On the Categorification problem for Motivic Donaldson-Thomas invariants

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CY varieties

- Y a Calabi-Yau variety of dimension 3 over $\mathbb{C}$, ie, $\omega_Y \simeq \mathcal{O}_Y$.

- Example: The Fermat quintic

$$X_5 = \{x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 = 0\} \subseteq \mathbb{P}^4_{\mathbb{C}}$$

or more generally, any smooth quintic in $\mathbb{P}^4_{\mathbb{C}}$, $\omega_Y \simeq \mathcal{O}(5-4-1) \simeq \mathcal{O}_Y$

Why CY?

- (Pre-history) Kaluza-Klein model: spacetime locally of the form $\mathbb{R}^{3+1} \times S^1$

  gravity + electromagnetism in $\mathbb{R}^{3+1} \leftrightarrow$ pure gravity in $\mathbb{R}^{3+1} \times S^1$.

  what about electroweak and strong interactions?

- (Candelas-Horowitz-Strominger-Witten) In string theory, spacetime is required to be of the form $\mathbb{R}^{3+1} \times Y$ where Y is a CY-manifold of real dimension 6.

  Physic's laws in $\mathbb{R}^{3+1} \leftrightarrow$ Geometric and topological properties of Y.

- Paths/interactions of string-particles through spacetime, define 2-dimensional real surfaces (1-dimensional algebraic curves) of genus g in Y.

Counting algebraic curves in a Calabi-Yau

- Counting **parametrized** curves $f : C \to Y$ (GW-invariants)

$$\underbrace{\overline{\mathcal{M}}_{g,n}(Y,\beta)}_{\text{moduli space (stack) of stable maps}} \qquad \text{quasi-smooth}, \; Vol = \int_{virt.\,class} \in \mathbb{Q} \; \checkmark$$

- Counting **embedded** curves $C \subseteq Y$:

$$\underbrace{\mathrm{Hilb}_{codim\,2}(Y)}_{\text{Hilbert scheme of codim 2 subschemes}} \qquad \text{not quasi-smooth}, \; Vol \; ✗$$

- Counting **ideal sheaves** $I_C \in \mathrm{Coh}(Y)$ (DT-invariants)

$$\underbrace{\mathcal{M}Coh(Y)}_{\text{Moduli of coherent sheaves}} \qquad \underbrace{\text{quasi-smooth}}_{\text{CY + Serre duality}}, \quad Vol = \int_{virt.\,class} \in \mathbb{Z} \; \checkmark$$

Behrend approach to DT-invariants

Observation: Serre duality + CY condition imposes a symmetry on the deformation theory of $\mathcal{M}Coh(Y)$: symmetric obstruction theory

$$\{1^{st\, order}\text{def. of } E \in Coh(Y)\} \simeq \{\text{Obstructions to def. of } E \in Coh(Y)\}^{\vee}$$

Theorem (K. Behrend)

Let X be a quasi-smooth algebraic stack with a symmetric obstruction theory (Ex: $X = \mathcal{M}Coh(Y)$). Then there is a function $\nu_{Behrend} : X \to \mathbb{Z}$ such that

$$Vol(X) := \int_{[X]^{vir}} = \chi(X, \nu_{Behrend}) = \sum_{n} n.\chi(\nu_{Behrend} = n)$$

Behind the scenes: This extra symmetry is a shadow of a *(-1)-shifted symplectic form* on $\mathcal{M}Coh(Y)$ [Pantev-Toën-Vaquié-Vezzosi].

In this talk: DT-theory $\leftrightarrow$ (-1)-shifted symplectic geometry

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Shifted Symplectic Geometry

- Deformation theory of a stack/derived stack X is controlled by its *cotangent complex* $\mathbb{L}_X \in \mathrm{DQcoh}_\infty(X)$. When X is locally of finite presentation, $\mathbb{L}_X$ is a *perfect complex*, with dual $\mathbb{T}_X = \mathbb{L}_X^\vee$ the *tangent complex*.

- X Smooth, $\mathbb{L}_X = \Omega^1_X$.

