Erratum to
PhD thesis "Théorie Homotopique Motivique des espaces non-commutatifs"
&
Article published Advances in Mathematics, "K-Theory and the bridge from Motives to non-commutative Motives"

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The author thanks M. Hoyois for pointing out the error in [3, Lemma 6.4.20]/[4, 3.25] and for several helpful remarks and suggestions.

Description of the Error:

The proof of [3, Lemma 6.4.20]/[4, 3.25] is wrong. As a consequence the conclusion of [3, 6.4.1.9]/[4, 3.24] that $\mathcal{H}_{nc}$ is stable, is unclear. Although we have the semi-orthogonal decomposition of $\mathrm{Perf}(\mathbb{P}^1)$, it does not follow that the fiber of ∞^* is $Perf(k)$ before we invert the topological circle. In this case, **we are forced to change the definition of** SH_{nc} **all over** [3]/[4] **as the** S^1**-stabilization of** $\mathcal{H}_{nc}$. After forcing this stabilization $\mathrm{Perf}(\mathbb{P}^1)$ becomes a direct sum of two copies of $Perf(k)$ and it is clear that ∞^* splits this sum and that the non-commutative motive of $(\mathbb{P}^1, \infty)$ is $Perf(k)$.

Consequences of the Error:

The error of [3, Lemma 6.4.20]/[4, 3.25] affects exclusively the proof of [3, Thm 7.0.32]/[4, Thm 4.7] in [3, Section 7.4]/[4, Section 4.5].

Correction: Proof of [3, Thm 7.0.32]/[4, Thm 4.7] **avoiding** [3, Lemma 6.4.20]/[4, 3.25]**. This replaces** [3, Section 7.4]/[4, Section 4.5]**.**

Recall $\mathrm{Fun}_{Nis}(\mathcal{D}g(k)^{ft}, \mathcal{S})$ has a zero object by construction. We show that:

Proposition 0.1. *There is a canonical equivalence in* $\mathrm{Fun}_{Nis}(\mathcal{D}g(k)^{ft}, S)$

$$\Sigma_{Nis} j(\mathrm{Perf}(k)) := l_{Nis}\Sigma^{naive} j(\mathrm{Perf}(k)) \simeq l_{Nis}(colim_{n\in\Delta^{op}} Seq_n)$$

where Seq is the simplicial object of [3, Rmk 7.4.3]/[4, Rmk 4.44] *giving the Waldhausen's S-construction of* [3, Construction 7.1.2]/[4, Construction 4.14]. *Moreover,* $\Sigma^N_{Nis}(Perf(k))$ *identifies with the Nisnevich localization of the N-iterated Waldhausen's construction.*

Proof. For each $n \geq 0$ we have short exact sequence in $\mathcal{D}g(k)^{idem}$ which are also pullbacks, given as

$$\begin{array}{ccc} \widehat{[n-1]}_k & \xrightarrow{V_n} & \widehat{[n]}_k \\ \downarrow & & \downarrow U_n \\ 0 & \longrightarrow & \widehat{[0]}_k \end{array} \qquad \begin{array}{l} V_n : \underline{0} \mapsto \underline{1}/\underline{0}, \quad \underline{1} \mapsto \underline{2}/\underline{0}, ..., \underline{n-1} \mapsto \underline{n}/\underline{0} \\ \\ U_n : \underline{0} \mapsto \underline{0}, \underline{1} \mapsto \underline{0}, ..., \underline{n} \mapsto \underline{0} \end{array} \tag{0.1}$$

The maps in the sequence admit retracts W_n and S_n with $W_n \circ V_n = Id$ and $U_n \circ S_n = Id$, given by $S_n : \underline{0} \mapsto \underline{0}$ and $W_n : \underline{0} \mapsto 0, \underline{1} \mapsto \underline{0}, ..., \underline{n} \mapsto \underline{n-1}$. Here $\underline{i}$ denotes the object of $[n-1]$. This notation is to make the distinction between $\underline{0}$ and the zero object 0. To check that (0.1) are exact one checks directly using Yoneda that V_n is fully faithful with essential image given by the full thick subcategory $< \underline{1}/\underline{0}, ..., \underline{n}/\underline{0} > \subseteq \widehat{[n]}_k$ spanned by the quotients $\underline{1}/\underline{0}, ..., \underline{n}/\underline{0}$. Moreover, one checks that U_n has the universal property of the Verdier quotient $\widehat{[n]}_k / < \underline{1}/\underline{0}, ..., \underline{n}/\underline{0} >$. Indeed, to give an dg-functor $\widehat{[n]}_k \to T$ such that all the objects $\underline{1}/\underline{0}, ..., \underline{n}/\underline{0}$ go to zero, as the functor is exact and the categories are stable, it follows that the maps $t_0 \to t_1, ..., t_0 \to t_n$ are equivalences in T. In this case the map is determined by the image of $t_0 = t_1 = ... = t_n$. The claim that the diagrams (0.1) are also pullbacks follows from [3, Corollary 2.1.12]/[4, Cor 1.18] because each $\widehat{[n-1]}_k$ has a compact generator $(\underline{0} \oplus ... \oplus \underline{n-1})$. By definition, we obtain Nisnevich diagrams of non-commutative spaces

$$\begin{array}{ccc} Perf(k) = \widehat{[0]}_k & \longrightarrow & 0 \\ {\scriptstyle U_n}\downarrow & & \downarrow \\ \widehat{[n]}_k & \xrightarrow[V_n]{} & \widehat{[n-1]}_k \end{array} \tag{0.2}$$

in $\mathcal{N}c\mathcal{S}(k)$. Now, as in [2, §3.3] and [5, §1.5] we remark that the exact sequences (0.1) are, as n varies, part of a co-simplicial diagrams in $\mathcal{D}g(k)^{idem}$ and therefore, simplicial diagram in $\mathcal{N}c\mathcal{S}(k)$

$$\begin{array}{ccc} \underbrace{Perf(k)}_{\text{constant dia.}} & \longrightarrow & 0 \\ {\scriptstyle U_n}\downarrow & & \downarrow \\ \underbrace{\widehat{[n]}_k}_{PSeq_\bullet} & \xrightarrow[V_n]{} & Seq_\bullet \end{array} \tag{0.3}$$

