \mathsf{Ker}$$

and the results of [Mou21:Theorem 1.8] show that the family $\mathrm{Ker}\,\mathscr{G}_p \to \mathbb{A}^1$ is the relative Cartier dual to $\mathrm{Def}_{\widehat{\mathbb{G}_{\mathrm{m}}}}$ with the action of $\mathbb{G}_{\mathrm{m}}$ on $\mathrm{Ker}\,\mathscr{G}_p$ corresponding to the action of $\mathbb{G}_{\mathrm{m}}$ on the deformation to the normal bundle so that

$$\mathsf{CartierDual}_{[\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]}([\mathrm{Def}_{\widehat{\mathbb{G}_{\mathrm{m}}}}/\mathbb{G}_{\mathrm{m}}]) \simeq [\mathrm{Ker}\,\mathscr{G}_p/\mathbb{G}_{\mathrm{m}})]$$

In particular, the splitting of the Proposition **2.3.80** is Cartier dual to the exponential map

$$\widehat{\mathbb{G}_{\mathsf{a}}} \xrightarrow[\sim]{\mathsf{exp}} \widehat{\mathbb{G}_{\mathrm{m}}}$$

See also [Bro21] for further discussion on Cartier duality for groups stacks.

In particular, the result of [Mou21:Theorem 1.8] shows that, ultimately, the origin of the HKR-filtration is the deformation to the normal bundle.

What this theorem tells us is that away from zero characteristic, the de Rham differential is *at best* the associated graded remainder of a filtered action of the circle.

Remark 2.3.82. The contents of the Theorem **2.3.67** can in fact be lifted from $\mathbb{Z}_{(p)}$-algebras to $\mathbb{Z}$-algebras replacing fixed points of Frob_p for the prime p alone, by the intersection of all fixed points of all Frobenius at all primes. See [Toë20].

Application 2.3.83. The discovery of the mixed graded circle $\mathsf{S}^1_{\epsilon-\mathsf{gr}}$ is at the origin of the theory of derived foliations by Toën-Vezzosi [TV20a; TV20b]

Further Reading 2.3.84. A similar universal property to the one stated in the Theorem **2.3.67** and similar corollaries to Proposition **2.3.70** have been independently obtain by Raksit [Arp20]. His approach avoids the construction of a geometric object implementing the filtered circle and constructs directly the algebra of global sections $\mathsf{C}^*(\mathsf{S}^1_{\mathsf{Fil}}, \mathscr{O})$. By doing so, he bypasses the use of Witt vectors and has the advantage of working directly over the sphere spectrum. The precise comparison between the two approaches remains unwritten. We sketch here a symmetric monoidal comparison of their categories of representations. Our filtered circle $\mathsf{S}^1_{\mathsf{Fil}}$ is of finite cohomologial dimension [MRT22: 3.4.10]. In particular, using base-change for the diagram (see [HLP23:A.1.10])

$$\begin{tikzcd} \mathsf{S}^1_{\mathsf{Fil}} \arrow[r] \arrow[d] & {[\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]} \arrow[d] \\ {[\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]} \arrow[r] & \mathsf{B}_{[\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]}\mathsf{S}^1_{\mathsf{Fil}} \end{tikzcd}$$

we find an equivalence of symmetric monoidal ∞-categories

$$\mathsf{QCoh}(\mathsf{BS}^1_{\mathsf{Fil}}) \simeq \mathsf{C}^*(\mathsf{S}^1_{\mathsf{Fil}}, \mathscr{O}) - \mathsf{CoMod}(\mathsf{Fil}(\mathsf{Mod}_{\mathsf{k}}))^{\otimes}$$

where the r.h.s requires the canonical bialgebra structure on $\mathsf{C}^*(\mathsf{S}^1_{\mathsf{Fil}}, \mathscr{O})$ induced from the group structure. Let now $\mathbb{T}^{\vee}_{\mathsf{Fil}}$ be the filtered bialgebra in $\mathsf{Fil}(\mathsf{Mod}_{\mathsf{k}})$ constructed by Raksit in [Arp20:Notation 6.1.4] base-changed from $\mathbb{Z}$ to $\mathsf{k} = \mathbb{Z}_{(p)}$. By construction, this is obtained as the Whitehead tower of the algebra of cochains $\mathsf{C}^*(\mathsf{S}^1, \mathsf{k})$ (see [Arp20:Notation 6.1.1]). The universal property of the Whitehead tower functor (see [MRT22:Construction 2.2.5]) induces an equivalence between the underlying filtered complexes

$$\mathbb{T}^{\vee}_{\mathsf{Fil}} = \tau_{\geq}(\mathbb{T}^{\vee}) \to \mathsf{C}^*(\mathsf{S}^1_{\mathsf{Fil}}, \mathscr{O})$$

as a consequence of the [MRT22:Prop. 3.3.2 and Theorem 3.4.17 and Remark 3.5.1]. Beware of the Warning **2.3.60**. Finally, it follows from the uniqueness statement in [Arp20:Theorem 6.1.5-(a)] that this map must also be an equivalence as bi-$\mathsf{E}^{\otimes}_{\infty}$ filtered bialgebras. In particular, we find equivalences of symmetric monoidal ∞-categories

$$\mathsf{QCoh}(\mathsf{BS}^1_{\mathsf{Fil}}) \simeq \mathsf{C}^*(\mathsf{S}^1_{\mathsf{Fil}}, \mathscr{O}) - \mathsf{CoMod}(\mathsf{Fil}(\mathsf{Mod}_{\mathsf{k}}))^{\otimes} \simeq \mathbb{T}^{\vee}_{\mathsf{Fil}} - \mathsf{CoMod}(\mathsf{Fil}(\mathsf{Mod}_{\mathsf{k}}))^{\otimes} \simeq \mathbb{T}_{\mathsf{Fil}} - \mathsf{Mod}(\mathsf{Fil}(\mathsf{Mod}_{\mathsf{k}}))^{\otimes}$$

See [Arp20:Proposition 2.2.1]. Notice however that in [Arp20:6.1.5-(b)], Raksit promotes $\mathbb{T}^{\vee}_{\mathsf{Fil}}$ with a structure of *derived commutative bialgebra*, extending the bi-$\mathsf{E}^{\otimes}_{\infty}$-structure used above. At the same time, by construction, the algebra of functions on the filtered circle admits a *cosimplicial enhancement* $\mathsf{C}^*_{\Delta}(\mathsf{S}^1_{\mathsf{Fil}}, \mathscr{O})$. We do not compare here these two enhancements.

Our interest in having a geometric pictured, in particular, a geometric version of the mixed graded circle, is motivated to the applications to foliations and enumerative geometry in positive characteristic.

Application 2.3.85. Shifted symplectict structures in positive characteristic. We will come back to this in Chapter **3**.

Application 2.3.86. The discussion in the Remark **2.3.81** summarizing the results of [Mou21] implies that the existence of a filtered circle $\mathsf{S}^1_{\mathsf{Fil},\mathbb{G}}$ associated to any abelian formal group $\mathbb{G}$ via the filtered group scheme

$$\mathsf{S}^1_{\mathsf{Fil},\mathbb{G}} := \mathsf{B}(\mathsf{CartierDual}_{[\mathbb{A}^1/\mathbb{G}_m]}(\mathrm{Def}_{\mathbb{G}}/\mathbb{G}_m)) \to [\mathbb{A}^1/\mathbb{G}_m]$$

encoding the deformation to the normal bundle at the unit of $\mathbb{G}$. These circles have been introduced in our work [MRT22:§6.3] to defined $\mathbb{G}$-twisted versions of Hochschild homology $\mathsf{HH}^{\mathbb{G}}$ and discussed in detail in [Mou21].

Future works 2.3.87. Together with B. Hennion and T. Moulinos we investigate the possibility of using the twisted version $\mathsf{HH}^{\mathbb{G}}$ of the Application **2.3.86** when $\mathsf{k} = \mathbb{C}$ and $\mathbb{G} = \widehat{\mathscr{E}}$ is the formal group of the universal elliptic curve, to recover the construction of the Witten genus in [Cos10; Cos11]. The Witten genus was introduced by Witten [Wit87] as the partition function of a 2-dimensional quantum field theory built from maps from an elliptic curve to a smooth manifold X. It was later re-interpreted in [Wit88] as the S^1-equivariant index of the Dirac operator on the loop space of a manifold. The computations of Zagier in [Zag88] exhibit the Witten class as formal power series

$$\mathsf{Witten}(X) = \mathsf{exp}(\sum_{k\geq 2} \frac{(2k-1)!}{(2\pi i)^{2k}} \mathsf{E}_{2k}\mathsf{Ch}_{2k}(\mathbb{T}_X)) \tag{113}$$

where the E_{2k} are the Eiseinstein series. The modular nature of $\mathsf{Witten}(X)$ thus depends on the vanishing of $\mathsf{Ch}_2(\mathbb{T}_X)$, so that the series E_2 has no contribution. The work of Ando-Hopkins-Strickland [AHS01] refined the Witten genus to a map of $\mathsf{E}_\infty^{\otimes}$-ring spectra $\mathsf{MString} \to \mathsf{TMF}$, deduced from a suitable cubical structure on the universal generalized elliptic curve $\overline{\mathscr{E}}$ over $\overline{\mathsf{M}}_{1,1}$ - the Deligne-Mumford compactification of the moduli stack of elliptic curves.

About 10 years ago, the works [Cos11; Cos10] provided a yet different construction of the Witten genus over $\mathbb{C}$, using tools from derived algebraic geometry, namely as a the difference of two (projective) volume forms on the formal derived moduli space

$$\mathbb{R}\widehat{\mathsf{Map}_{\overline{\mathsf{M}}_{1,1}}(\overline{\mathscr{E}}, X \times \overline{\mathsf{M}}_{1,1})} \tag{114}$$

of constant maps from the universal generalized elliptic curve to a target manifold X equipped with a fixed trivialization of its second chern character $\mathsf{Ch}_2(\mathbb{T}_X)$. More precisely, the work of Costello establishes a correspondance between the space of such trivializations, the space of BV-quantizations of $\mathsf{Map}_{\overline{\mathsf{M}}_{1,1}}(\overline{\mathscr{E}}, \mathsf{T}^*X \times \overline{\mathsf{M}}_{1,1})$ and the space of (projective volume forms) on (114). This strategy has also been applied by Grady-Gwilliam in [GG14; Gra12] to recover the Todd class, following the observation that the Witten genus at the nodal point gives the Todd class:

$$\lim_{\tau \to i.\infty} \mathsf{Wit}(X) = e^{-\frac{c_1(\mathbb{T}_X)}{2}}.\mathsf{Todd}_X \tag{115}$$

We believe that the Witten genus can thus be canonically recovered in the spirit of the Remark **2.3.58**. More precisely, we consider the universal elliptic curve $\mathscr{E} \to \overline{\mathsf{M}}_{1,1}$ (see Example **2.1.10**) and its formal completion $\widehat{\mathscr{E}}$ along the section $e : \overline{\mathsf{M}}_{1,1} \to \mathscr{E}$ corresponding to the unit element. This defines an abelian formal group scheme relatively to $\overline{\mathsf{M}}_{1,1}$ where at infinity we have the formal group of the Tate curve. The deformation to the normal bundle at the unit gives a filtered group stack relatively to $\overline{\mathsf{M}}_{1,1}$,

$$\mathrm{Def}_{\widehat{\mathscr{E}}} \to [\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}] \times \overline{\mathsf{M}}_{1,1}$$

which interpolates between $\widehat{\mathscr{E}}$ and formal group completion at the zero section of the linear bundle stack $\mathrm{Lin}_{\overline{\mathsf{M}}_{1,1}}(\overline{\Omega})$ with $\overline{\Omega}$ the Hodge bundle. We define the *universal elliptic filtered circle* as

$$\mathsf{S}^1_{\mathsf{Fil,ell}} := \mathsf{B}(\mathsf{CartierDual}_{[\mathbb{A}^1/\mathbb{G}_m]\times\overline{\mathsf{M}}_{1,1}}(\mathrm{Def}_{\widehat{\mathscr{E}}}))$$

and for any derived scheme X consider the relative mapping stack

$$\mathbb{R}\mathsf{Map}_{[\mathbb{A}^1/\mathbb{G}_m]\times\overline{\mathsf{M}}_{1,1}}(\mathsf{S}^1_{\mathsf{Fil,ell}}, X \times [\mathbb{A}^1/\mathbb{G}_m] \times \overline{\mathsf{M}}_{1,1}) \tag{116}$$

It is easy to show that underlying stack of this filtration is the stack of maps from the universal elliptic curve, formally completed along constant maps

$$[\mathbb{R}\mathsf{Map}_{[\mathbb{A}^1/\mathbb{G}_m]\times\overline{\mathsf{M}}_{1,1}}(\mathsf{S}^1_{\mathsf{Fil,ell}}, X \times [\mathbb{A}^1/\mathbb{G}_m] \times \overline{\mathsf{M}}_{1,1})]^{\mathrm{u}} \simeq \widehat{\mathbb{R}\mathsf{Map}_{\overline{\mathsf{M}}_{1,1}}(\mathscr{E}, X \times \overline{\mathsf{M}}_{1,1})} \tag{117}$$

and its associated graded is the twisted shifted tangent stack

$$[\mathbb{R}\mathsf{Map}_{[\mathbb{A}^1/\mathbb{G}_m]\times\overline{\mathsf{M}}_{1,1}}(\mathsf{S}^1_{\mathsf{Fil,ell}}, X \times [\mathbb{A}^1/\mathbb{G}_m] \times \overline{\mathsf{M}}_{1,1})]^{\mathsf{gr}} \simeq \mathsf{Spec}(\mathsf{Sym}_{X\times\overline{\mathsf{M}}_{1,1}}(\mathbb{L}_X[1]\otimes\Omega))/\mathbb{G}_m$$

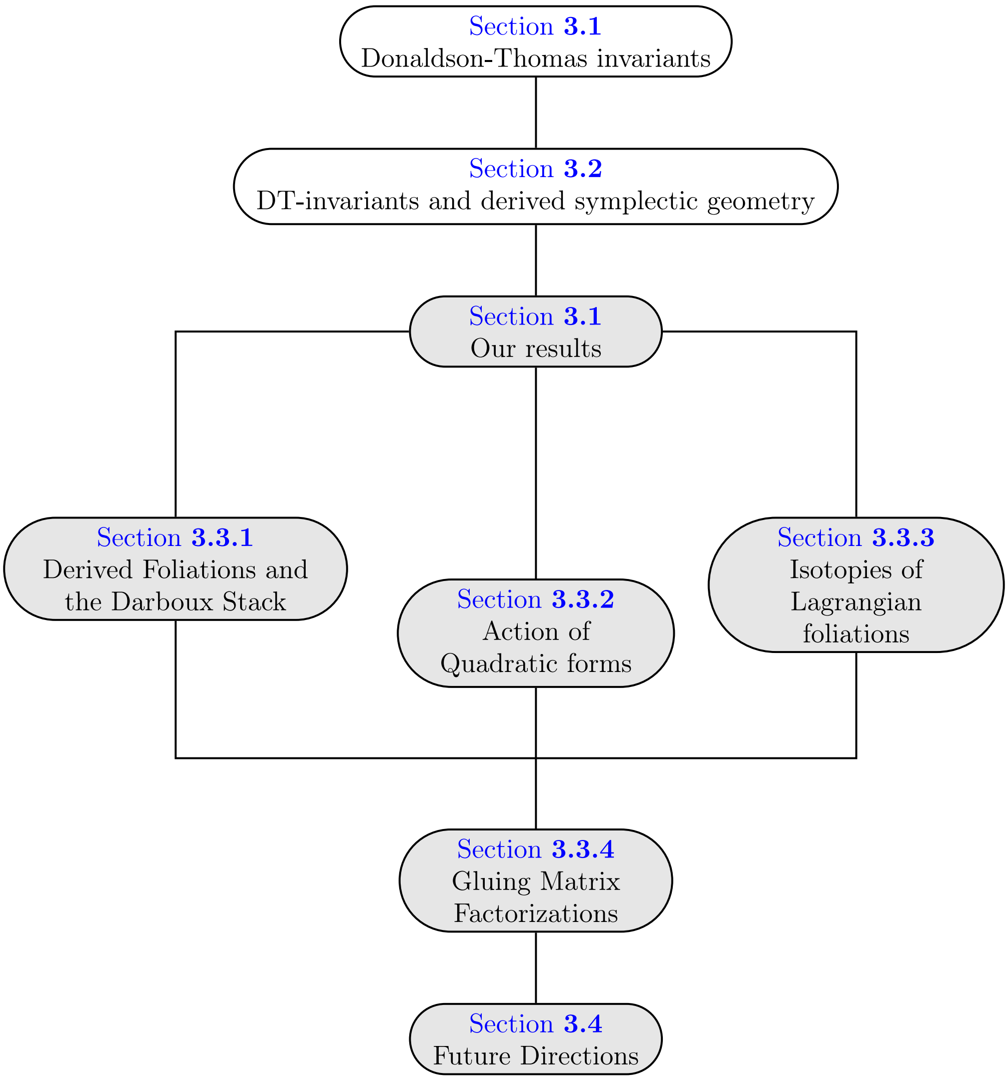
We use the cubical structure θ of [AHS01] to construct a splitting of the filtered stack (116). Notice that, at infinity, since the cohomology of the nodal curve coincides with the cohomology of the circle, we find the two sides of the HKR theorem in the Theorem **2.3.57**. We claim that the splitting induced by θ universally over $\overline{\mathsf{M}}_{1,1}$ coincides with the one of the Proposition **2.3.80** at the nodal curve. Furthermore, as explained in the Remark **2.3.58**, the Todd class appears as the difference between two trivializations of the canonical sheaf: the one produced by the loop group structure and the one produced by the abelian group structure on tangent vectors. We believe that the θ-splitting of (116) allows us to compare the difference of two natural trivializations of the canonical sheaves relative to $\overline{\mathsf{M}}_{1,1}$, namely, the one of the associated graded stack coming from vector bundle abelian group structure, and the one of (117) induced by the choice of a trivilization of the second chern character $\mathsf{Ch}_2(\mathbb{T}_X)$. The θ-splitting allows us to construct canonically a $\mathbb{G}_m$-equivariant function W on $\mathsf{Spec}(\mathsf{Sym}_{X\times\overline{\mathsf{M}}_{1,1}}(\mathbb{L}_X[1]\otimes\Omega))$. We believe our W is the Witten genus. So far we haven't been able to show this.