- n-shifted 2-forms $= \{\mathcal{O}_X \to \mathbb{L}_X \wedge \mathbb{L}_X[n]\} = \{\mathbb{T}_X \wedge \mathbb{T}_X \to \mathcal{O}_X[n]\}$

- de Rham diff. $DR(X) := [\mathcal{O}_X \xrightarrow{d_R} \mathbb{L}_X \xrightarrow{d_R} \mathbb{L}_X \wedge \mathbb{L}_X \longrightarrow \ldots]$

  [Connes, Toën-Vezzosi]: d_R should not be understood as an internal differential but rather as the action of an extra operator ϵ of degree 1

- n-shifted closed 2-forms: Need homotopy $d_R(\omega) \sim 0$

$$\{\text{maps} \quad \underbrace{k(2)[-2n-1]}_{\epsilon=0 \text{ action}} \to \underbrace{DR(X)}_{\epsilon=d_R}\}$$

Shifted Symplectic Geometry

Definition (Pantev-Toën-Vaquié-Vezzosi)

An n-shifted symplectic form on X is a n-shifted closed 2-form such that its underlying 2-form $\mathbb{T}_X \wedge \mathbb{T}_X \to \mathcal{O}_X[n]$ is non-degenerate , ie, induces an equivalence

$$\mathbb{T}_X \simeq \mathbb{L}_X[n]$$

- $X = T^*\mathbb{A}^1 = \mathbb{A}^2$ has 0-shifted symplectic form given by $\omega = dx \wedge dy$.
- $X = Perf$ the derived stack classifying perfect complexes has a 2-shifted symplectic form.

$$\mathbb{T}_{E,Perf} = REnd(E)[1] \simeq E \otimes E^\vee[1]$$

$$\mathbb{T}_{E,Perf} \wedge \mathbb{T}_{E,Perf} \simeq E \otimes E^\vee[1] \otimes E \otimes E^\vee[1] \to \mathcal{O}[2] \quad \text{evaluation map}$$

- (PTVV) Y a CY of dimension 3 over k. Then $X := \mathrm{Map}(\underbrace{Y}_{3}, \underbrace{Perf}_{2})$

  is (2-3=-1)-symplectic. In particular,

  $\mathcal{M}Coh(Y) \subseteq \mathrm{Map}(Y, Perf)$ is -1-symplectic (⇒ Behrend Symmetry)

Shifted Symplectic Geometry

- U a smooth k-scheme with a function $f : U \to \mathbb{A}^1_k$. The derived critical locus $X = dCrit(f)$ is the derived fiber product

$$\begin{array}{ccc} X := dCrit(f) & \xrightarrow{i} & U \\ \downarrow i & & \downarrow df \\ U & \xrightarrow{0} & T^*U \end{array}$$

$coh.deg$	-1	0	1		
		$i^*\mathbb{T}_U \xrightarrow{\text{Hess(f)}}$	$i^*\mathbb{L}_U$	$=$	$\mathbb{T}_X$
	$i^*\mathbb{T}_U \xrightarrow{\text{H(f)}^\vee}$	$i^*\mathbb{L}_U$		$=$	$\mathbb{L}_X$

symmetry of the Hessian $\Rightarrow \mathbb{T}_X \simeq \mathbb{L}_X[-1]$ is the underlying 2-form of a (-1)-shifted symplectic structure on X.

Example: $(U, f) = (\mathbb{A}^1, x^3)$ $\quad dCrit = Spec\, k[x]/(f' = 3x^2)$

Joyce's approach to DT-invariants

All examples are locally of this form:

Theorem (Brav-Bussi-Joyce (Darboux Lemma))

Let X be a (-1)-symplectic derived scheme. Then Zariski locally X is symplectomorphic to a derived critical locus $dCrit(U, f)$ with U smooth.

Consequence: Locally on X it makes sense to analyse the singularities of the function f on U via the perverse sheaf of vanishing cycles

$$P_{U,f} \in \mathrm{Perv}_{dCrit(f)}(U) = \mathrm{Perv}(dCrit(f)) = \mathrm{Perv}(Crit(f))$$

Problem: Ambiguity in the choice of local presentations:

$$dCrit(\mathbb{A}^1, x^3) = \mathrm{Spec}\, k[x]/(3x^2) \simeq \mathrm{Spec}\, k[x,y]/(3x^2, 2y) = dCrit(\mathbb{A}^2, x^3 + y^2)$$

$P_{(\mathbb{A}^1, x^3)}$ and $P_{(\mathbb{A}^2, x^3+y^2)}$ non-canonically isomorphic.

Joyce's approach to DT-invariants

Theorem (Brav-Bussi-Dupont-Joyce-Szendroi (BBDJS))

Let X be a (-1)-symplectic derived scheme. Assume that there exists a line bundle L together with an equivalence $L \otimes L \simeq \det(\mathbb{T}_X)$ (aka orientation data). Then:

- *The locally defined perverse sheaves of vanishing cycles $P_{U,f}$ glue to a globally defined perverse sheaf $P \in Perv(X)$.*
- *$\chi(P) = \nu_{Behrend}$ computing locally the Euler characteristic of vanishing cycles. Gives back DT-counting.*

Proof: Glue by hand using local presentations of the underlying classical scheme as classical critical loci.

