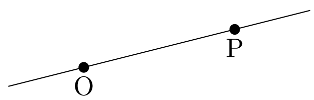
Simple example in derived geometry: counting lines in the plane in the non-generic situation

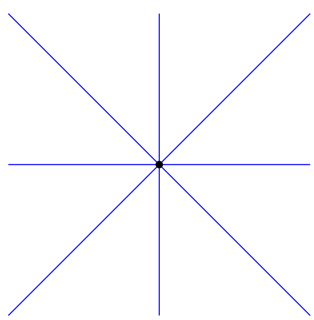
We describe another simple example illustrating how derived methods can solve counting problems, namely, we look at the problem of counting lines in the plane.

Given two points **P** and **O** in the plane, there exists a unique line ℓ passing through them:



Question 0.1. What happens when **P** gets closer and collides with **O** ?

Solution 0.2. In this case we obtain infinitely many lines, parametrized by the projective space $\mathbb{P}^1_{\mathbb{C}}$



Question 0.3. Why is it that when $\mathbf{P} = \mathbf{O}$ we jumped from 1 to infinitely many? How to recover the counting for *generic situation*?

Solution 0.4. To understand this we look at the *moduli space of lines*. More precisely, we look at the moduli of lines in the *projective plane* $\mathbb{P}^2_{\mathbb{C}}$ - $\mathbf{M}(\mathbb{P}^2_{\mathbb{C}})$. By fixing coordinates $[x : y : z]$ in $\mathbb{P}^2_{\mathbb{C}}$, a line is given by an equation of the form

$$ax + by + cz = 0$$

and is determined by the coefficients a, b and c.

Therefore, the moduli space of all lines is in bijection with the complex plane itself, via the identification

$$\begin{array}{rcl} \mathbf{M}(\mathbb{P}^2_{\mathbb{C}}) := \text{Moduli space of lines in} \mathbb{P}^2_{\mathbb{C}} & \simeq & \mathbb{P}^2_{\mathbb{C}} \\ \ell := \{[x : y : z] : ax + by + cz = 0\} & \leftarrow & [a : b : c] \end{array}$$

If we now fix the origin $\mathbf{O} = [0 : 0 : 1]$ we can look at the subspace

$$\mathbf{M_O} \subseteq \mathbf{M}(\mathbb{P}^2_{\mathbb{C}})$$

of all lines passing by $\mathbf{O}$. Under the above identification $\mathbf{M}(\mathbb{P}^2_{\mathbb{C}}) \simeq \mathbb{P}^2_{\mathbb{C}}$, this subspace is mapped to a copy of the line at infinity $\mathbb{P}^1_{\mathbb{C}} \subseteq \mathbb{P}^2_{\mathbb{C}}$

$$\begin{array}{rcl} \mathbf{M}(\mathbb{P}^2_{\mathbb{C}}) := \text{Moduli space of lines in } \mathbb{P}^2_{\mathbb{C}} & \simeq & \mathbb{P}^2_{\mathbb{C}} \\ \ell := \{[x : y : z] : ax + by + cz = 0\} & \leftarrow & [a : b : c] \\ \cup\!\shortmid & & \cup\!\shortmid \\ \mathbf{M}_O(\mathbb{P}^2_{\mathbb{C}}) := \text{Lines passing by } O = [0 : 0 : 1] & \simeq & \{c = 0\} = \mathbb{P}^1_{\mathbb{C}} \end{array}$$

More generally, for a given point $\mathbf{P}$ we can consider the subspace of lines passing by $\mathbf{P}$

$$\mathbf{M_O} \subseteq \mathbf{M}(\mathbb{P}^2_{\mathbb{C}})$$

Since we are interested in lines passing simultaneously through $\mathbf{P}$ and $\mathbf{O}$, we will look a the intersection $\mathbf{M}_O \cap \mathbf{M}_P$. Now:

$$If\ \mathbf{P} \neq \mathbf{O},\ \ \mathbf{M}_O \cap \mathbf{M}_P = * \ \text{(The unique line)}$$

$$If\ \mathbf{P} = \mathbf{O},\ \ \mathbf{M}_O \cap \mathbf{M}_O = \mathbb{P}^1_{\mathbb{C}} \cap_{\mathbb{P}^2_{\mathbb{C}}} \mathbb{P}^1_{\mathbb{C}} = \mathbb{P}^1_{\mathbb{C}} \ \text{(infinitely many lines)}$$

We now apply what we learned from the Apollonius-Toën example: in order to get the correct counting one must replace the naive intersection by the *derived intersection*. In this case, this means that when $\mathbf{P}=\mathbf{O}$ we should instead be taking the self-intersection of the line at infinity in $\mathbb{P}^2_{\mathbb{C}}$:

$$\mathbf{M}_O \cap^{\mathbf{d}} \mathbf{M}_O \simeq \mathbb{P}^1_{\mathbb{C}} \cap^{\mathbf{d}}_{\mathbb{P}^2_{\mathbb{C}}} \mathbb{P}^1_{\mathbb{C}}$$

A computation (see Example 1.2.35 for the details), shows that this is the derived scheme

$$= \underbrace{\mathsf{Spec}_{\mathbb{P}^1_{\mathbb{C}}}\left(\mathcal{O}^{\mathsf{der}}_{\mathbb{P}^1_{\mathbb{C}}} := [\ \overset{\text{deg }1}{\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(-1)} \xrightarrow{0} \overset{\text{deg }0}{\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}}\ ]\right)}_{\mathbb{P}^1_{\mathbb{C}} \text{ with derived structure}}$$

As in the Apollonius-Toën example, we obtain the correct counting by taking the chern character:

$$\mathsf{Ch}(\mathcal{O}^{\mathsf{der}}_{\mathbb{P}^1_{\mathbb{C}}}) \cap [\mathbb{P}^1_{\mathbb{C}}] = \mathbf{1}.[pt] \in \mathsf{H}_0(\mathbb{P}^1_{\mathbb{C}})$$

In many natural examples (Gromov-Witten theory, Donaldson-Thomas theory, etc) we are interested in counting problems that come to us in the most degenerate situation, precisely as in the example $\mathbf{P}=\mathbf{O}$. This example shows why looking for derived structures might be a good idea.