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Théorie Homotopique Motivique des Espaces Noncommutatifs

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"No. Try not. Do. Or do not. There is no try."

B.Toën (following Master Yoda)

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Théorie Homotopique Motivique des Espaces Noncommutatifs

Abstract

Cette thèse s'inscrit dans le cadre d'un projet de recherche visant à comparer la géométrie algébrique classique avec la nouvelle géométrie algébrique non commutative dans le sens de Kontsevich. Plus précisément, on veut comparer les niveaux motiviques des deux théories.

Dans la première partie de ce travail on s'occupe de donner la propriété universelle de l'$(\infty, 1)$-catégorie symétrique monoïdale $\mathcal{SH}$ sous-jacente à la théorie homotopique stable motivique des schémas, telle que construite par Voevodsky et Morel, avec sa structure monoïdale symétrique. Dans la deuxième partie de ce travail on introduit un analogue non-commutatif $\mathcal{SH}_{nc}$ à la théorie homotopique motivique stable pour les espaces non-commutatifs dans le sens de Kontsevich, que nous modélisons en utilisant la notion de dg-catégories de type fini de Toën-Vaquié. Pour cela on introduit un analogue non-commutatif approprié à la topologie de Nisnevich. En conséquence de notre propriété universelle pour $\mathcal{SH}$ on obtient gratuitement un pont entre les deux théories $\mathcal{SH} \to \mathcal{SH}_{nc}$. Ce pont admet un adjoint et notre second résultat principal est que cet adjoint envoie l'unité monoïdale de $\mathcal{SH}_{nc}$ vers l'objet KH dans $\mathcal{SH}$, représentant la K-théorie invariante par homotopie. Comme corollaire, principal on obtient que sur un corps de caractéristique nulle, le foncteur induit entre les KH-modules vers $\mathcal{SH}_{nc}$ est pleinement fidèle. On montre on passage que la K-théorie non-connective des dg-catégories est le faisceau associé à la K-théorie connective des dg-catégories, par rapport à là topologie de Nisnevich non-commutative.

Dans la troisième partie de ce travail, on étend nos résultats et constructions pour définir les motifs et les motifs non-commutatives sur une base quelconque. Nous utilisons ensuite des techniques récentes dues à Liu-Zheng pour établir un formalisme de six opérations pour $\mathcal{SH}$ dans le cadre des $(\infty, 1)$-catégories. Cela étend les résultats de Ayoub. Dans la dernière partie de cette thèse, on explique nos efforts pour établir un formalisme de six opérations dans le cadre des motifs non-commutatifs.

Mots-Clés : Motifs, Motifs Non-commutatifs, Catégories supérieures de motifs, Schémas Non-commutatifs, Théorie Homotopique Motivique des schémas non-commutatifs, K-Théorie

Motivic Homotopy Theory of Non-commutative Spaces

Abstract

This thesis aims to compare classical algebraic geometry with the new noncommutative algebraic geometry in the sense of Kontsevich. More precisely, we compare the motivic levels of both theories.

Our first main result in this thesis is a universal characterization for the symmetric monoidal $(\infty, 1)$-category $\mathcal{SH}$ underlying the motivic stable homotopy theory of schemes of Morel-Voevodsky. In the second part of this thesis we introduce a non-commutative analogue $\mathcal{SH}_{nc}$ of the Morel-Voevodsky motivic theory for the non-commutative spaces in the sense of Kontsevich, modeled by the dg-categories of finite type of Toen-Vaquié. For this purpose we introduce a non-commutative version of the Nisnevich topology. The universal characterization proved for the commutative theory allows us to obtain for free a bridge between the two theories $\mathcal{SH} \to \mathcal{SH}_{nc}$. This bridge has an adjoint and our second main result in this thesis is that this adjoint sends the tensor unit non-commutative motive to the commutative motivic ring spectrum KH representing homotopy invariant algebraic K-theory. As a corollary we deduce that over a field of characteristic zero the induced map from KH-modules in $\mathcal{SH}$ to non-commutative motives is fully-faithful. In the process we prove that the non-connective K-theory of dg-categories is the non-commutative Nisnevich sheafification of connective K-theory.

In the third part of this work we extend our results and constructions to define motives and non-commutative motives over general base schemes. Moreover, we use the recent techniques of multi-simplicial sets of Liu-Zheng to prove a formalism of six operations for $\mathcal{SH}$ in the setting of $(\infty, 1)$-categories, thus extending the results of J. Ayoub. In the last part of this thesis we explain our attempts to prove the existence of an analogous formalism of six operations for non-commutative motives.

Key Words : Motives, Non-commutative Motives, Higher Categories of Motives, Non-commutative schemes, Homotopy theory of non-commutative schemes, K-Theory

Prélude

Cette thèse s'inscrit dans le cadre d'un projet de recherche visant à comparer la géométrie algébrique classique avec la nouvelle géométrie algébrique non commutative dans le sens de Kontsevich [87]. Plus précisément, nous voulons comparer les niveaux motiviques des deux théories.

Motifs

Dans le programme initial envisagé par Grothendieck, le motif d'un objet géométrique X (par exemple X une variété projective lisse) était un nouvel objet mathématique conçu pour exprimer "le contenu arithmétique de X" [1]. Plus précisément, dans les années 60, Grothendieck et ses collaborateurs ont construit des exemples de ce qu'on appelle les théories cohomologiques de Weil, conçues pour capturer différentes informations sur l'arithmétique de X. En présence de plusieurs théories, il a envisagé l'existence d'une théorie universelle, celle qui rassemblerait toutes les informations arithmétiques. À l'époque, les théories cohomologiques ont été formulées de manière assez artificielle en utilisant des catégories abéliennes en tant ingrédient de base. La notion de catégorie triangulée est apparue pour fournir un nouveau cadre plus naturel pour les théories cohomologiques. Évidemment, la théorie de motifs a suivi ces innovations [13] et finalement, dans les années 90, V. Voevodsky [151] a construit ce qui est devenue la "cohomologie motivique". Plusieurs bonnes références introductives à ce programme arithmétique sont maintenant disponibles [3, 4, 102], avec le contexte historique présenté dans l'introduction de [30] ainsi que les dernières notes de cours par B. Kahn [75].

À la fin des années 90, Morel et Voevodsky [105] ont développé une théorie plus générale de motifs. Dans leur théorie, le motif de X est conçu pour être le squelette cohomologique de X, non seulement de la perspective d'une théorie cohomologique de Weil, mais pour toutes les théories cohomologiques généralisées pour les schémas (comme la K-théorie, le cobordisme algébrique et la cohomologie motivique). L'inspiration vient de la théorie homotopique stable des espaces où toutes les théories de cohomologie généralisées (des espaces) deviennent représentables. Un tel cadre pourrait fournir des définitions plus faciles pour la cohomologie motivique, la K-théorie algébrique, le cobordisme algébrique, etc, en se contentant de fournir leur spectre représentant. Leur construction comporte deux étapes principales : la première partie imite la théorie d'homotopie des espaces et sa stabilisation; la deuxième partie rend inversible le "motif de Tate" par rapport à la multiplication monoïdale. Le résultat final est connu sous le nom de *l'homotopie stable motivique des schémas* . Notre premier objectif dans cette thèse est de formuler une propriété universelle précise pour leur construction.

[1] par exemple, il doit saisir l'information sous-jacente de la L-fonction de X.

Géométrie Algébrique Non-commutative

En Géométrie Algébrique, et particulièrement après les travaux de Serre et de Grothendieck, l'étude d'un schéma X via sa catégorie abélienne des faisceaux quasi-cohérents $Qcoh(X)$ est devenu pratique courante. Cela s'explique par des raisons purement techniques, car à l'époque, les catégories abéliennes constituaient la seule façon de formuler des théories cohomologiques. En fait, l'objet $Qcoh(X)$ se révèle être un très bon remplacement pour l'objet géométrique X: grâce à [54, 120] nous savons que X peut être reconstruit à partir de $Qcoh(X)$. Cependant, les catégories abéliennes ne fournissent pas un cadre très naturel pour l'algèbre homologique. Ce fut Grothendieck qui remarque que ce cadre naturel serait ce que nous comprenons aujourd'hui comme étant la théorie homotopique des complexes dans la catégorie abélienne initiale. À cette époque, la norme pour traiter les théories homotopiques consistait à examiner leurs catégories homotopiques - l'inversion formelle stricte des équivalences faibles. C'est ainsi que nous obtenons la catégorie dérivée $D(X)$ du schéma X. Pour de nombreuses raisons, il était clair que l'on perdait trop d'information en passant de la théorie homotopique des complexes à la catégorie dérivée associée. La solution de ce problème est venue de deux directions différentes. D'abord, de la théorie de dg-catégories [20, 24, 25], et plus récemment, de la théorie des ∞-catégories [5, 16, 99, 100, 127, 145]. Le premier sujet a pris de l'ampleur avec les avancées de [22, 23, 43, 44, 78, 132, 139]. Le deuxième, bien que lancé tôt dans les années 80 avec le célèbre manuscrit [64], n'a atteint un état où son potentiel pourrait être pleinement exploité qu'au cours de ces dernières années, particulièrement grâce aux énormes efforts de [99, 100]. Les deux sujets constituent un moyen approprié pour codifier la théorie homotopique des complexes de faisceaux quasi-cohérents. En fait, les deux approches sont liées et, pour nos objectifs, devraient donner des réponses équivalentes (voir les résultats récents dans [36] et notre Section 6.2). Chaque schéma X (sur un anneau k) donne naissance à une k-dg-catégorie $L_{qcoh}(X)$ - *la dg-catégorie dérivée de* X - dont la catégorie homotopique est la catégorie derivéee de X. Pour les schémas raisonnables, cette dg-catégorie a une propriété essentielle profondément liée à son origine géométrique - elle dispose d'un générateur compact. En plus, ces objets compacts sont les complexes parfaits (voir [**?**] et [137]). En conséquence, la plus petite sous-dg-catégorie $L_{pe}(X)$ engendrée par les objets compacts est "affine", et suffisante pour récupérer la totalité de $L_{qcoh}(X)$.

Dans ses travaux [86, 87, 89], Kontsevich a entrepris une étude systématique des dg-catégories avec les mêmes propriétés formelles de $L_{pe}(X)$, en partant de l'observation qu'il existe de nombreux exemples de ces objets dans la nature : si A est une algèbre associative alors A peut être considérée comme une dg-catégorie avec un seul objet, et nous considérons $L(A)$ la dg-catégorie dérivée des complexes de A-modules, d'où nous prenons les objets compacts. Cela s'applique également à une algèbre différentielle graduée. La *catégorie de Fukaya* d'une varieté symplectique est un autre exemple [88]. Il y a aussi des exemples provenant de la géométrie complexe [110], de la théorie des représentations, des matrices de factorisation (voir [49]), et aussi des techniques de quantification par déformation. Cette variété d'exemples d'origine complètement différente a motivé la compréhension des dg-catégories comme les modèles naturels pour la notion d' *espace non-commutatif.* L'étude de ces dg-catégories peut être systématisée et la correspondance $X \mapsto L_{pe}(X)$ peut être rendue de façon fonctorielle

$$L_{pe} : \text{Obj. Géométriques}/k \longrightarrow \text{Espaces non-commutatifs}/k \tag{0.0.1}$$

En fait, le foncteur L_{pe} est défini pour une classe d'objets géométriques plus générale que cella des schémas, notamment, les *schémas dérivés* (voir [141, 14, 95]). Ce sont les objets géométriques naturels dans la théorie de la géométrie algébrique dérivée de [143, 144, 98, 146]. Ce fait est essentiel pour la géométrie non-commutative : grâce aux résultats de Toën-Vaquié dans [141], au niveau dérivé, L_{pe} admet un adjoint à droite qui nous permet d'associer un objet commutatif à un objet non-commutatif de façon naturelle.

Kontsevich propose également que, comme dans le contexte géométrique, ces espaces non-commutatives doivent admettre une théorie motivique. Le deuxième objectif de cette thèse est de proposer un candidat naturel à cette théorie qui étend d'une façon naturelle la théorie de Morel-Voevodsky. Le pont entre les deux théories sera donnée par une extension canonique du foncteur L_{pe}, fourni par notre résultat de caractérisation universelle de la théorie des motifs de Morel-Voevodsky.

Dans cette thèse

La construction motivique de Morel-Voevodsky a été initialement obtenue en utilisant les techniques de la théorie des catégories de modèles. Aujourd'hui on sait qu'une catégorie de modèles n'est qu'une façon stricte de présenter un objet plus fondamental, notamment, une $(\infty, 1)$-catégorie. Chaque catégorie de modèles a une $(\infty, 1)$-catégorie sous-jacente et c'est ce qui nous intéresse vraiment. On doit aussi remarquer que cette différence est au-delà de l'esthétique. Grâce aux travaux de J. Lurie [99, 100] on a maintenant les outils et les mécanismes pour montrer des théorèmes qui resterait inaccessible avec la langage des catégories de modèeles.

Dans la première partie de ce travail (Part I), on s'occupe de donner la propriété universelle de la $(\infty, 1)$-catégorie sous-jacente à la théorie homotopique stable motivique des schémas, telle que construite par Voevodsky et Morel, avec sa structure monoidale symétrique. Cette caractérisation est importante dèes lors que l'on veut comparer les motifs des schémas avec d'autres théories. Dans notre contexte, l'objectif principal est de concevoir une théorie de motifs non-commutatifs et de la relier à celle de Morel-Voevodsky. La propriété universelle fournit également une flèche monoïdale au niveau motivique rendant le diagramme suivant commutatif:

$$\begin{tikzcd}
\text{Classical Schemes}/k \arrow[r] \arrow[d] & \text{NC-Spaces}/k \arrow[d] \\
\text{Stable Motivic Homotopy}/k \arrow[r, dashed] & \text{NC-Stable Motivic Homotopy}/k
\end{tikzcd} \tag{0.0.2}$$

En général, ce type de flèches monoïdales sont extrêmement difficiles à obtenir en utilisant uniquement des méthodes constructives et les techniques de la théorie des catégories de modèles. Un autre avantage important de notre approche est la possibilité de travailler sur n'importe quelle base, pas nécessairement affine.

À ce point là, on se doit de signaler qu'une autre approche pour la théorie des motifs non-commutatifs est déjà présent dans la littérature dans les travaux de D-C. Cisinski and G. Tabuada [133, 35, 130, 134]. On dit que leur approche est de " nature cohomologique " alors que notre méthode pourrait être appelé "homologique" et plus proche de l'esprit de la théorie de l' homotopie stable. Plus tard, dans le Chapitre 8 on systématise la comparaison entre les deux approches et dévoile une forme de dualité entre eux. C'est exactement ce phénomène de dualité qui rend notre nouvelle approche comparable à la théorie de Morel-Voevodsky et qui permet à la flèche monoïdale d'exister d'une manière naturelle. Cette dualité est une obstruction à la comparaison des théories de Morel-Voevodsky et de Cisinski-Tabuada. On doit aussi mentionner que tous nos contenus mathématiques sont indépendants des leurs.