- method does not see the full derived structure.
- strategy works for perverse sheaves because:
  - they form a **1-category** (no higher homotopies needed to glue).
  - they have the $\mathbb{A}^1$-**homotopy invariance** property.

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Motivic DT and categorification

Different invariants capture vanishing cycles of f on U:

$$\underset{\text{Ayoub Motivic Van. Cycles}}{\mathcal{V}_{U,f}} \in \text{Motives} \xleftarrow[\text{Blanc-Toën-R.-Vezzosi, Pippi}]{} \text{dgcats}^{\text{2per}} \ni \underset{\text{Orlov/Eisenbud cat. of singularities}}{MF(U,f)}$$

$$\mathcal{V}_{U,f} \xrightarrow{\textit{Realization}} P_{U,f}, \qquad \text{dgcats}^{\text{2per}} \xrightarrow[\textit{Efimov, Sabbah}]{HP} Perv$$

$$P_{U,f} \in Perv$$

MF: $U_0 := f^{-1}(0)$, $M \in Coh(U_0)$, infinite resolution by projective modules becomes eventually 2-periodic [Serre-Auslander-Buchsbaum-Eisenbud]

$$\underbrace{\ldots \to F \to Q \to F \to Q}_{\in MF(U,f)} \to \underbrace{P_n \to \ldots \to P_2 \to P_1 \to P_0}_{\in Perf(U_0)} \to M$$

Motivic DT and categorification

Gluing Problem: Given a (-1)-symplectic derived scheme X, can we glue the Darboux locally defined dg-categories $MF(U, f)$ as a sheaf of dg-categories on X? Is Joyce's orientation data enough?

Rmk: Version of the gluing problem for the Fukaya category (Seidel, Kontsevich, Nadler, Shende, Ganatra, Pardon,...).

Complications: The gluing no longer takes place in a 1-category but in an ∞-category. Complicated coherences are required. Need a gluing mechanism.

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The Darboux Stack

Classical Picture: X a classical symplectic manifold, then locally X is of symplectomorphic to some T^*M (Darboux's lemma). We can analyse the moduli of such Darboux parametrizations:

$$Darb_X : S \subseteq X \text{ open} \mapsto \{M \text{ smooth manifold}, S \simeq T^*M \text{ symplectic}\}$$

The data of a symplectomorphism $S \simeq T^*M$ in particular implies:

- The fibers of the projection $S \simeq T^*M \to M$ define a smooth Lagrangian foliation $\mathcal{F}$ on S (ie, $\omega_{|_{\text{fibers}}} = 0$).
- The symplectic form on S is exact ie, there exists a 1-form α (Liouville form on T^*M) with $d_R(\alpha) = \omega$.

We call such $(\mathcal{F}, \alpha)$ a Darboux datum on S.

The Darboux Stack

(-1)-shifted geometry: These notions make sense thanks to the work of Toën-Vezzosi on derived foliations.

Theorem (Pantev-Toën)

S a (-1)-symplectic derived scheme. Then the following data are equivalent:

- *Darboux data on S, ie a globally defined smooth derived Lagrangian foliation $\mathcal{F}$ on S + an exact structure α.*
- *the data of an exact structure on S + a smooth formal scheme $\mathcal{U}$, and a Lagrangian fibration $S \to \mathcal{U}$ given by a closed immersion with $S_{red} = \mathcal{U}_{red}$.*
- *the data of a smooth formal scheme $\mathcal{U}$, a function f on $\mathcal{U}$ and a symplectomorphism $S \simeq dCrit(\mathcal{U}, f)$*

Classical Picture: Darboux data on $S \Leftrightarrow [S \subseteq T^*M \to M]$.

(-1)- **picture :** Darboux data on $S \Leftrightarrow [S \simeq dCrit(\mathcal{U}, f) \hookrightarrow \mathcal{U}]$.

Idea: $\mathcal{U} := S/\mathcal{F}$ the formal leaf space. f = exact struct. - isotropic struct.

The Darboux Stack

Example: $(\widehat{\mathbb{A}^1}, x^3)$ gives Darboux data

$$dCrit(x^3) = Spec(k[x]/(3x^2)) \hookrightarrow \widehat{\mathbb{A}^1}$$

Construction

The assignment:

$$S \to X \text{ étale} \mapsto \{(\alpha, \mathfrak{F}) : \text{Exact structure } \alpha + \text{smooth Lag. fol. } \mathfrak{F} \text{ on } S\}$$

defines a stack on the small étale site of a n-shifted symplectic derived scheme X. We call it the Darboux stack $Darb_X$.