CHAPTER 3

Ongoing work: Gluing invariants of Donaldson-Thomas type

This section presents an ongoing project with B. Hennion and J. Holstein on Donaldson-Thomas invariants (see Theorem **3.3.26** and Corollary **3.3.29**). A video lecture covering the materials in this section is available here

This project has been in development for the last two years and will soon appear on the arxiv. Its sequels will continue to drive my research in future projects. Here's the roadmap for this section:



3.1. Donaldson-Thomas invariants

Let us fix the base field $\mathsf{k} = \mathbb{C}$. In Section **2.1** we discussed how string theory leads to enumerative invariants of a Calabi-Yau 3-fold Y and how Gromov-Witten invariants appear as the counting of *parametrized curves* in Y.

In this section we discuss a different approach, namely, the counting of *embedded curves* in Y. Instead of looking at the moduli stack of stable maps $\overline{\mathsf{M}}_{g,n}(Y,\beta)$ (see Theorem **2.1.15**) we look at the moduli space of closed subschemes in Y of codimension 2, ie, the *Hilbert scheme*, $\mathsf{Hilb}_2(Y)$. Unfortunately, for a general embedding, these Hilbert schemes (or their derived versions) are neither smooth nor quasi-smooth (in the sense discussed in Section **2.1.4**): indeed, the deformation theory of an embedded curve is given by the cohomology of the shifted normal bundle of the embedding and when the embedding is not quasi-smooth neither will be the deformation theory. In particular, there is no hope of a virtual fundamental class to compute its volume. An observation by R. Thomas [Tho00] unlocked a new possibility: if we replace the embedded curves by their respective ideal sheaves, the deformation theory is better behaved, enough to define a virtual fundamental class and compute volumes. In this situation we are interested in the moduli space $\mathsf{MCoh}(Y)$ of coherent sheaves on Y (*), which through the ideal-subscheme correspondance, has essentially the same objects as the Hilbert scheme, but a different deformation theory. In loc. cit, Thomas also observed that if $E \in \mathsf{MCoh}(Y)$, then a combination of Serre duality and the CY-condition forces a duality on the deformation theory of E:

$$(118) \quad \{1^{st\,order}\text{def. of } E \in \mathsf{Coh}^{\mathsf{b}}(Y)\} \simeq \{\text{Obstructions to def. of } E \in \mathsf{Coh}^{\mathsf{b}}(Y)\}^{\vee}$$

between the set of obstructions to the existence of deformations and the space of deformations. This property guarantees that the volumes of the moduli spaces $\mathsf{MCoh}(Y)$ are quasi-smooth and therefore admit a virtual fundamental class (Section **2.1.4**).

The Donaldson-Thomas invariants of Y are defined as the volumes of the moduli spaces $\mathsf{MCoh}(Y)$:

$$\mathrm{DT}(Y) := \mathsf{Vol}(\mathsf{MCoh}(Y)) := \int_{[\mathsf{MCoh}(Y)]^{\mathsf{vir}}}$$

The symmetry latent in (118) has been further explored by K. Behrend [Beh09] who introduce the notion of *symmetric* perfect obstruction theory enhancing the notion in Definition **2.1.60** with Thomas's symmetry:

Theorem 3.1.1 (Behrend [Beh09]). *Let X be a proper Deligne-Mumford stack with a symmetric perfect obstruction theory. The there is a uniquely defined constructible function $\nu : X \to \mathbb{Z}$ such that*

(*)In fact we are looking at coherent sheaves with prescribed stability conditions. We decide to omit this from our notations in this survey.

$$\int_{[\mathsf{MCoh}(Y)]^{\mathsf{vir}}} = \chi(X,\nu)$$

where in the r.h.s we have the Euler characteristic of X weighted by ν. Moreover, the result is independent of the choice of symmetric obstruction theory.

3.2. DT-invariants and Shifted symplectic geometry

The starting point for our discussion is the discovery by Pantev-Toën-Vaquié-Vezzosi [PTVV13] that the extra symmetry (118), as the notion of symmetric obstruction theory, are really a shadow of a (-1)-shifted symplectic structure on the derived enhancement of $\mathsf{MCoh}(Y)$, in the sense discussed in Section **1.2.4**:

Lemma 3.2.1. *Let X be a (-1)-shifted derived scheme. Let $j : \mathsf{t}_0(X) \to X$ be the inclusion of the classical truncation. Then the canonical map given by the functoriality of cotangent complexes*

$$j^*\mathbb{L}_X \to \mathbb{L}_{\mathsf{t}_0(X)}$$

is a symmetric perfect obstruction theory.

Compare to the Lemma **2.1.61**. Finally, the starting point for our discussion is the following theorem:

Theorem 3.2.2 ([PTVV13])**.** *Let Y be a Calabi-Yau variety of dimension 3. Then the derived mapping stack $\mathbb{R}\mathsf{Map}(Y, \mathsf{Perf})$ inherits a $(2-3) = (-1)$-shifted symplectic form, obtained by integrating the 2-shifted symplectic form on the derived stack of perfect complexes Perf of the Example **1.2.40**.*

Finally, since Y is smooth, the derived moduli spaces $\mathbb{R}\mathsf{MCoh}(Y)$ can be found as open substacks in $\mathbb{R}\mathsf{Map}(Y, \mathsf{Perf})$ and therefore, symplectic.

Example 3.2.3. A simple example of a (-1)-shifted symplectic derived scheme is the derived critical locus discussed in the Example **1.2.36** and the Example **1.2.37**. In this case the symmetry (118) is quite explicit: due to the symmetry of the Hessian inducing (27), we have

$$\mathsf{H}^0(\mathbb{L}_{\mathsf{dCrit}(U,f)}) \simeq \mathsf{H}^{-1}(\mathbb{L}_{\mathsf{dCrit}(U,f)})^\vee$$

A result of Brav-Bussi-Joyce shows that the Example **3.2.3** is actually all there is, locally:

Theorem 3.2.4 (Darboux Lemma[BBJ19])**.** *Let X be a (-1)-shifted derived scheme. Then, Zariski locally, X is a derived critical locus of a function f on a smooth scheme U equipped with the shifted symplectic form of the Example **1.2.37**.*

Thanks to this result we have, locally defined on $\mathbb{R}\mathsf{MCoh}(Y)$, a perverse sheaf $P_{U,f}$ computing the cohomology of vanishing cycles for the function f.

Remark 3.2.5. By construction, the sheaf $P_{U,f}$ is supported on the critical locus

$$P_{U,f} \in \mathsf{Perv}_{\mathsf{dCrit}(U,f)}(U) = \mathsf{Perv}(\mathsf{dCrit}(U,f)) = \mathsf{Perv}(\mathsf{t}_0(\mathsf{dCrit}(U,f)))$$

Remark 3.2.6. It is easy to see that, even in the local case, the presentation of X as a derived critical locus is not unique. A simple enough example is

$$\mathsf{dCrit}(\mathsf{Spec}(\mathbb{C}), 0) = \mathsf{dCrit}(\mathbb{A}^1_{\mathbb{C}}, x^2)$$

More generally, for any choice of non-degenerate quadratic form q in $\mathbb{A}^n_{\mathbb{C}}$ we find symplectic identifications

$$\mathsf{dCrit}(\mathsf{Spec}(\mathbb{C}), 0) = \mathsf{dCrit}(\mathbb{A}^1, x^2) = \mathsf{dCrit}(\mathbb{A}^2_{\mathbb{C}}, x^2 + y^2) = \cdots = \mathsf{dCrit}(\mathbb{A}^n_{\mathbb{C}}, q)$$

Another example is obtained by adding a quadratic variable:

$$X = \mathsf{dCrit}(\mathbb{A}^1_{\mathbb{C}}, x^3) = \mathsf{dCrit}(\mathbb{A}^2_{\mathbb{C}}, x^3 + y^2) \tag{119}$$

More generally, if $X = \mathsf{dCrit}(U, f)$ then we also have

$$X = \mathsf{dCrit}(U \times \mathbb{A}^1_{\mathbb{C}}, f + x^2)$$

Remark 3.2.7. The ambiguity exhibited in Remark **3.2.6** transposes to the sheaves $P_{U,f}$. For instance, the sheaves $P_{\mathbb{A}^1_{\mathbb{C}},x^2}$ and $P_{\mathbb{A}^2_{\mathbb{C}},x^2+y^2}$ are isomorphic but not canonically: the identification requires the choice of an orientation of the Milnor sphere.

Remark 3.2.8. Let X be a (-1)-shifted symplectic derived scheme and $x \in X$ be a $\mathbb{C}$-point. The Theorem **3.2.4** tells us that there exists a Zariski open neighbourhood S of x in X, with $S = \mathsf{dCrit}(U, f)$. In this case, the value of Behrend's function ν (cf. Theorem **3.1.1**) at x has a very explicit description in terms of the Milnor numbers of f

$$\nu(x) = \mu_f = (-1)^{\dim U - 1}\chi(\mathscr{V}_f, x)$$

See [Beh09:§1.2].

A second main result by Brav-Bussi-Dupont-Joyce-Szendroi (BBDJS) [BBDJS15] establishes a condition under which the ambiguity in the Remark **3.2.7** can be solved and the gluing of the locally defined sheaves $P_{U,f}$ globally over $\mathbb{R}\mathsf{MCoh}(Y)$ is possible:

Theorem 3.2.9 ([BBDJS15])**.** *Let X be a (-1)-symplectic derived scheme. Assume that there exists a line bundle L together with an equivalence $L \otimes L \simeq \mathsf{det}(\mathbb{T}_X)$ (aka orientation data). Then:*

- *The locally defined perverse sheaves of vanishing cycles $P_{U,f}$ glue to a globally defined perverse sheaf $P_{Joyce} \in \mathsf{Perv}(X)$.*
- *The Euler characteristic of the perverse sheaf P_{Joyce} recovers Behrend's function of the Theorem **3.1.1***

$$\chi(P_{Joyce}) = \nu$$

and it particular it gives back the DT-counting.

Remark 3.2.10. The proof of the Theorem **3.2.9** is obtained gluing by hand along local presentations of the underlying classical scheme as classical critical loci. This methods has several drawbacks:

- the method does not see the full derived structure on X.
- it works for perverse sheaves because:
  - perverse sheaves form a **1-category** (no higher homotopies needed to glue).
  - perverse sheaves have the $\mathbb{A}^1$**-homotopy invariance** property.

The goal of the work in progress with B. Hennion and J. Holstein described in Section **3.3** below, is precisely to identify a more robust gluing mechanism, that allows us to glue more refined invariants of LG-pairs, such as the derived categories of singularities described in Section **2.2** which, thanks to the Corollary **2.2.43** and Theorem **2.2.63**, can be seen as a categorification of vanishing cohomology.

3.3. Our new results

3.3.1. Derived Foliations and the Darboux Stack. In order to explain our strategy, let us start with a construction in classical symplectic geometry:

Construction 3.3.1. Let X be a classical symplectic manifold. Then, by the classical Darboux's lemma, X is locally symplectomorphic to a cotangent bundle T^*M of smooth manifold M. We can analyse the functor of points parametrizing the choice of Darboux local models: for every open subset $S \subseteq X$ we consider the groupoid

$$\mathsf{Darb}_X(S) := \{M \text{ smooth manifold}, \phi : S \to \mathsf{T}^*M \text{ étale symplectomorphism}\}$$

Notice that the data of an étale symplectomorphism $\phi : S \to \mathsf{T}^*M$ in particular implies that

- the fibers of the projection $S \simeq \mathsf{T}^*M \to M$ define a *smooth Lagrangian foliation* $\mathscr{F}$ on S (ie, $\omega_{|\text{fibers}} = 0$);
- The symplectic form on S is *exact*, ie, there exists a 1-form α (the Liouville form on T^*M) with $d_R(\alpha) = \omega$.

In the setting of (-1)-shifted symplectic forms, we define

Definition 3.3.2. Let S be a (-1)-symplectic derived scheme and consider the map classifying the symplectic form together with its closure:

$$k(2)[3] \xrightarrow{\omega} \mathsf{HC}^-(S)$$

An exact structure α for ω consists of a null-homotopy of the composition

$$k(2)[3] \xrightarrow{\omega} \mathsf{HC}^-(S) \longrightarrow \mathsf{HP}(S)$$

i.e, an homotopy lifting rendering the commutativity of

$$\begin{tikzcd} & \mathsf{HC}(S)[1] \arrow[r] \arrow[d] & 0 \arrow[d] \\ k(2)[3] \arrow[ur, dashed, "\alpha"] \arrow[r, "\omega"] & \mathsf{HC}^-(S) \arrow[r] & \mathsf{HP}(S) \end{tikzcd}$$

Example 3.3.3. Let $\mathsf{dCrit}(U, f)$ be derived critical locus of a function f on a smooth scheme U, together with the (-1)-shifted symplectic structure of the Example **3.2.3**. Then, the symplectic structure is exact, as a consequence of the presentation of $\mathsf{dCrit}(U, f)$ as a Lagrangian intersection in T^*U where we have a Liouville form (cf. (28)).

Thanks to the works of Toën-Vezzosi and Pantev-Toën on derived foliations [PT14; TV20b; TV20a], the Construction **3.3.1** makes sense in the setting of (-1)-shifted symplectic derived schemes:

Theorem 3.3.4 (Pantev-Toën [PT14])**.** *Let $S = \mathsf{Spec}(A)$ be a (-1)-symplectic affine derived scheme. Then the following categories of data are equivalent:*

- *pairs $(\mathscr{F}, \alpha)$ with $\mathscr{F}$ a globally defined derived Lagrangian foliation on S with smooth leaf space $S/\mathscr{F}$, together with an exact structure α;*
- *the data of a smooth* formal *scheme U, a function f on U and a symplectomorphism $S \simeq \mathsf{dCrit}(U, f)$*

Proof. (Sketch) The correspondance between the two types of data works as follows: to a pair $(\mathscr{F}, \alpha)$ one assigns the formal scheme $U := S/\mathscr{F}$. The fact that this is a smooth formal scheme follows from the results of Bhatt in [Bha12]. The function f is obtained as the different between the isotropic structure and the exact structure. Conversely, if $S = \mathsf{dCrit}(U, f)$ we use the fact that the closed immersion $S = \mathsf{dCrit}(U, f) \hookrightarrow U$ is a Lagrangian fibration [Gra20; Saf20] so that its fibers define a Lagrangian foliation on S. Finally, fact that these assignments define an equivalence of categories is again a consequence of the results in [Bha12]. This is where it becomes crucial that U is a formal scheme so that it is coincides with the formal completion. $\square$

Remark 3.3.5. In terms of the Theorem **3.3.4** what the Darboux lemma Theorem **3.2.4** actually shows is that Zariski locally every (-1)-shifted derived scheme admits a Lagrangian foliation with smooth formal leaf space and an exact structure. The existence of an exact structure is actually a global statement: every (-1)-shifted derived scheme admits one as a consequence of a theorem of Deligne. See [Toë14:Corollary 5.3].

Remark 3.3.6. A version of the Theorem **3.3.4** in the case $n = -2$ recovers the local models of Borisov-Joyce [BJ17] and Thomas-Oh [OT23] for moduli spaces of sheaves on CY 4-folds.

Definition 3.3.7. Let $S = \mathsf{Spec}(A)$ be a (-1)-shifted symplectic affine derived scheme. A pair $(\mathscr{F}, \alpha)$ as in the Theorem **3.3.4**-(i) is called a *Darboux datum* on S. In terms of the Theorem **3.3.4**-(ii) it can also be presented as a pair (U, f) consisting of a smooth formal scheme and a function.

Example 3.3.8. Let $X = \mathsf{dCrit}(\mathbb{A}^1, x^3) = \mathsf{Spec}(\mathbb{C}[x]/(3x^2))$ as in the Example **1.2.36**. Then, the formal completion $\widehat{\mathbb{A}}^1_X = \widehat{\mathbb{A}}^1_{X_{\mathsf{red}}} = \widehat{\mathbb{A}}^1_0$ together with the Taylor expansion of the function x^3, defines a Darboux datum for X.