Pour arriver à la caractérisation universelle, nous aurons besoin de réécrire les constructions de Morel-Voevodsky dans le cadre des ∞-catégories. Grâce à [99] et [100], on dispose d'un dictionnaire entre ces deux mondes. En fait, les résultats dans [99] sont déjà suffisants pour caractériser l'∞-catégorie sous-jacente à la version non-stable et $\mathbb{A}^1$-invariante de la théorie homotopique des schémas. Le problème concerne la description du monde motivique stable avec sa structure monoıdale symétrique. Ce sera notre contribution principale. L'ingrédient-clé est la suivant:

Idée 0.0.1. *(Théorème 4.3.1):*
Soit $\mathcal{V}$ une catégorie de modèles simpliciale combinatoire avec un structure symétrique monoïdale compatible et un objet-unité cofibrant et soit $\mathcal{C}^{\otimes}$ son ∞-catégorie symétrique monoïdale sous-jacente. Soit

X un objet cofibrant dans $\mathcal{V}$ satisfaisant la condition suivante :

(*) la permutation cyclique $\sigma = (1,2,3) : X \otimes X \otimes X \to X \otimes X \otimes X$ est égale au morphisme d'identité dans la catégorie homotopique de $\mathcal{V}$.[2]

L'$(\infty,1)$-catégorie symétrique monoïdale sous-jacente à la catégorie de modèles $Sp^{\Sigma}(\mathcal{V}, X)$ des spectres symétriques dans $\mathcal{V}$, est la $(\infty,1)$-catégorie symétrique monoïdale universelle munie d'un foncteur monoïdale à partir de $\mathcal{C}^{\otimes}$, qui envoie l'objet X dans un objet tenseur-inversible.

On démontre ce résultat au Chapitre 4. La hypothèse (*) est bien connu: elle est déjà présente dans les travaux de Voevodsky ([150]) et apparaît également dans [71]. Signalons également que nous croyons ce résultat vrai même sans cette hypothèse (Remarque 4.3.2).

Dans le Chapitre 5 nous appliquons les résultats généraux du Chapitre 4 à la théorie homotopique stable motivique des schémas :

Corollaire 0.0.2. *(Corollary 5.3.2) Soit S un schéma de base et soit $Sm^{ft}(S)$ la catégorie des schémas lisses séparées et de type fini sur S. L' $(\infty,1)$-catégorie $\mathcal{SH}(S)$ sous-jacente à la théorie homotopique stable motivique des schémas est stable, présentable et admet une structure monoïdale symétrique canonique $\mathcal{SH}(S)^{\otimes}$. En plus, la construction de Morel-Voevodsky fournit un foncteur $Sm^{ft}(S)^{\times} \to \mathcal{SH}(S)^{\otimes}$ monoïdal par rapport au produit cartésien des schémas, et possédant la propriété universelle suivant:*

$(*)$ *pour toute l'$(\infty,1)$-catégorie symétrique monoïdale présentable et pointée $\mathcal{D}^{\otimes}$, le morphisme de composition* [3]

$$Fun^{\otimes,L}(\mathcal{SH}(S)^{\otimes}, \mathcal{D}^{\otimes}) \to Fun^{\otimes}(Sm^{ft}(S)^{\times}, \mathcal{D}^{\otimes}) \tag{0.0.3}$$

est pleinement fidèle et son image est composée de ces foncteurs monoïdaux $Sm^{ft}(S)^{\times} \to \mathcal{D}^{\otimes}$ satisfaisant la descente Nisnevich, l'invariance par $\mathbb{A}^1$-homotopie et tels que la cofibre de l'image du point à l'infini $S \xrightarrow{\infty} \mathbb{P}^1_S$ est un objet tenseur-inversible dans $\mathcal{D}^{\otimes}$. En plus, toute $(\infty,1)$-catégorie monoïdale symétrique présentable et pointée qui admet un foncteur monoïdal satisfaisant ces conditions, est stable.

Ce résultat apporte une réponse au problème de construire des réalisations motiviques monoidales. Leur existence a des conséquences profondes. Voir [76] pour une introduction.

Exemple 0.0.3. Soit $S = Spec(k)$ un corps de caractéristique nulle. L'application $X \mapsto \Sigma^{\infty}(X(\mathbb{C}))$ fournit un foncteur $Sm^{ft}(S) \to Sp$ où Sp est l'$(\infty,1)$-catégorie des spectres topologiques. Cette application est bien connue pour être monoïdal et pour satisfaire toutes les conditions dans le corollaire précédent. Par conséquent, il s'étend d'une manière unique en un foncteur monoïdal $\mathcal{SH}(S)^{\otimes} \to Sp^{\otimes}$;

Exemple 0.0.4. Soit $S = Spec(k)$ comme dans l'exemple précédent. Un autre exemple d'une réalisation motivique monoïdale est la réalisation de Hodge. Bien définie, l'application $X \mapsto C_{DR}(X)$ qui envoie un schéma sur son complexe de De Rham, fournit un foncteur $Sm^{ft}(S) \to \mathcal{D}(k)$ où $\mathcal{D}(k)$ est la version ∞-catégorique de la catégorie dérivée de k. Cette application est connu pour être monoïdale par rapport au produit cartésien des schémas (formule de Künneth) et pour satisfaire toutes les conditions du corollaire. Par conséquent, il s'étend d'une manière unique en un foncteur monoïdal - la réalisation de Hodge motivique - entre $\mathcal{SH}(S)^{\otimes} \to \mathcal{D}(k)^{\otimes}$ (où sur la gauche, on a la structure monoïdale induite par le produit tensoriel dérivé de complexes). Cet exemple a été récemment travaillé en détail dans la thèse de B. Drew [42]. Un de ses corollaires est une nouvelle preuve de la correspondance de Riemann-Hilbert pour D-modules holonomes ([42, Thm 3.4.1]).

[2] Plus précisément on demande l'existence d'une homotopie dans $\mathcal{V}$ entre la identité et la permutation cyclique.
[3] avec les notations de 2.1

Dans la deuxième partie de ce travail (Part II) on établit une comparaison entre les mondes motiviques commutatifs et non-commutatifs. Dans le Chapitre 6, après quelques préliminaires sur les dg-catégories, on introduit l'$(\infty,1)$-catégorie des espaces non-commutatifs lisses $\mathcal{N}cS(k)$ comme l'opposé de l'$(\infty,1)$-catégorie des dg-catégories idempotente complètes de type fini $\mathcal{D}g(k)^{ft} \subseteq \mathcal{D}g(k)^{idem}$ qui ont étées introduites par Toën-Vaquié dans [141]. En introduisant un analogue non-commutatif approprié pour la topologie de Nisnevich (Def. 6.4.7) et en considérant la version non-commutative de la droit affine $L_{pe}(\mathbb{A}^1)$, on construit une nouvelle $(\infty,1)$-catégorie symétrique monoïdale stable $\mathcal{SH}_{nc}(S)^{\otimes}$ qui encode une théorie homotopique motivique stable pour ces espaces non-commutatifs. Cela fournit une nouvelle approche aux motifs non-commutatifs.

La première étape pour comparer les deux mondes est la construction d'un foncteur L_{pe} de la théorie des schémas affines lisses sur k vers l'$(\infty,1)$-catégorie $\mathcal{N}cS(k)$. Celui-ci doit être monoïdal (Prop. 6.3.8). En conséquence de notre résultat principal dans le corollaire précédant, on a gratuitement une extension monoïdale de ce foncteur au monde motivique :

$$\begin{tikzcd} N(\mathit{AffSm}^{ft}(k)) \arrow[r, "L_{pe}"] \arrow[d] & \mathcal{N}cS(k) \arrow[d] \\ \mathcal{SH}(k) \arrow[r, dashed] & \mathcal{SH}_{nc}(k) \end{tikzcd} \tag{0.0.4}$$

Dans le Chapitre 7 on commence à explorer les propriétés de ce pont. Plus précisément, on explique comment l'utiliser pour comprendre les différentes versions de la K-théorie algébrique des dg-catégories et des schémas.

Pour préciser nos résultats, nous aurons besoin de quelques remarques préliminaires. Comme le lecteur le verra plus tard (Section 5.4 et Remarque 6.4.23), les $(\infty,1)$-catégories $\mathcal{SH}(k)^{\otimes}$ et $\mathcal{SH}_{nc}(k)^{\otimes}$ peuvent être construitess comme une suit de localisations réflexives monoïdales des préfaisceux spectraux suivie par des inversions du cercle algébrique $\mathbb{G}_m$ par rapport au produit tensoriel. Plus précisément, on a une suit d'étapes monoïdales

$$\begin{tikzcd} N(\mathit{AffSm}^{ft}(k))^{\times} \arrow[r, "L_{pe}^{\otimes}"] \arrow[d, hook, "(\Sigma_+^{\infty}\circ j)^{\otimes}"] & \mathcal{N}cS(k)^{\otimes} \arrow[d, hook, "(\Sigma_+^{\infty}\circ j_{nc})^{\otimes}"] \\ \mathit{Fun}(N(\mathit{AffSm}^{ft}(k))^{op}, \mathit{Sp})^{\otimes} \arrow[r, dashed, "\mathcal{L}_1^{\otimes}"] \arrow[d, "l_{Nis}^{\otimes}"] & \mathit{Fun}(\mathcal{D}g(k)^{ft}, \mathit{Sp})^{\otimes} \arrow[d, "l_{Nis}^{nc,\otimes}"'] \\ \mathit{Fun}_{Nis}(N(\mathit{AffSm}^{ft}(k))^{op}, \mathit{Sp})^{\otimes} \arrow[r, dashed, "\mathcal{L}_2^{\otimes}"] \arrow[d, "l_{\mathbb{A}^1}^{\otimes}"] & \mathit{Fun}_{Nis}(\mathcal{D}g(k)^{ft}, \mathit{Sp})^{\otimes} \arrow[d, "l_{\mathbb{A}^1}^{nc,\otimes}"'] \\ \mathit{Fun}_{Nis,\mathbb{A}^1}(N(\mathit{AffSm}^{ft}(k))^{op}, \mathit{Sp})^{\otimes} \arrow[r, dashed, "\mathcal{L}_3^{\otimes}"] \arrow[d, "\Sigma_{\mathbb{G}_m}^{\otimes}"] & \mathit{Fun}_{Nis,L_{pe}(\mathbb{A}^1)}(\mathcal{D}g(k)^{ft}, \mathit{Sp})^{\otimes} \arrow[d, "\sim"] \\ \mathcal{SH}(k)^{\otimes} \arrow[r, dashed, "\mathcal{L}^{\otimes}"] & \mathcal{SH}_{nc}(k)^{\otimes} \end{tikzcd} \tag{0.0.5}$$

où chaque foncteur est induit par une propriété universelle. Pour des raisons formelles, chacun de ces foncteurs admet un adjoint à droite, que nous noterons respectivement $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M}_3$ et $\mathcal{M}$.

Ce mécanisme permet de restreindre les invariants non-commutatifs au monde commutatif.

Exemple 0.0.5. Un exemple important d'un invariant non-commutative est l'homologie de Hochschild des dg-catégories. D'après les travaux de B. Keller dans [82] et comme on explique dans la Remarque 6.4.24, cet invariant peut être vu comme un ∞-foncteur $HH : \mathcal{D}g(k)^{ft} \to \widehat{Sp}$. Un autre exemple important est l' *homologie cyclique périodique des dg-catégories HP*. Il résulte du célèbre théorème HKR que la restriction de HP au monde commutatif est la cohomologie de De Rham classique des schémas.

Exemple 0.0.6. Un autre exemple important récemment introduit par A. Blanc dans sa thèse [**?**] est la K-théorie topologique des dg-catégories. Il s'agit d'un candidat pour la version non-commutative de la réalisation de Betti.

Dans le Chapitre 7 on s'intéresse à comprendre les restrictions des différentes versions de la K-théorie des dg-catégories. Chacun de ces objets vit dans l' $(\infty, 1)$-catégorie $Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp})$. On s'intéresse en particulier à :

- K^c - la K-théorie *connective* de Waldhausen. (Section 7.1.2).
- K^S - la K-théorie *non-connective* des dg-catégories telle que défini dans [34] avec la méthode de Schlichting [122] (Section 7.1.3). Par construction, ce foncteur vient naturellement munie d'une transformation naturelle canonique $K^c \to K^S$ qui est une équivalence dans la partie connective.

Pour le premier cas, il résulte immédiatement de la version spectrale du lemma de Yoneda (Remarque 5.4.1) et de la définition dans [137, Section 3] que $\mathcal{M}_1(K^c)$ n'est autre que la K-théorie algébrique connective des schémas. Dans le deuxième cas, par le résultat de comparaison de [122, Theorem 7.1], on trouve la K-théorie non-connective des schémas telle que définit par Bass-Thomason-Trobaugh dans [137]. La construction de K^S dans [34] en utilisant les méthodes de [122] est en quelque sort ad-hoc. Dans l'un de nos résultats on explique comment K^S apparait de façon canonique comme le faisceaux associé à K^c par rapport à la notion de topologie de Nisnevich pour les espaces non-commutatives introduit dans cette thèse. Le résultat suivant résume nos principaux résultats dans ce domaine :

Théorème 0.0.7.

(i) (Thm. 7.0.29) Le morphisme canonique $K^c \to K^S$ présent la K-théorie non-connective comme le faisceau associé a la K-théorie connective par rapport à la notion de topologie Nisnevich pour les espaces non-commutatifs;

(ii) (Thm. 7.0.31) La localisation (noncommutative) $\mathbb{A}^1$-invariante $l^{nc}_{\mathbb{A}^1}(K^S)$ est l'unité 1_{nc} pour la structure monoïdale dans $\mathcal{SH}_{nc}(k)^{\otimes}$;

(iii) (Thm. 7.0.32) L'image de $l^{nc}_{\mathbb{A}^1}(K^S)$ par l'adjoint à droit $\mathcal{M}$ est le objet KH dans $\mathcal{SH}(k)$ qui représente la version $\mathbb{A}^1$-invariante de la K-théorie algébrique non-connective des schémas telle que définie par Weibel et étudiée dans [150] et [29]. En particulier, comme $\mathcal{M}$ est faiblement monoïdal (c'est l'adjoint d'un foncteur monoïdal) il envoie la structure d'algèbre trivial de 1_{nc} vers une structure d'algébre commutative sur l'objet KH. En plus, le foncteur monoïdal $\mathcal{L}^{\otimes}$ se factorise par les KH-modules comme

$$\mathcal{SH}(k)^{\otimes} \xrightarrow{-\otimes KH} Mod_{KH}(\mathcal{SH}(k))^{\otimes} \dashrightarrow \mathcal{SH}_{nc}(k)^{\otimes}$$

On doit signaler que la première partie de ce théorème (i) n'est pas vrais dans le monde commutatif. Le phénomène qui permet ce résultat dans le monde non-commutatif est le fait que la nouvelle notion de carré Nisnevich combine les revêtements d'origine géometrique (ceux formées via L_{pe}) et les revêtements d'origine catégorique, à savoir, ceux qui sont induits par des collections exceptionnelles.

