Applications of derived algebraic geometry in enumerative geometry and invariants of singularities

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The thesis describes the content of three publications

- Categorification of Gromov-Witten invariants
  (joint with Mann);

- Motives and Categories of Singularities
  (joint with Blanc-Toën-Vezzosi);

- Universal Hochschild-Kostant-Rosenberg theorem
  (joint with Moulinos-Toën).

Plan

1. What is derived geometry?
2. What kind of enumerative problems?
3. Research results
4. Ongoing and Future works

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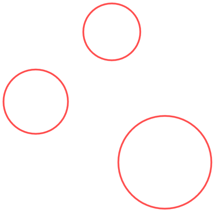
What is derived geometry?

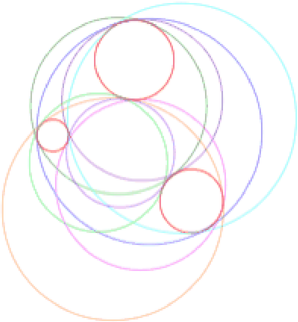
The Toën-Apollonius Example

Algebraic Geometry

The Apollonius Problem: How many circles simultaneously tangent to three fixed circles?



Algebraic Geometry



What happens when the radius go to zero?

Algebraic Geometry

A circle is determined by three parameters (center(a, b), r).

To fix a circle C is to fix a point in $\mathbb{P}^3$.

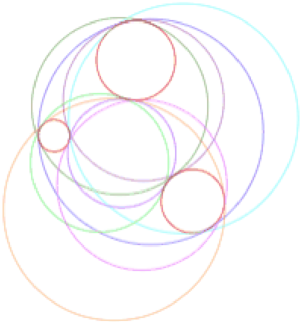
Every circle C, determines a subspace $Z_C \subseteq \mathbb{P}^3$ of all circles *tangent* to C.

Apollonius: Fix three circles C_1, C_2, C_3. Understand the intersection

$$Z_{C_1} \cap Z_{C_2} \cap Z_{C_3} \qquad \text{inside } \mathbb{P}^3$$

Algebraic Geometry

7 became algebra:



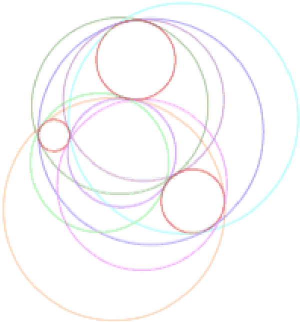
$$\mathbb{C}[\epsilon_x, \epsilon_y, \epsilon_z] = \mathbb{C} \oplus \underbrace{\mathbb{C}.\epsilon_x \oplus \mathbb{C}.\epsilon_y \oplus \mathbb{C}.\epsilon_z \oplus \mathbb{C}.\epsilon_x\epsilon_y \oplus \mathbb{C}.\epsilon_x\epsilon_z \oplus \mathbb{C}.\epsilon_y\epsilon_z \oplus \mathbb{C}\epsilon_x.\epsilon_y.\epsilon_z}_{\text{algebraic infinitesimals}}$$

(Grothendieck, 1960's) Every commutative ring A has a geometric interpretation as the ring of functions of a space, the affine scheme $Spec(A)$.

$$Z_{C_1} \cap^{Sch} Z_{C_2} \cap^{Sch} Z_{C_3} = \mathrm{Spec}(\mathbb{C}[\epsilon_x, \epsilon_y, \epsilon_z])$$

Algebraic Geometry

The **Toën-Appolonius** case: what if we collapse all to one point?



Algebraic Geometry

Infinitely many tangent circles!

How did 8 became infinitely many?

Answer

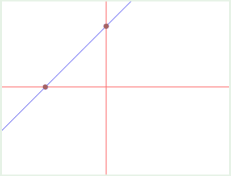
Redundancies!

Interlude I: Serre's formula (1957)

Example

Intersect the axis in 4-dimensions $R = \mathbb{C}[x, y, z, w]$, with the diagonal

$$\text{Axis} := \begin{cases} xz = 0 \\ xw = 0 \\ yz = 0 \\ yw = 0 \end{cases} \qquad \text{Diag} := \begin{cases} x - z = 0 \\ y - w = 0 \end{cases}$$



$$R/(xz, xw, yz, yw) \otimes_R R/(x - z, y - w) \simeq \mathbb{C}[x, y]/(x^2, xy, y^2)$$

$$= \underbrace{\mathbb{C} \oplus \mathbb{C}.\epsilon_x \oplus \mathbb{C}.\epsilon_y}_{3 \neq 2} \quad \text{too many!}$$

Example

Problem: room for redundancies!

$$f := xw - yz = w(x-z) - z(y-w)$$

vanishes for two reasons.