Remark: $Darb_X := Exact_X \times LagFol_X^{sm}$

Comment: In the case where X is (-2)-symplectic, this recovers the local data used by Borisov-Joyce and Oh-Thomas to glue DT-invariants for Calabi-Yau 4-folds.

The Darboux Stack

Construction

Both MF and Joyce's construction have $Darb_X$ *as a natural domain, and define natural transformations of sheaves on the small étale site of* X*:*

$$P : Darb_X(S) \ni (\mathcal{U}, f) \to P_{\mathcal{U},f} \in Perv_X(S) := Perv(S)^{\simeq}$$

$$MF : Darb_X(S) \ni (\mathcal{U}, f) \to MF(\mathcal{U}, f) \in dgcat^{2per}_{X_{dR}}(S) := \underbrace{(dgcat^{2per}_{S_{dR}})^{\simeq}}_{categorical\ \ crystals}$$

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Action of Quadratic Bundles

Ambiguity Problem: in the choice of local presentations:

$$dCrit(\mathbb{A}^1, x^3) = \mathrm{Spec}\, k[x]/(3x^2) \simeq \mathrm{Spec}\, k[x,y]/(3x^2, 2y) = dCrit(\mathbb{A}^2, x^3 + y^2)$$

Definition

$Quad_{dR}(S) := \{(Q, q) :$ (loc. trivial) quadratic vector bundles on $S_{dR}\}$

Construction

X a (-1)-symplectic derived scheme. Then:

- *The assignment S/X étale $\mapsto Quad_{dR}(S)$ defines a sheaf of monoids $Quad_{X_{dR}}$ on X_{et} for the sum of quadratic bundles,*
- *$Quad_{X_{dR}}(S)$ acts on $Darb_X(S)$,*

$$dCrit(\mathcal{U}, f) \simeq S \simeq dCrit(\mathcal{U} \underset{S_{dR}}{\times} Q, f + q)$$

Rmk: Every morphism of Darboux foliations on X, $(\mathcal{U}, f) \to (\mathcal{V}, g)$ is étale locally of the form $(\mathcal{U}, f) \to (\mathcal{U} \underset{S_{dR}}{\times} Q, f + q)$

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Recovering the perverse gluing of BBDJS:

Fact: $(M, q) \in Quad_{X_{dR}}(S)$ then $det(M)$ is a 2-torsion line bundle over S, ie, $det(M)^2 \simeq \mathcal{O}_S$. This follows from the non-degeneracy of the Hessian.

Construction

X a (-1)-symplectic derived scheme. Then:

- *$det : Quad_{X_{dR}} \to B\mu_{2,X} = fib(B\mathbb{G}_{m,X} \underset{2}{\to} B\mathbb{G}_{m,X})$ is a map of monoids.*
- *$P : Darb_X \to Perv_X$ comes with **homotopy coherent data** rendering the actions compatible on both sides (on the right the action of $B\mu_2$ is defined by BBDJS).*
- *In particular, P descends to the quotients*

$$\overline{P} : Darb_X / Quad_{X_{dR}} \to Perv_X / B\mu_{2,X}$$

Recovering the perverse gluing of BBDJS:

Warning: $Quad_{X_dR}$ is not a group but $B\mu_{2,X}$ is. The construction factors through the group completion

$$Quad^{+}_{X_dR}, \qquad Darb_X/Quad_{X_{dR}} := Darb^{+}_X/Quad^{+}_{X_{dR}}$$

Theorem (Hennion-Holstein-R. as a reformulation of BBDJS)

X a (-1)-symplectic derived scheme.Then the quotient map

$$\overline{P} : Darb_X/Quad_{X_{dR}} \to Perv_X/B\mu_{2,X}$$

is ***null-homotopic****, ie, it admits a factorization through X (seen as a the final object of the étale topos X):*

$$\begin{array}{ccc} Darb_X/Quad_{X_{dR}} & \xrightarrow{\overline{P}} & Perv_X/B\mu_{2,X} \\ \big\downarrow{\scriptstyle final} & \nearrow & \\ X & & \end{array}$$

In other words, the gluing of the perverse sheaves $P_{U,f}$ is always well-defined in the quotient $Perv_X/B\mu_{2,X}$.

Recovering the perverse gluing of BBDJS:

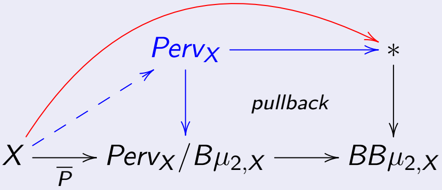
Remark

The composition

$$X \to Perv_X/B\mu_{2,X} \to */B\mu_{2,X} = BB\mu_{2,X}$$

is the class in $H^2(X, \mathbb{Z}/2\mathbb{Z})$ of the bundle classifying square roots of $det(\mathbb{T}_X)$.