Our first main observation is the following:

Proposition 3.3.9. *Let X be a (n)-shifted symplectic derived scheme. The assignment:*

$$S \to X \text{ étale} \mapsto \{(\alpha, \mathscr{F}) : \text{ Darboux data on } S\}$$

defines a hypercomplete stack *on the small étale site of X. We call it the Darboux stack and denote it by $\underline{\mathsf{Darb}}_X$.*

Remark 3.3.10. The Darboux stack is actually a product of stacks on the small étale site $X_{\text{ét}}$

$$\underline{\mathsf{Darb}}_X = \underline{\mathsf{Exact}}_X \times \underline{\mathsf{LagFol}}_X$$

where $\underline{\mathsf{Exact}}_X$ is the stack parametrizing exact structures on X and $\underline{\mathsf{LagFol}}_X$ is the stack parametrizing Lagrangian foliations with smooth leaf space.

Construction 3.3.11. Thanks to the Remark **3.3.5** we always have an exact structure on a (-1)-shifted symplectic scheme X. Given a particular choice of exact structure α we define $\underline{\mathsf{Darb}}^{\alpha}_X$ to be the fiber over α of $\underline{\mathsf{Darb}}_X$ along the projection

$$\underline{\mathsf{Exact}}_X \times \underline{\mathsf{LagFol}}_X \to \underline{\mathsf{Exact}}_X$$

The invariants that we care about are actually defined on the Darboux stack:

Example 3.3.12. Let X be a (-1)-shifted symplectic derived scheme with exact structure α. The assignment given by the Milnor number (cf. (59))

$$\underline{\mathsf{Darb}}_X(S) \ni (U, f) \to \mu_f \in \mathbb{Z}$$

defines a morphism of sheaves on the small étale site of X

$$\underline{\mathsf{Darb}}_X \to \mathbb{Z}_X \tag{120}$$

where $\mathbb{Z}_X$ is the constant sheaf. Notice that the data of globally defined $\mathbb{Z}$-valued function on X is equivalent to the data of a morphism of sheaves on $X_{\text{ét}}$

$$*_X \to \mathbb{Z}_X$$

where $*_X$ is the final object in the small étale topos of X [(*)]. In particular, the existence of Behrend's function in the Theorem **3.1.1** is equivalent to the existence of a factorization of (120) through $*_X$

$$\begin{tikzcd} \underline{\mathsf{Darb}}_X \arrow[r] \arrow[d] & \mathbb{Z}_X \\ *_X \arrow[ur, dashed, "\exists\, \nu_{Behrend}"'] & \end{tikzcd} \tag{121}$$

Example 3.3.13. Let X be a (-1)-shifted symplectic derived scheme. Perverse sheaves form a stack on the small étale site of X - $\underline{\mathsf{Perv}}_X$ - defined by sending an étale neighborhood $S \to X$ to the maximal groupoid $\mathsf{Perv}_X(S)^{\simeq}$. The locally defined perverse sheaves given Darboux's lemma Theorem **3.2.4** define a natural transformation of stacks[(†)]:

$$\underline{\mathsf{Darb}}_X \to \underline{\mathsf{Perv}}_X \tag{122}$$

defined on an étale neighborhood $S \to X$ by the assignment

$$(U, f) \mapsto P_{U,f}$$

Similarly to the Example **3.3.12**, give a globally defined perverse sheaf on X is equivalent to the data of a morphism of étale stacks on $X_{\text{ét}}$

[(*)]Indeed, if F is a stack in the small étale topos $\mathsf{Sh}_{\text{ét}}(X)$ we have $F(X) \simeq \mathbb{R}\mathsf{Map}(*_X, F)$.

[(†)]In fact, to define the sheaves of vanishing cycles on the formal completions directly, one needs to use Berkovich's formalism of vanishing cycles on formal schemes [Ber94]

$$*_X \to \underline{\mathsf{Perv}}_X$$

Theorem **3.2.9** amounts to say that in the presence of an orientation data, the morphism of stacks (122) admits a factorization

$$\begin{tikzcd} \underline{\mathsf{Darb}}_X \arrow[r] \arrow[d] & \underline{\mathsf{Perv}}_X \\ *_X \arrow[ur, dashed, "\exists\, P_{Joyce}"'] & \end{tikzcd}$$

Example 3.3.14. Recall the construction of the 2-periodic dg-categories $\mathrm{Sing}^{2-\mathsf{per}}$ in Section **2.2.3**. For our applications, we will need to consider an extra piece of data on $\mathrm{Sing}^{2-\mathsf{per}}(U, f)$ when U is a formal completion of an affine derived scheme S, namely, a flat connection seen as an action of the category of 2-periodic $\mathscr{D}$-modules over S on $\mathrm{MF}(U, f)$[(*)]. The assignment

$$[S \to X] \in X_{\text{ét}} \mapsto \mathsf{dgCat}_S^{2-\mathsf{per},\mathsf{idem},\nabla}$$

defines a sheaf of the étale topology, which we denote as $\underline{\mathsf{dgCat}}_X^{2-\mathsf{per},\mathsf{idem},\nabla}$. Moreover, the assignment

$$(U, f) \mapsto \mathrm{Sing}^{2-\mathsf{per}}(U, f)$$

defines a natural transformation of sheaves on $X_{\text{ét}}$

$$\underline{\mathsf{Darb}}_X \to \underline{\mathsf{dgCat}}_X^{2-\mathsf{per},\mathsf{idem},\nabla} \tag{123}$$

Our main purpose in this work is to describe the type of orientation data necessary for the locally defined sheaves $\mathrm{Sing}^{2-\mathsf{per}}(U, f)$ to glue globally on X, in other words, the conditions and extra information, necessary for the morphism of stacks (123) to admit a factorization

$$\begin{tikzcd} \underline{\mathsf{Darb}}_X \arrow[r] \arrow[d] & \underline{\mathsf{dgCat}}_X^{2-\mathsf{per},\mathsf{idem},\nabla} \\ *_X \arrow[ur, dashed, "\exists\, ???"'] & \end{tikzcd}$$

3.3.2. Action of Quadratic Forms. We now return to the ambiguity problem of the Remark **3.2.6** and Remark **3.2.7**. In fact, as suggested in the Remark **3.2.6**, part of this problem is created by an action of quadratic bundles:

(*) Alternatively, one can simply look at 2-periodic dg-categories over de dR-stack of [Sim96b; ST97].

Construction 3.3.15. Let X a derived scheme. We consider the stack $\underline{\mathsf{Quad}}_X^{\nabla}$ on $X_{\text{ét}}$ classifying quadratic bundles on X with an étale locally trivial flat connection. More precisely, given an étale neighborhood $S \to X$ we consider the groupoid $\underline{\mathsf{Quad}}_X^{\nabla}(S)$ classifying triples (Q, q, ∇) where (Q, q) is a non-degenerate quadratic vector bundle on S with ∇ a compatible flat connection along S which is étale locally trivial on S. In practice it will be more convenient to reformulate this definition using the dR-stack, identifying the groupoid $\underline{\mathsf{Quad}}_X^{\nabla}(S)$ with the full subcategory of $\mathsf{Quad}(S_{\mathsf{dR}})$ classifying étale locally trivial connections.

Remark 3.3.16. Consider $\mathsf{Quad}_{\mathsf{k}}$ the groupoid of non-degenerate quadratic forms over the base field k. Let $p : X \to \mathsf{Spec}(\mathsf{k})$ be the canonical projection. Then the stack $\underline{\mathsf{Quad}}_X^{\nabla}$ coincides with $p^*\mathsf{Quad}_{\mathsf{k}}$, where $p^* : \mathsf{Sh}_{\text{ét}}(\mathsf{Spec}(\mathsf{k})) \to \mathsf{Sh}_{\text{ét}}(X)$ is the sheafification of the naive inverse image p^{-1}. Indeed, for any $S \to X$ in $X_{\text{ét}}$ we have a map of prestacks

$$(p^{-1}\mathsf{Quad}_{\mathsf{k}})(S) = \mathsf{Quad}_{\mathsf{k}} \to \underline{\mathsf{Quad}}_X^{\nabla}(S)$$

sending $(\mathsf{k}^{\oplus n}, q)$ to $Q = \mathbb{A}^n \times S_{\mathsf{dR}}$ with the trivial flat connection. This is an isomorphism on π_1 because in the inverse direction we can extract flat sections as part of the Riemann-Hilbert correspondence. The defining condition of $\underline{\mathsf{Quad}}_X^{\nabla}$ implies that after sheafification, it becomes an isomorphism on the π_0-sheaves showing that the induced map

$$p^*\mathsf{Quad}_{\mathsf{k}} \to \underline{\mathsf{Quad}}_X^{\nabla}$$

is an isomorphism of stacks.

Construction 3.3.17. Let X be a derived scheme. The stack $\underline{\mathsf{Quad}}_X^{\nabla}$ of the Construction **3.3.15** carries a monoid structure with respect to the direct sum of quadratic bundles. Assume now X is (-1)-shifted symplectic. The monoidal stack $\underline{\mathsf{Quad}}_X^{\nabla}$ acts on the Darboux stack $\underline{\mathsf{Darb}}_X$ (cf. Proposition **3.3.9**). Objectwise, this action can be described as follows: if $S \to X$ is an étale neighbourhood of X, (Q, q) is a quadratic form on S_{dR}, and (U, f) is a Darboux datum for S, ie, $S \simeq \mathsf{dCrit}(U, f)$, then the pair

$$(U \underset{S_{\mathsf{dR}}}{\times} \widehat{Q}, f + q)$$

is again a Darboux local model for S, since

$$\mathsf{dCrit}(U, f) = \mathsf{dCrit}(U \underset{S_{\mathsf{dR}}}{\times} \widehat{Q}, f + q)$$

because the quadratic term has trivial critical locus.

This defines a morphism of étale stacks

$$\underline{\mathsf{Quad}}_X^{\nabla} \times \underline{\mathsf{Darb}}_X \to \underline{\mathsf{Darb}}_X$$

compatible with the monoid structure on $\underline{\mathsf{Quad}}_X^{\nabla}$.

Remark 3.3.18. By construction, the action of the Construction **3.3.17** preserves the exact structure: namely, when presenting a Darboux local model as a pair $(\mathscr{F}, \alpha)$ the action of $\underline{\mathsf{Quad}}^{\nabla}$ only changes $\mathscr{F}$ leaving the contribution of α fixed.

Remark 3.3.19. The action of quadratic bundles of the Construction **3.3.17** controls the ambiguity expressed in the Remark **3.2.6**.

Example 3.3.20. The action of quadratic bundles of the Construction **3.3.17** is compatible with the morphism of stacks (122) in the Example **3.3.13**:

$$\underline{\mathsf{Darb}}_X \to \underline{\mathsf{Perv}}_X$$

More precisely, we consider (μ_2) the étale locally constant sheaf of roots of unit on X. Its classifying stack $\mathsf{B}\mu_2$ is the fiber of the map $\mathsf{B}\mathbb{G}_{\mathrm{m}} \to \mathsf{B}\mathbb{G}_{\mathrm{m}}$ classifying the map sending a line bundle L to its square $L \otimes L$. By definition $\mathsf{B}\mu_2$ is the classifying stack of line bundles L together with an isomorphism $L \otimes L \simeq \mathscr{O}$. If (Q, q) is a quadratic bundle on S, its determinant $\mathsf{det} Q$ is a line bundle. The determinant of the quadratic form induces an isomorphism $\mathsf{det}\, Q \simeq \mathscr{O}$. This observation defines a morphism of monoidal stacks

$$\mathsf{det} : \underline{\mathsf{Quad}}_X^{\nabla} \to \mathsf{B}(\mu_2)_X$$

given by the determinant map. Moreover, the stack $\mathsf{B}(\mu_2)_X$ acts on the stack of perverse sheaves Perv_X (see [BBDJS15]). The Thom-Sebastiani theorem for vanishing cycles expresses the fact that this action is compatible with the one of $\underline{\mathsf{Quad}}_X^{\nabla}$ on $\underline{\mathsf{Darb}}_X$: namely, we have an equivariant structure

$$\underline{\mathsf{Quad}}_X^{\nabla} \circlearrowright \underline{\mathsf{Darb}}_X \to \underline{\mathsf{Perv}}_X \circlearrowleft \mathsf{B}\mu_2$$

Finally, the morphism of stacks (122) descends to the quotients

$$\underline{\mathsf{Darb}}_X / \underline{\mathsf{Quad}}_X^{\nabla} \to \underline{\mathsf{Perv}}_X / \mathsf{B}\mu_2 \tag{124}$$

Remark 3.3.21. The monoid stack $\underline{\mathsf{Quad}}_X^{\nabla}$ is not a group stack. We denote by $\underline{\mathsf{Quad}}_X^{\nabla,+}$ its group completion in the small étale topos of X. By definition, the quotient $\underline{\mathsf{Darb}}_X / \underline{\mathsf{Quad}}_X^{\nabla}$ is the quotient of $\underline{\mathsf{Darb}}_X^{+} / \underline{\mathsf{Quad}}_X^{\nabla,+}$. Notice also that since $\mathsf{B}\mu_2$ is a group, the determinant map $\underline{\mathsf{Darb}}_X \to \mathsf{B}\mu_2$ factors through the group completion $\underline{\mathsf{Darb}}_X^{+}$.

Example 3.3.22. The action of quadratic bundles of the Construction **3.3.17** is also compatible with the functor of matrix factorizations of the Example **3.3.14**: if (Q, q, ∇) is a quadratic bundle over S with flat connection, the Thom-Sebastiani theorem for matrix factorizations Proposition **2.2.27** together with Knörrer periodicity (see [Pre11: 9.1.7]) imply that $\mathrm{Sing}^{2-\mathsf{per}}(Q, q)$ is a 2-periodic *2-torsion* dg-category, ie

$$\mathrm{Sing}^{\mathsf{2-per}}(Q,q) \underset{S[u,u^{-1}]}{\otimes} \mathrm{Sing}^{\mathsf{2-per}}(Q,q) \simeq \mathrm{Sing}^{\mathsf{2-per}}(S,0)$$

In particular, $\mathrm{Sing}^{\mathsf{2-per}}(Q,q)$ defines a 2-torsion 2-periodic derived Azumaya algebra over S (see [Toë12]). Such Azumaya algebras are parametrized by a derived stack $\mathsf{Az}^{\mathsf{2-per,2-tor}}$ that we can restrict to the small étale site of X - $\mathsf{Az}_X^{\mathsf{2-per,2-tor}}$. Moreover, since (Q,q) carries a flat connection, these can be promoted to $\underline{\mathsf{Az}}_X^{\mathsf{2-per,2-tor},\nabla}$. Under tensor products, this is a monoidal stack that acts on 2-periodic dg-categories over X:

$$\underline{\mathsf{Az}}^{\mathsf{2-per,2-tor},\nabla} \circlearrowleft \underline{\mathsf{dgCat}}_X^{\mathsf{2-per,idem},\nabla}$$

The Thom-Sebastiani theorem provides equivariance data for the morphism

$$\underline{\mathsf{Quad}}_X^{\nabla} \circlearrowright \underline{\mathsf{Darb}}_X \to \underline{\mathsf{dgCat}}_X^{\mathsf{2-per,idem},\nabla} \circlearrowleft \underline{\mathsf{Az}}^{\mathsf{2-per,2-tor},\nabla}$$

and a factorization through the quotient:

$$\underline{\mathsf{Darb}}_X/\underline{\mathsf{Quad}}_X^{\nabla} \to \underline{\mathsf{dgCat}}_X^{\mathsf{2-per,idem},\nabla}/\underline{\mathsf{Az}}^{\mathsf{2-per,2-tor},\nabla} \tag{125}$$

The Examples **3.3.20** and **3.3.22** give us a first insight of the globalization mechanism behing the Theorem **3.2.9**. Indeed, using the reformulation in Example **3.3.13** by means of the factorization (121), part of the mystery would be solved if the quotient $\underline{\mathsf{Darb}}_X^{+}/\underline{\mathsf{Quad}}_X^{\nabla,+}$ was contractible, ie, equivalent to $*_X$. Our first result goes in that direction:

Proposition 3.3.23 (Hennion-Holstein-R.)**.** *Let X be (-1)-shifted symplectic derived scheme with exact structure α[(*)]. Then, the action of $\underline{\mathsf{Quad}}^{\nabla,+}$ on $\underline{\mathsf{Darb}}_X^{\alpha,+}$ is transitive. In particular, the π_0 sheaf $\pi_0(\underline{\mathsf{Darb}}_X^{\alpha,+}/\underline{\mathsf{Quad}}_X^{\nabla,+})$ is trivial.*

However, $\underline{\mathsf{Darb}}_X^{\alpha,+}/\underline{\mathsf{Quad}}_X^{\nabla,+}$ is not contractible: as we shall see in the Section **3.3.3** below, there is another source of ambiguity that we haven't considered yet, namely, arising from morphisms of local models.

3.3.3. $\mathbb{A}^1$-Contractibility of the Darboux Stack.

Example 3.3.24. Let $X = \mathsf{dCrit}(\widehat{\mathbb{A}}^2_0, f(x,y) := x^3 + y^4)$ and consider the automorphism $\phi : \widehat{\mathbb{A}}^2_0 \to \widehat{\mathbb{A}}^2_0$ obtained by sending

$$x \mapsto x + y^4, \qquad y \mapsto h(x,y).y$$

where $h = \sqrt[4]{1 - 3x^2 - 3xy^4 - y^8} \in \mathbb{C}[[x,y]]$. This automorphism is the identity on the derived critical locus and in fact preserves the shifted symplectic structure. It provides a non-trivial element in the automorphism sheaf

$$\pi_1(\underline{\mathsf{Darb}}_X^{\alpha,+}/\underline{\mathsf{Quad}}_X^{\nabla,+}, (\widehat{\mathbb{A}}^2_0, f(x,y))$$

[(*)]See the Remark **3.3.5** and Construction **3.3.11**

Implicit in the proof of Theorem **3.2.9** the authors already face the ambiguity posed by the map ϕ. In [BBDJS15:Prop. 3.4] the authors solve this issue realizing that it is possible be build a 1-parameter family (isotopy) interpolating ϕ to the identiy map. In our example, this amounts to the interpolation ϕ_t defined by

$$x \mapsto x + ty^4, \qquad y \mapsto \sqrt[4]{1 - 3tx^2 - 3xt^2y^4 - t^3y^8}.y$$

When $t = 0$ we have $\phi_0 = \mathsf{id}$ and for $t = 1$, $\phi_1 = \phi$.