$$f \in \underbrace{(x-z, y-w)}_{I_{Diag}} \cap \underbrace{(xz, xz, yz, yw)}_{I_{Axis}}$$

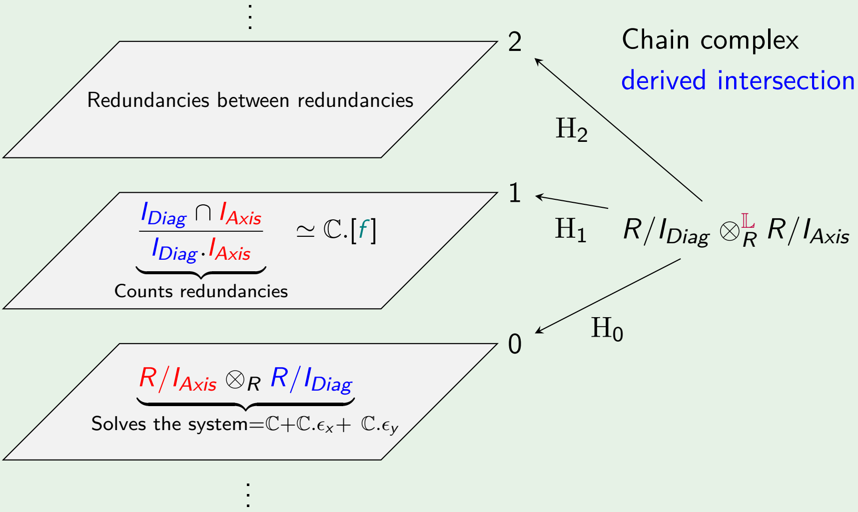
but the two reasons are different:

$$f \notin I_{Diag}.I_{Axis}$$

$$\frac{I_{Diag} \cap I_{Axis}}{I_{Diag}.I_{Axis}} \simeq \mathbb{C}.[f]$$

Correct counting: $\underbrace{\mathbb{C} \oplus \mathbb{C}.\epsilon_x \oplus \mathbb{C}.\epsilon_y - \mathbb{C}.[f]}_{3-1=2}$

Example



Serre's corrected excess = + dim. of even floors - dim. of odd floors

In Serre's computation: $R/I_{Diag} \otimes^{\mathbb{L}}_{R} R/I_{Axis}$ is seen as a linear object. Lacks direct geometric interpretation.

End of Interlude I

Interlude II: Derived Geometry (2000)

What is derived geometry?

$A = R/I_{Diag} \otimes^{\mathbb{L}}_R R/I_{Axis}$ is a ring-object in the chain complexes (**cdga**).

(Toën-Vezzosi, Lurie, 2002) Any **cdga** has a geometric interpretation

$A \in$ **cdga** $\mapsto \mathrm{Spec}(A)$ affine <u>derived</u> scheme, Functions$(\mathrm{Spec}(A)) = A$

Construction (Toën-Vezzosi, Lurie, 2002)

Affine derived schemes can be glued up to quasi-isomorphisms of cdga's.

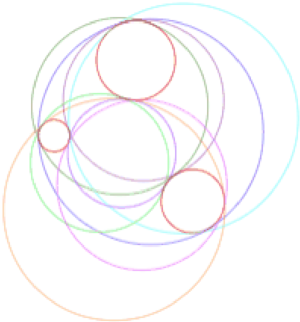
Result: $\underbrace{(\infty, 1) - \text{category}}_{\text{See appendix HDR}}$ of <u>non-affine</u> derived schemes **dSch**

Intersections in **dSch** automatically account for Serre's formula.

End of the Interludes.

Back to the circles.

Algebraic Geometry

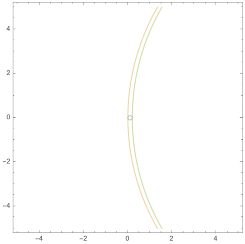


What is derived geometry?

Collapsed all circles C_1, C_2 and C_3 to a point $\rightsquigarrow$ jumped from 8 tangent circles to the whole plane of possibilities.

What is derived geometry?

In fact: two infinite planes of possibilities!



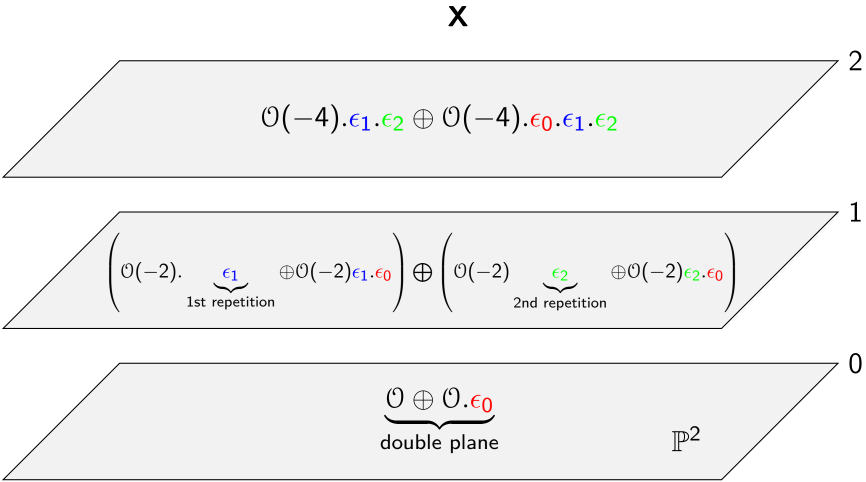
How to retrace 8?

