# Motives of singularity categories

SUMMER school IHES

(Thank you for the invitation to give this lecture)

# Motives of singularity categories

- Please stop me if you have questions
- Results obtained with Blanc-Toën-Vezzosi
- survey of more recent results (PhD thesis Massimo Pippi; Bloch conductor formula (Toën-Vezzosi))

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1) ~~Singularity~~ Review singularity categories (and Matrix Factorizations)
2) Motives of singularity categories
3) Relation with vanishing cycles
4) survey of recent results

# ① Review of singularity categories and Matrix Factorizations

# 1) Review: singularity categories

## 1.1 Sing & MF

Narrative of this talk starts with a Theorem of

Serre

$$\boxed{X \text{ scheme is } \underline{\text{regular}} \iff \underbrace{\mathrm{Perf}(X) = D^b_{coh}(X)}_{\text{derived categories}}}$$

Rmk $\subseteq$ always* (need $\mathcal{O}_X \in coh^b(X)$, automatic in this talk)

$\supseteq$ regularity

Key lemma: $X$ regular $\iff$ Any $M \in coh(M)$ admits a finite resolution by vectr bndls

$$\underbrace{0 \to P_n \to \cdots P_2 \to P_1 \to P_0}_{\in \mathrm{Perf}(X)} \to M$$

when $X$ has singularities?

$\hookrightarrow$ $\mathrm{Perf}(X) \subseteq D^b\mathrm{coh}(X)$ ok

$\hookrightarrow$ what is the excess?

(Orlov)

Definition: $\text{Singularity category} := D^b\mathrm{coh}(X) \,/\, \mathrm{Perf}\, X$ (Verdier quotient)

(Eisenbud): suppose

$X$ = zero locus of a function (hypersurface)

$$\begin{array}{c} X \\ \cap\!\mid \\ \underbrace{U}_{\text{smooth}} \xrightarrow{\ f\ } \mathbb{A}^1 \end{array}$$

Then one can control the excess: $\forall M \in \mathrm{coh}(X) \,\exists$

$$\text{etc} \to P_{n+4} \to P_{n+3} \to P_{n+2} \to P_{n+1} \to \underbrace{P_n \to \cdots P_2 \to P_1 \to P_0}_{\in\, \mathrm{Perf}(X)} \to M$$

infinite length but 2-periodic

formula (Auslander–Buchsbaum)

**Definition** (Eisenbud)

vectr bundls on $U$

$$\underbrace{MF(U,f)}_{\text{category of matrix factorizations}} = \left\{ Q \underset{\delta_1}{\overset{\delta_0}{\rightleftarrows}} P \;\middle|\; \begin{matrix} \delta_0\delta_1 = \text{multp. by } f \\ \| \\ \delta_1\delta_0 \end{matrix} \right\}$$

**Theorem** (Orlov) exact sequence

$$\mathrm{Perf}(X) \hookrightarrow D^b\mathrm{Coh}(X) \longrightarrow \boxed{\begin{matrix} MF(U,f) \\ \text{is} \\ \mathrm{Sing}(X). \end{matrix}}$$

2 periodic

$$\underbrace{[P_n \to \cdots P_2 \to P_1 \to P_0]}_{\in \mathrm{Perf}(X)} \longleftarrow\!\shortmid M \longmapsto \underbrace{\left[ Q \underset{\delta_1}{\overset{\delta_0}{\rightleftarrows}} P \right]}_{\text{2-periodic ``infinite length''}}$$

In particular, by Serre

$$MF(U,f) = 0 \iff X \text{ smooth}$$

$\Downarrow$ controls the singularities

Example : (Knörrer periodicity)

$$MF(\mathbb{A}^2, x^2+y^2) = MF(\mathrm{Spec}(\mathbb{C}), 0) \overset{\text{def}}{=} \{\text{2-periodic complexes}\}$$

$\mathbb{C}$

2per. Morita equivalence

$$MF(\mathbb{A}^2, x^2+y^2) = MF(\mathbb{A}^1, x^2) \underset{\text{2-periodic}}{\otimes} MF(\mathbb{A}^1, y^2)$$

(Thom-Sebastiani Preygel)

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So, where is this going?