The Example **3.3.24** explains us that if we want to solve the ambiguities at the level of morphisms, we need, at the very least, to identity isotopic families:

Construction 3.3.25 (Foliations up to integrable homotopy)**.** We consider the colimit

$$\underline{\mathsf{Darb}}_X^{\mathbb{A}^1} := \mathsf{colim}\ \Big(\underline{\mathsf{Darb}}^{\mathsf{const}}_{X\times\mathbb{A}^1/\mathbb{A}^1} \underset{ev_0}{\overset{ev_1}{\rightrightarrows}} \underline{\mathsf{Darb}}_X \Big)$$

where $\underline{\mathsf{Darb}}^{\mathsf{const}}_{X\times\mathbb{A}^1/\mathbb{A}^1}$ denotes the stack classifying constant families of lagrangian foliations on X parametrized by $\mathbb{A}^1$.

Finally, here's our main theorem:

Theorem 3.3.26 (Hennion-Holstein-R.)**.** *Let X be a (-1)-shifted symplectic derived scheme with exact structure α. Then the final morphism*

$$\underline{\mathsf{Darb}}_X^{\alpha,\mathbb{A}^1,+}/\underline{\mathsf{Quad}}^{\nabla,\mathbb{A}^1,+} \to *_X$$

is an equivalence of stacks on $X_{\text{ét}}$. In other words, the quotient stack is contractible.

As a first application, we explain how to recover Theorem **3.2.9** from our Theorem **3.3.26**:

Application 3.3.27. Consider again the quotient map (124):

$$\underline{\mathsf{Darb}}_X^{\alpha}/\underline{\mathsf{Quad}}_X^{\nabla} \to \underline{\mathsf{Perv}}_X/\mathsf{B}\mu_2$$

The crucial property we need about perverse sheaves is their $\mathbb{A}^1$-invariance: perverse sheaves identify isotopic families (see [BBDJS15:Prop. 2.8]). This implies that (124) factors through the quotient

$$\begin{tikzcd} \underline{\mathsf{Darb}}_X^{\alpha}/\underline{\mathsf{Quad}}_X^{\nabla} \arrow[r] \arrow[d] & \underline{\mathsf{Perv}}_X/\mathsf{B}\mu_2 \\ \underline{\mathsf{Darb}}_X^{\alpha,\mathbb{A}^1,+}/\underline{\mathsf{Quad}}^{\nabla,\mathbb{A}^1,+} \arrow[ur, dashed] & \end{tikzcd}$$

But our Theorem **3.3.26** implies that this last quotient is contractible

$$\begin{tikzcd}
\underline{\mathsf{Darb}}^{\alpha}_X/\underline{\mathsf{Quad}}^{\nabla}_X \arrow[r] \arrow[d] & \underline{\mathsf{Perv}}_X/\mathsf{B}\mu_2 \\
*_X \simeq \underline{\mathsf{Darb}}^{\alpha,\mathbb{A}^1,+}_X/\underline{\mathsf{Quad}}^{\nabla,\mathbb{A}^1,+} \arrow[ur, dashed, "\widetilde{P}"'] &
\end{tikzcd}$$

and the resulting dashed arrow defines a global section of the stack $\underline{\mathsf{Perv}}_X/\mathsf{B}\mu_2$ on X, in other words, a globally defined $\mathsf{B}\mu_2$-twisted perverse sheaf.

In this framework, it becomes clear the role of the orientation data in the Theorem **3.2.9**: the question is whether this globally defined twisted perverse sheaf can be *untwisted*, ie, when do we have a lifting

$$\begin{tikzcd}
& \underline{\mathsf{Perv}}_X \arrow[d] \\
*_X \arrow[r, "\widetilde{P}"'] \arrow[ur, dotted, "\exists P?"] & \underline{\mathsf{Perv}}_X/\mathsf{B}\mu_2
\end{tikzcd} \tag{126}$$

Fortunately, the diagram (126) is part of a pullback diagram of stacks

$$\begin{tikzcd}
& \underline{\mathsf{Perv}}_X \arrow[r] \arrow[d] & *_X \arrow[d] \\
*_X \arrow[r, "\widetilde{P}"'] \arrow[ur, dotted, "\exists P?"] & \underline{\mathsf{Perv}}_X/\mathsf{B}\mu_2 \arrow[r] & *_X/\mathsf{B}\mu_2 \simeq \mathsf{B}(\mathsf{B}\mu_2)
\end{tikzcd} \tag{127}$$

induced by the equivariance of the projection $\underline{\mathsf{Perv}}_X \to *_X$. In particular, the existence of the untwisting lift P is equivalent to the data of a trivialization of the composition

$$*_X \to \underline{\mathsf{Perv}}_X/\mathsf{B}\mu_2 \to \mathsf{B}(\mathsf{B}\mu_2) \tag{128}$$

One can check that this arrow classifies precisely the $\mathsf{B}\mu_2$-gerbe of square roots of the canonical bundle of X and the choice of a trivialization amounts to selecting one.

3.3.4. Gluing categories of Matrix Factorizations. Our Theorem **3.3.26** provides a general mechanism for gluing local invariants on a (-1)-shifted derived scheme. Our main goal is to use the factorization (125) to glue sheaves of categories of matrix factorizations.

Problem 3.3.28. Contrary to the situation with perverse sheaves explained in the Application **3.3.27**, we do not know if the functor $\mathrm{Sing}^{2-\mathsf{per}}$ identifies isotopic morphisms as in the Example **3.3.24**. We believe this to be true but we don't have a counterexample.

However, our methods suffice to glue the *motivic realizations* of $\mathrm{Sing}^{2-\mathsf{per}}$ as explained in Construction **2.2.52**:

Corollary 3.3.29 (Hennion-Holstein-R.). *The composition with the motivic realization functor* [a] *:*

$$\underline{\mathsf{Darb}}_X/\underline{\mathsf{Quad}}_X^{\nabla} \to \underline{\mathsf{dgCat}}_X^{2-\mathsf{per},\mathsf{idem},\nabla}/\underline{\mathsf{Az}}^{2-\mathsf{per},2-\mathsf{tor},\nabla} \to \underline{\mathsf{Mod}_{\mathbb{H}(\eta,\mathsf{KGL})}(\mathsf{SH})^{\nabla}}_X/\underline{\mathsf{Az}}^{2-\mathsf{per},2-\mathsf{tor},\nabla}$$

descends to the quotient under isotopies

$$\begin{tikzcd} \underline{\mathsf{Darb}}_X/\underline{\mathsf{Quad}}_X^{\nabla} \arrow[r] \arrow[d] & \underline{\mathsf{Mod}_{\mathbb{H}(\eta,\mathsf{KGL})}(\mathsf{SH})^{\nabla}}_X/\underline{\mathsf{Az}}^{2-\mathsf{per},2-\mathsf{tor},\nabla} \\ *_X \simeq \underline{\mathsf{Darb}}_X^{\alpha,\mathbb{A}^1,+}/\underline{\mathsf{Quad}}^{\nabla,\mathbb{A}^1,+} \arrow[ur, dashed] & \end{tikzcd}$$

and the dashed arrow defines a globally defined $\underline{\mathsf{Az}}^{2-\mathsf{per},2-\mathsf{tor},\nabla}$-twisted *sheaf of motives* $\widetilde{\mathrm{MF}}$ *over* X*.*

Furthermore, the obstruction to untwist $\widetilde{\mathrm{MF}}$ *is controlled by three cohomology classes capturing the only non-trivial homotopy groups of* $\mathsf{B}\underline{\mathsf{Az}}^{2-\mathsf{per},2-\mathsf{tor},\nabla}$

- $\pi_1\mathsf{B}\underline{\mathsf{Az}}^{2-\mathsf{per},2-\mathsf{tor},\nabla} = \pi_0\underline{\mathsf{Az}}^{2-\mathsf{per},2-\mathsf{tor},\nabla} \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \to \{\mathrm{Sing}^{2-\mathsf{per}}(*,0), \mathrm{MF}(\mathbb{A}^1, x^2)\}$
- $\pi_2\mathsf{B}\underline{\mathsf{Az}}^{2-\mathsf{per},2-\mathsf{tor},\nabla} = \pi_1\underline{\mathsf{Az}}^{2-\mathsf{per},2-\mathsf{tor},\nabla} \simeq \mathbb{Z}/2\mathbb{Z} \simeq \{\mathsf{id}, [1]\}$
- $\pi_3\mathsf{B}\underline{\mathsf{Az}}^{2-\mathsf{per},2-\mathsf{tor},\nabla} = \pi_2\underline{\mathsf{Az}}^{2-\mathsf{per},2-\mathsf{tor},\nabla} \simeq \mathbb{Z}/2\mathbb{Z} \simeq \mathrm{Ker}(z^2 : \mathbb{C}^* \to \mathbb{C}^*)$

[a] See Definition **2.2.59** for the notation $\mathbb{H}(\eta, \mathsf{KGL})$.

Remark 3.3.30. It was quite surprising to us to find that only three classes need to be trivialized in order to untwist $\widetilde{\mathrm{MF}}$. Originally we expected an infinite amount of homotopies to be necessary. The magic behind this is the 2-torsion aspect: indeed, the fiber sequence

$$\underline{\mathsf{Az}}^{2-\mathsf{per},2-\mathsf{tor},\nabla} \to \underline{\mathsf{Az}}_X^{2-\mathsf{per},\nabla} \underset{2}{\to} \underline{\mathsf{Az}}_X^{2-\mathsf{per},\nabla}$$

keeps the infinitely many homotopies away from the 2-torsion part.

Remark 3.3.31. Partial results to the gluing of the dg-categories MF have been obtained by [Tod19; Tod23] in the case of (-1)-shifted cotangent stacks.

Remark 3.3.32. A question that we do not address in our work is that of the *existence* of the orientation data required to untwist $\widetilde{\mathrm{MF}}$ in Corollary **3.3.29**. Notice that the class in $\pi_2\mathsf{B}\underline{\mathsf{Az}}^{2-\mathsf{per},2-\mathsf{tor},\nabla}$ is the class that appears in Joyce's Theorem **3.2.9**, classifying the $\mathsf{B}\mu_2$-gerbe of square roots of the canonical bundle, given by the map (128). In this case the existence of orientation data is established by Joyce-Upmeier in [JU21]. Very recently in [NS23] Naef-Safronov provide a canonical way to produce orientation data for derived mapping stacks of local systems on compact closed 3-dimensional topological manifolds via torsion volume forms.

3.4. Future Directions

We believe that our gluing strategy of in the Theorem **3.3.26** can be used to attack other problems in the field:

3.4.1. Derived 2-periodic Azumaya algebras. The obstruction to *untwist* the sheaf $\widetilde{\mathrm{MF}}$ of the Corollary **3.3.29** is morphism

$$*_X \to \mathsf{B}\,\underline{\mathsf{Az}}^{2-\mathsf{per},2-\mathsf{tor},\nabla}$$

For the moment we do not know what kind of objects this $\underline{\mathsf{Az}}^{2-\mathsf{per},2-\mathsf{tor},\nabla}$-principal bundle classifies. We intent to investigate this in the near future.

3.4.2. Joyce's conjecture. Joyce [JS19:Conjecture 1.1] conjectures that every choice of oriented lagrangian L of a oriented (-1)-shifted symplectic derived scheme X, should produce a global section of the glued perverse sheaf P_{Joyce} of the Theorem **3.2.9**. Our Darboux stack can be modified to incorporate local models with a choice of Lagrangians. We believe that this modification of the Darboux stack, together with arguments similar to the ones used to prove the Theorem **3.3.26** will allow us to prove Joyce's conjecture in full generality. Partial results to this conjecture have also been recently announced by H. Park, A. Khan and T. Kinjo in the case of shifted cotangent stacks. Moreover, we believe that the same methods will allow us to show a categorified version of this conjecture, namely, that to every oriented lagrangian L of a oriented (-1)-shifted symplectic derived scheme X, we can assign an object in the glued matrix factorization category MF. This is being currently investigated.

3.4.3. Donaldson-Thomas invariants in positive characteristic. The techniques used in Section **2.3** to defined the graded circle, allow us to define notions of Lagrangian foliations in positive characteristic. These notions are currently the object of the study of the Phd thesis of Victor Alfieri in a co-supervision with B. Toën. The first step is to define to define the shifted symplectic structure on the derived moduli space $\mathsf{MCoh}(Y)$ via a reformulation of the the AKSZ construction of the Theorem **3.2.2**. We believe that some of the techniques used to prove Theorem **3.3.26** can be extrapolated to this setting.

3.4.4. Calabi-Yau 4-folds. We believe that a version of our Theorem **3.3.26** for (-2)-shifted symplectic forms can be used to recover the results of [BJ17] and Thomas-Oh [OT23] gluing of the virtual fundamental class for moduli of sheaves on

Calabi-Yau 4-folds [(*)]. In the future we intend to go even further and develop some of the remaining foundational issues related to derived foliations for the applications to Donaldson-Thomas theory for Calabi-Yau manifolds of dimension n, in particular, the constructions of leaf-spaces for foliations transversality in tor-amplitude $[0, n]$.

3.4.5. Grothendieck-Witt groups and Mirror symmetry. Because of the $\mathbb{A}^1$-invariance property for Grothendieck Witt groups [Sch17] we the group completion $\underline{\mathsf{Quad}}^{\nabla,\mathbb{A}^1,+}$ that appears in Theorem **3.3.26** maps naturally to the Grothendieck-Witt stack (see [Sch17; BKØ11]) of quadratic bundles on the small étale topos of X

$$\underline{\mathsf{Quad}}^{\nabla,\mathbb{A}^1,+} \to \underline{\mathsf{GW}}_X^{[0]}$$

The composition

$$*_X \simeq \underline{\mathsf{Darb}}_X^{\alpha,\mathbb{A}^1,+}/\underline{\mathsf{Quad}}^{\nabla,\mathbb{A}^1,+} \to \mathsf{B}\underline{\mathsf{Quad}}^{\nabla,\mathbb{A}^1,+} \to \Sigma\underline{\mathsf{GW}}_X^{[0]} \tag{129}$$

can be identified with the following construction: recall from [Sch17:Thm 6.1] the existence of a Bott sequence in algebraic Hermitian K-theory

$$\mathsf{GW}^{[n]} \to \mathsf{K} \to \mathsf{GW}^{[n+1]} \tag{130}$$

where the $\mathsf{GW}^{[n]}$ stands for the Grothendieck-Witt spectra of chain complexes with a n-shifted quadratic form. The first map is the map induced by forgeting the quadratic form and the second map is given by the sending a complex E to the trivial metabolic form. This construction can be reproduced at the level of stacks on the small étale site of X and we obtain a cofiber/fiber sequence

$$\underline{\mathsf{GW}}_X^{[n]} \to \mathsf{K}_X \to \underline{\mathsf{GW}}_X^{[n+1]}$$

Now, whenever X is a $n < 0$ shifted symplectic derived scheme, instead of interpreting the tangent bundle $(\mathbb{T}_X, \omega)$ as a (-1)-shifted symplectic complex, we can instead, up to shifting by 2, interpret $(\mathbb{T}_X[2], \omega)$ as a 1-shifted quadratic complex, and in particular, classified by a map in $\mathsf{Sh}(X_{\text{ét}})$

$$\omega : *_X \to \underline{\mathsf{GW}}_X^{[1]}$$

which we can compose with the boundary map of the sequence (130)

$$*_X \to \underline{\mathsf{GW}}_X^{[1]} \to \Sigma\underline{\mathsf{GW}}_X^{[0]} \tag{131}$$

We claim that

$$(129) = (131)$$

.

[(*)] See the Remark **3.3.6**.

In future works we intent to explore the relation between this construction and mirror symmetry. Indeed, recall that by definition of the Grothendieck-Witt for n-shifted quadratic modules ([Sch17:Remark 5.9]), we have a 4-periodic behavior

$$\mathsf{GW}^{[n]} \simeq \mathsf{GW}^{[n+4]} \tag{132}$$

In particular, the isomorphism

$$\mathsf{GW}^{[-4]} \simeq \mathsf{GW}^{[0]} \tag{133}$$

gives us a specific way to relate invariants for Calabi-Yau 3-folds to invariants of Calabi-Yau 7-folds.

Remark 3.4.1. In [CDP93; CHSW85] the authors investigate an example that fits this 4-periodic picture, namely, a rigid Calabi-Yau 3-fold Z whose mirror is a 7-fold $\widehat{Z}$ and the moduli of coherent sheaves on $\widehat{Z}$ has a Calabi-Yau semi-orthogonal piece (see [SS21; Kuz19]).

We intend to investigate this in the future.