Answer

The ***Toën-Appolonius derived intersection***

$$\mathbf{X} = Z_C \cap^{\mathbf{dSch}} Z_C \cap^{\mathbf{dSch}} Z_C$$

is a derived projective plane. The derived structure subtracts the double infinity and retraces the 8 circles algebraically.



(Kontsevich 95, Fontanine-Kapranov 2009, Khan 2019)

$$\underbrace{[\mathbf{X}]}_{\textbf{Derived Fundamental Class}} := \mathrm{Ch}[2\mathcal{O} - 4\mathcal{O}(-2) + 2\mathcal{O}(-4)] \cap [\mathbb{P}^2] = 8.[pt] \in H_0(\mathbb{P}^2)$$

What is derived geometry?

The Toën-Appolonius Example

$\mathbf{X} = Z_C \cap^{\mathbf{dSch}} Z_C \cap^{\mathbf{dSch}} Z_C$ is a derived projective plane:

- of virtual dimension zero, which means it behaves like a point
- the fundamental class indicates the point has multiplicity 8.

Upshot: The derived structure corrects the counting.

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From Physics to Enumerative Geometry

Motivation: String theory unifies general relativity and quantum mechanics at a cost:

$$\text{Spacetime} = \mathbb{R}^{3+1} \times \underbrace{Y}_{\textbf{Extra dimensions}}$$

(Candelas-Horowitz-Strominger-Witten 85):

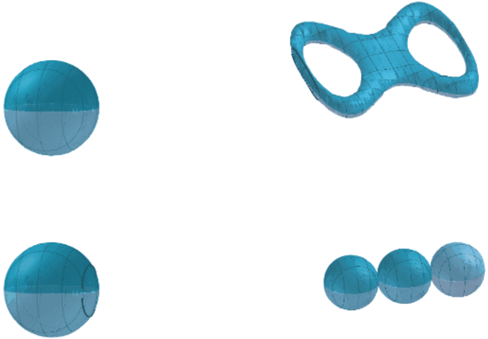
- Y is an algebraic variety of complex dimension 3;
- Calabi-Yau, ie, $\omega_Y \simeq \mathcal{O}_Y$;
- Example: the Fermat quintic $Y = \{x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 = 0\} \subseteq \mathbb{P}^4_{\mathbb{C}}$

Principle

Physical forces in 4-dim are consequence of the geometric and topological properties of the extra dimensions Y.

From Physics to Enumerative Geometry

Motivation: Paths of strings in Y define complex algebraic curves:



Feynman path integral for strings $= \underbrace{\text{Need to count algebraic curves in } Y}_{\text{Physics problem becomes geometric problem!}}$.

From Physics to Enumerative Geometry

Counting curves **in Y**:three strategies

- Hilbert approach: Ignore histories:

  $\mathrm{Hilb}_{codim\,2}(Y)$=\{ all curves  ,  , etc, in Y \}

- Gromov-Witten approach (94): Include successive histories

  $\overline{\mathcal{M}}_{g,n}(Y,d)$ = moduli space of stable maps of genus g, degree d and n marked points.

- Donaldson-Thomas approach (2000):
  Replace curves by their functions (ideal sheaves I_C)

  $\mathcal{M}Coh^{\tau}(Y)$ = moduli space of coherent sheaves with stability τ

From Physics to Enumerative Geometry

In full generality: the three moduli spaces

$$\mathrm{Hilb}_{codim\,2}(Y), \quad \overline{\mathcal{M}}_{g,n}(Y,d), \quad \mathcal{M}Coh^{\tau}(Y)$$

are too singular. Need further correction for counting.

Theorem (Toën-Vezzosi, 2005)

The three moduli spaces, admit canonical derived structures.

$$\mathbb{D}\mathrm{Hilb}_{codim\,2}(Y), \quad \mathbb{D}\overline{\mathcal{M}}_{g,n}(Y,d), \quad \mathbb{D}\mathcal{M}Coh^{\tau}(Y)$$

My research: Exploit the derived geometry of these moduli spaces.