$\mathrm{Spec} R \hookleftarrow \mathrm{Spec}(R/f)$

Dyckerhoff/Preygel

regular local ring

$$HH_*(MF(\mathrm{Spec}(R), f)) \simeq \frac{R}{(\partial_1 f, \dots, \partial_n f)}$$

Jacobian ring

concentrated in degree $d = \dim R$

explicit computation by compact generator of MF

Thm

Efimov, Dyckerhoff

$$\underbrace{HP}_{\text{periodic cyclic homology}}\left(MF(U,f)\right) = H^{\bullet}_{\mathbb{Z}/2}\Big(X, \big(\overbrace{\Omega^{\bullet}_{U}, d + df \wedge -}^{\text{twisted de Rham}}\big)\Big)^{\text{2 per}}$$

periodic cyclic homology

(Connes, Tsygan, Keller)

$\wr\!\wr$ Sabbah, Kontsevich

(cohomology of vanishing cycles)$^{2}$ periodized of $f$ + monodromy action

Main result of this talk ($1^{st}$ formulation)

Motive of $MF(U,f)$ $\longleftrightarrow$ (Motivic vanishing cycles) Galois-inv

Ayoub's thesis

Lift the previous results to motives

## (2) Motives of singularity categories

# 2) Motives of singularity categories

## 2.1 Digression: NC-Motives (Kontsevich)

↳ previous talks: cohomological approach (Tabuada; Cisinski-Tabuada)

This Talk: "homological approach" (R.)

⇝ closer to Morel-Voevodsky Motivic $\mathbb{A}^1$-Hmt theory

idea

$$\begin{tikzcd}
\mathrm{Sm}_S^{\times} \arrow[r, "\Sigma^\infty", "\text{Morel Voevodsky}"'] \arrow[d, "\mathrm{Perf}"'] & \mathrm{SH}_S^{\otimes} \arrow[d, dashed] \\
\mathrm{dgcat}_S^{op} \arrow[r, "\text{NC-version of Morel-Voevodsky}"'] & \mathrm{SH}^{nc,\otimes}
\end{tikzcd}$$

} $\infty$-category

NC-spaces $:= \mathrm{dgcat}_S^{op}$

(No dualization needed!)

$\mathrm{Sm}^{op} \to \mathrm{Ab}$

$\mathrm{Sm} \to \mathrm{SH}_S$

$\mathrm{Sm}$

# Universal property of Morel-Voevodsky

**Thm** (R) $\underline{\underline{\Sigma}}^{\infty} : Sm_S^{\times} \longrightarrow SH_S^{\otimes}$ is universal among those symmetric monoidal functors

$$Sm_S^{\times} \xrightarrow{F} D^{\otimes}$$

symmetric monoidal $\infty$-cat.

such that

- $D \in$ stable, presentable
- $F$ ( Nisnevich cover ) = pushout square
- $F(X \times \mathbb{A}^1) \simeq F(X)$ ( $\mathbb{A}^1$-invariance )
- $\underbrace{F(\mathbb{P}^1, \infty) := \text{cofib}\big( F(\infty) \to F(\mathbb{P}^1) \big)}$ $\underbrace{\text{is } \otimes\text{-invertible}}_{\text{(Poincaré duality)}}$

$$\begin{tikzcd} Sm^{\times} \arrow[r, "F"] \arrow[d, "\Sigma^{\infty}"'] & D^{\otimes} \\ SH^{\otimes} \arrow[ur, dashed, "\otimes"'] & \end{tikzcd}$$

## construction :

h. finite cellular object

- $\mathrm{NCspaces} = \mathrm{dgcat}^{op}_{\text{finite type}}$

- Nisnevich square of $\text{dg-cat}_{\text{ftype}}$

$$\begin{tikzcd} K' \arrow[r, hook] \arrow[d, "\text{IS}"'] & T' \arrow[r] \arrow[d] & U' \arrow[d] \\ K \arrow[r, hook] & T \arrow[r] & U \end{tikzcd}$$

with compact generators

localizations = "open immersion"

Perf(Nisnevich) is Nisnevich in this sense.

ex: Perf(Nisnevich square)