An orientation data of BBDJS corresponds precisely to the choice of a ***null-homotopy*** *of this composition*



Such a null-homotopy provides a lifting through the fiber product and defines a well-defined glued perverse sheaf.

What about gluing MF:

Fact: $(M,q) \in Quad_{X_{dR}}(S)$ then $MF(M,q)$ has a structure of 2-torsion 2-periodic derived Azumaya algebra over S_{dR}. This is a consequence of Preygel-Thom-Sebastiani followed by Knörrer periodicity

$$MF(M,q) \otimes MF(M,q) \simeq MF(M \times M, q \boxplus -q) \simeq MF(S_{dR}, 0)$$

Construction

X a (-1)-symplectic derived scheme. Then:

- $MF : Quad_{X_{dR}} \to Az_{X_{dR}}^{2per, 2-tor}$ *is a map of monoids.*
- $MF : Darb_X \to dgcat_{X_{dR}}^{2per}$ *comes with* ***homotopy coherent data*** *rendering the actions compatible on both sides (on the right the action of* $Az_{X_{dR}}^{2per, 2-tor}$ *is given by tensor products of dg-categories).*
- *In particular, MF descends to the quotients*

$$\overline{MF} : Darb_X / Quad_{X_{dR}} \to dgcat_{X_{dR}}^{2per} / Az_{X_{dR}}^{2per, 2-tor}$$

What about gluing MF:

Dishonest Corollary

Let X be a (-1)-shifted symplectic derived scheme. Assume X is equipped with a section

$$X \to Darb_X / Quad_{X_{dR}}$$

Then the locally defined categories $MF(\mathcal{U}, f)$ glue as a sheaf of 2-periodic dg-categories on X under the prescription of a categorical orientation data, ie, a trivialization of the composition

$$X \to Darb_X / Quad_{X_{dR}} \to dgcat^{2per}_{X_{dR}} / Az^{2per, 2-tor}_{X_{dR}} \to BAz^{2per, 2-tor}_{X_{dR}}$$

New Orientation data

The orientation data of BBDJS is (a priori) not enough to glue MF. A categorical orientation provides new obstruction classes coming from the fibration sequence

$$Az_{X_{dR}}^{2per, 2-tor} \to Az_{X_{dR}}^{2per} \underset{2}{\to} Az_{X_{dR}}^{2per}$$

- $\pi_0(Az_{X_{dR}}^{2per,2-tor}) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \to \{MF(*,0), MF(\mathbb{A}^1, x^2)\}\}.$
- $\pi_1(Az_{X_{dR}}^{2per,2-tor}) \simeq \underbrace{\mathbb{Z}/2\mathbb{Z}}_{BBDJS} \simeq \{Id, [1]\}$
- $\pi_2(Az_{X_{dR}}^{2per,2-tor}) = \mathbb{Z}/2\mathbb{Z} \simeq \mathrm{Ker}(z^2 : \mathbb{C}^* \to \mathbb{C}^*)$
- $\pi_n(Az_{X_{dR}}^{2per,2-tor}) = 0 \ \ n \geq 3,$

What about gluing MF:

Honest gluing of MF: requires a factorization

$$\begin{array}{ccc} Darb_X/Quad_{X_{dR}} & \xrightarrow{\overline{MF}} & dgcat^{2per}_{X_{dR}}/Az^{2per,2-tor}_{X_{dR}} \\ \downarrow{\scriptstyle \text{final}} & \nearrow & \\ X & & \end{array}$$

Intermediate Results (Obstructions)

- $\pi_0^{sheaf}(Darb_X/Quad_{X_{dR}}) = Exact_X$ *(transitivity of the action).*
- $\pi_1^{sheaf}(Darb_X/Quad_{X_{dR}}) \neq 0$ *in general*
- *eventually truncated because X is quasi-smooth*
- *The map $\overline{MF}$ is zero on all higher π_n^{sheaf} for $n \geq 2$.*

Gluing

Work in progress

$$\frac{Darb_X}{(\mathbb{A}^1 - \ invariance \ on \ morphisms \ + \ Quad_{X_{dR}} - action)} \simeq X$$

Question: What happens through Mirror symmetry? Relation to the works of Nadler, Shende, Ganatra, Pardon gluing Fukaya categories of 0-shifted Weinstein manifolds via Kashiwara-Schapira microlocal sheaves?

Thank you for your time.