Plan

Result 1: the Gromov-Witten approach $\overline{\mathcal{M}}_{0,n}(Y,d)$

Toy model: $\overline{\mathcal{M}}_{0,n}(Y=\mathbb{P}^2,d)$ is smooth of dimension $3d+n-1$

Fix $n=3d-1$ points $p_1,\cdots p_{3d-1}\in\mathbb{P}^2$

$$\underbrace{\{\text{rational curves of degree } d \text{ through } p_1,\cdots,p_{3d-1}\}}_{N_d}\subset \quad \overline{\mathcal{M}}_{0,n=3d-1}(\mathbb{P}^2,d)$$

Theorem (Kontsevich-Manin relations 94)

Relations arise d varies in $\overline{\mathcal{M}}_{0,3ⓓ-1}(\mathbb{P}^2,ⓓ)$

$$N_{ⓓ}=\sum_{\substack{d_A+d_B=ⓓ\\ d_A\geq 1;\, d_B\geq 1}} N_{d_A}.N_{d_B}.d_A^2.d_B\left(d_B\binom{3ⓓ-4}{3d_A-2}-d_A\binom{3ⓓ-4}{3d_A-1}\right)$$

Result 1: the Gromov-Witten approach $\overline{\mathcal{M}}_{0,n}(Y, d)$

Induction basis: $N_1 = 1$ (one line through two points);

```
d= 2 ;n= 5 ;N= 1
d= 3 ;n= 8 ;N= 12
d= 4 ;n= 11 ;N= 620
d= 5 ;n= 14 ;N= 87304
d= 6 ;n= 17 ;N= 26312976
d= 7 ;n= 20 ;N= 14616808192
d= 8 ;n= 23 ;N= 13525751027392
d= 9 ;n= 26 ;N= 19385778269260800
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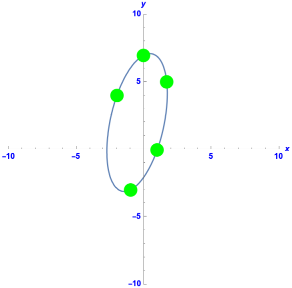
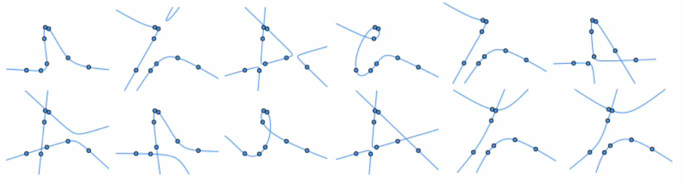
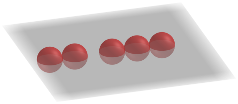




See HDR Section 2.1.3.4.

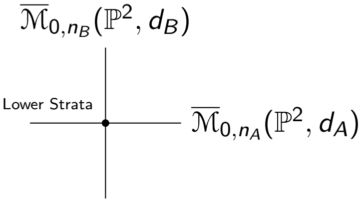
Result 1: the Gromov-Witten approach $\overline{\mathcal{M}}_{0,n}(Y,d)$

Formula reflects how curves glue (Kapranov-Getzler)



$$\coprod_{\substack{d_1+d_2=d\\ n_1+n_2-2=n}} \overline{\mathcal{M}}_{0,n_1}(\mathbb{P}^2,d_1)\times_{\mathbb{P}^2}\overline{\mathcal{M}}_{0,n_2}(\mathbb{P}^2,d_2)\xrightarrow[\text{glue}]{}\underbrace{\overline{\mathcal{M}}^{split}_{0,n_1+n_2-2}(\mathbb{P}^2,d_1+d_2)}_{\text{subspace of broken histories}}\subseteq\overline{\mathcal{M}}_{0,n}(\mathbb{P}^2,d)$$

Birational, surjective, but **not iso.**



Result 1: the Gromov-Witten approach $\overline{\mathcal{M}}_{0,n}(Y,d)$

Setup:

$$\overline{\mathcal{M}}_{0,n}(\mathbb{P}^2,d)$$

forget $\mathbb{P}^2$, keep the curve: $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^2,d) \to \overline{\mathcal{M}}_{0,n}$ = Moduli of curves

$ev_1 \times \cdots \times ev_n$: $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^2,d) \to \mathbb{P}^2 \times \cdots \times \mathbb{P}^2$

$$N_d \underbrace{=}_{\text{Poincaré duality}} \left(\bigcup_{i=1}^{3d-1} ev_i^*(\underbrace{\gamma_i}_{\text{Poincaré dual of } p_i}) \right) \cap [\overline{\mathcal{M}}_{0,n}(\mathbb{P}^2,d)], \qquad I_{0,n}^d := push \circ pull : H^*(\mathbb{P}^2)^{\otimes n} \to H^*(\overline{\mathcal{M}}_{0,n})$$

Theorem (Kontsevich-Manin, 95)

The pull-push makes $H_*(\mathbb{P}^2)$ a graded-module over $H_*(\overline{M}_{0,n})$ (CohFT). The associativity constrains