- $\mathbb{A}^1_{nc} = \mathrm{Perf}(\mathbb{A}^1)$ $\quad F(T \otimes \mathrm{Perf}(\mathbb{A}^1)) \simeq F(T)$

semi-orthogonal $= \mathrm{Perf}(k) \oplus \mathrm{Perf}(k)$

- $\mathbb{P}^1_{nc} = \mathrm{Perf}(\mathbb{P}^1)$ $\quad \mathrm{Perf}(\mathbb{P}^1, \infty)$ $\otimes$-invertible

$$\mathrm{NC\text{-}Spcs} \xrightarrow[\Sigma^\infty_{nc}]{\text{Nisnevich}_{nc} + \mathbb{A}^1_{nc} + \text{Poincaré}_{nc}} SH^{\otimes}_{nc}$$

formal construction

same universal property

<u>Rmk</u>: $SH_{nc}^{\otimes}$ defined this way is $\underline{Pr^L_R\text{-}\otimes\text{-dual}}$ to the version of Tubuada's talk

$$\Downarrow \qquad SH^{\otimes}_{nc} = \mathrm{Fun}^L(M_{Tab}, S)$$

upshot:

lax monoidal $\infty$-functors

$$\begin{tikzcd} Sm_S \arrow[r, "\Sigma^\infty"] \arrow[d, "Perf"] & SH^{\otimes}_S \arrow[d, "L^{\otimes}"', "\text{universal property}"] \\ Nc\text{-}spaces_S \arrow[r, "\Sigma^\infty_{nc}"] & SH^{\otimes}_{nc,S} \arrow[u, bend right, dashed, "M^{\otimes\, lax}"'] \end{tikzcd}$$

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<u>Thm</u> (Tabuada; R.) $\quad \mathrm{Hom}_{SH_{nc}}(\Sigma^\infty_{nc} T, 1_{nc}) = \underbrace{KH^{nc}(T)}$ schlichting | Waldhausen

<u>in particular</u>:

$$\begin{tikzcd} SH \arrow[d, shift right, "L"'] & KH \\ SH_{nc} \ni 1_{nc} \arrow[u, shift right, "M"'] \arrow[ur, mapsto] & \end{tikzcd}$$

$KH$ = Motivic ring spectrum representing Algebraic K-Theory.

and $M$ lands in <u>KH-Modules</u>

$$SH_{nc} \xrightarrow{M} \mathrm{Mod}_{KH}(SH) \xrightarrow{\text{forget}} SH$$

20 mins

## 2.2 Motives of dg-categories

**Construction:**

$$T \in \text{dg-cat} \equiv \mathrm{NCSpc}^{op} \xrightarrow[\Sigma^\infty_{nc}]{} SH_{nc}^{op} \xrightarrow{\underline{\mathrm{Hom}}(-,\mathbb{1}_{nc})} sH_{nc} \xrightarrow{M} SH$$

$$T \longmapsto \text{(Notation)} \ [T] \in SH$$

is $\otimes$ lax-monoidal

Presheaf of spectra

$$= KH(T \otimes \mathrm{Perf}(-))$$

**explicitly:**

$$\mathrm{Maps}_{SH}\left(X, M(\underline{\mathrm{Hom}}(\Sigma^\infty_{nc} T, \mathbb{1}_{nc}))\right)$$

$$\parallel$$

$$\mathrm{Maps}_{sH_{nc}}\left(\mathrm{Perf}(X), \underline{\mathrm{Hom}}(\Sigma^\infty_{nc} T, \mathbb{1}_{nc})\right)$$

$$\parallel$$

$$\mathrm{Maps}\left(\mathrm{Perf}(X) \otimes \Sigma^\infty_{nc} T, \mathbb{1}_{nc}\right)$$

$$\parallel$$

$$KH\left(\mathrm{Perf}(X) \otimes_s T\right)$$

<u>Exemple</u>

$$\begin{array}{c} X \\ p \downarrow \\ S \end{array} \quad \text{smooth scheme}/S \qquad\qquad \begin{array}{c} SH_X \\ \downarrow p_* \\ SH_S \end{array}$$

$$[\mathrm{Perf}\, X] = \underbrace{p_* KH_X}_{\text{six operations for } SH}$$

(Cisinski-Déglise; Ayoub)