3.4.6. Relation with Gluing Fukaya Categories and Mirror symmetry. As discussed in Section **2.2.1**, categories of matrix factorizations are mirrors to Fukaya categories. The gluing problem addressed in the Corollary **3.3.29** has analogous versions for Fukaya categories. This is the issue of recent progresses in the field [PS22; NS20; She21; GPS18b; GPS17; GPS18a; ACGK20]. The ingredients and arguments used in the Fukaya side are reminiscent of the arguments used to glue matrix factorizations in Corollary **3.3.29**, such as the role of quadratic forms and the Grothendieck-Witt groups. A research goal in the nearest future is to further explore the relationship between the gluing procedures via Mirror symmetry.

APPENDIX A

A brief study guide to $(\infty, 1)$-categories

The following notes do not concern my research results. Instead, we compile an introduction to ∞-categories, intended for a graduate course delivered at the school "Géométrie algébrique dérivée et interactions" in Toulouse, 2017. Far from aiming for an exhaustive introduction, the main goal was to provide a study guide with precise references to [Lu-HTT; Lur23; Cis19].

Platonic Form A.0.1. An ∞-category is a mathematical object that assembles:

- *objects*,
- *1-morphisms between objects*,
- for every $n \geq 2$, a notion of *n-morphisms* between *$n-1$-morphisms*;
- for every $n \geq 1$, (weak) composition laws of n-morphisms only well-defined up to the data of higher morphisms.
- Associativity of compositions up to homotopy:

Example A.0.2. Whatever definition we pick, it should accommodate the construction of a fundamental ∞-groupoid for a CW-complex X, such that:

- objects = points of X,
- 1-morphisms=paths in X,
- 2-morphisms=homotopy of paths (2-cells);
- 3-morphisms= homotopies between homotopies of paths (3-cells)
- ...

Problem A.0.3. There is no direct definition of higher categories simultaneously operational and close to the platonic form described above (infinitely many axioms!).

Solution A.0.4. The document [Gro22] describes Grothendieck's attempts to solve Problem **A.0.3**. It was only a decade later that C. Simpson [Sim97] explained that a direct definition of ∞-categories is not really a necessity: for all working purposes one can avoid the Problem **A.0.3** by modeling ∞-categories using "exaggerated" templates/models that contain more structure than what the platonic form requires. This means, finding a *model category* whose objects serve as templates for ∞-categories. Today we know several of such models: Quasi-categories, Segal Categories, Simplicial Categories, etc). In fact, thanks to Quillen years before [Qui67], modelling became a common practice in all forms of homotopy theory:

- Topological spaces, simplicial sets, categories, etc, all provide models for the study of homotopy types;
- Both simplicial algebras and differential graded algebras provides models for homotopy-commutative differential graded algebras over $\mathbb{Q}$;
- Simplicial presheaves provide models for higher and derived stacks.

We do not review here the theory of model categories. Instead we refer to [Lu-HTT:Appendix A.2] and [Hov99]. Our goal, following the works of Joyal, is to explain how to use simplicial sets as templates for ∞-categories.

Here's the roadmap for these notes:

Appendix **A.1**
Modeling classical categories with simplicial sets

Appendix **A.2**
Modeling Homotopy types with simplicial sets

Appendix **A.3**
Modeling ∞-categories with simplicial sets

Appendix **A.4**
Examples of ∞-categories

Appendix **A.5**
Colimits and Localizations

A.1. Modeling classical categories with simplicial sets

In this section we explain how to use simplicial sets as templates for classical 1-categories.

Definition A.1.1. [Lur23:Subsection 0008] The category Δ has objects given by finite ordered sets $[n] = \{0 < 1 < ... < n\}$ and morphisms given by order-preserving maps. The category of simplicial sets is by definition the category of presheaves of sets on Δ, ie:

$$\mathsf{SSets} := \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Sets})$$

Remark A.1.2. Unwinding the definitions, one can exhibit a simplicial set $S \in \mathsf{SSets}$ as a diagram of sets

$$\begin{tikzcd}
\cdots & S_3 \arrow[r, shift left=9, "\partial_3"] \arrow[r, shift left=3, "\partial_2"] \arrow[r, shift right=3, "\partial_1"] \arrow[r, shift right=9, "\partial_0"'] & S_2 \arrow[l, shift left=6, "\epsilon_2"'] \arrow[l, "\epsilon_1"'] \arrow[l, shift right=6, "\epsilon_0"'] \arrow[r, shift left=6, "\partial_2"] \arrow[r, "\partial_1"] \arrow[r, shift right=6, "\partial_0"'] & S_1 \arrow[l, shift left=3, "\epsilon_1"'] \arrow[l, shift right=3, "\epsilon_0"'] \arrow[r, shift left=3, "\partial_1"] \arrow[r, shift right=3, "\partial_0"'] & S_0 \arrow[l, "\epsilon_0"']
\end{tikzcd}$$

The boundary maps ∂_i are obtained as image of the injective maps in Δ and the degeneration maps ϵ_j, the image of the surjective maps. The combinatorics of ∂ and ϵ are the standard simplicial relations [Lur23:Exercise 000G].

Notation A.1.3. [Lur23:Construction 000L] We denote by Δ^n the image through the Yoneda embedding of $[n]$.

$$\Delta \hookrightarrow \mathsf{SSets} \qquad [n] \mapsto \Delta^n$$

Remark A.1.4. Let S be a simplicial set. By the Yoneda's lemma, we also have $S([n]) = \mathrm{Hom}_{\mathsf{SSets}}(\Delta^n, S)$;

Remark A.1.5. It follows formally from the Definition **A.1.1** that every simplicial set S is a colimit of simplicial sets of the form Δ^n. Indeed, for any category C, every presheaf $F : \mathsf{C}^{\mathsf{op}} \to \mathsf{Sets}$ is a colimit of representables.

Example A.1.6. The simplicial set Δ^2 has three 0-simplexes, given by Yoneda by the three different maps $0, 1, 2 : [0] \to [2]$ in Δ. It has three 1-simplexes, given by the three maps $[1] \to [2]$ corresponding to $0 \leq 1$, $1 \leq 2$ and $0 \leq 2$. Finally, it has a 2-simplex given by the identity $[2] \to [2]$. All other n-simplexes for $n \geq 3$ are *degenerated*, ie, they are in the image of one of the surjective maps ϵ_i. We picture the non-degenerated simplexes of Δ^2 as

$$\Delta^2 := \begin{tikzcd}
 & 2 & \\
0 \arrow[ur, "\partial_1"] \arrow[rr, "\partial_2"'] & & 1 \arrow[ul, "\partial_0"']
\end{tikzcd}$$

Notation A.1.7. [Lur23:Construction 000U] We define the horn Λ^j_n to be Δ^n deprived of its j^{th}-face and the interior. We can picture its non-degenerated simplexes as

$$\Lambda^1_2 = \begin{tikzcd} & 2 & \\ 0 \arrow[rr, "\partial_2"'] & & 1 \arrow[ul, "\partial_0"'] \end{tikzcd}$$

Notation A.1.8. We denote by Cat 1-category of (small) classical 1-categories with morphisms given by functors.

Construction A.1.9. The category Δ can be found as a subcategory of Cat: $[n]$ can be identified with the category associated to the linearly ordered set $\{0 < 1 < \cdots < n\}$ with a unique morphism $i \to j$ if and only if $i \leq j$. Morphisms of linearly ordered sets are in bijection with functors $[n] \to [m]$. We write

$$\Delta \subseteq \mathsf{Cat}$$

for the inclusion. In particular, since Cat has all colimits, the universal property of presheaves of sets guarantees the existence of a left Kan extension τ

$$\begin{tikzcd} \Delta \arrow[r, hook] \arrow[d, hook] & \mathsf{Cat} \\ \mathsf{SSets} \arrow[ur, dashed, "\tau"'] & \end{tikzcd} \tag{134}$$

that sends Δ^n to $[n]$ and commutes with colimits. See [Lur23:Corollary 004N].

Construction A.1.10. Let C be a classical 1-category. Then we encode the information of C in a simplicial set called the *nerve of* C, denote as $\mathsf{N}(\mathsf{C})$, defined by

$$\mathsf{N}(\mathsf{C})([n]) := \mathrm{Hom}_{\mathsf{Cat}}([n], \mathsf{C}) =$$

$$= \{\text{composable chains of morphism in } \mathsf{C}, \quad X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} .... \xrightarrow{f_{n-1}} X_n \ \}$$

The boundary maps ∂_i encode the composition law and the degeneracy maps ϵ_i are given by adding identity maps to the chain. See [Lur23:Remark 002S].

The assignment sending a category C to its nerve $\mathsf{N}(\mathsf{C})$ defines a functor from the category of 1-categories to simplicial sets.

$$\mathsf{N} : \mathsf{Cat} \to \mathsf{SSets}$$

By construction, it provides a right adjoint to τ.

Example A.1.11. [Lur23:Example 002W]. We have a canonical isomorphism of simplicial sets $\mathsf{N}([n]) \simeq \Delta^n$. Indeed, for every $[m] \in \Delta$

$$\mathrm{Hom}_{\mathsf{SSets}}(\Delta^m, \mathsf{N}([n])) \underset{\text{definition}}{=} \mathrm{Hom}_{\mathsf{Cat}}([m],[n]) =$$

$$= \mathrm{Hom}_{\Delta}([m],[n]) \underset{\text{Yoneda}}{=} \mathrm{Hom}_{\mathsf{SSets}}(\Delta^m, \Delta^n)$$

The following result is originally due to Grothendieck [Gro22]

Lemma A.1.12.

(i) The Nerve functor is fully faithful, ie, it induces a bijection:

$$\{\textit{Functors } \mathsf{C} \to \mathsf{D}\} \simeq \{\textit{Maps of simplicial sets } \mathrm{N}(\mathsf{C}) \to \mathrm{N}(\mathsf{D})\}$$

(ii) A simplicial set X is the nerve of a category C if and only if it satisfies the following condition: $\forall n \geq 2$, $\forall\ 0<i<n$, $\forall\, u : \Lambda^i_n \to X$, there exists a unique *factorization:*

$$\begin{tikzcd} \Lambda^i_n \arrow[r, "u"] \arrow[d, hook] & X \\ \Delta^n \arrow[ur, dashed, "\exists!\ \textit{factorization}"'] & \end{tikzcd}$$

Proof. For (i) see [Lur23:Tag 002L]. For (ii) see [Lur23:Subsection 0030]. $\square$

Construction A.1.13. The category of simplicial sets SSets admits an internal-hom for morphisms: given K and S simplicial sets, we define a simplicial set of morphisms $\underline{\mathrm{Hom}}(K,S)$ as the simplicial set with n-simplexes given by

$$\underline{\mathrm{Hom}}_{\mathsf{SSets}}(K,S)([n]) := \mathrm{Hom}_{\mathsf{SSets}}(K \times \Delta^n, S)$$

Since every simplicial set P is a colimit of simplicial sets of the form Δ^n,we have a canonical bijection

$$\mathrm{Hom}_{\mathsf{SSets}}(P, \underline{\mathrm{Hom}}_{\mathsf{SSets}}(K,S)) \simeq \mathrm{Hom}_{\mathsf{SSets}}(P \times K, S)$$

Notation A.1.14. Let C and D be 1-categories. We denote by $\mathsf{Fun}(\mathsf{C},\mathsf{D})$ the category whose objects are functors $\mathsf{C} \to \mathsf{D}$ and morphisms are given by natural transformations.

Corollary A.1.15. *[Lur23:Proposition 0062] Let C and D be 1-categories. Then we have a canonical isomorphism of simplicial sets*

$$\mathsf{N}(\mathsf{Fun}(\mathsf{C},\mathsf{D})) \simeq \underline{\mathrm{Hom}}_{\mathsf{SSets}}(\mathsf{N}(\mathsf{C}), \mathsf{N}(\mathsf{D}))$$

Corollary A.1.16. *[Lur23:Corollary 0065] The functor τ commutes with finite products.*

Example A.1.17. Let us illustrate how to interpret compositions of morphisms in C using the language of simplicial sets in $\mathsf{N}(\mathsf{C})$. By definition, to give a composable chain of morphisms in C

$$X \xrightarrow{f} Y \xrightarrow{g} Z \quad \text{in } \mathsf{C}$$

is equivalent to give a morphism of simplicial sets

$$(g, f) : \Lambda^1_2 = \begin{tikzcd} & 2 & \\ 0 \arrow[rr, "\partial_2"'] & & 1 \arrow[ul, "\partial_0"'] \end{tikzcd} \to \mathrm{N}(\mathsf{C})$$

with $u(\partial_2) = f$ and $u(\partial_0) = g$. To generate the composition of g and f, is equivalent to extend the map (g, f) along the inclusion

$$\begin{tikzcd} & 2 & \\ 0 \arrow[rr, "\partial_2"'] & & 1 \arrow[ul, "\partial_0"'] \end{tikzcd} \subseteq \Delta^2 := \begin{tikzcd} & 2 & \\ 0 \arrow[ur, "\partial_1"] \arrow[rr, "\partial_2"'] & & 1 \arrow[ul, "\partial_0"'] \end{tikzcd}$$

In other words, the problem of generating the composition $g \circ f$ is equivalent to the extension problem:

$$\begin{tikzcd} \Lambda^1_2 \arrow[r, "{(g,f)}"] \arrow[d, hook] & \mathrm{N}(\mathsf{C}) \\ \Delta^2 \arrow[ur, dashed, "\exists!\text{ extension}"'] & \end{tikzcd} \qquad \begin{tikzcd} & Z & \\ X \arrow[ur, "\exists!\, g\circ f := \partial_1"] \arrow[rr, "f"'] & & Y \arrow[ul, "g"'] \end{tikzcd}$$

In particular, the uniqueness of the lifting is equivalent to the uniqueness of the composition.

Example A.1.18. Let us now illustrate how to process the associativity law for compositions in C in terms of the lifting property of $\mathsf{N}(\mathsf{C})$. Following the Example **A.1.17**, to give a composable chains of morphisms

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W \quad \text{in } \mathsf{C}$$

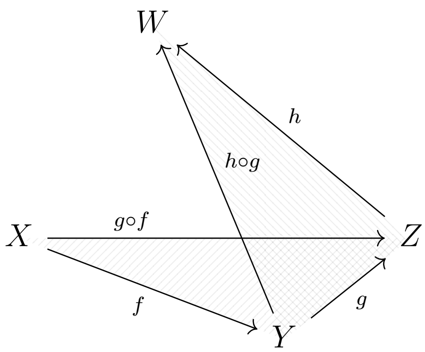
amounts to the data of two maps of simplicial sets

$$\Lambda^1_2 \xrightarrow{(g,f)} \mathrm{N}(\mathsf{C}) \qquad \Lambda^1_2 \xrightarrow{(h,g)} \mathrm{N}(\mathsf{C})$$

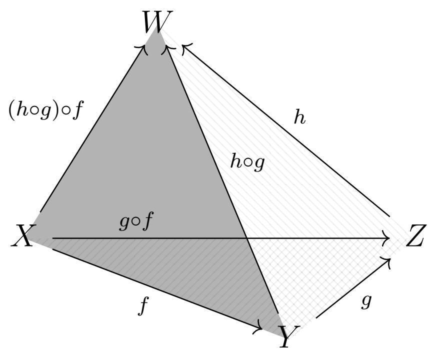
As in the Example **A.1.17** we generate compositions using the lifting properties, and obtain two maps of simplicial sets

$$\Delta^2 \xrightarrow{(g,f)} \mathrm{N}(\mathsf{C}) \qquad \Delta^2 \xrightarrow{(h,g)} \mathrm{N}(\mathsf{C})$$

which we can picture together as



We can again use the unique lifting property to fill the face XYW, by generating the composition $(h \circ g) \circ f$:



and obtain this way a map $\Lambda^1_3 \to \mathsf{N}(\mathsf{C})$. Finally, we can use the unique lifting property along the inclusion $\Lambda^1_3 \subseteq \Delta^3$ to obtain

$$\begin{tikzcd}
\Lambda^1_3 \arrow[r] \arrow[d, hook] & \mathrm{N}(\mathsf{C}) \\
\Delta^3 \arrow[ur, dashed, "\exists!\text{ extension}"'] &
\end{tikzcd}$$

and the resulting filling of the face XZW is a witness of the relation

$$h \circ (g \circ f) = (h \circ g) \circ f$$

Let us now illustrate how simplicial sets can also be used to model classical groupoids.

Lemma A.1.19. *A simplicial set X is the nerve of a groupoid C if and only if $\forall n \geq 2$, $\forall\ 0 \leq i \leq n$ and $\forall\, u : \Lambda^i_n \to X$, we have*

$$\begin{tikzcd} \Lambda^i_n \arrow[r, "u"] \arrow[d, hook] & X \\ \Delta^n \arrow[ur, dashed, "\exists!\ \textit{factorization}"'] & \end{tikzcd}$$

Proof. The assumption that the lifting property holds for $\forall\ 0<i<n$ implies by the Lemma **A.1.12** that X is the nerve of a category C. It remains to show that C is a groupoid if we assume the lifting property for $i = 0, n$. In other words, that every morphism in C, has an inverse. Let us show that if the lifting property holds for $i = 0$, then every morphism has a left inverse. For $i = 0$, we are interested in the lifting property along the inclusion

$$\Lambda^0_2 = \begin{tikzcd} & 2 & \\ 0 \arrow[ur, "\partial_1"] \arrow[rr, "\partial_2"'] & & 1 \end{tikzcd} \subseteq \Delta^2 = \begin{tikzcd} & 2 & \\ 0 \arrow[ur, "\partial_1"] \arrow[rr, "\partial_2"'] & & 1 \arrow[ul, "\partial_0"'] \end{tikzcd}$$

In this case, let us consider a diagram of the form

$$\Lambda^0_2 \to \mathsf{N}(\mathsf{C})$$

corresponding to morphisms in C

$$\begin{tikzcd} & X & \\ X \arrow[ur, "Id_X"] \arrow[rr, "f"'] & & Y \end{tikzcd}$$

The lifting property allows us to extend this map to

$$\begin{tikzcd} \Lambda^0_2 \arrow[r, "u"] \arrow[d, hook] & \mathsf{N}(\mathsf{C}) \\ \Delta^2 \arrow[ur, dashed, "\exists!\ \text{factorization}"'] & \end{tikzcd}$$

corresponding to a commutative diagram

$$\begin{tikzcd} & X & \\ X \arrow[ur, "Id_X"] \arrow[rr, "f"'] & & Y \arrow[ul, "g"'] \end{tikzcd}$$

exhibiting g as a left inverse to f. The same argument for $i = n$ shows that every morphism has a right inverse.