Finally, composition with Yoneda $j : \mathcal{N}c\mathcal{S}(k) \hookrightarrow \mathrm{Fun}(\mathcal{D}g(k)^{ft}, \mathcal{S})$ we obtain a sequence of simplicial objects

$$j(Perf(k)) \to j(PSeq_\bullet) \to Seq_\bullet$$

where $Seq_\bullet$ is the simplicial object giving Waldhausen's S-construction explained in [3, Rmk 7.4.3]/[4, Rmk 4.44] giving the Waldhausen's S-construction of [3, Const. 7.1.2]/[4, Const. 4.14]. As explained in [5, Lemma 1.5.1] the simplicial object $PSeq_\bullet$ is simplicially homotopy equivalent to the constant associated to the zero dg-category and therefore, it is contractible. In this case we obtain a sequence,

$$j(Perf(k)) \to |PSeq_\bullet| := colim_{\Delta^{op}} j(PSeq_\bullet) \simeq * \to |Seq_\bullet| := colim_{\Delta^{op}} Seq_\bullet$$

which, because of the Nisnevich localization (i) forces each level of (0.2) to become pushouts, (ii) forces the zero dg-category to remain a zero object and (iii) preserves sifted colimits, induces a pushout square in Nisnevich sheaves of spaces

$$\begin{array}{ccc} j(Perf(k)) & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & l_{Nis}(|Seq_\bullet|) \end{array} \quad (0.4)$$

which exhibits $l_{Nis}(|Seq_\bullet|)$ as the (Nisnevich local) suspension of $Perf(k)$. This concludes the proof of the first claim. For the iterated S-construction, notice that

$$\Sigma_{Nis}\Sigma_{Nis}(Perf(k)) \simeq \Sigma_{Nis} colim_{\Delta^{op}} Seq_n = \Sigma_{Nis} colim_{\Delta^{op}} j([n-1]_k) \simeq l_{Nis} colim_{\Delta^{op}} \Sigma_{Nis} j([n-1]_k)$$

and that using the fact that the category of Nisnevich sheaves of spaces has a zero object (by convention) and the monoidal structure is compatible with colimits in each variable, we have $\Sigma_{Nis} j([n-1]_k) \simeq (\Sigma_{Nis} j([n-1]_k)) \otimes j(Perf(k)) \simeq j([n-1]) \otimes \Sigma_{Nis} Perf(k)$, and in the continuation of the chain of equivalences above, we obtain

$$\simeq l_{Nis} colim_{(n,m)\in \Delta^{op}\otimes\Delta^{op}} j([n-1]) \otimes j([m-1]) = l_{Nis}|Seq_{\bullet,\bullet}|$$

This is exactly the formula for the 2-iterated S-construction of Waldhausen. We conclude by induction. □

Corollary 0.2. *(*[3, Thm 7.0.32]/[4, Thm 4.7]*) We have an equivalence in $Fun_{Nis}(\mathcal{D}g(k)^{ft}, \mathrm{Sp})$*

$$\Sigma^\infty_{Nis} j(\mathrm{Perf}(k)) = K^S.$$

In particular, after forcing $\mathbb{A}^1$-invariance we have an equivalence between the unit non-commutative motive and $l_{\mathbb{A}^1}(K^S) = KH$ in $\mathcal{SH}_{nc}$

Proof. By definition [5] the presheaf of spectra given connective K-theory spectrum K^C_{Spec} is represented by the sequence of presheaves of spaces given by the iterated S-construction, $(j(Perf(k)), |Seq_\bullet|, |Seq_{\bullet,\bullet}|, ...)$ which thanks to Waldhausen's additivity theorem is an Ω-spectrum for $n \geq 1$. As a presheaf of spectra it is equivalent to the filtered colimit in $\mathrm{Fun}(\mathcal{D}g(k)^{ft}, \mathrm{Sp})$ given by $colim_{n\geq 0}\Omega^n\Sigma^\infty|Seq_{\underbrace{\bullet, ...\bullet}_{n}}|$. Now we compute $l_{Nis}(K^C_{Spec})$. Because of [3, Thm. 7.0.29]/[4, Thm. 4.4] we know that $l_{Nis}(K^C_{Spec}) = K^S$. But the formula above combined with Prop. 0.1 gives also (because l_{Nis} for presheaves of spectra is exact and commutes with colimits)

$$l_{Nis}(K^C_{Spec}) = l_{Nis} colim_{n\geq 0}\Omega^n\Sigma^\infty|Seq_{\underbrace{\bullet, ...\bullet}_{n}}| \simeq colim_{n\geq 0}\Omega^n\Sigma^\infty_{Nis}\Sigma^n_{Nis} j(Perf(k)) \simeq$$

$$\simeq colim_n \Omega^n \Sigma^n_{Nis}\Sigma^\infty_{Nis} j(Perf(k)) \simeq \Sigma^\infty_{Nis} j(\mathrm{Perf}(k)$$

We obtain [3, Thm 7.0.32]/[4, Theorem 4.7] by forcing $\mathbb{A}^1$-invariance. □

References

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