<u>Definition</u> $X/S$, $X/S \xrightarrow{f} \mathbb{A}^1_S$ function

$X_0 = \{f = 0\}$

$\mathrm{Sing}(X_0) \in \mathrm{dgcat}_S$

$$\boxed{\text{Motive of } \mathrm{Sing}(X_0) := [\mathrm{Sing}\, X_0]}$$

$$\underbrace{\mathrm{Mod}_{KH}(SH) \text{ encodes } KH(\mathrm{Sing}(X_0) \otimes \mathrm{Perf}(-))}_{1^{st} \text{ ingredient}}$$

ok

<u>Prop</u>: $MF : \text{Pairs} \longrightarrow$ 2-periodic dg.categorys
$(U,f)$

is <u>lax monoidal</u> (Thom-Sebastiani)

unit $= (*, 0) \longmapsto MF(*,0) =$ 2-periodic complexes

$\Rightarrow \quad [MF(U,f)] = [\operatorname{Sing} X_0]$ is <u>a module</u> over $[MF(*,0)]$

action

<u>Corollary</u>: $[MF(*,0)] \curvearrowright [\operatorname{Sing}(*,0)]$

$\Rightarrow [\operatorname{Sing}(X_0)] \in$ <u>2 periodic</u> KU-modules in SH

- can we compute $[\mathrm{Sing}\, X_0]$?

**Setting:**

$$\begin{tikzcd}
X_0 \arrow[r] \arrow[d] & X \arrow[d, "p"] \arrow[dd, bend right, "f"'] & X_\eta \arrow[l] \arrow[d] \\
\sigma \arrow[r, hook, "i_\sigma"] \arrow[d] & S \text{ base} \arrow[d, "\pi = \text{uniformizer}"] & \eta \arrow[l, "j_\eta"'] \arrow[d] \\
0 \arrow[r, hook] & \mathbb{A}^1_S & \mathbb{G}_{m,S} \arrow[l]
\end{tikzcd}$$

$S \to \mathbb{C}[[T]]$, $S \to \mathbb{F}_p[[T]]$, $S \to \mathbb{Z}_p$

More generally (excellent trait)

**Construction**

$$\mathbb{H}^{\text{motivic}}_*(\eta) = \text{cohomology of the punctured disk}$$

$$\text{is a commutative algebra object} \quad SH_\sigma \ni \; \overset{\shortparallel}{i_\sigma^* j_{\eta *} \mathbb{1}_\eta} \quad /\mathbb{Q} \text{ coeff}$$

$$\shortparallel$$

$$\text{Motivic purity} \quad \mathbb{1}_\sigma \oplus \underbrace{\mathbb{1}_\sigma(-1)}_{\text{Tate twist}}[-1]$$

generator $\theta$ in degree $(-1)[-1]$

- $IH^*(\eta)$ acts on $IH^*(X_\eta)$
  (pullback)
  = multiplication by $\Theta$

- $IH^*(X_0)$ also acts on $H^*(X_\eta)$ (specialization)

$\underbrace{\qquad\qquad\qquad\qquad}$

combine the two

$$IH^*(\eta) \otimes_\sigma IH^*(X_0) \xrightarrow{\text{map of algebras}} IH^*(X_\eta)$$

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Rmk : can repeat previous construction with KH-modules

$$IH^*(\eta) \otimes_\sigma IH^*_{KH}(X_0) \longrightarrow IH^*_{KH}(X_\eta)$$

monodromy action

30 min

## Proposition

$$[MF(X,f)] = \mathrm{hfiber}\left( \mathbb{H}^*_\sigma(\eta) \otimes_\sigma \mathbb{H}^*_{KH}(X_0) \xrightarrow[\text{monodromy}]{} \mathbb{H}^*_{KH}(X_\eta) \right)$$

• **Key ingredients.**

1)
$$\underbrace{\mathbb{H}^*_\sigma(\eta)\otimes_\sigma KH}_{\text{coh of punctured disk}} \underset{\text{iso of algebras}}{\simeq} \overbrace{\Big[\underbrace{\overbrace{\mathrm{Sing}}^{\text{symmetric monoidal category}}(S,0)}_{\text{2-periodic complexes}}\Big]}^{\text{comm. algebra object}}$$