La première partie de ce théorème est prouvé en montrant que la $(-)^B$-construction de Bass-Thomason donné dans [137] est un modèle explicite pour la version non-commutative de la localisation Nisnevich pour les prefaisceux à valeur dans les spectres connectifs et qui envoient les carrés Nisnevich des espaces non-commutatifs vers des produit fibrés de spectres connectives. Rappelons que l'inclusion des spectres connectives dans les spectres ne préserve pas les produits fibrés. Plus généralement, on démontre que le foncteur de troncation induit une équivalence d'$(\infty, 1)$-catégories entre les préfaisceaux spectraux Nisnevich locaux et les préfaisceaux à valeurs dans les spectres connectifs qui ont la descente Nisnevich au sens connective. La construction $(-)^B$ donne un inverse explicite à la troncation dans ce contexte. Le deuxième résultat utilise un résultat fondamental dans la thèse de A. Blanc [17, Prop. 4.6]: la version scindée de la S-construction de Waldhausen est $\mathbb{A}^1$-équivalente à la S-construction complétée.

Le corollaire suivant fournit une nouvelle formalisation d'un résultat prévu par Kontsevich [87, 89] et également vérifié par le formalisme de Cisinski-Tabuada.

Corollaire 0.0.8. *(Cor. 7.0.33) Soient $\mathcal{X}$ et $\mathcal{Y}$ des espaces non-commutatifs et supposons $\mathcal{Y}$ propre et lisse. Dans ce cas on a une équivalence de spectres*

$$Map_{\mathcal{SH}_{nc}(k)}(\mathcal{X}, \mathcal{Y}) \simeq l_{\mathbb{A}^1}^{nc}(K^S)(\mathcal{X} \otimes \mathcal{Y}^{op})$$

où $\mathcal{Y}^{op}$ est le dual de $\mathcal{Y}$ et où on a identifié $\mathcal{X}$ et $\mathcal{Y}$ avec leurs images dans $\mathcal{SH}_{nc}(k)$.

Finalement, le résultat suivant a été le but principal de cette thèse. En combinant le théorème précédent avec la description des objets compacts dans $\mathcal{SH}(k)$ sur un corps avec des résolutions des singularités donné par J.Riou dans [117], on déduit :

Corollaire 0.0.9. *(Cor. 7.0.37) Soit k un corps avec des résolutions des singularités. Alors la factorisation canonique*

$$Mod_{KH}(\mathcal{SH}(k))^{\otimes} \dashrightarrow \mathcal{SH}_{nc}(k)^{\otimes}$$

est pleinement fidéle.

Les experts (je pense en particulier à B. Toen, M. Vaquié et G.Vezzosi et aussi à D-C. Cisinski et G. Tabuada) s'attendaient à ce resultat.

On passe maintenant à la description de la troisième et dernière partie de cette thèse (Part III). Le but original a été d'étendre la pleine fidélité du corollaire précédent aux motifs sur une base plus générale. Dans le Chapitre 9 on explique comment étendre les théories des motifs pour un schéma de base quelconque, et aussi comment prolonger le pont entre les deux théories de façon naturelle. La stratégie était de démontrer l'existence d'un formalisme de six opérations pour la théorie des motifs non-commutatives, dans l'esprit des travaux de thèse de J. Ayoub [6, 7] dans le cadre commutatif. Pour cela nous avons besoin d'avoir un formalisme de six opérations dans le monde des ∞-catégories et de demontrer que les résultats d' Ayoub peuvent être étendus à ce monde. On s'occupe de cela dans le Chapitre 9-Section 9.4 en utilisent les techniques multi-simpliciales récemment développées dans [93, 94] (voir le Théorème 9.4.36). Dans le monde commutatif, c'était une importante observation de Voevodsky [41] qu'un tel formalisme serait une conséquence d'une propriété de localisation très simple.(Voir les Théorèmes 9.4.25 et 9.4.37).

Malheureusement nous ne sommes pas parvenu à démontrer cette propriété dans le monde non-commutatif et le dernier chapitre de cette thèse - Chapitre 10 - est devenu une description l'état actuel de nos efforts. Nous avons néanmoins réussi à réduire la preuve à deux problèmes fondamentaux. Le premièr (Conjecture 10.0.41) est complètement indépendant de la théorie motivique et ne concerne que la théorie des dg-catégories de type fini. On la vérifie dans le cas d'un dg-catégorie propre et lisse et aussi le cas d'un dg-catégorie d'origine géométrique. Le cas d'une dg-catégorie de type fini général reste inconnu. Le deuxième problème (Open Problem 10.0.45) concerne notre notion de topologie Nisnevich pour les dg-catégories. Contrairement à la situation géométrique, on n'a pas à notre disposition une description pratique des points de cette topologie. Le but du Chapitre 10 est d'expliquer comment une solution positive à ces deux problèmes implique l'existence du formalisme des six opérations dans le monde non-commutatif (Proposition 10.0.47). De plus, on verra que la conjecture est suffisant pour déduire la pleine fidélité entre les motifs et les motifs non-commutatifs sur une base lisse S sur un corps de caractéristique nulle (Corollaire 10.0.48).

Pour conclure, nous présentons au lecteur quelques directions de recherche nous aimerions poursuite dans l'avenir, en utilisant le formalism et les résultats développés dans cette thèse. Voir la section 1.5.

Contents

Chapter **1**

Introduction

1.1 Motivation

This thesis is part of a research project aiming to compare classical algebraic geometry with the new noncommutative algebraic geometry in the sense of Kontsevich [87]. More precisely, we want to compare the motivic levels of both theories.

1.2 Motives

In the original program envisioned by Grothendieck, the motive of a geometric object X (eg. X a projective smooth variety) was a new mathematical object designed to express “the arithmetical content of X” [1]. More precisely, in the sixties, Grothendieck and his collaborators started a quest to construct examples of the so-called Weil cohomology theories, designed to capture different arithmetic information about X. In the presence of many such theories he envisioned the existence of a universal one, which would gather all the arithmetic information. At that time, cohomology theories were formulated in a rather artificial way using abelian categories as the basic input. The notion of triangulated categories appeared as an attemptive to provided a new, more natural setting for cohomology theories. Of course, the subject of motives followed these innovations [13] and finally, in the 90's, V. Voevodsky [151] constructed what became known as "motivic cohomology". Many good introductory references to this arithmetic program are now available [3, 4, 102], together with the historical background given in the introduction of [30] as well as the recent course notes by B. Kahn [75].

In the late 90's, Morel and Voevodsky [105] developed a more general theory of motives. In their theory, the motive of X is designed to be the cohomological skeleton of X, not only in the eyes of a Weil cohomology theory, but for all the generalized cohomology theories for schemes (like K-theory, algebraic cobordism and motivic cohomology) at once. The inspiration comes from the stable homotopy theory of spaces where all generalized cohomology theories (of spaces) become representable. Such a setting would provide easier definitions for the motivic cohomology, algebraic K-theory, algebraic cobordism, and so on, by merely providing their representing spectra. Their construction has two main steps: the first part mimics the homotopy theory of spaces and its stabilization; the second part forces the "Tate motive" to become invertible with respect to the monoidal multiplication. The final result is known as the *motivic stable homotopy theory of schemes.* Our first main goal in this thesis is to formulate a precise universal property for their construction.

[1] for instance, it should capture the information underlying the L-function of X.

1.3 Noncommutative Algebraic Geometry

In Algebraic Geometry, and specially after the works of Serre and Grothendieck, it became a common practice to study a scheme X via its abelian category of quasi-coherent sheaves $Qcoh(X)$. The reason for this is in fact purely technical for at that time, abelian categories were the only formal background to formulate cohomology theories. In fact, the object $Qcoh(X)$ turns out to be a very good replacement for the geometrical object X: thanks to [54, 120] we know that X can be reconstructed from $Qcoh(X)$. However, it happens that abelian categories do not provide a very natural framework for homological algebra. It was Grothendieck who first noticed that this natural framework would be, what we nowadays understand as, the homotopy theory of complexes in the abelian category. At that time, the standard way to deal with homotopy theories was to consider their homotopy categories - the formal strict inversion of the weak-equivalences. This is how we obtain the derived category of the scheme $D(X)$. For many reasons, it was clear that the passage from the whole homotopy theory of complexes to the derived category loses too much information. The answer to this problem appeared from two different directions. First, from the theory of dg-categories [20, 24, 25]. More recently, an ultimately, with the theory of ∞-categories [5, 16, 99, 100, 127, 145]. The first subject became very popular specially with all the advances in [22, 23, 43, 44, 78, 132, 139]. The second, although initiated in the 80s with the famous manuscript [64], only in the last ten years reached a state where its full potential can be explored. This is specially due to the tremendous efforts of [99, 100]. Both subjects provide an appropriate way to encode the homotopy theory of complexes of quasi-coherent sheaves. In fact, the two approaches are related and, for our purposes, should give equivalent answers (see the recent results in [36] and our Section 6.2). Every scheme X (over a ring k) gives birth to a k-dg-category $L_{qcoh}(X)$ - *the dg-derived category of* X - whose "zero level" recovers the classical derived category of X. For reasonable schemes, this dg-category has an essential property deeply related to its geometrical origin - it has a compact generator and the compact objects are the perfect complexes (see [23] and [137]). It follows that the smaller sub-dg-category $L_{pe}(X)$ spanned by the compact objects is "affine", and enough to recover the whole $L_{qcoh}(X)$.

In his works [86, 87, 89], Kontsevich initiated a systematic study of the dg-categories with the same formal properties of $L_{pe}(X)$, with the observation that many different examples of such objects exist in nature: if A is an associative algebra then A can be considered as a dg-category with a single object and we consider $L(A)$ the dg derived category of complexes of A-modules and take its compact objects. The same works with a differential graded algebra. The *Fukaya* category of a sympletic manifold is another example [**?**]. There are also examples coming from complex geometry [110], representation theory, matrix factorizations (see [49]), and also from the techniques of deformation quantization. This variety of examples with completely different origins motivated the understanding of dg-categories as natural *noncommutative spaces*. The study of these dg-categories can be systematized and the assignment $X \mapsto L_{pe}(X)$ can be properly arranged as a functor

$$L_{pe} : \text{Classical Schemes}/k \longrightarrow \text{Noncommutative Spaces}/k \tag{1.3.1}$$

In fact, the functor L_{pe} is defined not only for schemes but for a more general class of geometrical objects, so called *derived stacks* (see [141, 14, 95]). They are the natural geometric objects in the theory of derived algebraic geometry of [143, 144, 98, 146]. For the purposes of noncommutative geometry, this fact is crucial: thanks to the results of Toën-Vaquié in [141], at the level of derived stacks, L_{pe} admits a right adjoint, providing a canonical mechanism to construct a geometric object out of a noncommutative one.

Kontsevich proposes also that similarly to schemes, these noncommutative spaces should admit a motivic theory. Our second main goal in this thesis is to provide a natural candidate for this theory, that extends in a natural way the theory of Morel-Voevodsky. The bridge between the two theories is a canonical extension of the map L_{pe} given by our universal characterization of the theory for schemes.

1.4 In this Thesis

The motivic construction of Morel-Voevodsky was originally obtained using the techniques of model category theory. Nowadays we know that a model category is merely strict presentation of a more fundamental object - an $(\infty, 1)$-category. Every model category has an underlying $(\infty, 1)$-category and the later is what really matters. It is important to say that the need for this passage overcomes the philosophical reasons and that thanks to the techniques of [99, 100] we now have the ways to do and prove things which would remain out of range only with the highly restrictive techniques of model categories.

The first part of our quest (Part I) concerns the universal characterization of the $(\infty, 1)$-category underlying the stable motivic homotopy theory of schemes, as constructed by Voevodsky and Morel, with its symmetric monoidal structure. The characterization becomes meaningful if we want to compare the motives of schemes with other theories. In our case, the goal is to conceive a theory of motives of noncommutative spaces and to relate it to the theory of Voevodsky-Morel. By providing such a universal characterization we will be able to ensure, for free, the existence of a (monoidal) dotted arrow at the motivic level

$$\begin{tikzcd} \text{Classical Schemes}/k \arrow[r] \arrow[d] & \text{NC-Spaces}/k \arrow[d] \\ \text{Stable Motivic Homotopy}/k \arrow[r, dashed] & \text{NC-Stable Motivic Homotopy}/k \end{tikzcd} \tag{1.4.1}$$

In general, monoidal maps such as the one here presented are extremely hard to obtain only by constructive methods and the techniques of model category theory. Other important advantage is that it allows us to work over any base scheme, not necessarily a field.

At this point we should also emphasize that a different approach to non-commutative motives already exists in the literature, due to D-C. Cisinski and G. Tabuada (see [133, 35, 130] and [134] for a pedagogical overview). Their approach is essentially of "'cohomological nature"' while our method could be said "'homological" and follows the spirit of stable homotopy theory. Later on in Chapter 8 we systematize the comparison between the two approaches and unveal a form of duality between them. It is exactly this duality phenomenon that makes our new approach comparable to the theory of Morel-Voevodsky and allows the dotted monoidal map to exist in a natural way. The same duality blocks a direct comparison in their case. We should also mention that all our mathematical contents and proofs are independent of theirs.

To achieve the universal characterization we will need to rewrite the constructions of Morel-Voevodsky in the setting of ∞-categories. The dictionary between the two worlds is given by the techniques of [99] and [100]. In fact, [99] already contains all the necessary results for the characterization of the $\mathbb{A}^1$-*homotopy theory of schemes* and its *stable non-motivic* version. The problem concerns the description of the stable motivic world with its symmetric monoidal structure. This is our main contribution in this subject. The key ingredient is the following:

Insight 1.4.1. *(see the Theorem 4.3.1 for the precise formulation):*
Let $\mathcal{V}$ be a combinatorial simplicial symmetric monoidal model category with a cofibrant unit and let $\mathcal{C}^{\otimes}$ denote its underlying symmetric monoidal ∞-category. Let X be a cofibrant object in $\mathcal{V}$ satisfying the following condition:

(*) the cyclic permutation of factors $\sigma = (123) : X \otimes X \otimes X \to X \otimes X \otimes X$ is equal to the identity map in the homotopy category of $\mathcal{V}$.[2]

[2] More precisely we demand the existence of an homotopy in $\mathcal{V}$ between the cyclic permutation and the identity.

Then the underlying symmetric monoidal ∞-category of $Sp^{\Sigma}(\mathcal{V}, X)$ is the universal symmetric monoidal $(\infty,1)$-category equipped with a monoidal map from $\mathcal{C}^{\otimes}$, sending X to an invertible object.