$$I_{0,n}^d = \sum_{d_1+d_2=d} \left(I_{0,n_1}^{d_1}(-,h^0) \otimes I_{0,n_2}^{d_2}(h^2,-) + I_{0,n_1}^{d_1}(-,h^1) \otimes I_{0,n_2}^{d_2}(h^1,-) + I_{0,n_1}^{d_1}(-,h^2) \otimes I_{0,n_2}^{d_2}(h^1,-) \right)$$

are responsible for the formula of Kontsevich-Manin.

Result 1: the Gromov-Witten approach $\overline{\mathcal{M}}_{0,n}(Y, d)$

Problem: Formula works on $\mathbb{P}^2$ because $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^2, d)$ is smooth.

Need correction for general Y.

Behrend-Fantechi 97: modify the fundamental class $[\overline{\mathcal{M}}_{0,n}(Y, d)]^{BF}$

Our strategy: Use derived geometry

Theorem (Mann-R. 2015)

Let Y be a smooth projective varierty. Then:

- The derived structures $\mathbb{D}\overline{\mathcal{M}}_{0,n}(Y, d)$ are compatible with the gluings of curves.
- $[\mathbb{D}\overline{\mathcal{M}}_{0,n}(Y, d)] = [\overline{\mathcal{M}}_{0,n}(Y, d)]^{BF}$

Result 1: the Gromov-Witten approach $\overline{\mathcal{M}}_{0,n}(Y,d)$

$$\mathbb{D}\overline{\mathcal{M}}_{0,n}(Y,d)$$

$$\overline{\mathcal{M}}_{0,n} \times Y^{n-1} \xleftarrow{\text{forget}\times ev_1\times\cdots\times ev_{n-1}} \mathbb{D}\overline{\mathcal{M}}_{0,n}(Y,d) \xrightarrow{ev_n} Y$$

Corollary (Mann-R. 2015)

The push-pull through $\mathbb{D}\overline{\mathcal{M}}_{0,n}(Y,d)$ makes $D^b_{Coh}(Y)$ a graded $D^b_{Coh}(M_{0,\bullet})$-module in dg-categories. the formulas of Kontsevich-Manin lift to the derived categories.

- Recover Givental-Lee formulas in K-theory (2001).
- Recover Kontsevich-Manin formulas in homology (94).

Further works: Porta-Yu non-Archimedean Mirror Symmetry

Result 2 & 3: Donaldson-Thomas $\mathcal{M}Coh^{\tau}(Y)$

$$\textbf{DT} - \text{number} := "Volume(\mathcal{M}Coh^{\tau}(Y))"$$

Observation(Thomas 2000): Serre duality + CY forces a symmetry

$$\{1^{st\ order}\text{def. of } E \in Coh(Y)\} \simeq \{\text{Obstructions to def. of } E \in Coh(Y)\}^{\vee}$$

Behind the scenes: *(-1)-shifted symplectic form*:

Theorem (Pantev-Toën-Vaquié-Vezzosi, 2011)

Y a 3-CY. Then $\mathbb{D}\mathcal{M}Coh^{\tau}(Y)$ is (-1)-shifted symplectic.

Recall: Symplectic form $\quad \mathbb{T} \simeq \mathbb{T}^*$

Shifted Symplectic form $\quad \mathbb{T} \simeq \mathbb{T}^*[-1]$

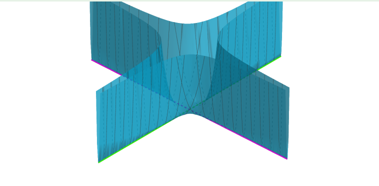
Warning: Shifted forms cannot exist when the moduli is smooth!

Result 2 & 3: Donaldson-Thomas $\mathcal{M}Coh^{\tau}(Y)$

Example (Lagrangian intersections)

$U = \mathbb{A}^2$ $\qquad f = x^2y^2$

$$\begin{array}{ccc} \mathbb{D}Crit(f) & \longrightarrow & U \\ \downarrow & & \downarrow {\scriptstyle df=2xy^2dx+2yx^2dy} \\ U & \xrightarrow{0} & T^*U \end{array}$$