2) use the exact sequence

$$\begin{array}{ccccc} [\mathrm{Perf}\, X_0] & \longrightarrow & [D^b\mathrm{Coh}\, X_0] & \longrightarrow & [\mathrm{Sing}\, X_0] \\ \| & & \| & & \| \\ \mathrm{Pr}\, KH_{X_0} & & i_* i^! KH_X & & \end{array}$$

$\leftarrow$ G-theory of X with support on $X_0$ + Devissage

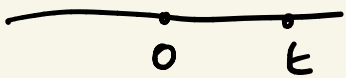
## ③ Relation with vanishing cycles

# 3) Relation with vanishing cycles

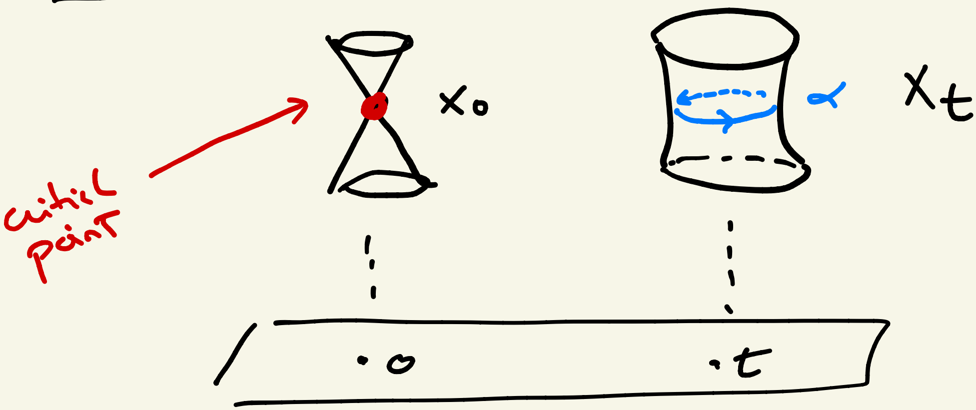


## 3.1) vanishing cycles (Milnor, SGA)

$$\begin{array}{c} X \\ \downarrow f \\ \mathbb{A}^1 \end{array} \qquad X_0 \quad X_t$$



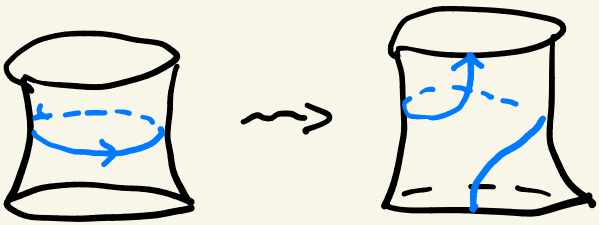

**Example** : $\mathbb{C}^2 \xrightarrow{x^2+y^2} \mathbb{C}$




$$0 \longleftarrow t$$

collapses $\longleftarrow \alpha$

**Monodromy** :



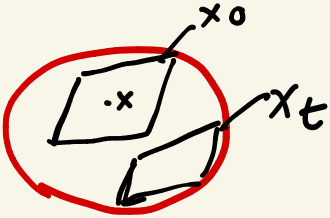
Constuction:

$X_0$  For each $x \in X_0$ take ball in $\mathbb{C}^n$ around $x$

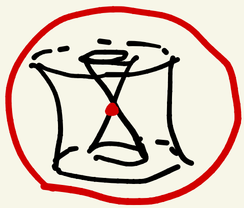
$F_{\varepsilon,x} := X_t \cap \text{Ball}$

$0 < t << \varepsilon$

$x \neq (0,0)$




$x = (0,0)$



**Thm** (SGA7) $\underset{\in X_0}{x} \longmapsto H^*_{sing}(F_{\varepsilon,x}, \mathbb{Z})$ is a

① sheaf on $X_0$ (sheaf of Nearby cycles) ④

② $x \longmapsto \tilde{H}^*_{sing}(F_{\varepsilon,x}, \mathbb{Z})$

is also the stalk of a sheaf on $X_0$, ②

(sheaf of vanishing cycles)