It is the goal of Chapter 4 to prove this theorem. This extra assumption on X is not new. It is already present in the works of Voevodsky ([150]) and it also appears in [71]. We must point out that we believe our result to be true even without this extra assumption on X. We will explain this in the Remark 4.3.2.

In Chapter 5 we apply the general results of Chapter 4 to the Motivic stable homotopy theory of schemes:

Corollary 1.4.2. *(Corollary 5.3.2) Let S be a base scheme and let $Sm^{ft}(S)$ denote the category of smooth separated schemes of finite type over S. The $(\infty,1)$-category $\mathcal{SH}(S)$ underlying the stable motivic homotopy theory of schemes is stable, presentable and admits a canonical symmetric monoidal structure $\mathcal{SH}(S)^{\otimes}$. Moreover, the construction of Morel-Voevodsky provides a functor $Sm^{ft}(S)^{\times} \to \mathcal{SH}(S)^{\otimes}$ monoidal with respect to the cartesian product of schemes, and endowed with the following universal property:*

$(*)$ for any pointed presentable symmetric monoidal $(\infty,1)$-category $\mathcal{D}^{\otimes}$, the composition map [3]

$$Fun^{\otimes,L}(\mathcal{SH}(S)^{\otimes}, \mathcal{D}^{\otimes}) \to Fun^{\otimes}(Sm^{ft}(S)^{\times}, \mathcal{D}^{\otimes}) \tag{1.4.2}$$

is fully faithful and its image consists of those monoidal functors $Sm^{ft}(S)^{\times} \to \mathcal{D}^{\otimes}$ satisfying Nisnevich descent, $\mathbb{A}^1$-invariance and such that the cofiber of the image of the point at ∞, $S \xrightarrow{\infty} \mathbb{P}^1_S$ is an invertible object in $\mathcal{D}^{\otimes}$. Moreover, any pointed presentable symmetric monoidal $(\infty,1)$-category $\mathcal{D}^{\otimes}$ admitting a monoidal map in this image, is necessarily stable.

This result trivializes the problem of finding motivic monoidal realizations. The existence of these have deep consequences. See [76] for an overview.

Example 1.4.3. Let $S = Spec(k)$ be field of characteristic zero. The assignment $X \mapsto \Sigma^{\infty}(X(\mathbb{C}))$ provides a functor $Sm^{ft}(S) \to Sp$ with Sp the $(\infty,1)$-category of spectra (see below). This map is known to be monoidal, to satisfy all the descent conditions in the previous corollary and to invert $\mathbb{P}^1$ in the required sense. Therefore, it extends in a essentially unique way to a monoidal map of stable presentable symmetric monoidal $(\infty,1)$-categories $\mathcal{SH}(S)^{\otimes} \to Sp^{\otimes}$;

Example 1.4.4. Again, let $S = Spec(k)$. Another immediate example of a monoidal motivic realization is the Hodge realization. Properly constructed, the map $X \mapsto C_{DR}(X)$ sending a scheme to its De Rham complex provides a functor $Sm^{ft}(S) \to \mathcal{D}(k)$ with $\mathcal{D}(k)$ the $(\infty,1)$-derived category of k. This map is known to be monoidal with respect to the cartesian product of schemes (Kunneth formula), satisfies all the descent conditions and inverts $\mathbb{P}^1$ in the sense above. Because of the universal characterization, it extends in a essentially unique way to a monoidal motivic Hodge Realization $\mathcal{SH}(S)^{\otimes} \to \mathcal{D}(k)^{\otimes}$ (where on the left we have the monoidal structure induced by the derived tensor product of complexes). This example has recently been worked out in detail in the PhD thesis of B. Drew [42]. One of its corollaries is a new Riemann-Hilbert correspondence for holonomic D-modules (see [42, Thm 3.4.1]).

Example 1.4.5. Our main theorem also provides a universal characterization for the G-equivariant version of motivic homotopy theory (in the sense of [73]). As proved in [73, Section 2.2 Lemma 2] we also fall in the situation of $\otimes$-inverting a symmetric object.

In the second part of this work (Part II) we systematize the comparison between the commutative and noncommutative motivic worlds. In Chapter 6, after some preliminaries on dg-categories, we

[3] see the notations in 2.1

introduce the $(\infty,1)$-category of smooth noncommutative spaces $\mathcal{N}cS(k)$ as the opposite of the $(\infty,1)$-category of idempotent dg-categories of finite type $\mathcal{D}g(k)^{ft} \subseteq \mathcal{D}g(k)^{idem}$ introduced by Toën-Vaquié in [141]. By introducing an appropriate noncommutative analogue for the Nisnevich topology (Def. 6.4.7) and considering the noncommutative version of the affine line $L_{pe}(\mathbb{A}^1)$, we construct a new stable presentable symmetric monoidal $(\infty,1)$-category $\mathcal{SH}_{nc}(S)^{\otimes}$ encoding a stable motivic homotopy theory for these noncommutative spaces. This provides a new approach to noncommutative motives.

The first step to compare the commutative and the non-commutative world is to encode the map $X \mapsto L_{pe}(X)$ as a functor L_{pe} from smooth affine schemes towards $\mathcal{N}cS(k)$ (see the Prop. 6.3.8). Our universal characterization of the stable motivic homotopy theory of schemes allows us to extend it to a monoidal colimit preserving functor

$$\begin{tikzcd} N(\mathit{AffSm}^{ft}(k)) \arrow[r, "L_{pe}"] \arrow[d] & \mathcal{N}cS(k) \arrow[d] \\ \mathcal{SH}(k) \arrow[r, dashed] & \mathcal{SH}_{nc}(k) \end{tikzcd} \tag{1.4.3}$$

In Chapter 7 we explain how this bridge can now be used to understand the different algebraic K-theories of dg-categories and schemes. To explain our main results in this topic we need some technical background. First, and as the reader shall later see (Section 5.4 and Remark 6.4.23), both $\mathcal{SH}(k)^{\otimes}$ and $\mathcal{SH}_{nc}(k)^{\otimes}$ can be obtained as a sequence of monoidal reflexive localizations of $(\infty,1)$-categories of spectral presheaves followed by the $\otimes$-inversion of the algebraic circle $\mathbb{G}_m$. With this in mind, the construction of the comparison map in the previous commutative diagram can be explained in a sequence of steps

$$\begin{tikzcd} N(\mathit{AffSm}^{ft}(k))^{\times} \arrow[r, "L_{pe}^{\otimes}"] \arrow[d, hook, "(\Sigma^{\infty}_{+}\circ j)^{\otimes}"] & \mathcal{N}cS(k)^{\otimes} \arrow[d, hook, "(\Sigma^{\infty}_{+}\circ j_{nc})^{\otimes}"] \\ \mathit{Fun}(N(\mathit{AffSm}^{ft}(k))^{op}, \mathit{Sp})^{\otimes} \arrow[r, dashed, "\mathcal{L}_1^{\otimes}"] \arrow[d, "l_{Nis}^{\otimes}"] & \mathit{Fun}(\mathcal{D}g(k)^{ft}, \mathit{Sp})^{\otimes} \arrow[d, "l_{Nis}^{nc,\otimes}"'] \\ \mathit{Fun}_{Nis}(N(\mathit{AffSm}^{ft}(k))^{op}, \mathit{Sp})^{\otimes} \arrow[r, dashed, "\mathcal{L}_2^{\otimes}"] \arrow[d, "l_{\mathbb{A}^1}^{\otimes}"] & \mathit{Fun}_{Nis}(\mathcal{D}g(k)^{ft}, \mathit{Sp})^{\otimes} \arrow[d, "l_{\mathbb{A}^1}^{nc,\otimes}"'] \\ \mathit{Fun}_{Nis,\mathbb{A}^1}(N(\mathit{AffSm}^{ft}(k))^{op}, \mathit{Sp})^{\otimes} \arrow[r, dashed, "\mathcal{L}_3^{\otimes}"] \arrow[d, "\Sigma_{\mathbb{G}_m}^{\otimes}"] & \mathit{Fun}_{Nis,L_{pe}(\mathbb{A}^1)}(\mathcal{D}g(k)^{ft}, \mathit{Sp})^{\otimes} \arrow[d, "\sim"] \\ \mathcal{SH}(k)^{\otimes} \arrow[r, dashed, "\mathcal{L}^{\otimes}"] & \mathcal{SH}_{nc}(k)^{\otimes} \end{tikzcd} \tag{1.4.4}$$

where each dotted map is induced by a universal property. By formal abstract nonsense these functors admit right-adjoints which we shall, respectively, denote as $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M}_3$ and $\mathcal{M}$.

This mechanism allows us to restrict noncommutative invariants to the commutative world.

Example 1.4.6. An important example of a noncommutative invariant is the Hochschild homology of dg-categories. Thanks to the works of B. Keller in [82] and as explained in the Remark 6.4.24 this invariant can be completely encoded by means of an ∞-functor $HH : \mathcal{D}g(k)^{ft} \to \widehat{Sp}$. Another important example is the so called *periodic cyclic homology of dg-categories HP*. It follows from the famous HKR theorem that the restriction of HP to the commutative world recovers the classical de Rham cohomology of schemes. For more details see the discussion in [17, Section 3.1].

Example 1.4.7. Another important example recently introduced by A. Blanc in his thesis [17] is the topological K-theory of dg-categories. This is a candidate for the non-commutative version of the Betti realization.

In Chapter 7 we will be interested in the restriction of the various algebraic K-theories of dg-categories. As we shall explain below, all of them live as objects in $Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp})$. There are two of primary relevance to us:

- K^c - encoding the *connective* K-theory given by Waldhausen's S-construction. See the discussion in Section 7.1.2 below.
- K^S - encoding the *non-connective* K-theory of dg-categories as defined in [34] using the adaptation of the Schlichting's framework of [122] to the context of dg-categories. (see the discussion in Section 7.1.3). By construction, this functor comes naturally equipped with a canonical natural transformation $K^c \to K^S$ which is an equivalence in the connective part.

For the first one, it follows immediately from the spectral version of Yoneda lemma (see the Remark 5.4.1) and from the definition in [137, Section 3] that $\mathcal{M}_1(K^c)$ recovers the connective algebraic K-theory of schemes. The second one, by the comparison result [122, Theorem 7.1], recovers the non-connective K-theory of schemes of Bass-Thomason-Trobaugh of [137]. The construction of K^S in [34] using the methods of [122] is somehow ad-hoc. We explain how the non-connective version of K-theory K^S can be canonically obtained from the connective version K^c as a result of enforcing our noncommutative-world version of Nisnevich descent. The following theorem summarizes our main technical results in this topic:

Theorem 1.4.8.

(i) (Thm. 7.0.29) The canonical morphism $K^c \to K^S$ presents non-connective K-theory as the (noncommutative) Nisnevich sheafification of connective K-theory;

(ii) (Thm. 7.0.31) The further (noncommutative) $\mathbb{A}^1$-localization $l^{nc}_{\mathbb{A}^1}(K^S)$ is a unit 1_{nc} for the monoidal structure in $\mathcal{SH}_{nc}(k)^{\otimes}$;

(iii) (Thm. 7.0.32) The image of $l^{nc}_{\mathbb{A}^1}(K^S)$ along the right-adjoint $\mathcal{M}$ recovers the object KH in $\mathcal{SH}(k)$ representing $\mathbb{A}^1$-invariant algebraic K-theory of Weibel (also known as homotopy invariant K-theory) studied in [150] and in [29]. In particular, since $\mathcal{M}$ is lax monoidal (it is right-adjoint to a monoidal functor) it sends the trivial algebra structure in 1_{nc} to a commutative algebra structure in KH so that the monoidal map $\mathcal{L}^{\otimes}$ factors as

$$\mathcal{SH}(k)^{\otimes} \xrightarrow{-\otimes KH} Mod_{KH}(\mathcal{SH}(k))^{\otimes} \dashrightarrow \mathcal{SH}_{nc}(k)^{\otimes}$$

Let us emphasize that the part (i) of this theorem is not true if we restrict ourselves to the non-connective K-theory of schemes. The phenomenon that makes it possible in the noncommutative world is the fact the new notion of Nisnevich squares of noncommutative spaces combines at the same time coverings of geometrical origin (namely, those coming via L_{pe} from classical Nisnevich squares) and coverings of categorical origin, namely, the ones induced by exceptional collections.

The first part of this theorem is proved by showing that the Bass-construction $(-)^B$ given in Thomason's paper [137] is an explicit model for the (noncommutative) Nisnevich localization of presheaves with values in connective spectra and sending Nisnevich squares of noncommutative spaces to pullback squares in connective spectra. Recall that the inclusion of connective spectra in all spectra does not preserve pullbacks. More generally, we prove that the connective truncation functor induces an equivalence of $(\infty,1)$-categories between the $(\infty,1)$-category of Nisnevich local spectral presheaves and the $(\infty,1)$-category of spectral presheaves with values in connective spectra and satisfying connective Nisnevich descent. The $(-)^B$-construction is an explicit inverse to this truncation. The second result uses a fundamental result of A. Blanc in his Phd Thesis [17, Prop. 4.6], namely, that the split version of the Waldhausen S-construction is $\mathbb{A}^1$-homotopy equivalent to the full S-construction. To prove this last part we also show that the commutative and noncommutative versions of the $\mathbb{A}^1$-localizations are compatible with the right adjoints.

The following corollary provides a new formalization of a result understood by Kontsevich [87, 89] long ago and also already satisfied by the formalism of Cisinski-Tabuada.

Corollary 1.4.9. *(see Cor. 7.0.33) Let $\mathcal{X}$ and $\mathcal{Y}$ be two noncommutative smooth spaces and assume that $\mathcal{Y}$ is smooth and proper. Then we have an equivalence of spectra*

$$Map_{\mathcal{SH}_{nc}(k)}(\mathcal{X},\mathcal{Y}) \simeq l^{nc}_{\mathbb{A}^1}(K^S)(\mathcal{X}\otimes\mathcal{Y}^{op})$$

where $\mathcal{Y}^{op}$ is the dual of $\mathcal{Y}$ and we have identified $\mathcal{X}$ and $\mathcal{Y}$ with their images in $\mathcal{SH}_{nc}(k)$.

Finally, the following result was the main goal of this thesis. It follows from the previous theorem together with the results of J. Riou describing the compact generators in $\mathcal{SH}(k)$ over a field with resolutions of singularites (see [117]).

Corollary 1.4.10. *(see Cor. 7.0.37) If k is a field admitting resolutions of singularities then the canonical factorization*

$$Mod_{KH}(\mathcal{SH}(k))^{\otimes} \dashrightarrow \mathcal{SH}_{nc}(k)^{\otimes}$$

is fully faithful.

This result has been expected and known to some people after a while. I think particularly of B. Toen, M. Vaquie and G.Vezzosi and also D-C. Cisinski and G. Tabuada.