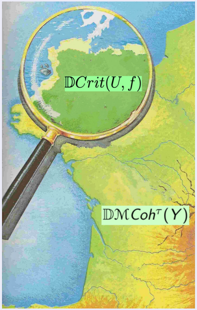
$$\begin{array}{ccccc} -1 & 0 & 1 & & \\ & \mathbb{T}_U & \xrightarrow{\text{Hess(f)}} \mathbb{T}_U^* & = & \mathbb{T} \\ \mathbb{T}_U & \xrightarrow{\text{Hess(f)}^\vee} \mathbb{T}_U^* & & = & \mathbb{T}^* \end{array}$$

Symmetry of the Hessian $\Rightarrow \mathbb{T} \simeq \mathbb{T}^*[-1]$ (-1)-shifted symplectic form

Result 2 & 3: Donaldson-Thomas $\mathcal{M}Coh^{\tau}(Y)$

Theorem (Brav-Bussi-Joyce (Darboux Lemma) 2013)

Locally any (-1)-shifted symplectic form is symplectomorphic to a $\mathbb{D}Crit(U, f)$.



Local geometry of $\mathbb{D}\mathcal{M}Coh^{\tau}(Y)$ = local geometry of the singularities of f.

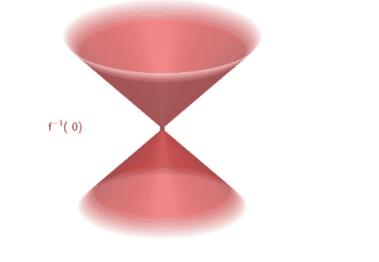
Result 2 - Invariants of Singularities

Capture the singularities of f:

- Milnor number, when f has isolated singularities

$$\mu(f) = \dim_{\mathbb{C}} \underbrace{\mathcal{O}_U/(\partial_1 f, \cdots \partial_n f)}_{\mathcal{O}_{Crit(f)}} \in \mathbb{Z}$$

Example ($U = \mathbb{A}^2$, $f = x^2 + y^2$)

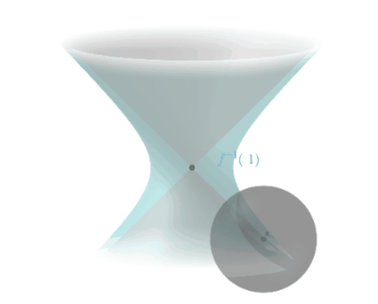


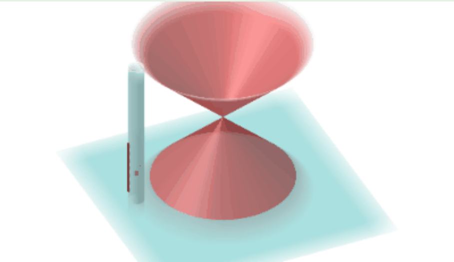
$$\mu(f) = \dim_{\mathbb{C}} \mathbb{C}[x, y]/(2x, 2y) = 1$$

Result 2 - Matrix Factorizations

- Vanishing cohomology sheaf (Milnor, Deligne, Grothendieck)

Example ($U = \mathbb{A}^2$, $f = x^2 + y^2$)





$$x \in f^{-1}(0) \mapsto H^{2-1}(\underbrace{B(x,\epsilon) \cap f^{-1}(\epsilon)}_{\text{Milnor Fiber at } x}) \simeq \begin{cases} \mathbb{Z}^{\mu(f)} & x = 0 \\ 0 & x \neq 0 \end{cases}$$

are stalks of a **sheaf** $\mathbf{P_{U,f}}$ supported on critical points, with monodromy.

Result 2 - Matrix Factorizations

Invariants: Capture the singularities of f

- Categorification:

  Serre-Auslander-Buchsbaum-Eisenbud: Every $M \in Coh(f^{-1}(0))$ admits infinite resolution that becomes 2-periodic

$$\underbrace{\ldots \to F \to Q \to F \to Q}_{\in MF(U,f)} \to \underbrace{P_n \to \ldots \to P_2 \to P_1 \to P_0}_{\in Perf(f^{-1}(0))} \to M$$

Definition (Eisenbud-Orlov)

$$MF(U,f) := D^b_{Coh}(f^{-1}(0)) \underbrace{/}_{dg-quotient} Perf(f^{-1}(0))$$

Result 2 - Invariants of Singularities

Invariants: Capture the singularities of f

- $MF(U, f)$

Example

Knörrer periodicity

$$MF(\mathbb{A}^2, x^2 + y^2) \simeq \{\underbrace{\cdots \to E_{-1} \to E_0 \to E_{-1} \to E_0 \to \cdots}_{\text{2-periodic chain complexes}}\}$$

$$MF(U \times \mathbb{A}^2, f + x^2 + y^2) \simeq MF(U, f)$$

$$MF(U \times \mathbb{A}^1, f + x^2) \neq MF(U, f)$$

Result 2 - Invariants of Singularities

Theorem (Over $\mathbb{C}$, Dyckerhoff 2009, Sabbah 2010, Preygel 2011, Efimov 2015)

$$HP(MF(U,f)) \leftrightarrow (H^*(f^{-1}(0), P_{U,f}), Monodromy)$$