$\square$

Summary A.1.20. Simplicial sets and their morphisms provide a faithful model of the theory of 1-categories and functors. Allowing the lifting property for the extremities $i = 0, n$ gives inverses. For terms in the middle $0<i<n$, gives compositions.

A.2. Modeling Homotopy Types with simplicial sets

In order to explain how to pass to ∞-categories, we need first to review how to use simplicial sets to model homotopy types:

Notation A.2.1. We denote by Top the 1-category of topological spaces.

Construction A.2.2. Consider Δ_n the topological n-simplex, ie the subspace of $\mathbb{R}^{n+1}$ defined by the equation $x_0 + \cdots + x_n = 1$ and the condition $x_i \geq 0, \forall i$. The different continuous inclusions and surjections of topological simplexes satisfy the relations in the category Δ. More precisely, the assignment $[n] \to \Delta_n$ defines a functor

$$\Delta \to \mathsf{Top}$$

Since Top has all colimits, the universal property of presheaves of sets provides a left Kan extension

$$\begin{tikzcd} \Delta \arrow[r] \arrow[d] & \mathsf{Top} \\ \mathsf{SSets} \arrow[ur, dashed, "|-|"'] & \end{tikzcd}$$

called *topological realization*. See [Lur23:Subsection 001X]. By abstract-nonsense, this admits a right adjoint Sing sending a topological space T to the simplicial set $\mathsf{Sing}(T)$ whose n-simplexes are the continuous maps $\Delta_n \to T$. See [Lur23:Subsection 001Q].

Definition A.2.3. [Lur23:Subsection 002G] A *Kan complex* is a simplicial set X satisfying the following lifting property:$\forall n \geq 2$, $\forall\ 0 \leq i \leq n$ and $\forall\, u : \Lambda^i_n \to X$, there exists a factorization

$$\begin{tikzcd} \Lambda^i_n \arrow[r, "u"] \arrow[d, hook] & X \\ \Delta^n \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

Warning A.2.4. The factorization in the Definition **A.2.3** is not necessarily unique.

Lemma A.2.5. *(See [Lur23:Proposition 002K]) The simplicial sets of the form* $\mathsf{Sing}(T)$ *are Kan complexes.*

Kan complexes are exhaustive templates for homotopy types of topological spaces in the following sense:

Proposition A.2.6. *(See [Hov99:Theorem 3.6.7]). The adjunction* $(\mathsf{Sing}, |-|)$ *is a Quillen equivalence of model categories between:*

(i) the Quillen model structure on Top *where the weak-equivalences are given by continuous maps* $f : T \to T'$ *such that* $\pi_0(f) : \pi_0(T) \to \pi_0(T')$ *is an isomorphism of sets and for every* $x \in \pi_0(T)$*, the induced maps* $\pi_n(T, x) \to \pi_n(T', f(x))$ *are isomorphisms for all* $n \geq 1$*,*

(ii) the Kan model structure on SSets *with weak-equivalences given by those maps of simplicial sets that induce weak-equivalences of topological spaces (as defined in (i)) after taking topological realization (cf. Construction* ***A.2.2****).*

A.3. Modeling ∞-categories with simplicial sets

Finally, we can explain how to use simplicial sets to model ∞-categories. We achieved this by relaxing the definition of Kan complexes in order to accommodate, simultaneously, nerves of classical categories, and Kan complexes:

Definition A.3.1. [Lur23:Definition 003A] A quasi-category is a simplicial set $\mathscr{C}$ with the following property: $\forall n \geq 2$, $\forall\, 0<i<n$, and $\forall\, u : \Lambda^i_n \to \mathscr{C}$ there exists a lifting

$$\begin{tikzcd} \Lambda^i_n \arrow[r] \arrow[d, hook] & \mathscr{C} \\ \Delta^n \arrow[ur, dashed, "\exists \text{ factorization } (\textit{not necessarily unique!})"'] & \end{tikzcd}$$

Example A.3.2. [Lur23:Section 0039]

- The nerve of a category $\mathrm{N}(\mathsf{C})$ is a quasi-category with the unique lifting property;
- A Kan complex is a quasi-category.
- Let $\mathscr{C}$ and $\mathscr{D}$ be quasi-categories. Then the product simplicial set $\mathscr{C} \times \mathscr{D}$ is a quasi-category.

Convention A.3.3. We call the 0-simplexes of a Quasi-category $\mathscr{C}$, the objects of $\mathscr{C}$. We call the 1-simplexes of $\mathscr{C}$ the 1-morphisms in $\mathscr{C}$.

Construction A.3.4. As in the Example **A.1.17** let us illustrate how to *generate* compositions of 1-morphisms in a quasi-category $\mathscr{C}$. By definition, to give a composable chain of 1-morphisms in $\mathscr{C}$

$$X \xrightarrow{f} Y \xrightarrow{g} Z \quad \text{in } \mathscr{C}$$

is to give a map of simplicial sets

$$u : \Lambda_2^1 = \begin{tikzcd} & & 2 \\ 0 \arrow[rr, "\partial_2"'] & & 1 \arrow[u, "\partial_0"'] \end{tikzcd} \to \mathscr{C}$$

with $u(\partial_2) = f$ and $u(\partial_0) = g$. To generate *a* composition, we consider a lifting

$$\begin{tikzcd} \Lambda_2^1 \arrow[r] \arrow[d, hook] & \mathscr{C} \\ \Delta^2 \arrow[ur, dashed] & \end{tikzcd}$$

Contrary to the situation with classical categories, this lifting is not unique. The image of the 2-simplex corresponding to the identity $\Delta^2 \to \Delta^2$ along $\Delta^2 \to \mathscr{C}$ gives us a filling

$$\begin{tikzcd} & Z & \\ X \arrow[ur, "\partial_1"] \arrow[rr, "f"'] & & Y \arrow[ul, "g"'] \end{tikzcd}$$

The green color for the filling represents the data of the 2-simplex in C that witnesses the commutativity of the diagram. Another choice of lifting would give a different 2-simplex in C, a different witness, of the commutativity

$$\begin{tikzcd} & Z & \\ X \arrow[ur, "\partial_1"] \arrow[rr, "f"'] & & Y \arrow[ul, "g"'] \end{tikzcd}$$

In this sense, compositions exist, but are not uniquely defined. Each 2-simplex in $\mathscr{C}$ is a different witness of a form of commutativity". There can be many! In the Example **A.3.5** below we explain how to relate the different choices.

Example A.3.5. Consider two 2-simplexes in quasi-category $\mathscr{C}$

$$u_1, u_2 : \Delta^2 \to \mathscr{C}$$

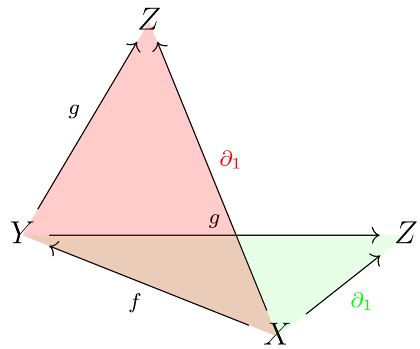
expressing the commutativity of two diagrams with the same 0 and 1 simplexes:

$$\begin{tikzcd} & Z & \\ X \arrow[ur, "\partial_1"] \arrow[rr, "f"'] & & Y \arrow[ul, "g"'] \end{tikzcd} \qquad \begin{tikzcd} & Z & \\ X \arrow[ur, "\partial_1"] \arrow[rr, "f"'] & & Y \arrow[ul, "g"'] \end{tikzcd}$$

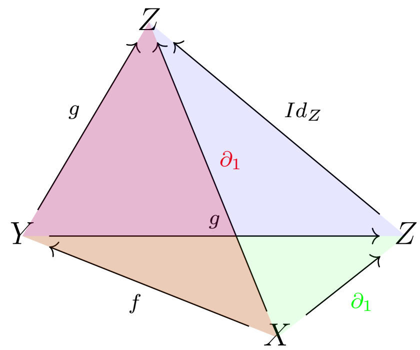
As in the Example **A.1.18**, let us consider the gluing of simplicial sets

$$\Delta^2 \coprod_{\Delta^1} \Delta^2 \to \mathscr{C}$$

given by



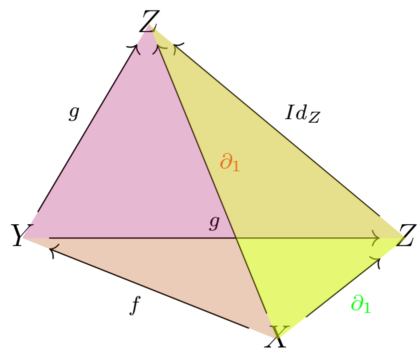
We can add one degenerated face (in blue) expressing the identity of Z and obtain this way



corresponding to a map of simplicial sets

$$\Lambda^1_3 \to \mathscr{C}$$

Finally, using the lifting property for the inclusion $\Lambda^1_3 \subseteq \Delta^3$ we can choose a map of simplicial sets $\Delta^3 \to \mathscr{C}$ corresponding to a filling of the volume



The new 2-simplex in yellow, corresponding to a map of simplicial sets $\Delta^2 \to \mathscr{C}$ can be interpreted as a homotopy between the two compositions $\partial_1 \sim \partial_1$. Iterating this lifting mechanism one obtains higher compatibilities between compositions encoding by n-simplexes.

Example A.3.6. Let us now illustrate how to express associativity relations for compositons. We apply the same mechanism as in Example **A.1.18**: to give two composable chains of morphisms

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W \quad \text{in } \mathscr{C}$$

is to give two maps of simplicial sets

$$\Lambda^1_2 \xrightarrow{(g,f)} \mathscr{C} \qquad \Lambda^1_2 \xrightarrow{(h,g)} \mathscr{C}$$

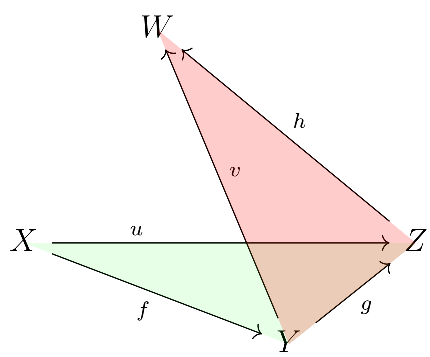
Using the lifting property, we can choose compositions

$$\Delta^2 \xrightarrow{-(g,f)} \mathscr{C} \qquad \Delta^2 \xrightarrow{-(h,g)} \mathscr{C}$$

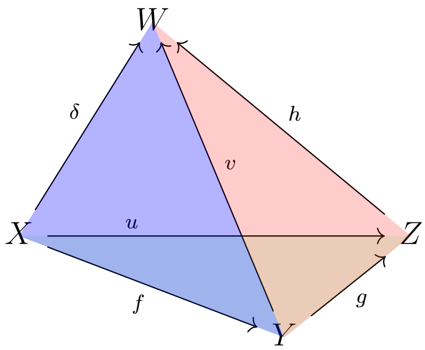
that one can patch together as a map of simplicial sets

$$\Delta^2 \coprod_{\Delta^1} \Delta^2 \to \mathscr{C}$$

and visualize as



The lifting properties allow us to fill the volume



where again the filling of the face XZW witnesses the associativity of the compositions.

This collection of examples motivates the following definition:

Definition A.3.7. An $(\infty, 1)$-category is a quasi-category. An ∞-functor is a map of simplicial sets between quasi-categories.

Remark A.3.8. Functors between classical categories are essentially assignments that preserve commutative diagrams. In the world of quasi-categories, this is precisely what a map of simplicial sets does, by sending n-simplexes to n-simplexes and preserving boundaries.

Lemma A.3.9. *(See [Lur23:Theorem 0066] and ([Lur23:Corollary 01GM]). Let $\mathscr{C}$ be an ∞-category, K any simplicial set. Then the internal-hom in* SSets *(cf. Construction* ***A.1.13****)* $\underline{\mathrm{Hom}}_{\mathsf{SSets}}(K, \mathscr{C})$ *is a ∞-category. We call it the ∞-category of ∞-functors and use the notation* $\mathsf{Fun}(K, \mathscr{C})$ *to denote it. Moreover, when $\mathscr{C}$ is a Kan-complex, so is* $\mathsf{Fun}(K, \mathscr{C})$.

Example A.3.10. Let $\mathscr{C}$ be an ∞-category. A *commutative square* in $\mathscr{C}$ is a ∞-functor $\Delta^1 \times \Delta^1 \to \mathscr{C}$. One can decompose the square $\Delta^1 \times \Delta^1$ as a gluing of two *commutative triangles* that agree along the diagonal, ie, we have an isomorphism of simplicial sets

$$\Delta^1 \times \Delta^1 \simeq \Delta^2 \coprod_{\Delta^1} \Delta^2$$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] \arrow[dr] & Y \arrow[d] \\ X \arrow[r, "g"'] & Y \end{tikzcd}$$

Construction A.3.11. [Lur23:Subsection 0049] One can extract a classical category from an ∞-category $\mathscr{C}$, the so-called homotopy category, denoted by $\mathrm{h}\mathscr{C}$:

- Objects of $\mathrm{h}\mathscr{C}$: 0-simplexes of $\mathscr{C}$;

- Morphisms of h$\mathscr{C}$:= homotopy classes of 1-morphisms: f, g are equivalent if there exists a 2-morphism $u : \Delta^2 \to \mathscr{C}$

$$\begin{tikzcd} & Z & \\ X \arrow[ur, "g"] \arrow[rr, "f"'] & & Y \arrow[ul, "Id_Y"'] \end{tikzcd}$$ rendering f homotopic to g

- Compositions: Well-defined using the lifting property.

Lemma A.3.12. *[Lur23:Proposition 004H] Let $\mathscr{C}$ be a quasi-category. Then $\mathsf{h}\mathscr{C}$ agrees with $\tau\mathscr{C}$ of the Construction* ***A.1.9****.*

Definition A.3.13. A subcategory of a quasi-category $\mathscr{C}$ is a sub-simplicial set $\mathscr{D}$ obtained as a fiber product in simplicial sets

$$\begin{tikzcd} \mathscr{D} \arrow[r] \arrow[d] & \mathscr{C} \arrow[d] \\ \mathsf{N}(\mathsf{D}) \arrow[r] & \mathsf{N}(\mathsf{h}\mathscr{C}) \end{tikzcd}$$

with D a subcategory of $\mathsf{h}\mathscr{C}$.

Definition A.3.14. [Lur23:Subsection 004Q] A 1-morphisms $f : X \to Y$ of $\mathscr{C}$ is said to be an *equivalence* (or *invertible*) if its homotopy class $[f]$ in $\mathsf{h}\mathscr{C}$ is an isomorphism, ie, if there exists two 2-simplex in $\mathscr{C}$

$$\begin{tikzcd} & X & \\ X \arrow[ur, "Id_X"] \arrow[rr, "f"'] & & Y \arrow[ul, "\exists"'] \end{tikzcd} \qquad \begin{tikzcd} & Y & \\ Y \arrow[ur, "Id_Y"] \arrow[rr, "\exists"'] & & X \arrow[ul, "f"'] \end{tikzcd}$$

rendering f invertible. An ∞-category is an ∞-*groupoid* if all its 1-morphisms are equivalences.

Lemma A.3.15. *(See [Lur23:Proposition 019D]) An ∞-category $\mathscr{C}$ is an ∞-groupoid if and only if it is a Kan-complex.*

Construction A.3.16. [Lur23:Subsection 01CZ] Let $\mathscr{C}$ be an ∞-category and consider the fiber product of simplicial sets as in the Definition **A.3.13** with

$$\begin{tikzcd} \mathscr{C}^{\simeq} \arrow[r] \arrow[d] & \mathscr{C} \arrow[d] \\ \mathsf{N}((\mathsf{h}\mathscr{C})^{\simeq}) \arrow[r] & \mathsf{N}(\mathsf{h}\mathscr{C}) \end{tikzcd}$$

where $(\mathsf{h}\mathscr{C})^{\simeq}$ is the subcategory of $\mathsf{h}\mathscr{C}$ spanned by isomorphisms.

Lemma A.3.17. *[Lur23:Corollary 01H1] The simplicial set $\mathscr{C}^{\simeq} \subseteq \mathscr{C}$ of the Construction **A.3.16** is a Kan-complex. We call it the* maximal ∞-groupoid *of $\mathscr{C}$.*

Example A.3.18. The simplicial set $\Delta^1 = \mathsf{N}([1])$ is not a ∞-groupoid but is an ∞-category .