Let us now describe the goals of the third and final part of this thesis (Part III). Our original aim in this part was to extend the fully-faithfulness result to the bridge between motives and non-commutative motives over a more general basis. In Chapter 9 we explain how to define motives and non-commutative motives over a general base scheme and how to naturally extend the bridge. Our strategy was to proof the existence of a formalism of six operations for the theory of non-commutative motives, in the spirit of the same result well-known for commutative motives proved in J. Ayoub's thesis [6, 7]. For that purpose we needed to have a formalism of six operations in the world of ∞-categories and to prove that the results of Ayoub lift to this world. This is done in Chapter 9-Section 9.4 using the techniques of multi-simplicial sets recently developed in [93, 94][4]. See the Theorem 9.4.36. It was a key insight of Voevodsky [41] that such a formalism follows almost entirely from a very basic localization property (see the Theorems 9.4.25 and 9.4.37).

Unfortunately we were not able to accomplish the proof of this property in the non-commutative world and the last chapter of this thesis - Chapter 10 - became an attempt to describe the current status of our efforts. We were able to reduce the proof to two basic statements. The first one (Conjecture 10.0.41) is completely independent of the motivic theory and concerns only the theory of dg-categories of finite type in the sense of [141]. We could confirm it for smooth and proper dg-categories and also for dg-categories of geometric origin but the general case of a dg-category of finite type remains unknown. The second statement (Open Problem 10.0.45) concerns our notion of Nisnevich topology for dg-categories. Contrary to the geometric situation, we don't know any nice description of its points. We explain how a positive solution to the two problems implies the existence of the formalism of six operations in the non-commutative world

Proposition 1.4.11. *(Proposition 10.0.47) The Conjecture 10.0.41 together with a positive solution to the Open Problem 10.0.45 implies the existence of a formalism of six operations for non-commutative motives over smooth quasi-projective schemes over a field of characteristic zero.*

Moreover,

[4] A different approach is currently being developed by D. Gaitsgory and Nick Rozenblyum. Their main strategy is to use the universal property of the $(\infty, 2)$-category of correspondences.

Proposition 1.4.12. *(Prop. 10.0.48) The Conjecture 10.0.41 alone implies that for any regular scheme S over a field of characteristic zero the natural map from motives to non-commutative motives*

$$Mod_{KH}(\mathcal{SH}(S))^{\otimes} \dashrightarrow \mathcal{SH}_{nc}^{\otimes}(S)$$

is fully faithful.

This last result is also a reason why our approach to non-commutative motives seems necessary. The methods of Cisinski-Tabuada cannot be extended to general base schemes because of the duality phenomenom that distinguishes the two approaches.

1.5 Future Research Directions

To conclude this introduction we would like to outline some possible research directions that we would like to investigate in the future, using the materials developed in this thesis.

The first application we would like to mention concerns the study of motives of Deligne-Munford stacks. If X is a Deligne-Munford stack over a field k of characteristic zero then our methods allow us to assign to it an object in $\mathcal{SH}(k)$. Namely, X has a naturally associated non-commutative motive - the non-commutative motive of the dg-category $L_{pe}(X)$. Our framework allows us to restrict this object to the commutative world and produce a module over K-theory, $\mathcal{M}(L_{pe}(X)) \in \mathcal{SH}(k)$. In particular, we hope that the decomposition of the inertia stack of X used in [138] can be also applied in this motivic context. This would provide a decomposition of the non-commutative motive of X in terms of pieces of geometric origin. The advantage of this decomposition is that it does not depend on any assumptions on the existence of semi-orthogonal decompositions for $L_{pe}(X)$. The important new ingredient is the fully-faithfulness of the brigde between the two motivic worlds (Cor. 7.0.37).

The other research directions we have in mind make a crutial use of our new theory of non-commutative motives over general base schemes.

The first application concerns the construction of a non-commutative mixed Hodge realization functor. In the commutative case this was studied in the [42] already usng the new universal property proved in this thesis (Corollary 5.3.2). In [81], the authors introduced the notion of a *noncommutative Hodge Structure*. Recall that thanks to the famous theorem HKR, the *Periodic Cyclic Homology* $HP_\bullet(X)$ provides the correct noncommutative analogue of the classical de Rham cohomology. They formulate the following conjecture:

(*) If X is a "good enough" noncommutative space then $HP_\bullet(X)$ carries a noncommutative Hodge Structure;

Said in a different way, $HP_\bullet$ should provide a functor from noncommutative spaces to noncommutative Hodge-structures. We should then expect this functor to factor through our new noncommutative version of the motivic stable homotopy theory because of its universal property. More generally, we expect our main commutative diagram to fit in a larger one

$$\begin{tikzcd}
\text{Classical Schemes}/k \arrow[r, "L_{pe}"] \arrow[d] \arrow[ddr, "H_{DR}(-)"] & \text{NC-Schemes}/k \arrow[d] \arrow[ddr, "HP_\bullet(-)"] & \\
\text{Stable Motivic Homotopy}/k \arrow[r, dashed] \arrow[dr, dashed, "univ\ prop."] & \text{NC-Stable Motivic Homotopy}/k \arrow[dr, dashed, "univ\ prop."] & \\
 & \text{Classical Hodge-Structures} \arrow[r] & \text{NC-Hodge Structures}
\end{tikzcd} \tag{1.5.1}$$

where the map from the classical to the noncommutative Hodge structures was introduced in [81][5]. The diagonal maps are known as the *Hodge-realizations functors*: the commutative case is known to the experts (see [118] for a survey of the main results); the noncommutative case is given by the conjecture $(*)$. This conjecture can be divided in two parts: the first concerns the de Rham part (see [79]) and the second is related to the Betti part. In his thesis A. Blanc constructed a candidate for the second [17].

The general methods developed in this thesis can now be use to talk about a mixed version of these Hodge structures. In particular, we can hope this non-commutative mixed Hodge realization to be compatible with the one of [42].

A second application we have in mind is the construction a formalism of vanishing cycles for non-commutative motives, in the spirit of the one developped in J. Ayoub's thesis [7] for the motivic stable homotopy theory of schemes. Part of this formalism needs the existence of the six operations and this was actually one of the main reasons for our interest in them. In the commutative world the motivic vanishing cycles formalism is compatible with the Betti and the l-adic realizations [8]. One expects the same compatibilitity in the non-commutative world. Moreover, there are deep reasons to believe that such a formalism is closely related to the theory of 2-periodic dg-categories of matrix factorization. See [111] and the results in [50].

The last research direction that we would like to mention concerns the relation between Azumaya algebras, twisted forms of K-theory and motives. Here it is also crutial our formalism of non-commutative motives over a general basis. Let X be a scheme quasi-compact and quasi-separated over a field of characteristic zero having an ample line bundle. By a theorem of O.Gabber every torsion element in $H^2_{et}(X, \mathbb{G}_m)$ is represented by an Azumaya algebra - a sheaf of $\mathcal{O}_X$-algebras A, locally free and of finite rank with the property that the natural map $A^{op} \otimes_{\mathcal{O}_X} A \to Hom_{\mathcal{O}_X}(A, A)$ is an equivalence. In [77] B. Kahn and M. Levine introduced two different objects in $\mathcal{SH}(k)$ attached to an Azumaya algebra A over $X = Spec(k)$ - from one side a twisted form of the spectrum representing algebraic K-theory and in a more geometric flavour, the motive of the Severi-Brauer variety associated to A. They compare the motivic slices of these two objects. At the same time, also over a $X = Spec(k)$, the approach of non-commutative motives of Cisinski-Tabuada allows us to attach a non-commutative motive over k to every Azumaya algebra (over k) and it is immediate to see that the restriction of this motive to the commutative world recovers the twisted form of K-theory of Levine-Kahn [136].

Our results in this thesis allow us to vastly extend these results to base schemes more general than a field. Recently, in [140, Theorem 0.1], B. Toën extended the key result of O. Gabber: every element in $H^2_{et}(X, \mathbb{G}_m)$ (not necessarily torsion) is represented by a *derived Azumaya algebra* on X. As defined in loc. cit, these are sheaves A of $\mathcal{O}_X$-dg-algebras such that: (i) the underlying complex of A is perfect complex on X; (ii) for every affine $U = Spec(R)$ open subscheme of X the restriction of A to U is a compact generator of the derived category of R and (iii) the natural morphism of complexes of $\mathcal{O}_X$-modules $A^{op} \otimes^{\mathbb{L}}_{\mathcal{O}_X} A \to \mathbb{R}\underline{Hom}_{\mathcal{O}_X}(A, A)$ is a quasi-isomorphism of complexes. As explained in [140, Prop. 1.5] this definition is equivalent to that of $\otimes$-invertible dg-category over X. Every such object, as explained in the Remark 9.3.6, produces then a $\otimes$-invertible non-commutative motive over X which we can restrict to $\mathcal{SH}(X)$. This procedure attaches a motive over X to every element in $H^2_{et}(X, \mathbb{G}_m)$, namely, a twisted form of the algebraic K-theory spectrum. One can now, as in [77], hope to understand the geometric features of this motive.

[5] Of course we should only expected the part of the diagram concerning the Hodge Theory to work if we restrict to a good class of schemes over k

Preliminaries

Chapter **2**

Preliminaries I: Higher Category Theory

In this chapter we set our notations and review the main notions and tools from Higher Category Theory of [99], together with the mechanism to pass from the world of model categories to $(\infty, 1)$-categories. These tools will be used all along this thesis. Classical categories will be called 1-categories.

2.1 Notations and Categorical Preliminaries

2.1.1 Quasi-Categories

The theory of $(\infty, 1)$-categories has been deeply explored over the last years and we now have many different models to access them. In this article we follow the approach of [99, 100], using the model provided by Joyal's theory of Quasi-Categories [74]. In this sense, the two notions of *quasi-category* and $(\infty, 1)$-*category* will be identified throughout the text. Recall that the Joyal's model structure is a combinatorial, left proper, cartesian closed model structure in the category of simplicial sets $\hat{\Delta}$, for which the cofibrations are the monomorphisms and the fibrant objects are the quasi-categories - by definition, the simplicial sets $\mathcal{C}$ with the lifting property

$$\begin{tikzcd} \Lambda^k[n] \arrow[r, "f"] \arrow[d, hook] & \mathcal{C} \\ \Delta[n] \arrow[ur, dashed] & \end{tikzcd} \tag{2.1.1}$$

for any inclusion of an inner-horn $\Lambda^k[n] \subseteq \Delta[n]$ with $0 < k < n$ and any map f.

For a quasi-category $\mathcal{C}$, we will follow [99] and write $Obj(\mathcal{C})$ for the set of zero-simplexes of $\mathcal{C}$; given two objects $X, Y \in Obj(\mathcal{C})$ we let $Map_{\mathcal{C}}(X, Y)$ denote the Kan complex *Mapping Space between X and Y* and finally we let $h(\mathcal{C})$ denote the homotopy 1-category of $\mathcal{C}$. Moreover, as in [74, 99] the term *categorical equivalence* will refer to a weak-equivalence of simplicial sets for the Joyal's model structure.

2.1.2 Universes

In order to deal with the set-theoretical issues we will follow the approach of Universes (our main reference being the Appendix by Nicolas Bourbaki in [1]). We will denote them as $\mathbb{U}$, $\mathbb{V}$, $\mathbb{W}$, etc. Moreover, we adopt a model for set theory where every set is *artinian*. In this case, for every strongly inaccessible cardinal [1] κ, the collection $\mathbb{U}(\kappa)$ of all sets of rank [2] $< \kappa$ is a set and satisfies the axioms

[1]Recall that a cardinal κ is called strongly inaccessible if it is regular (meaning, the sum $\sum_i \alpha_i$ of strictly smaller cardinals $\alpha_i < \kappa$ with $i \in I$ and $card(I) < \kappa$, is again strictly smaller than κ, which is the same as saying that κ is not the sum of cardinals smaller than κ) and if for any strictly smaller cardinal $\alpha < \kappa$, we have $2^\alpha < \kappa$

[2]Recall that a set X is said to have rank smaller than κ if the cardinal of X is smaller than κ and for any succession of memberships $X_n \in X_{n-1} \in ... \in X_0 = X$, every X_i has cardinal smaller than κ.

of a Universe. The correspondence $\kappa \mapsto \mathbb{U}(\kappa)$ establishes a bijection between strongly inaccessible cardinals and Universes, with inverse given by $\mathbb{U} \mapsto card(\mathbb{U})$. We adopt the *axiom of Universes* which allows us to consider every set as a member of a certain universe (equivalently, every cardinal can be strictly upper bounded by a strongly inaccessible cardinal). We will also adopt the *axiom of infinity*, meaning that all our universes will contain the natural numbers $\mathbb{N}$ and therefore $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$. Whenever necessary we will feel free to enlarge the universe $\mathbb{U} \in \mathbb{V}$. This is possible by the axiom of Universes.

Let $\mathbb{U}$ be an universe. As in [1] we say that a mathematical object T is $\mathbb{U}$-small (or simply, *small*) if all the data defining T is collected by sets isomorphic to elements of $\mathbb{U}$. For instance, a set is $\mathbb{U}$-small if it is isomorphic to a set in $\mathbb{U}$; a category is $\mathbb{U}$-small if both its collection of objects and morphisms are isomorphic to sets in $\mathbb{U}$; a simplicial set X is $\mathbb{U}$-small if all its level sets X_i are isomorphic to elements in $\mathbb{U}$, etc. A mathematical object T is called *essentially small* if is equivalent (in a context to be specified) to a $\mathbb{U}$-small object. A category $\mathcal{C}$ is called *locally* $\mathbb{U}$*-small* (resp. locally essentially $\mathbb{U}$-small) if its hom-sets between any two objects are $\mathbb{U}$-small (resp. essentially small).[3] We define the category of $\mathbb{U}$-sets as follows: the collection of objects is $\mathbb{U}$ and the morphisms are the functions between the sets in $\mathbb{U}$. It is locally small. Another example is the category of $\mathbb{U}$-small categories $Cat_{\mathbb{U}}$ whose objects are the $\mathbb{U}$-small categories and functors between them. Another important example is given by $\hat{\Delta}_{\mathbb{U}}$ the category of $\mathbb{U}$-small simplicial sets. Again, it is locally small and, together with the Joyal model structure (see [74]) it forms a $\mathbb{U}$-combinatorial model category (in the sense of [143]) and its cofibrant-fibrant objects are the $\mathbb{U}$-small $(\infty,1)$-categories.