③ exact sequence of sheaves on $X_0$

$$\mathbb{Z}_{X_0} \longrightarrow \Psi \longrightarrow V$$

④ $$R\Gamma(X_0, \Psi) = H^*_{sing}(\underbrace{X_t}_{\text{smooth fiber}}, \mathbb{Z})$$

$$R\Gamma(X_0, V) =: \text{vanishing cycles}$$

<u>fact</u>: $X_0$ smooth $\iff$ vanishing coh $= 0$

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<u>Moreover</u>

$$\begin{tikzcd}
\mathbb{Z}_{X_0} \arrow[r] \arrow[loop below, "\text{trivial action}"] & \Psi \arrow[r] \arrow[loop below, "\text{monodromy}"] & V \arrow[loop below, "\underline{\underline{\text{action}}}"]
\end{tikzcd}$$

$\pi_1(\mathbb{C}\setminus\{0\})$

exact sequence of sheaves on $X_0$

In this talk: We want to use the $\ell$-adic / Motivic version of this construction

↙ $\nu_\ell, \psi_\ell$ ↘ $\nu_{mot}, \psi_{mot}$

SGA 7 Ayoub

$$\bar\eta \to \eta$$

Construction:

$$\mathbb{Q}_{\ell, X_0} \to \psi_\ell \to \nu_\ell \quad \Big| \quad \mathbb{1}_{X_0} \to \psi_{mot} \to \nu_{mot}$$

$\circlearrowleft \mathrm{Gal}(\bar\eta/\eta)$ → exact sequence in $\mathrm{Sh}_{\mathbb{Q}_\ell}(X_0 \times_k \bar k)^{\mathrm{Gal}(\bar\eta/\eta)}$

exact sequence

- $\underbrace{I}_{\text{inertia subgroup.}} := \mathrm{Gal}(\bar\eta/\eta^{unr}) \subseteq \mathrm{Gal}(\bar\eta/\eta) \to \mathrm{Gal}(\bar\sigma/\sigma)$

$\Downarrow$ homotopy fixed points

$$(\mathbb{Q}_\ell, X_0)^{hI} \to \psi_\ell^{hI} \to \nu_\ell^{hI}$$

Sequence of $\mathbb{Q}_{\ell,\sigma}^{hI}$-modules

40 min

<u>LEMMA</u>:

Deligne SGA

$$(\mathbb{Q}_\ell, \sigma)^{hI} \simeq \mathbb{H}^*_{\mathbb{Q}_\ell}(\eta)$$

cohomology of the punctured disk.

(Topological analogy) $\eta = S^1$ $\qquad \bar{\eta} \to \eta$ $\qquad \mathbb{C} = \sigma$

$*$ " $\to S^1$ universal cover

$$C^*(S^1, \mathbb{C}) \simeq \mathbb{C}^{h\mathbb{Z}}$$

<u>Construction</u>: (Ayoub, Cisinski - Déglise)

$\ell$-adic Realization functor $\qquad SH^{\otimes}_{\mathbb{Q}} \xrightarrow{R^\ell} Sh^{\otimes}_{\mathbb{Q}_\ell}$

<u>Thm</u> (Ayoub) The formalism of vanishing cycles is compatible under the $\ell$-adic realization

$$\underbrace{SH^{\otimes}_{\mathbb{Q}}}_{\mathbb{Q}\text{-coefficients}} \xrightarrow{R_\ell} Sh_{\mathbb{Q}_\ell}$$