$$\mu(f) \overset{\chi}{\longleftarrow\!\!\shortmid} P_{U,f} \overset{HP}{\longleftarrow\!\!\shortmid} MF(U,f)$$

Motivic explanation of this comparison over any characteristic:

Theorem (Blanc-R.-Toën-Vezzosi, 2018)

Let U be a regular scheme with a function f. Then

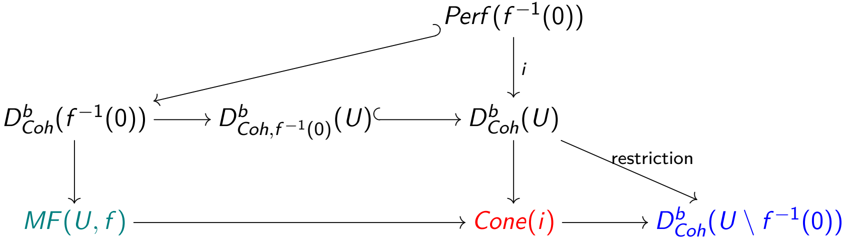
$$\text{Motive of } \mathrm{MF}(U,f) = (\text{Motivic Vanishing cycles of } f)^{\text{Inertia-fixed}}_{2-per}[-1]$$

as objects in the 2-periodized Morel-Voevodsky category SH.

Result 2 - Invariants of Singularities

Heuristic argument: Construct isomorphism between two sequences:

- From the definition of MF:



- the exact triangle of vanishing cycles:

$$\text{Inertia-Inv. Motivic Vanishing Cycles}[-1] \to \mathbb{H}_{Mot}(f^{-1}(0)) \otimes H^*(\mathrm{S}^1) \underbrace{\to}_{Monodromy} \mathbb{H}_{Mot}(U \setminus f^{-1}(0))$$

Result 2 - Invariants of Singularities

Applications: Bloch conductor formula:

- Toën-Vezzosi 2018
- Beraldo-Pippi 2022

Result 3 - Universal HKR

Recall: $\mathbb{DM}Coh^{\tau}(Y)$ is (-1)-shifted symplectic

Long term: DT invariants for CY in positive and mixed characteristic

Problem: definition of closed form.

Example

X a smooth projective algebraic variety

$$\Omega_X^{\geq p} := [\, 0 \longrightarrow 0 \longrightarrow \underbrace{\Omega_X^p}_{\deg\ p} \xrightarrow{d_R} \underbrace{\Omega_X^{p+1}}_{\deg\ p+1} \xrightarrow{d_R} \cdots \xrightarrow{d_R} \underbrace{\Omega_X^n}_{\deg\ n} \,]$$

$$\text{closed } p-\text{forms} = H^0(X, \Omega_X^{\geq p}[p])$$

Construction

For a non-smooth $X = \mathrm{Spec}(A)$ *over* k*, must replace* $\Omega^1_{A/k}$ *by* $\mathbb{L}_{A/k}$

$$DR(X) := Sym_A(\mathbb{L}_{A/k}[1]) = \oplus_{p\geq 0} \left(\wedge_A^p \mathbb{L}_{A/k}\right)[p], \qquad d_R : \left(\wedge_A^p \mathbb{L}_{A/k}\right)[p] \to \left(\wedge_A^{p+1} \mathbb{L}_{A/k}\right)[p+1][-1]$$

Result 3 - Universal HKR

Theorem (Toën-Vezzosi 2009, BenZvi-Nadler 2010, re-interpretation of Hochschild-Kostant-Rosenberg theorem 1962)

Over a field k of characteristic zero:

$$Spec(DR(X/k))) = \underline{Map}_k(S^1, X)$$

- *the canonical S^1-action on the r.h.s induces the de Rham differential on the lhs.*
- *Closed forms are fixed points for the S^1-action.*

Remark: This is used by Pantev-Toën-Váquie-Vezzosi to define the (-1)-shifted symplectic form in $\mathbb{DM}Coh^{\tau}(Y)$.

Remark: This theorem fails in positive and mixed characteristic.

Result 3 - Universal HKR

Theorem (Moulinos-R.-Toën, 2020)

Over any ring k there exists a filtered group stack S^1_{FIL} (the filtered circle), such that for any derived scheme X over k, the derived mapping stack