Consider now an enlargement of universes $\mathbb{U} \in \mathbb{V}$. In this case, it follows from the axiomatics that every $\mathbb{U}$-small object is also $\mathbb{V}$-small. With a convenient choice for $\mathbb{V}$, the collection of all $\mathbb{U}$-small $(\infty,1)$-categories can be organized as a $\mathbb{V}$-small $(\infty,1)$-category, $Cat_{\infty}^{\mathbb{U}}$ (See [99, Chapter 3] for the details). We have a canonical inclusion $\hat{\Delta}_{\mathbb{U}} \subseteq \hat{\Delta}_{\mathbb{V}}$ which is compatible with the Joyal Model structure. Again, through a convenient enlargement of the universes $\mathbb{U} \in \mathbb{V} \in \mathbb{W}$, we have an inclusion of $\mathbb{W}$-small $(\infty,1)$-categories $Cat_{\infty}^{\mathbb{U}} \subseteq Cat_{\infty}^{\mathbb{V}}$[4]. We say that a $\mathbb{V}$-small $(\infty,1)$-category is *essentially* $\mathbb{U}$*-small* if it is weak-equivalent in $\hat{\Delta}_{\mathbb{V}}$ to a $\mathbb{U}$-small simplicial set. Thanks to [99, 5.4.1.2], the following conditions are equivalent for a $\mathbb{V}$-small $(\infty,1)$-category $\mathcal{C}$: (i) $\mathcal{C}$ is essentially $\mathbb{U}$-small; *(ii)* $card(\pi_0(\mathcal{C})) < card(\mathbb{U})$ and $\mathcal{C}$ is locally small, which means that for any two objects X and Y in $\mathcal{C}$, we have $card(\pi_i(Map_{\mathcal{C}}(X,Y))) < card(\mathbb{U})$; *(iii)* $\mathcal{C}$ is a $card(\mathbb{U})$-compact object in $Cat_{\infty}^{\mathbb{V}}$ (see 2.1.13 below).

Some constructions require us to control "how small" our objects are. Given a cardinal τ in the universe $\mathbb{U}$, we will say that a small simplicial set K is τ-small if a fibrant-replacement $\mathcal{C}$ of K satisfies the conditions above, replacing $card(\mathbb{U})$ by τ.

The category of $\mathbb{U}$-small simplicial sets can also be endowed with the standard Quillen model structure (see [69]) and it forms a $\mathbb{U}$-combinatorial simplicial model category in which the fibrant-cofibrant objects are the $\mathbb{U}$-small *Kan-complexes*. They provide models for the homotopy types of $\mathbb{U}$-small spaces and following the ideas of the Section 2.2 we can collect them in a new $(\infty,1)$-category $\mathcal{S}_{\mathbb{U}}$. Again we can enlarge the universe $\mathbb{U} \in \mathbb{V}$ and produce inclusions $\mathcal{S}_{\mathbb{U}} \subseteq \mathcal{S}_{\mathbb{V}}$.

Throughout the text we will fix three universes $\mathbb{U} \in \mathbb{V} \in \mathbb{W}$ with $\mathbb{V}$ chosen conveniently large and $\mathbb{W}$, very large. In general, we will work with the $\mathbb{V}$-small simplicial sets and the $\mathbb{U}$-small objects will be refered to simply as *small*. In order to simplify the notations we write Cat_{∞} (resp. $\mathcal{S}$) to denote the $(\infty,1)$-category of small $(\infty,1)$-categories (resp. spaces). With our convenient choice for $\mathbb{V}$, both of them are $\mathbb{V}$-small. The third universe $\mathbb{W}$ is assumed to be sufficiently large so that we have $\mathbb{W}$-small simplicial sets Cat_{∞}^{big} (resp. $\widehat{\mathcal{S}}$) to encode the $(\infty,1)$-category of all the $\mathbb{V}$-small $(\infty,1)$-categories (resp.

[3] Notice that this definition is not demanding any smallness condition on the collection of objects and therefore a locally small category does not need to be small

[4] $Cat_{\infty}^{\mathbb{U}}$ is $\mathbb{V}$-small and so it is also $\mathbb{W}$-small

spaces).

2.1.3 Fibrations of Simplicial Sets

Let $p : X \to Y$ map of simplicial sets. We say p is a *trivial fibration* if it has the right-lifting property with respect to every monomorphism of simplicial sets. We say p is a categorical fibration if it is a fibration for the Joyal model structure. We say p is an $inner fibration$ if it has the right-lifting property with respect to every inclusion $\Lambda^k[n] \subseteq \Delta[n]$, with $0 < k < n$. We have

$$\{\text{trivial fibrations}\} \subseteq \{\text{categorical fibrations}\} \subseteq \{\text{inner fibrations}\} \tag{2.1.2}$$

2.1.4 Categories of Functors

The Joyal model structure is cartesian closed (see [74] or [99, Cor. 2.3.2.4]). In particular, if $\mathcal{C}$ and $\mathcal{D}$ are $(\infty, 1)$-categories in a certain universe, the internal-hom in $\hat{\Delta}$, $Fun(\mathcal{C}, \mathcal{D}) := \underline{Hom}_{\hat{\Delta}}(\mathcal{C}, \mathcal{D})$ is again an $(\infty, 1)$-category in the same universe (See [99, Prop. 1.2.7.3]). It provides the good notion of $(\infty, 1)$-category of functors between $\mathcal{C}$ and $\mathcal{D}$;

2.1.5 Diagrams

Let $\mathcal{C}$ be an $(\infty, 1)$-category and let K be a simplicial set. A diagram in $\mathcal{C}$ indexed by K is a map of simplicial sets $d : K \to \mathcal{C}$. We denote by $K^{\triangleright}$ (resp. $K^{\triangleleft}$) the simplicial set $K * \Delta[1]$ (resp. $\Delta[1] * K$) where $*$ is the join operation of simplicial sets (see [99, Section 1.2.8]).

If $\mathcal{C}$ is an $(\infty, 1)$-category, a commutative square in $\mathcal{C}$ is a diagram $d : \Delta[1] \times \Delta[1] \to \mathcal{C}$. This is the same as a map $\Lambda^0[2]^{\triangleright} \to \mathcal{C}$. One can easily check that $\Delta[1] \times \Delta[1]$ has four 0-simplexes A,B,C, D; five non-degenerated 1-simplexes f,g,h,u,v and four important 2-simplexes α, β, γ, σ (from which α and σ are non-degenerated) which we can picture together as

$$\begin{tikzcd}
A \arrow[r, equal, "id"] \arrow[d, "u"'] \arrow[dr] \arrow[d, phantom, shift left=2, "\alpha"] & A \arrow[r, "f"] \arrow[d, "\gamma"'] \arrow[dr] & B \arrow[d, "g"] \arrow[d, phantom, shift right=3, "\sigma"] \\
D \arrow[r, "v"] \arrow[ur, Rightarrow] & C \arrow[r, equal, "\beta", "id"'] \arrow[ur, Rightarrow] & C
\end{tikzcd} \tag{2.1.3}$$

where all the inner 1-simplexes are given by h. Since $\mathcal{C}$ is an $(\infty, 1)$-category, we can use the lifting property to show that the data of a commutative diagram in $\mathcal{C}$ is equivalent to the data of two triangles

$$\begin{tikzcd}
A \arrow[d, "u"'] \arrow[dr, "v \circ u"] \arrow[d, phantom, shift left=2, "\beta"] & \\
D \arrow[r, "v"] \arrow[ur, Rightarrow] & C
\end{tikzcd} \qquad \begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "g \circ f"'] & B \arrow[d, "g"] \arrow[dl, Rightarrow, "\alpha"'] \\
& C
\end{tikzcd} \tag{2.1.4}$$

together with a map $r : A \to C$ and two-cells providing homotopies between $g \circ f \simeq r \simeq v \circ u$.

2.1.6 Comma-Categories

If $\mathcal{C}$ is an $(\infty, 1)$-category and X is an object in $\mathcal{C}$, there are $(\infty, 1)$-categories $\mathcal{C}_{/X}$ and $\mathcal{C}_{X/}$ where the objects are, respectively, the morphisms $A \to X$ and $X \to A$. More generally, if $p : K \to \mathcal{C}$ is a diagram in $\mathcal{C}$ indexed by a simplicial set K, there are $(\infty, 1)$-categories $\mathcal{C}_{/p}$ and $\mathcal{C}_{p/}$ of cones (resp cocones) over the diagram. These $(\infty, 1)$-categories are characterized by an universal property - see for instance [99, 1.2.9.2].

2.1.7 Limits and Colimits

Let $\mathcal{C}$ be an $(\infty, 1)$-category. An object $E : \Delta[0] \to \mathcal{C}$ is said to be *initial* (resp. *final*) if for every object Y in $\mathcal{C}$ the mapping space $Map_{\mathcal{C}}(E, Y)$ (resp. $Map_{\mathcal{C}}(Y, E)$) is weakly contractible (see [99, 1.2.12.1, 1.2.12.3, 1.2.12.5]).

Let $\mathcal{C}$ be an $(\infty, 1)$-category and let $K \to \mathcal{C}$ be a diagram in $\mathcal{C}$. A *colimit* (resp. *limit*) for a diagram $d : K \to \mathcal{C}$ is an initial (resp. final) object in the category $\mathcal{C}_{p/}$ (resp. $\mathcal{C}_{/p}$). By the universal property defining the comma-categories, this corresponds to the data of a new diagram $\bar{d} : K^{\triangleright} \to \mathcal{C}$ (resp. $K^{\triangleleft} \to \mathcal{C}$) extending d and satisfying the universal property of [99, 1.2.13.5]. Whenever appropriate, we will also use the relative notions of limits and colimits (see [99, 4.3.1.1]).

Following [99, 4.1.1.1, 4.1.1.8], we say that a map of simplicial sets $\phi : K' \to K$ is *cofinal* if for every $(\infty, 1)$-category $\mathcal{C}$ and every colimit diagram $K^{\triangleleft} \to \mathcal{C}$, the composition with ϕ, $(K')^{\triangleleft} \to \mathcal{C}$ remains a colimit diagram.

We will say that an $(\infty, 1)$-category *has all small colimits* (resp. *limits*) if every diagram in $\mathcal{C}$ indexed by a small simplicial set has a colimit (resp. limit) in $\mathcal{C}$. As in the classical situation, $\mathcal{C}$ has all κ-small colimits (resp. limits) if and only if it has all κ-small coproducts and all pushouts exist [99, 4.4.2.6] (resp. κ-small products and pullbacks). In particular, it has an initial (resp. final) object.

If $\mathcal{C}$ is an $(\infty, 1)$-category having colimits of a certain kind, then for any simplicial set S, the $(\infty, 1)$-category $Fun(S, \mathcal{C})$ has colimits of the same kind and they can computed objectwise in $\mathcal{C}$ [99, 5.1.2.3].

If $\mathcal{C}$ and $\mathcal{D}$ are $(\infty, 1)$-categories with colimits we will denote by $Fun^L(\mathcal{C}, \mathcal{D})$ the full subcategory of $Fun(\mathcal{C}, \mathcal{D})$ spanned by those functors which commute with colimits.

We say that an $(\infty, 1)$-category is *pointed* if it admits an object which is simultaneously initial and final. Given an arbitrary $(\infty, 1)$-category with a final object $*$, we consider the comma-category $\mathcal{C}_* := \mathcal{C}_{*/}$. This is pointed. Moreover there is a canonical forgetful morphism $\mathcal{C}_* \to \mathcal{C}$ which commutes with limits.

2.1.8 Subcategories

If $\mathcal{C}$ is an $(\infty, 1)$-category, and $\mathcal{O}$ is a subset of objects and $\mathcal{F}$ is a subset of edges between the objects in $\mathcal{O}$, the *subcategory of* $\mathcal{C}$ spanned by the objects in $\mathcal{O}$ together with the edges in $\mathcal{F}$ (closed under composition and containing all identity maps) is the new $(\infty, 1)$-category $\mathcal{C}_{\mathcal{O},\mathcal{F}}$ obtained as the pullback of the diagram

$$\begin{tikzcd} \mathcal{C}_{\mathcal{O},\mathcal{F}} \arrow[r, "i"] \arrow[d] & \mathcal{C} \arrow[d] \\ N(h(\mathcal{C})_{\mathcal{O},\mathcal{F}}) \arrow[r] & N(h(\mathcal{C})) \end{tikzcd} \tag{2.1.5}$$

where the lower map is the nerve of the inclusion of the subcategory $h(\mathcal{C})_{\mathcal{O},\mathcal{F}}$ of $h(\mathcal{C})$, spanned by the objects in $\mathcal{O}$ together with the morphisms in $h(\mathcal{C})$ represented by the edges in $\mathcal{F}$. The right-vertical map is the unit of the adjunction (h, N). It follows immediately from the definition that $\mathcal{C}_{\mathcal{O},\mathcal{F}}$ will also be an $(\infty, 1)$-category.

2.1.9 Grothendieck Construction

We recall the existence of a *Grothendieck Construction* for $(\infty, 1)$-categories (See [99, Chapter 3]). Thanks to this, we can present a functor between two $(\infty, 1)$-categories $\mathcal{C} \to \mathcal{D}$ as a *cocartesian* *fibration* (see [99, 2.4.2.1]) $p : \mathcal{M} \to \Delta[1]$ with $p^{-1}(\{0\}) = \mathcal{C}$ and $p^{-1}(\{1\}) = \mathcal{D}$. Using this machinery, the data of an adjunction between $\mathcal{C}$ and $\mathcal{D}$ corresponds to a *bifibration* $\mathcal{M} \to \Delta[1]$ (see the proof of [99, 5.2.1.4] to understand how to extract a pair of functors out of a bifibration, using the model structure on marked simplicial sets);

2.1.10 Adjointable Squares

Let $\sigma :=$

$$\begin{tikzcd} \mathcal{C}_1 \arrow[r, "f_1"] \arrow[d, "g_1"] & \mathcal{C}_2 \arrow[d, "g_2"] \\ \mathcal{D}_1 \arrow[r, "f_2"] & \mathcal{D}_2 \end{tikzcd} \tag{2.1.6}$$

be a homotopy commutative diagram of $(\infty, 1)$-categories. Recall that σ is said to be left *adjointable* (see [99, 7.3.1.2] or [100, 4.7.5.13]) if the functors f_1 and f_2 admit left-adjoints u_1, u_2 and the diagram $\sigma' :=$

$$\begin{tikzcd} \mathcal{C}_1 \arrow[d, "g_1"] & \mathcal{C}_2 \arrow[l, "u_1"'] \arrow[d, "g_2"] \\ \mathcal{D}_1 & \mathcal{D}_2 \arrow[l, "u_2"'] \end{tikzcd} \tag{2.1.7}$$

commutes by means of the natural transformations

$$u_2 \circ g_2 \simeq (u_2 \circ g_2) \circ Id \to u_2 \circ g_2 \circ (f_1 \circ u_1) \simeq u_2 \circ (f_2 \circ g_1) \circ u_1 \simeq (u_2 \circ f_2) \circ g_1 \circ u_1 \to Id \circ (g_1 \circ u_1)$$

where $Id \to f_1 \circ u_1$ and $u_2 \circ f_2 \to Id$ are the unit and the co-unit of the adjuntions.

In this case we say that the square σ' is left adjoint to the square σ.

One omits the obvious notion of *right adjointable*.