## 3.2 Singularity categories & vanishing cycles

$$
\begin{tikzcd}
\text{dg-cat}_S \arrow[r] \arrow[rr, bend left, "{[-]}"] \arrow[drr, dashed, "{[\cdot]}"'] & SH_{nc,S} \arrow[r] & SH_S \\
 & & \mathrm{Mod}_{KH}(SH) \arrow[u] \arrow[d, "R^{\ell}", "\text{Riou ($\gamma$-filtration, BoTT-periodicity)}"'] \\
 & & \mathrm{Mod}_{R^{\ell}(KH)}(Sh_{\mathbb{Q}_\ell})
\end{tikzcd}
$$

$$R^{\ell}(KH) = \underbrace{\bigoplus_{n\in\mathbb{Z}} \mathbb{Q}_\ell(n)[2n]}_{\text{2-periodized } \ell\text{-adic coh.}} = \text{notation } \mathbb{Q}_\ell(\beta)$$

$$= \mathrm{Free}_{E_\infty\text{-alg}}\left(\mathbb{Q}_\ell(1)[2]\right)$$

**Main Theorem** (Blanc - R - Toën - Vezzosi)

$$\mathbb{IH}^*\left(R^\ell\left([\mathrm{Sing}\,X_0]\right)\right) \simeq \mathbb{IH}^\vee_{\mathbb{Q}_\ell}\left(\nu^{hI}[-1]\right)(\beta)$$

$SH_{X_0}$

Motivic singularity categories

Motivic vanishing cycles

Tate - 2 periodized

**Proof** :

Key ingredient = isomorphisms of algebras

$$R^\ell\left[\mathrm{Sing}(S,0)\right] \overset{\text{previous slide}}{\simeq} \mathbb{IH}^\vee_{\mathbb{Q}_\ell}(\eta)(\beta) \overset{\text{previous slide}}{\simeq} \left(\mathbb{Q}_{\ell,\sigma}^{hI}\right)(\beta)$$

singularity categories

motivic monodromy

Galois action

+ playing with exact sequences $\nearrow$ $\mathrm{Perf} \to \mathrm{Coh} \to \mathrm{Sing}$
$\searrow$ $\mathbb{Q}_\ell \to \Psi \to \nu$ $\square$

quick

(4) survey of more recent results

# 4.1) Pippi's PhD thesis

(arxiv 2006.06301)

→ Extension of this results to $(X, \underbrace{f_1, \dots, f_n}_{\{f_i\}})$

$\longrightarrow$ multiple functions

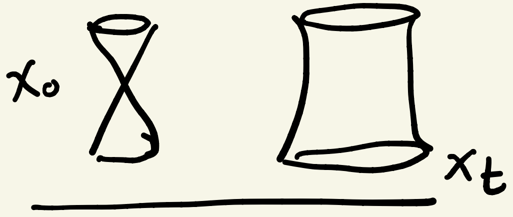
1. New Definition of Matrix Factorizations and Singularity categories
2. extension of Orlov Theorem MF ⟷ Sing
3. comparison $[MF(X,\{f_i\})] \longleftrightarrow$ Vanishing cycles for a global section of a line bundle

uses result of Orlov & Burke-Walker

$$\mathrm{Sing}(X,\{f_i\}) \simeq \mathrm{Sing}(\mathbb{P}^{n-1}_X, W_{\{f_i\}})$$

section of a line bundle

## 4.2 Toën-Vezzosi Program to Bloch conductor formula




**over $\mathbb{C}$**

Deligne-Milnor number

$$\mu = \chi(X_t) - \chi(X_0)$$

$$\chi(X_t) = \chi(\Psi), \qquad \chi(X_0) = \chi(\mathbb{Z}_{X_0})$$

$$\underbrace{\text{dim of vanishing cycles}}_{\chi(\nu)}$$

exact sequence

$$\mathbb{Z}_{X_0} \longrightarrow \Psi \longrightarrow \nu$$

**over $\mathbb{Z}_p$** (arithmetic disk)

Bloch conjecture

$$\text{Swan conductor} + \mu = \underbrace{\chi(\Psi) - \chi(\mathbb{Q}_{\ell, X_0})}_{\chi(\nu)}$$

$(-)^{hI}$ is $\overset{\text{lax}}{\checkmark}$ symmetric monoidal

$$\Downarrow$$

$$\chi(V) = \chi(V^{I})$$

$$\Vert \text{ thm}$$

$$\chi([MF])$$

<u>Toën-Vezzosi</u>: Compute $\chi([MF])$ and

show $\chi(MF) = \underline{\mu} + \underline{\text{swan}}$

(Theorem when $I$ acts unipotently)

—————— $h$ ——————

Thank you!