Example A.3.19. Let J denote the category with two objects 0 and 1 and a unique morphism $f : 0 \to 1$ admitting a inverse $g : 1 \to 0$. J is a groupoid, obtained from $[1]$ by inverting the unique arrow $0 \to 1$. In particular, J contains $[1]$ as a full subcategory. The simplicial set $\mathsf{N}(J)$ is an ∞-groupoid .

Example A.3.20. Let T be a topological space. Then its singular simplicial set $\mathrm{Sing}(T)$ is an ∞-groupoid. See [Lur23:Proposition 002K]. We call it the fundamental ∞-groupoid of T

Construction A.3.21. [Lur23:Construction 01J5]Let us fix $\mathscr{C}$ an ∞-category and X and Y two objects. We construct a simplicial set of morphisms $\mathsf{Map}_{\mathscr{C}}(X,Y)$ whose points are the 1-morphisms $X \to Y$ in $\mathscr{C}$. We define $\mathsf{Map}_{\mathscr{C}}(X,Y)$ as the fiber product of the diagram of simplicial sets

$$\begin{tikzcd}
\Delta^0 \arrow[dr, "X"] & & \mathsf{Fun}(\Delta^1, \mathscr{C}) \arrow[dl, "ev_0"'] \arrow[dr, "ev_1"] & & \Delta^0 \arrow[dl, "Y"'] \\
& \mathscr{C} & & \mathscr{C} &
\end{tikzcd}$$

By design,

- 0-simplexes:$=$ $f : \Delta^1 \to \mathscr{C}$ with $\partial_1(f) = X$, $\partial_0(f) = Y$.
- 1-simplexes are given by commutative squares, $\Delta^1 \times \Delta^1 \to \mathscr{C}$
- n-simplexes:$=$ $\Delta^n \times \Delta^1 \to \mathscr{C}$

Lemma A.3.22. *(See [Lur23:Subsection 01J4]). Let $\mathscr{C}$ be an ∞-category. Then $\mathsf{Map}_{\mathscr{C}}(X,Y)$ is a ∞-groupoid*

Terminology A.3.23. Let $\mathscr{C}$ be an ∞-category and X and Y objects of $\mathscr{C}$. The ∞-groupoid $\mathsf{Map}_{\mathscr{C}}(X,Y)$ of the Lemma **A.3.22** and Construction **A.3.21** is called the *mapping space* of morphisms from X to Y.

Remark A.3.24. Notice that $\mathrm{Hom}_{\mathsf{h}\mathscr{C}}(X,Y) = \pi_0(|\mathsf{Map}_{\mathscr{C}}(X,Y)|)$.

Lemma A.3.25. *([Lu-HTT: 1.2.4.1] and [Lur23:Theorem 019F]) Let $\mathscr{C}$ be a ∞-category and $f : \Delta^1 \to \mathscr{C}$ a 1-morphism $f : X \to Y$. The following are equivalent:*

*(i) f is an equivalence (in the sense of Definition **A.3.14**);*

(ii) *For every object $Z \in \mathscr{C}$, composition with f induces a weak-homotopy equivalence of mapping spaces*[(*)]

$$\mathsf{Map}_{\mathscr{C}}(Z, X) \to \mathsf{Map}_{\mathscr{C}}(Z, Y)$$

(iii) *For every object $Z \in \mathscr{C}$, composition with f induces a weak homotopy equivalence of mapping spaces*

$$\mathsf{Map}_{\mathscr{C}}(Y, Z) \to \mathsf{Map}_{\mathscr{C}}(X, Z)$$

(iv) *For every $n \geq 2$ and for every map $f_0 : \Lambda^n_0 \to \mathscr{C}$ such that $(f_0)_{|_{\Delta^{0,1}}} = f$, there exists an extension of f_0 to Δ^n*

Warning A.3.26. If $\mathscr{C}$ is an ∞-category, there is a priori no strict manifestation of the "composition law" in $\mathscr{C}$ defined as a map of simplicial sets using the definition of mapping spaces given in the Construction **A.3.21**.

$$\mathrm{Map}_{\mathscr{C}}(X, Y) \times \mathrm{Map}_{\mathscr{C}}(Y, Z) - \times - > \mathrm{Map}_{\mathscr{C}}(X, Z)$$

There is, however, a procedure to rectify composition laws, built on the formalism of simplicial categories, ie, categories enriched in simplicial sets, which we will use in Appendix **A.4** below.

To conclude this section we discuss the notion of equivalence for ∞-categories:

Definition A.3.27. Let $f : \mathscr{C} \to \mathscr{D}$ be a ∞-functor between ∞-categories. We say that f is:

(i) *essentially surjective* if the induced functor $\mathrm{h}\mathscr{C} \to \mathrm{h}\mathscr{D}$ is essentially surjective in the sense of classical 1-categories. See [Lur23:Remark 01JS];

(ii) *fully faithful* if for every pair of objects X, Y in $\mathscr{C}$, the induced map of the Kan complexes in the Lemma **A.3.22**

$$\mathsf{Map}_{\mathscr{C}}(X, Y) \to \mathsf{Map}_{\mathscr{C}}(f(X), f(Y))$$

is weak-equivalences of Kan complexes in the sense of the model structure of the Proposition **A.2.6**-(ii). See [Lur23:Definition 01JH].

(iii) an *equivalence* if there exists an ∞-functor $g : \mathscr{D} \to \mathscr{C}$ such that $f \circ g$ and $g \circ f$ are isomorphic to the identity functors, respectively, in the 1-categories $\mathsf{hFun}(\mathscr{D}, \mathscr{D})$ and $\mathsf{hFun}(\mathscr{C}, \mathscr{C})$. See [Lur23:Definition 01DY]

Definition A.3.28. [Lur23:Definition 01E8] Let $f : S \to T$ be a morphism of simplicial sets. We say that f is a *weak categorical equivalence* if for every ∞-category $\mathscr{C}$ the induced composition functor of classical 1-categories

$$\mathsf{hFun}(T, \mathscr{C}) \to \mathsf{hFun}(S, \mathscr{C})$$

(*) ie, in the sense of the Kan model structure of the Proposition **A.2.6**-(ii).

induces a bijection between isomorphisms classes of objects. [(*)].

Remark A.3.29. [Lur23:Example 01EA] Let $\mathscr{C}$ and $\mathscr{D}$ be ∞-categories and let $f : \mathscr{C} \to \mathscr{D}$ be a ∞-functor. Then f is an equivalence in the sense of Definition **A.3.27**-(iii) if and only if it is a weak categorical equivalence of simplicial sets (in the sense of Definition **A.3.28**).

Lemma A.3.30. *[Lur23:Theorem 01JX] Let $f : \mathscr{C} \to \mathscr{D}$ be a functor between ∞-categories. Then f is an equivalence of ∞-categories (in the sense of Definition **A.3.27**-(iii)) if and only if f is fully faithful and essentially surjective (Definition **A.3.27**-(i) and (ii)).*

Example A.3.31. [Lur23:Example 01E1] Let $f : \mathsf{C} \to \mathsf{D}$ be a functor between classical 1-categories. Then f is an equivalence if and only if $\mathsf{N}(f) : \mathsf{N}(\mathsf{C}) \to \mathsf{N}(\mathsf{D})$ is an equivalence of ∞-categories.

A.4. Examples of ∞-categories

The goal of this section is to explain how to use simplicial categories to produce examples of ∞-categories.

Definition A.4.1. [Lur23:Subsection 00JQ] A simplicial category A is a category enriched in simplicial sets, ie, for every two objects X and Y, there is a simplicial set $\mathsf{Map}_{\mathsf{A}}(X, Y)$ of morphisms $X \to Y$. Compositions define maps of simplicial sets

$$\mathsf{Map}_{\mathsf{A}}(X, Y) \times \mathsf{Map}_{\mathsf{A}}(Y, Z) \to \mathsf{Map}_{\mathsf{A}}(X, Z)$$

We denote by Cat^{Δ} the 1-category of simplicial categories.

We use simplicial categories to construct models of the categories $[n]$, where compositions are coherently "relaxed":

Example A.4.2. Consider the category [2] with objects $\{0, 1, 2\}$. We construct a simplicial category, denoted $\mathfrak{C}[2]$, where the composition of the morphisms corresponding to $0 < 1$ and $1 < 2$, ie, $0 < 1 < 2$, does not agree on the nose with $0 < 2$. We define:

- $\mathfrak{C}[2]$ has three objects: 0,1 and 2;
- There is a unique morphism $0 \to 1$, corresponding to $0 < 1$. In other words we attribute the simplicial enrichment

$$\mathsf{Map}_{\mathfrak{C}[2]}(0, 1) := \Delta^0$$

- There is a unique morphism $1 \to 2$, corresponding to $1 < 2$. Again, in terms of simplicial enrichement we have

[(*)] The notion of categorical equivalence in [Lur23] is the notion of *weak categorical equivalence* in [Lu-HTT]. We decided to use weak categorical equivalence to match [Lu-HTT].

$$\mathsf{Map}_{\mathfrak{C}[2]}(1,2) := \Delta^0$$

- There are two morphisms $0 \to 2$, one corresponding to $0 < 1 < 2$ and another one corresponding to $0 < 2$. These two morphisms are related by a 1-simplex corresponding to the inclusion $\{0,2\} \subseteq \{0,1,2\}$. We set

$$\mathsf{Map}_{\mathfrak{C}[2]}(0,2) := \Delta^1$$

- The composition map is given by

$$\Delta^0 \times \Delta^0 \to \Delta^1$$

  given by the inclusion of the vertice at 1.

We now explain how to generalize to $[n]$:

Construction A.4.3. ([Lur23:Subsection 00KM] and [Lur23:Notation 00KN]) Let $[n] \in \Delta$. We define a simplicial category $\mathfrak{C}[n]$ as follows:

- The objects of $\mathfrak{C}[n]$ are the elements of $[n]$, ie the set $\{0, 1, 2, \cdots, n\}$;

- For each $0 \leq i \leq j \leq n$, the enrichment is given by

$$\mathsf{Map}_{\mathfrak{C}[n]}(i,j) := \begin{cases} \emptyset & \text{if } j < i \\ \mathsf{N}(P_{i.j}) & \text{if } i \leq j \end{cases}$$

  where $P_{i,j}$ denotes the 1-category associated to the partially ordered set of subsets $I \subseteq [n]$ such that $i, j \in I$, for all $k \in I$, $i \leq k \leq j$. The partial order is given by inclusion.

- If $0 \leq i \leq ij \leq k \leq n$, the composition law

$$\mathsf{Map}_{\mathfrak{C}[n]}(i,j) \times \mathsf{Map}_{\mathfrak{C}[n]}(j,k) \to \mathsf{Map}_{\mathfrak{C}[n]}(i,k)$$

  is given by the map of partially ordered sets

$$P_{i,j} \times P_{j,k} \to P_{i,k}$$

  given by inclusion

$$(I, J) \mapsto I \cup J$$

The construction $[n] \mapsto \mathfrak{C}[n]$ defines a functor

$$\Delta \to \mathsf{Cat}^{\Delta}$$

By the universal property of presheaves, since Cat^{Δ} has all colimits, this left Kan extends to a functor

$$\mathfrak{C} : \mathsf{SSets} \to \mathsf{Cat}^{\Delta}$$

This functor is called the *rectification functor*. Moreover, by the adjoint functor theorem in classical category theory, it admits a right adjoint $\mathsf{N}_\Delta : \mathsf{Cat}^\Delta \to \mathsf{SSets}$ - the homotopy coherent nerve [Lur23:Definition 00KS] .

Remark A.4.4. On simplicial categories with trivial simplicial enrichment (ie, constant simplicial sets, the homotopy coherent nerve agrees with the Construction **A.1.10**. See [Lur23:Example 00KZ]. However, contrary to Construction **A.1.10**, N_Δ is not fully faithful in general.

Lemma A.4.5. *[Lur23:Theorem 00LJ] If* A *is a simplicial category enriched in Kan complexes,* $\mathsf{N}_\Delta(\mathsf{A})$ *is an* ∞*-category.*

Let us now recall some terminology specific to simplicial categories:

Construction A.4.6. If A is a simplicial category, we define its homotopy category hA to be the 1-category with the same objects and morphisms obtained by applying $\pi_0 \circ |-|$ to the simplicial enrichments, ie

$$\mathrm{Hom}_{\mathsf{hA}}(X, Y) := \pi_0\, |\mathsf{Map}_{\mathsf{A}}(X, Y)|$$

See Construction **A.2.2** for the notation. To define compositions we use functoriality of the construction $\pi_0 \circ |-|$.

Terminology A.4.7. Let $F : \mathsf{A} \to \mathsf{B}$ be a simplicially enriched functor. We say that F is:

- *homotopically essentially surjective* if the induced functor $\mathsf{hA} \to \mathsf{hB}$ is essentially surjective;
- *homotopically fully faithful* if it induces weak-equivalences on the simplicially enriched homs with respect to the Kan model structure (see Proposition **A.2.6**);
- a *weak-equivalence of simplicial categories* if it is simultaenously homotopically essentially surjective and fully faithful.

Definition A.4.8. Let $f : S \to T$ be a map of simplicial sets. We say that f is a *categorical equivalence* if the induced map of simplicial categories $\mathfrak{C}[S] \to \mathfrak{C}[T]$ is a weak-equivalence of simplicial categories in the sense of the Terminology **A.4.7**.

The following result explains the reason why $\mathfrak{C}$ is called rectification:

Theorem A.4.9. *[Lu-HTT: 2.2.5.1]*

(i) (Bergner) There exists a model structure on Cat^{Δ} *with cofibrant-fibrant objects given by simplicial categories enriched in Kan-complexes and weak-equivalences of simplicial categories as in Terminology* ***A.4.7****.*

(ii) (Lurie-Joyal) There exists a model structure on the category SSets *with weak-equivalences given by those maps whose simplicial sets* $S \to T$ *that are weak categorical equivalences in the sense of the Definition* ***A.3.28****, every object is cofibrant and fibrant objects are precisely* ∞*-categories.*

(iii) (Lurie) The adjunction $(\mathsf{N}_{\Delta}, \mathfrak{C})$ *is a Quillen equivalence.*

Remark A.4.10. Part of the proof of the Theorem **A.4.9** requires showing that if A is a simplicial category and $\mathsf{N}_{\Delta}(\mathsf{A})$ is its homotopy coherent nerve, then for any pair of objects X and Y in A, the mapping space $\mathsf{Map}_{\mathsf{N}_{\Delta}(\mathsf{A})}(X, Y)$ of the Terminology **A.3.23** is weakly equivalent to the original simplicial enrichment $\mathsf{Map}_{\mathsf{A}}(X, Y)$. See [Lur23:Subsection 01LA].

We can now produce new non-trivial examples of ∞-categories. The main mechanism is the following:

Construction A.4.11. Let M be a simplicial model category (cf. [Lu-HTT:A.3.1.7]). Then the full simplicial subcategory $\mathsf{A} := M^{\circ}$ spanned by cofibrant-fibrant objects is naturally enriched in Kan complexes. See [Lu-HTT:A.3.1.8]. Therefore, by the Lemma **A.4.5**, $\mathsf{N}_{\Delta}(M^{\circ})$ is an ∞-category with mapping spaces weakly-equivalent to those of M° (cf. Remark **A.4.10**).

Terminology A.4.12. We say that $\mathsf{N}_{\Delta}(M^{\circ})$ is the *underlying* ∞*-category of the simplicial model category* M.

Example A.4.13. We construct the ∞-category of spaces $\mathcal{S}$, by setting

$$\mathcal{S} := \mathrm{N}_{\Delta}(\mathsf{SSets}^{\circ})$$

where SSets° is the simplicial category of cofibrant-fibrant objects in the simplicial model category of simplicial sets with the Kan model structure of the Proposition **A.2.6**. In this case cofibrant-fibrant objects are precisely given by Kan-complexes (aka ∞-groupoids) and the mapping spaces in the sense of the Construction **A.3.21** given by

$$\mathsf{Map}_{\mathcal{S}}(X, Y) = \underline{\mathrm{Hom}}_{\mathsf{SSets}}(X, Y)$$

where the r.h.s is the simplicial enrichement given by internal-homs of simplicial sets of the Construction **A.1.13** (which in this case, and thanks to the Lemma **A.3.9** are also Kan-complexes). The equivalences in $\mathcal{S}$ are precisely the weak-homotopy equivalences of Kan-complexes in the sense of the Proposition **A.2.6**-(ii). See [Lur23:Subsection 01YX].

Example A.4.14. The category of simplicial sets SSets equipped with the Joyal model structure of the Theorem **A.4.9** is not a *simplicial model category*. In [Lu-HTT: 3.1.3.7] Lurie constructs a variation of this model structure on SSets, namely a model structure on the category of *marked simplicial sets* SSets^+ which is Quillen equivalent to the model structure of Joyal [Lu-HTT: 3.1.5.1]. The advantage of this new model is that it defines a simplicial model category. We define the ∞-category of (small) ∞-categories

$$\mathsf{Cat}_\infty = \mathsf{N}_\Delta((\mathsf{SSets}^+)^\circ)$$

The objects of Cat_∞ are ∞-categories and the mapping spaces $\mathsf{Map}_{\mathsf{Cat}_\infty}(\mathscr{C},\mathscr{D})$ are given by the maximal ∞-groupoid in $\mathsf{Fun}(\mathscr{C},\mathscr{D})$. Equivalences are equivalences of ∞-categories in the sense of the Definition **A.3.14**.