2.1.11 Localizations

There is a theory of localizations for $(\infty, 1)$-categories. If $(\mathcal{C}, W)$ is an $(\infty, 1)$-category together with a class of morphisms W closed under homotopy, composition and containing all equivalences, then we can produce a new $(\infty, 1)$-category $\mathcal{C}[W^{-1}]$ together with a map $\mathcal{C} \to \mathcal{C}[W^{-1}]$ with the universal property of sending the edges in W to equivalences. To construct this localization we can make use of the model structure on the marked simplicial sets of [99, Chapter 3]. Recall that every marked simplicial set is cofibrant and the fibrant ones are precisely the pairs $\mathcal{C}^\sharp := (\mathcal{C}, eq)$ with $\mathcal{C}$ a quasi-category and eq the collection of all equivalences in $\mathcal{C}$. Therefore, $\mathcal{C}[W^{-1}]$ can be obtained as a fibrant-replacement of the pair $(\mathcal{C}, W)$. We recover the desired universal property from the fact that the marked structure is simplicial. Following [100, Cons. 4.1.3.1], this procedure can be presented in more robust terms. Namely, it is possible to construct an $(\infty, 1)$-category $\mathcal{W}Cat_\infty$ where the objects are the pairs $(\mathcal{C}, W)$ with $\mathcal{C}$ a quasi-category and W a class of morphisms in $\mathcal{C}$. Moreover, the mapping $\mathcal{C} \mapsto \mathcal{C}^\sharp$ provides a fully faithful functor

$$Cat_\infty \subseteq \mathcal{W}Cat_\infty \tag{2.1.8}$$

and the upper localization procedure $(\mathcal{C}, W) \mapsto \mathcal{C}[W^{-1}]$ provides a left adjoint to this inclusion (see [100, Prop. 4.1.3.2].

Let $\mathcal{C}$ be an $(\infty,1)$-category and let $\mathcal{C}_0$ be a full subcategory of $\mathcal{C}$. We say that $\mathcal{C}_0$ is a *reflexive localization of* $\mathcal{C}$ if the fully faithful inclusion $\mathcal{C}_0 \subseteq \mathcal{C}$ admits a left adjoint $L : \mathcal{C} \to \mathcal{C}_0$. A reflexive localization is a particular instance of the notion in the previous item, with W the class of edges in $\mathcal{C}$ which are sent to equivalences through L (see [99, Prop. 5.2.7.12]);

2.1.12 Presheaves

If $\mathcal{C}$ is a small $(\infty,1)$-category, the $(\infty,1)$-category of ∞-presheaves over $\mathcal{C}$ is defined as $\mathcal{P}(\mathcal{C}) := Fun(\mathcal{C}^{op}, \mathcal{S})$. It is not small anymore because $\mathcal{S}$ is not small. It comes naturally equipped with a fully faithful analogue of the classical Yoneda map $\mathcal{C} \to \mathcal{P}(\mathcal{C})$, endowed with the following universal property: for any $(\infty,1)$-category $\mathcal{D}$ having all colimits indexed by small simplicial sets, the composition

$$Fun^L(\mathcal{P}(\mathcal{C}), D) \to Fun(\mathcal{C}, \mathcal{D}) \tag{2.1.9}$$

induces an equivalence of $(\infty,1)$-categories, where the left-side denotes the full-subcategory of all colimit preserving functors (see [99, Thm. 5.1.5.6]).

2.1.13 κ-filtered categories and κ-compact objects

Let κ be a small cardinal. A simplicial set S is called κ-filtered if there is an $(\infty,1)$-category $\mathcal{C}$ together with a categorical equivalence $\mathcal{C} \to S$, such that for any κ-small simplicial set K, any diagram $K \to \mathcal{C}$ admits a cocone $K^{\triangleright} \to \mathcal{C}$ (see [99, Notation 1.2.8.4]). We use the terminology *filtered* when $\kappa = \omega$. Notice that if $\kappa \leq \kappa'$ and $\mathcal{C}$ is κ'-filtered then it is also κ-filtered.

It follows from [99, 4.2.3.11] that an $(\infty,1)$-category $\mathcal{C}$ has all small colimits if and only if there exists a regular cardinal κ such that $\mathcal{C}$ has all κ-small colimits together with all κ-filtered colimits.

Let object X in a big $(\infty,1)$-category $\mathcal{C}$. We say that X is *completely compact* if the associated map $Map_{\mathcal{C}}(X,-) : \mathcal{C} \to \mathcal{S}^{big}$ commutes with all small colimits. We say that X is κ-compact (for κ a small regular cardinal) if $Map_{\mathcal{C}}(X,-)$ commutes with colimits indexed by κ-filtered simplicial sets. We denote by $\mathcal{C}^{\kappa}$ the full subcategory of $\mathcal{C}$ spanned by the κ-compact objects in $\mathcal{C}$. We use the terminology *compact* when $\kappa = \omega$. Notice that if $\kappa \leq \kappa'$ and X is κ-compact it is also κ'-compact.

2.1.14 Ind-Completion

Let $\mathcal{C}$ be a small $(\infty,1)$-category and choose a regular cardinal κ with $\kappa < card(\mathbb{U})$. Following [99, Section 5.3.5], it is possible to formally complete $\mathcal{C}$ with all small colimits indexed by small κ-filtered simplicial sets. More precisely, we can construct a new $(\infty,1)$-category $Ind_{\kappa}(\mathcal{C})$ (which is not small anymore), together with a canonical map $\mathcal{C} \to Ind_{\kappa}(\mathcal{C})$ having the following universal property: for any $(\infty,1)$-category $\mathcal{D}$ having all colimits indexed by a small κ-filtered simplicial set, the composition

$$Fun^{\kappa}(Ind_{\kappa}(\mathcal{C}), D) \to Fun(\mathcal{C}, \mathcal{D}) \tag{2.1.10}$$

induces an equivalence of $(\infty,1)$-categories, where the left-side denotes the full-subcategory spanned by the functors commuting with colimits indexed by a κ-filtered small simplicial set (see of [99, Thm. 5.3.5.10]). In the case $\kappa = \omega$ we write $Ind(\mathcal{C}) := Ind_{\omega}(\mathcal{C})$.

2.1.15 Completion with colimits

Following [99, Section 5.3.6], given an arbitrary $(\infty,1)$-category $\mathcal{C}$ together with a collection $\mathcal{K}$ of arbitrary simplicial sets and a collection of diagrams $\mathcal{R} = \{p_i : K_i \to \mathcal{C}\}$ with each $K_i \in \mathcal{K}$, we can form a new $(\infty,1)$-category $\mathcal{P}^{\mathcal{K}}_{\mathcal{R}}(\mathcal{C})$ together with a canonical map $\mathcal{C} \to \mathcal{P}^{\mathcal{K}}_{\mathcal{R}}(\mathcal{C})$ such that for any $(\infty,1)$-category $\mathcal{D}$, the composition map

$$Fun_{\mathcal{K}}(\mathcal{P}^{\mathcal{K}}_{\mathcal{R}}(\mathcal{C}), \mathcal{D}) \to Fun_{\mathcal{R}}(\mathcal{C}, \mathcal{D}) \tag{2.1.11}$$

is an equivalence of $(\infty,1)$-categories, where the left-side denotes the full subcategory of $\mathcal{K}$-colimit preserving functors and the right-side denotes the full-subcategory of functors sending diagrams in the collection $\mathcal{R}$ to colimit diagrams in $\mathcal{D}$. This allows us to formally adjoin colimits of a given type to a certain $(\infty,1)$-category. We denote by $Cat_{\infty}^{big}(\mathcal{K})$ the (non-full) subcategory of Cat_{∞}^{big} spanned by the $(\infty,1)$-categories which admit all the colimits of diagrams indexed by simplicial sets in $\mathcal{K}$, together with the $\mathcal{K}$-colimit preserving functors. The intersection $Cat_{\infty}^{big}(\mathcal{K})\cap Cat_{\infty}$ is denoted as $Cat_{\infty}(\mathcal{K})$. In the particular case when $\mathcal{K}$ is the collection of κ-small simplicial sets, we will use the notation $Cat_{\infty}(\kappa)$.

If $\mathcal{K}\subseteq\mathcal{K}'$ are two collections of arbitrary simplicial sets and $\mathcal{C}$ is an arbitrary $(\infty,1)$-category having all $\mathcal{K}$-indexed colimits, we can let $\mathcal{R}$ be the collection of all $\mathcal{K}$-colimit diagrams in $\mathcal{C}$. The result of the previous paragraph $\mathcal{P}_{\mathcal{R}}^{\mathcal{K}}(\mathcal{C})$ will in this particular case, be denoted as $\mathcal{P}_{\mathcal{K}}^{\mathcal{K}'}(\mathcal{C})$. By ignoring the set-theoretical aspects, the universal property defining $\mathcal{P}_{\mathcal{K}}^{\mathcal{K}'}(\mathcal{C})$ allows us to understand the formula $\mathcal{C}\mapsto\mathcal{P}_{\mathcal{K}}^{\mathcal{K}'}(\mathcal{C})$ as an informal left adjoint to the canonical (non-full) inclusion of the collection of $(\infty,1)$-categories with all the $\mathcal{K}'$-indexed colimits together with the $\mathcal{K}'$-colimit preserving functors between them, into the collection of $(\infty,1)$-categories with all the $\mathcal{K}$-indexed colimits together with the $\mathcal{K}$-colimit preserving functors.

By combining the universal properties, we find that if $\mathcal{K}$ is the empty collection and $\mathcal{K}'$ is the collection of all small simplicial sets, $\mathcal{P}_{\mathcal{K}}^{\mathcal{K}'}(\mathcal{C})$ is simply given by $\mathcal{P}(\mathcal{C})$. In the case $\mathcal{K}$ is the empty collection and $\mathcal{K}'$ is the collection of all κ-small filtered simplicial sets (for some small cardinal κ), we obtain an equivalence $\mathcal{P}_{\mathcal{K}}^{\mathcal{K}'}(\mathcal{C})\simeq Ind_{\kappa}(\mathcal{C})$. Another important example is when $\mathcal{K}$ is again the empty collection and $\mathcal{K}'$ is the collection of κ-small simplicial sets. In this case we have a canonical equivalence $\mathcal{P}_{\mathcal{K}}^{\mathcal{K}'}(\mathcal{C})\simeq\mathcal{P}(\mathcal{C})^{\kappa}$. Following the fact that an $(\infty,1)$-category has all small colimits if and only it has κ-small colimits and κ-filtered colimits, we find a canonical equivalence between $\mathcal{P}(\mathcal{C})$ and $Ind_{\kappa}(\mathcal{P}(\mathcal{C})^{\kappa})$.

2.1.16 Sifted Colimits and Geometric Realizations

Following [99, 5.5.8.1], a simplicial set K is said to be *sifted* if it is nonempty and if the diagonal map $K\to K\times K$ is cofinal. The main examples are given by filtered simplicial sets and by the simplicial set $N(\Delta)^{op}$ - the opposite of the nerve of the category Δ (see [99, 5.5.8.4]).

A simplicial object in an $(\infty,1)$-category $\mathcal{C}$ is, by definition, a diagram $N(\Delta)^{op}\to\mathcal{C}$. We say that $\mathcal{C}$ admits *geometric realizations of simplicial objects* if every simplicial object in $\mathcal{C}$ has a colimit.

For a small $(\infty,1)$-category $\mathcal{C}$, we let $\mathcal{P}_{\Sigma}(\mathcal{C})$ denote the formal completion of $\mathcal{C}$ under sifted colimits (as in the previous section). Thanks to [99, 5.5.8.14], if $\mathcal{C}$ has finite coproducts, the formal completion $\mathcal{P}_{\Sigma}(\mathcal{C})$ is equivalent to the completion of $Ind(\mathcal{C})$ under geometric realizations of simplicial objects. Moreover, by [99, 5.5.8.17], if $\mathcal{C}$ has small colimits, a functor $\mathcal{C}\to\mathcal{D}$ commutes with sifted colimits if and only if it commutes with filtered colimits and geometric realizations.

2.1.17 Accessibility

Sometimes an arbitrary $(\infty,1)$-category $\mathcal{C}$ is not small but it is completely determined by small information. Let κ be a small regular cardinal. We say that a big[5] $(\infty,1)$-category $\mathcal{C}$ is *κ-accessible* if there exists a small $(\infty,1)$-category $\mathcal{C}^0$ together with an equivalence

$$Ind_{\kappa}(\mathcal{C}^0)\to\mathcal{C} \tag{2.1.12}$$

By [99, 5.4.2.2] a big $(\infty,1)$-category is κ-accessible if and only if it is locally small, admits small κ-filtered colimits, $\mathcal{C}^{\kappa}$ is essentially small and generates $\mathcal{C}$ under small κ-filtered colimits. In this case, by [99, 5.4.2.4], $\mathcal{C}^{\kappa}$ is the idempotent completion of $\mathcal{C}^0$.

[5] We can also define the notion of accessibility for the small $(\infty,1)$-categories. In this case, by [99, Cor. 5.4.3.6], a small $(\infty,1)$-category is accessible if and only if it is idempotent complete - see the next subsection.

We say that a big $(\infty,1)$-category is *accessible* if it is κ-accessible for some small regular cardinal κ. Given two small cardinals $\kappa<\kappa'$, a κ-accessible $(\infty,1)$-category is not necessarily κ'-accessible. However, by [99, Prop. 5.4.2.11], this is the case if κ' satisfies the following condition: for any cardinals $\tau<\kappa$ and $\pi<\kappa'$, we have $\pi^\tau<\kappa'$.

An important example of accessibility comes from the theory of presheaves: if $\mathcal{C}$ is a small $(\infty,1)$-category, $\mathcal{P}(\mathcal{C})$ is accessible (see Proposition 5.3.5.12 of [99]).

The natural morphisms between the accessible $(\infty,1)$-categories are the functors $f:\mathcal{C}\to\mathcal{D}$ which are again determined by the small data. More precisely, if $\mathcal{C}$ and $\mathcal{D}$ are κ-accessible, a functor f is called κ-accessible if it preserves small κ-filtered colimits and sends κ-compact objects in $\mathcal{C}$ to κ-compact objects in $\mathcal{D}$. The crucial result is that the information of the restriction $f|_\kappa:\mathcal{C}^\kappa\to\mathcal{D}^\kappa$ determines f in a essentially unique way (see [99, 5.3.5.10]).

2.1.18 Idempotent Complete $(\infty,1)$-categories

Let $\mathcal{C}$ be a classical 1-category and let X be an object in $\mathcal{C}$. A morphism $f:X\to X$ is said to be an idempotent if $f\circ f=f$. If we want to extend this notion to the setting of higher category theory, we need to specify a 2-cell σ rendering the diagram

$$\begin{tikzcd} X \arrow[d, "f"'] \arrow[dr, "f"] & \\ X \arrow[r, "f"'] \arrow[ur, phantom, "\sigma"] & X \end{tikzcd} \tag{2.1.13}$$

commutative. Moreover, we should be able to glue together different copies of σ to built up a 3-cell encoding the relation $f\circ f\circ f\simeq f$. This continues for every positive n. In [99, Section 4.4.5] the author introduces a simplicial set *Idem* suitable to encode all this kinds of coherences. It has a unique nondegenerate cell on each dimension $n\geq 0$. To give a diagram $Idem\to\mathcal{C}$ is equivalent to the data of an object $X\in\mathcal{C}$, together with a morphism $f:X\to X$ and all the expected coherences that make f an idempotent.