$$\underline{Map}_{FIL}(S^1_{FIL}, X)$$

defines a filtration on $\underline{Map}_k(S^1, X)$ with ass. graded $Spec(DR(X/k))$.

- the canonical S^1_{FIL}-action induces the de Rham differential
- Closed forms are fixed points for the S^1_{FIL}-action.

Corollary (Moulinos-R.-Toën, 2020)

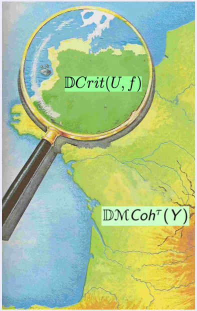
There exists a sensible notion of (-1)-shifted symplectic forms on derived stacks in positive and mixed characteristic.

Plan

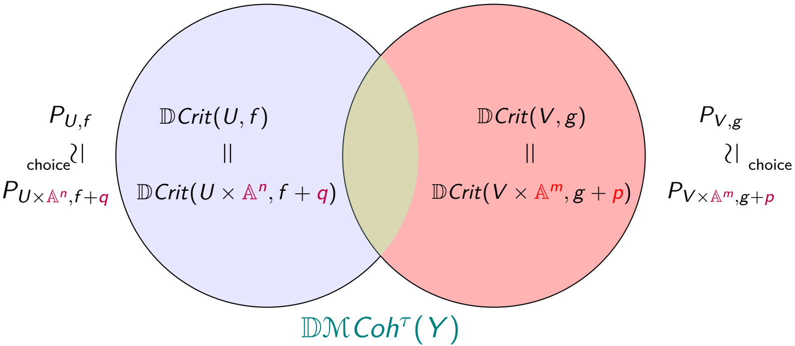
Categorical Donaldson-Thomas Invariants

Theorem (Brav-Bussi-Joyce (Darboux Lemma) 2013)

Locally any (-1)-shifted symplectic form is symplectomorphic to a $\mathbb{D}Crit(U, f)$.



Categorical Donaldson-Thomas Invariants

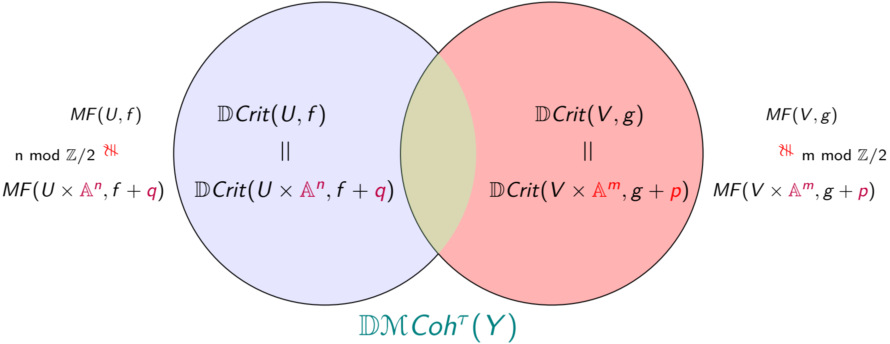


Theorem (Brav-Bussi-Dupont-Joyce-Szendroi 2015)

Assume there exists a square root of the canonical bundle of $\mathbb{D}\mathcal{M}Coh^{\tau}(Y)$*.*
Then the local ambiguities can be solved and the $P_{U,f}$ *glued .*

Remark: $\chi(global\ P) = DT$

Categorical Donaldson-Thomas Invariants



Conjecture (Kontsevich-Soibelman, Toda): The local categories $MF(U,f)$ glue upon the prescription of an orientation data to be identified.

Theorem (Hennion-Holstein-R., 2024)

The conjecture is true for the associated nc-motives.

The orientation corresponds to the trivialization of three obstruction classes: $\alpha \in H^1(X,\mathbb{Z}/2)$, $\beta_{Joyce} \in H^2(X,\mathbb{Z}/2)$, $\gamma \in H^3(X,\mathbb{Z}/2)$

Categorical Donaldson-Thomas Invariants

Main new idea: Study the **Darboux stack** of a (-1)-shifted symplectic derived scheme X

$$\mathbf{Darb}_X : \qquad \text{Open subsets of } X \longrightarrow \text{Spaces}$$

$$S \subseteq X \longmapsto \{\text{ Symplectic identifications } S \simeq \mathbb{D}Crit(U, f)\}$$

and its quotient by the action of the stack of quadratic bundles $\mathbf{Quad}_X$

$$\mathbf{Darb}_X/\mathbf{Quad}_X \xrightarrow{MF} \mathrm{dgCats}, \qquad \mathbf{Darb}_X/\mathbf{Quad}_X \xrightarrow{P} Perv, \qquad \mathbf{Darb}_X/\mathbf{Quad}_X \xrightarrow{\mu} \mathbb{Z}$$

Other ongoing projects

- Joyce conjecture on Lagrangians in (-1)-shifted symplectic forms:

Conjecture

X a (-1)-shifted symplectic derived scheme with square root of canonical bundle. Then any oriented Lagrangian L in X defines a global section of the glued perverse sheaf (idem, a global section of the glued MF).

We believe our strategy to glue MF also allows us to solve this conjecture (Hennion-Holstein-R.).

- Witten genus via derived geometry (Hennion-Moulinos-R-Safranov) (See HDR).

Thank you for your time.

From Physics to Enumerative Geometry