Example A.4.15. [Lu-SAG: 25.1.1.3] The category of simplicial commutative algebras over a (classical) commutative ring R, forms a simplicial model category M. The ∞-category of simplicial commutative rings SCR_R is by definition the underlying ∞-category

$$\mathsf{SCR}_R := \mathsf{N}_\Delta(M^\circ)$$

See the Remark **1.1.12**.

We now build new examples starting from Example **A.4.13** and Example **A.4.14** above:

Construction A.4.16. [Lur23:Subsection 003L] Reversing the linear order in the objects of Δ, defines a functor $\mathrm{Op}:\Delta\to\Delta$. If S is a simplicial set seen as a functor $S:\Delta^{\mathsf{op}}\to\mathsf{Sets}$ we define a new simplicial set S^{op} to be the composition $S\circ\mathrm{Op}^{\mathsf{op}}$.

Lemma A.4.17. *[Lur23:Proposition 003S] Let $\mathscr{C}$ be an ∞-category. Then $\mathscr{C}^{\mathsf{op}}$ is an ∞-category.*

Example A.4.18. Let $\mathscr{C}$ be an ∞-category. The ∞-category of *presheaves of spaces* in $\mathscr{C}$ - denoted $\mathscr{P}(\mathscr{C})$ - is by definition

$$\mathscr{P}(\mathscr{C}) := \mathsf{Fun}(\mathscr{C}^{\mathsf{op}},\mathcal{S})$$

cf. Lemma **A.3.9**. It comes with Yoneda functor $j:\mathscr{C}\to\mathscr{P}(\mathscr{C})$ sending an object X in $\mathscr{C}$ to the presheaf $\mathsf{Map}_{\mathscr{C}}(-,X)$[Lur23:Proposition 03N2]. Moreover, as in classical category theory, j is fully faithful (see [Lur23:Proposition 03M5] and [Lur23:Theorem 03NJ]).

Example A.4.19. Let K be a simplicial set and $\mathscr{C} := \mathsf{N}_\Delta(M^\circ)$ the underlying ∞-category of a simplicial model category M of the Construction **A.4.11**. Then the ∞-category of functors $\mathsf{Fun}(K,\mathscr{C})$ can also be presented as the underlying model category of a simplicial model category. Namely, we consider $\mathfrak{C}[K]$ the simplicial category of the Construction **A.4.11** and the category $M^{\mathfrak{C}[K]}$ of simplically enriched functors $\mathfrak{C}[K] \to M$. This has a model structure of its own - the projective model structure [Lu-HTT:Proposition A.3.3.2] - where weak-equivalences are objectwise weak-equivalences in M. The [Lu-HTT:Thm 4.2.4.4] establishes an equivalence of ∞-categories

$$\mathsf{Fun}(K, \mathscr{C}) \simeq \mathsf{N}_{\Delta}((M^{\mathfrak{C}[K]})^{\circ})$$

In the particular case where $M = \mathsf{SSets}$ with the Kan model structure and $K = \mathsf{N}(\mathsf{D})$ is the nerve of a 1-category, this establishes simplicial presheaves as a model for the ∞-presheaves of the Example **A.4.18**. This result was first established in [SH01].

A.5. Colimits and Localizations

Definition A.5.1. [Lur23:Subsection 02H6] Let $\mathscr{C}$ be a ∞-category. We say that an object $X \in \mathscr{C}$ is a *initial* object if for every object $Z \in \mathscr{C}$ the Kan-complex $\mathrm{Map}_{\mathscr{C}}(X, Z)$ is contractible. Similarly, one says that X is *final* if for every object Z, the Kan complex $\mathsf{Map}_{\mathscr{C}}(Z, X)$ is contractible.

Remark A.5.2. The universal property of initial and final objects is only defined only up to a contractible space of choices. Moreover, the collection of candidates forms a contractible space [Lur23:Corollary 02HM].

Definition A.5.3. Let $\mathscr{C}$ be an ∞-category . A *zero object* of $\mathscr{C}$ is an object which is simultaneously initial and final. We say that $\mathscr{C}$ is pointed if it contains a zero object.

Terminology A.5.4. Let K be a simplicial set and $\mathscr{C}$ an ∞-category . We will call a map of simplicial sets $d : K \to \mathscr{C}$ a K-index diagram in $\mathscr{C}$. When K is itself an ∞-category , this coincides with the notion of ∞-functor of the Definition **A.3.7**.

Construction A.5.5. [Lur23:Subsection 016K] Let K be a simplicial set and $d : K \to \mathscr{C}$ a diagram. We construct a new simplicial set $K^{\triangleright}$ by formally adding an exterior vertex to K by means of a pushout in SSets

$$K^{\triangleright} := (K \times \Delta^1) \coprod_{K \times \{1\}} \Delta^0$$

By construction, it comes with a inclusion $K \simeq K \times \{0\} \subseteq K^{\triangleright}$. Similarly, we define a *cone under* d is a map of simplicial sets $\tilde{d} : K^{\triangleright} \to \mathscr{C}$ whose restriction to K is d. Similarly, one can define the a "cone above d" replacing $K^{\triangleright}$ by $K^{\triangleleft}$:

$$K^{\triangleleft} := (K \times \Delta^1) \coprod_{K \times \{0\}} \Delta^0$$

It comes with a inclusion $K \simeq K \times \{1\} \subseteq K^{\triangleleft}$. See [Lur23:Notation 01HR], [Lur23:Theorem 01HV] and [Lur23:Construction 0177].

Example A.5.6. We have:

- If $K = \Delta^0$, then $K^\triangleright = \Delta^1$;
- if $K = \Delta^1$, then $K^\triangleright = \Delta^2$;
- if $K = \Lambda^0_2$, then $K^\triangleright = \Delta^1 \times \Delta^1$.

Construction A.5.7. Let $\mathscr{C}$ be an ∞-category and $d : K \to \mathscr{C}$ a diagram with K a simplicial set. We define the simplicial sets of *cones under d* - $\mathscr{C}_{d/.}$ (respectively, *cones above d* - $\mathscr{C}_{./d}$) to be the simplicial sets obtained by the pullbacks in SSets

$$\begin{tikzcd} \mathscr{C}_{d/.} \arrow[r] \arrow[d] & \mathsf{Fun}(K^\triangleright, \mathscr{C}) \arrow[d, "\text{restriction along } K \to K^\triangleright"] \\ \Delta^0 \arrow[r, "d"] & \mathsf{Fun}(K, \mathscr{C}) \end{tikzcd} \qquad \begin{tikzcd} \mathscr{C}_{./d} \arrow[r] \arrow[d] & \mathsf{Fun}(K^\triangleleft, \mathscr{C}) \arrow[d, "\text{restriction along } K \to K^\triangleleft"] \\ \Delta^0 \arrow[r, "d"] & \mathsf{Fun}(K, \mathscr{C}) \end{tikzcd}$$

Lemma A.5.8. *[Lur23:Proposition 018F] Let $d : K \to \mathscr{C}$ be as above. Then the simplicial sets $\mathscr{C}_{d/.}$ and $\mathscr{C}_{d/.}$ are ∞-categories.*

Example A.5.9. When $K = \Delta^0$, a diagram $d : K \to \mathscr{C}$ is the data of an object X in $\mathscr{C}$. In this case Construction **A.5.7** defines the ∞-categories of objects over and under X, $\mathscr{C}_{./X}$ and $\mathscr{C}_{X/.}$

Definition A.5.10. [Lur23:Definition 02JM] and [Lur23:Variant 02JN]) A *colimit* (resp. *limit*) of d is an initial object of $\mathscr{C}_{d/.}$ (resp. final object of $\mathscr{C}_{./d}$).

Finally, the following result establishes the equivalence between homotopy colimits in model categories (see [Lu-HTT:A.2.8]) and colimits in ∞-categories in the sense of Definition **A.5.10**:

Lemma A.5.11. *[Lu-HTT:Theorem 4.2.4.1] Let $F : J \to \mathsf{A}$ be a simplicial functor between simplicial categories enriched in Kan complexes. Let X be an object of A together with a compatible family of maps $\{\eta_j : F(j) \to X\}_{j \in J}$. Then X is a homotopy colimit of F iff the induced map of simplicial sets $\mathsf{N}_\Delta(J)^\triangleright \to \mathsf{N}_\Delta(\mathsf{A})$ is a colimit diagram in the ∞-category $\mathsf{N}_\Delta(\mathsf{A})$ (cf. Lemma **A.4.5**).*

Terminology A.5.12. Let $\mathscr{C}$ be an ∞-category:

- We say that a commutative square $K^\triangleright = \Delta^1 \times \Delta^1 \to \mathscr{C}$ is a *pushout square* if it is a colimit diagram for its restriction to $K = \Lambda^0_2$;
- Similarly, we say that a commutative square $K^\triangleleft = \Delta^1 \times \Delta^1 \to \mathscr{C}$ is a *pullback square* if it is a limit diagram for its restriction to $K = \Lambda^2_2$;

Example A.5.13. The pushout in the ∞-category of spaces $\mathscr{S}$

$$\mathsf{colim}^{\mathscr{S}} \left(\begin{tikzcd} * \coprod * \arrow[r] \arrow[d] & * \\ * & \end{tikzcd} \right)$$

can be computed using the Lemma **A.5.11** as a homotopy colimit in the model category of topological spaces replacing the two vertical maps by cofibrations and computing the naive colimit. See [Lu-HTT:A.2.4.4, A.2.4.5] and [Hov99:Prop 2.4.18.]:

$$= \mathsf{hcolim} \left(\begin{tikzcd} * \coprod * \arrow[r] \arrow[d] & * \\ * & \end{tikzcd} \right) = \mathsf{colim} \left(\begin{tikzcd} * \coprod * \arrow[r] \arrow[d] & \Delta^1 \\ \Delta^1 & \end{tikzcd} \right) = \mathsf{S}^1$$

Example A.5.14. Let $x : * \to X$ in $\mathscr{S}$. The pullback in $\mathscr{S}$ of

$$\begin{tikzcd} & * \arrow[d, "x"] \\ * \arrow[r, "x"] & X \end{tikzcd}$$

recovers the based loop space of X at x,

$$* \overset{h}{\underset{X}{\times}} * \simeq \Omega_x X$$

This is obtained replacing the inclusion of the point x in X by the homotopy equivalent path-space fibration $P_x \to X$.

Terminology A.5.15. [Lu-HAlg: 1.1.1.4] Let $\mathscr{C}$ be an ∞-category . A commutative square $\Delta^1 \times \Delta^1 \to \mathscr{C}$ of the form

(135)
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d, "g"] \\ 0 \arrow[r] & Z \end{tikzcd}$$

with 0 a zero object in C, is called a *triangle*. We say that a triangle is a *cofiber sequence* (respectively, *fiber sequence*) if the underlying commutative square is a pushout square (respectively, pullback square). In this case we say that Z is a *cofiber* of f (respectively, X is a *fiber* of g).

Definition A.5.16. [Lu-HAlg: 1.1.1.9] Let $\mathscr{C}$ be an ∞-category with a zero object and such that every morphism admits a fiber and a cofiber. We say that $\mathscr{C}$ is *stable* if a triangle in $\mathscr{C}$ is a fiber sequence if and only if it is a cofiber sequence.

Lemma A.5.17. *[Lu-HAlg: 1.1.2.14] The homotopy category* $\mathsf{h}\mathscr{C}$ *of a stable* ∞-*category* $\mathscr{C}$ *is triangulated.*

Definition A.5.18. [Lu-HAlg: 1.3.4.1] Let $\mathscr{C} \in \mathsf{Cat}_\infty$ and W a set of morphisms. A localization of $\mathscr{C}$ along W, it is exists, is ∞-category $\mathscr{C}[W^{-1}] \in \mathsf{Cat}_\infty$ together with an ∞-functor $L : \mathscr{C} \to \mathscr{C}[W^{-1}]$ satisfying the following universal property: for every $\mathscr{D} \in \mathsf{Cat}_\infty$, composition with L

$$\mathsf{Fun}(\mathscr{C}[W^{-1}], \mathscr{D}) \to \mathsf{Fun}(\mathscr{C}, \mathscr{D})$$

is a fully faithful functor with essential image given by $\mathsf{Fun}^W(\mathscr{C}, \mathscr{D}) \subseteq \mathsf{Fun}(\mathscr{C}, \mathscr{D})$ the full subcategory of $\mathsf{Fun}(\mathscr{C}, \mathscr{D})$ spanned by those functors $\mathscr{C} \to \mathscr{D}$ sending W to equivalences in $\mathscr{D}$.

Lemma A.5.19. *Let $\mathscr{C} \in \mathsf{Cat}_\infty$ and W a set of morphisms in $\mathscr{C}$. Then $\mathscr{C}[W^{-1}]$ exists.*

Proof. This is a consequence of the fact Cat_∞ admits all colimits. The pushout diagram

$$\begin{tikzcd} \coprod_{w \in W} \Delta^1 \arrow[r] \arrow[d] & \mathscr{C} \arrow[d, "L"] \\ \coprod_{w \in W} \mathsf{N}(J) \arrow[r] & \mathscr{C}[W^{-1}] \end{tikzcd} \qquad \mathsf{N}(J) \text{ cf. Example } \textbf{A.3.19}$$

satisfies the required universal property. □

Terminology A.5.20. [Lu-HAlg: 1.3.4.15] Let C be a 1-category with a class of weak-equivalences W. The *underlying ∞-category of the pair* (C, W) is the localization $\mathsf{N}(\mathsf{C})[W^{-1}]$ in the sense of the Lemma **A.5.19**.

Remark A.5.21. Let C be a 1-category with a set of morphisms W. Then the homotopy category $\mathsf{h}(\mathsf{N}(\mathsf{C})[W^{-1}])$ recovers the Gabriel-Zisman localization of [GZ67].

Theorem A.5.22. *[Lu-HAlg: 1.3.4.20] Let M be a simplicial model category*(*) *with class of weak-equivalences W. Let:*

- $\mathsf{N}(M)$ *denote the nerve of the underlying category spanned by cofibrant objects, where we forget the simplicial enrichment. (cf. Construction **A.1.10**) and let $\mathsf{N}(M)[W^{-1}]$ denote the ∞-category obtained via the Lemma **A.5.19**;*
- $\mathsf{N}_\Delta(M^\circ)$ *denote the ∞-category of the Construction **A.4.11**.*

Then the composition of cofibrant and fibrant replacement functors(†)

$$\mathsf{N}(M) \to \mathsf{N}_\Delta(M^\circ)$$

(*)Assume the existence of functorial fibrant and cofibrant replacements - see [Lu-HAlg: 1.3.4.16].
(†)See [Lu-HAlg: 1.3.4.18].

induces an equivalence of ∞-categories

$$\mathsf{N}(M)[W^{-1}] \simeq \mathsf{N}_\Delta(M^\circ)$$

*ie, in this case the notions of underlying ∞-category in Terminology **A.4.12** and Terminology **A.5.20** agree.*

Finally, we conclude this appendix with the construction of the ∞-derived category of modules and the ∞-category of derived rings used in Section **1.1**:

Construction A.5.23. [Lu-HAlg: 1.3.5.15] Let R be a (classical) commutative ring. We denote by $\mathrm{Ch}(R)$ the 1-category of chain complexes of R-modules and consider W_{qiso} the class of quasi-isomorphisms. The derived ∞-category of R-modules Mod_R is the underlying ∞-category of the pair $(\mathrm{Ch}(R), W_{qiso})$ (cf. Terminology **A.5.20**), ie, the localization(*)

$$\mathsf{Mod}_R := \mathsf{N}(\mathrm{Ch}(R))[W_{qiso}^{-1}]$$

The 1-category of chain complexes admits a dg-enhancement that can be used to obtain a more explicit description. For more details see [Lu-HAlg:§1.3.5].

Furthermore, the tensor product of (cofibrant) chain complexes is compatible with quasi-isomorphisms [Lu-HAlg: 7.1.2.11] and therefore induces a symmetric monoidal structure in Mod_R given by derived tensor products $-\overset{\mathbb{L}}{\underset{R}{\otimes}}-$ (see [Lu-HAlg: 4.1.7.6]). Consult [Lu-HAlg: 7.1.2.13] for a symmetric monoidal comparison with modules in spectra.

Construction A.5.24. Let now R be a commutative $\mathbb{Q}$-algebra. Consider again $\mathrm{Ch}(R)$ the 1-category of chain complexes of R-modules. The 1-category of commutative differential graded algebras (cdga) is by definition, the category of commutative algebra-objects in $\mathrm{Ch}(R)$ with respect to the tensor product of chain complexes, ie, $\mathsf{CAlg}(\mathrm{Ch}(R))$. A quasi-isomorphism of cdga's is a morphism of cdga's whose underlying morphism of chain complexes is a quasi-isomorphism. The ∞-category of cdga's is by definition, the localization(†)

$$\mathsf{cdga}_R := \mathsf{N}(\mathsf{CAlg}(\mathrm{Ch}(R))^c)[W_{qiso}^{-1}]$$

By [Lu-HAlg: 4.5.4.7, 7.1.4.11], this is equivalent to the ∞-category of commutative algebra objects in the symmetric monoidal ∞-category Mod_R of the Construction **A.5.23**.

(*)To be exact, one needs here to enlarge the universes since $\mathrm{Ch}(R)$ is not a small category. The localization takes place in $\mathsf{Cat}_\infty^{\mathrm{big}}$.

(†)As in the Construction **A.5.23**, the localization must be taken in $\mathsf{Cat}_\infty^{\mathrm{big}}$

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