Recall now that an object $Y\in\mathcal{C}$ is said to be a *retract* of an object $X\in\mathcal{C}$ if the identity of Y factors as a composition $Y\to X\to Y$. Every decomposition likes this provides a morphism $f:X\to Y\to X$ which by [99, 4.4.5.7], can be extended to a diagram $Idem\to\mathcal{C}$. It follows that if d has a colimit in $\mathcal{C}$, this colimit is canonically equivalent to Y [99, 4.4.5.14]. Following this, $\mathcal{C}$ is said to be *idempotent complete* if every diagram $d:Idem\to\mathcal{C}$ has a colimit. In this case, there is a bijective correspondence between retracts and idempotents. In particular, every functor $\mathcal{C}\to\mathcal{D}$ between idempotent complete $(\infty,1)$-categories preserves colimits indexed by the simplicial set *Idem*, because the functoriality will send retracts to retracts.[6]

Remark 2.1.1. Since the simplicial set *Idem* is not finite, the fact that an $(\infty,1)$-category $\mathcal{C}$ has all finite colimits does not imply that $\mathcal{C}$ is idempotent complete. However, even though *Idem* is not filtrant, if κ is a regular cardinal and $\mathcal{C}$ admits small κ-filtered colimits then $\mathcal{C}$ is idempotent complete [99, 4.4.5.16].

We denote by Cat_∞^{idem} the full subcategory of Cat_∞ spanned by the small $(\infty,1)$-categories which are idempotent complete. By [99, 5.1.4.2] every $(\infty,1)$-category $\mathcal{C}$ admits an idempotent completion $Idem(\mathcal{C})$ given by the full subcategory of $\mathcal{P}(\mathcal{C})$ spanned by the completely compact objects (which by [99, 5.1.6.8] are exactly the retracts of objects in the image of the Yoneda embedding). The formula $\mathcal{C}\mapsto Idem(\mathcal{C})$ provides a left adjoint to the full inclusion

[6] We can rewrite this definition in more simpler terms. Since $\mathcal{P}(\mathcal{C})$ has all colimits, we can easily see that an $(\infty,1)$-category $\mathcal{C}$ is idempotent complete if and only if the image of the Yoneda embedding $\mathcal{C}\to\mathcal{P}(\mathcal{C})$ is stable under retracts.

$$Cat_{\infty}^{idem} \subseteq Cat_{\infty} \tag{2.1.14}$$

Following the discussion in 2.1.15, we can also identify Cat_{∞}^{idem} with $Cat_{\infty}(\mathcal{K})$ where $\mathcal{K} = \{Idem\}$. Moreover, we have a canonical equivalence of functors $\mathcal{P}^{\{Idem\}}(-) \simeq Idem(-)$. By [99, 5.4.2.4], $Idem(\mathcal{C})$ can also be identified with $Ind_{\kappa}(\mathcal{C})^{\kappa}$, the full subcategory of κ-compact objects in $Ind_{\kappa}(\mathcal{C})$, for any small regular cardinal κ.

Let now $\mathcal{C}$ be a small $(\infty,1)$-category and let $\mathcal{C} \to \mathcal{C}'$ be an idempotent completion of $\mathcal{C}$. Then, by [99, 5.5.1.3], for any regular cardinal κ, the induced morphism $Ind_{\kappa}(\mathcal{C}) \to Ind_{\kappa}(\mathcal{C}')$ is an equivalence of $(\infty,1)$-categories. Thus, if $\mathcal{C}$ is a κ-accessible $(\infty,1)$-category, with $\mathcal{C} \simeq Ind_{\kappa}(\mathcal{C}_0)$ for some small $(\infty,1)$-category $\mathcal{C}_0$ and some regular cardinal κ, then, since $\mathcal{C}_0 \to Ind_{\kappa}(\mathcal{C}_0)^{\kappa} \simeq \mathcal{C}^{\kappa}$ is an idempotent completion of $\mathcal{C}$, the canonical morphism $Ind_{\kappa}(\mathcal{C}^{\kappa}) \to \mathcal{C}$ is an equivalence. The converse is immediate by definition.

2.1.19 Presentable $(\infty,1)$-categories

We say that an $(\infty,1)$-category $\mathcal{C}$ is *presentable* if it is accessible and admits all colimits indexed by small simplicial sets. Again, we have a good criterion to understand if an $(\infty,1)$-category $\mathcal{C}$ is presentable. By [99, Thm. 5.5.1.1]), the following are equivalent: (i) $\mathcal{C}$ is presentable; (ii) there exists a small $(\infty,1)$-category $\mathcal{D}$ such that $\mathcal{C}$ is an accessible reflexive localization of $\mathcal{P}(\mathcal{D})$[7]; (iii) $\mathcal{C}$ is locally small, admits small colimits and there exists a small regular cardinal κ and a small S set of κ-compact objects in $\mathcal{C}$ such that every object of $\mathcal{C}$ is a colimit of a small diagram with values in the full subcategory of $\mathcal{C}$ spanned by S.

The natural morphisms between the presentable $(\infty,1)$-categories are the colimit preserving functors. We let $\mathcal{P}r^L$ (resp. $\mathcal{P}r^R$) denote the (non full!) subcategory of Cat_{∞}^{big} spanned by *presentable* $(\infty,1)$-categories together with colimit (resp. limit) preserving functors. As Cat_{∞}^{big}, $\mathcal{P}r^L$ is only a $\mathbb{W}$-small $(\infty,1)$-category. By the *Adjoint Functor Theorem* (see [99, Cor. 5.5.2.9]) a functor between presentable $(\infty,1)$-categories commutes with colimits (resp. limits) if and only if it admits a right (resp. left) adjoint (resp. and is accessible) and therefore we have a canonical equivalence $\mathcal{P}r^L \simeq (\mathcal{P}r^R)^{op}$. By [99, 5.5.3.13, 5.5.3.18] we know that both $\mathcal{P}r^L$ and $\mathcal{P}r^R$ admit all small limits and the inclusions $\mathcal{P}r^L, \mathcal{P}r^R \subseteq Cat_{\infty}^{big}$ preserve them. In particular, colimits in $\mathcal{P}r^L$ are computed as limits in $\mathcal{P}r^R$ using the natural equivalence $\mathcal{P}r^L \simeq (\mathcal{P}r^R)^{\circ}$.

2.1.20 κ-compactly generated $(\infty,1)$-categories

Although each presentable $(\infty,1)$-category is determined by small information, not all the information in the study of $\mathcal{P}r^L$ is determined by small data. This is mainly because the morphisms in $\mathcal{P}r^L$ are all kinds of colimit preserving functors without necessarily having a compatibility condition with the small information. Again, as in the accessible setting, if we want to isolate what is determined by small information, we consider for each small regular cardinal κ, the (non-full!) subcategory $\mathcal{P}r^L_{\kappa} \subseteq \mathcal{P}r^L$ spanned by the presentable κ-accessible $(\infty,1)$-categories together with those colimit preserving functors that preserve κ-compact objects. By definition, we say that an $(\infty,1)$-category is κ-compactly generated if it is an object of $\mathcal{P}r^L_{\kappa}$. The idea that κ-compactly generated $(\infty,1)$-categories are determined by smaller information can now be made precise: by the Propositions [99, 5.5.7.8, 5.5.7.10], the correspondence $\mathcal{C} \mapsto \mathcal{C}^{\kappa}$ sending a κ-compactly generated $(\infty,1)$-category to the full subcategory $\mathcal{C}^{\kappa} \subseteq \mathcal{C}$ spanned by the κ-compact objects, determines a fully faithful map of $(\infty,1)$-categories $\mathcal{P}r^L_{\kappa} \to Cat_{\infty}^{big}(\kappa)$ whose image is the full subcategory $Cat_{\infty}(\kappa)^{idem}$ of $Cat_{\infty}^{big}(\kappa)$ spanned by those big $(\infty,1)$-categories $\mathcal{C}$ with all κ-small colimits, which are essentially small and idempotent complete. Following the discussion in 2.1.18, $Cat_{\infty}(\kappa)^{idem}$ can be identified with $Cat_{\infty}(\mathcal{K})$ with $\mathcal{K}$

[7] The reflexive localization $\mathcal{C} \subseteq \mathcal{P}(\mathcal{D})$ is accessible if $\mathcal{C}$ is accessible for some cardinal. Using the universal property of the ind-completion (see [99, 5.5.1.2]) this is equivalent to ask for the composition $\mathcal{P}(\mathcal{D}) \to \mathcal{C} \subseteq \mathcal{P}(\mathcal{D})$ to be κ-accessible for some small regular cardinal κ

the collection of all κ-small simplicial sets together with the simplicial set $Idem$. The construction $Ind_\kappa : Cat_\infty(\mathcal{K}) \to \mathcal{P}r^L_\kappa$ provides an inverse to this map. [8]. Moreover, and following the discussion in 2.1.1, in case $\kappa > \omega$ we can drop the idempotent considerations because the full inclusion $Cat_\infty(\kappa)^{idem} \subseteq Cat_\infty(\kappa)$ is an equivalence.

Following [100, 5.3.2.9], $\mathcal{P}r^L_\kappa$ is presentable and the inclusion $\mathcal{P}r^L_\kappa \subseteq \mathcal{P}r^L$ preserves colimits.

We will be particularly interested in $\mathcal{P}r^L_\omega$, the study of the presentable $(\infty,1)$-categories of the form $Ind(\mathcal{C}_0)$ with $\mathcal{C}_0$ having all finite colimits. These are called *compactly generated.*

2.1.21 Localizations of Presentable $(\infty, 1)$-categories

The theory of presentable $(\infty,1)$-categories admits a very friendly internal theory of localizations. By [99, Prop. 5.5.4.15 and 5.5.4.20], if $\mathcal{C}$ is a presentable $(\infty,1)$-category and W is strongly saturated class of morphisms in $\mathcal{C}$ generated by a set S (as in [99, 5.5.4.5]), then the localization $\mathcal{C}[W^{-1}]$ is again a presentable $(\infty,1)$-category equivalent to the full subcategory of $\mathcal{C}$ spanned by the S-local objects and the localization map is a left adjoint to this inclusion.

2.1.22 Postnikov Towers

Recall that a space $X \in \mathcal{S}$ is said to be n-truncated if the homotopy groups $\pi_i(X,x)$ are all trivial for $i \geq n$. It is said to be n-connective if all the homotopy groups $\pi_i(X,x)$ are trivial for $i < n$. If $\mathcal{C}$ is an $(\infty,1)$-category we say that an object $X \in \mathcal{C}$ is n-truncated if for every object Y in $\mathcal{C}$ the mapping spaces $Map_{\mathcal{C}}(Y,X)$ are n-truncated. This notion agrees with the previous definition when $\mathcal{C} = \mathcal{S}$. Let $\tau_{\leq n}\mathcal{C}$ denote the full subcategory of $\mathcal{C}$ spanned by the n-truncated objects. A morphism $f : X \to X'$ in $\mathcal{C}$ is said to exhibit X' as an n-truncation of X if for every n-truncated object Y in $\mathcal{C}$ the composition with f induces an homotopy equivalence $Map_{\mathcal{C}}(Y,X) \simeq Map_{\mathcal{C}}(Y,X')$. By definition a *Postnikov tower* in $\mathcal{C}$ is a diagram $X : (N(\mathbb{Z}_{\geq 0})^{op})^{\triangleleft} \to \mathcal{C}$ such that for every n the map $X_* \to X_n$ exhibits X_n as an n-truncation of X_*. In particular, this implies that $X_n \to X_m$ exhibits X_m as a m-truncation of X_n. We say that Postnikov towers converge in $\mathcal{C}$ if the forgetful map $Fun((N(\mathbb{Z}_{\geq 0})^{op})^{\triangleleft}, \mathcal{C}) \to Fun(N(\mathbb{Z}_{\geq 0}), \mathcal{C})$ induces an equivalence when restricted to the full subcategory spanned by the Postnikov towers. In particular, if $\mathcal{C}$ admits all limits, Postnikov towers converge in $\mathcal{C}$ if and only if every Postnikov tower is a limit diagram [99, 5.5.6.26].

If $\mathcal{C}$ is presentable, the inclusions $\tau_{\leq n}\mathcal{C} \subseteq \mathcal{C}$ admits a left adjoint for every $n \geq 0$. This follows from the Adjoint functor theorem together with the fact that this inclusion commutes with all limits [99, 5.5.6.5]. In this case, we can find a sequence of functors

$$... \to \tau_{\leq 2}\mathcal{C} \to \tau_{\leq 1}\mathcal{C} \to \tau_{\leq 0}\mathcal{C} \tag{2.1.15}$$

and Postnikov towers converge in $\mathcal{C}$ if and only if $\mathcal{C}$ is the limit of this sequence [99, 3.3.3.1].

2.1.23 Stable $(\infty, 1)$-categories

We now discuss another important topic. In the classical setting, the notion of *triangulated category* seems to be of fundamental importance. *Stable ∞-categories* are the proper providers of triangulated structures - for any stable ∞-category $\mathcal{C}$ the homotopy category $h(\mathcal{C})$ carries a natural triangulated structure, where the exact triangles rise from the fiber sequences and the shift functor is given by the suspension (see [100, 1.1.2.15]). In practice, most triangulated categories are of this form. Grosso modo, a stable ∞-category is an ∞-category with a zero object, finite limits and colimits, satisfying the stronger condition that every pushout square is a pullback square and vice-versa (see [100, Def. 1.1.1.9, Prop. 1.1.3.4]). In particular this implies that finite sums are equivalent to finite products [100, 1.1.2.10]. If $\mathcal{C}$ is a pointed $(\infty,1)$-category with finite colimits, one equivalent way to formulate

[8] See also the Proposition [100, 5.3.2.9] for a direct proof of this result

the notion of stability is to ask for the suspension functor $X \mapsto \Sigma(X) := * \coprod_X *$ and its adjoint $Y \mapsto \Omega(Y) := * \times_X *$ to form an equivalence $\mathcal{C} \to \mathcal{C}$ (see [100, Cor. 1.4.2.27]). It is important to remark that stability is a property rather than an additional structure. The canonical example of a stable $(\infty,1)$-category is the $(\infty,1)$-category of spectra Sp. The appropriate maps between stable ∞-categories are the functors commuting with finite limits (or equivalently, with finite colimits - see [100, 1.1.4.1]). The collection of small stable ∞-categories together with these functors (so called *exact*) can be organized in a new ∞-category $Cat_\infty^{\mathcal{E}x}$. Thanks to [100, 1.1.4.4] $Cat_\infty^{\mathcal{E}x}$ has all small limits and the inclusion in Cat_∞ preserves them. Moreover, if K is a simplicial set and $\mathcal{C}$ is stable then $Fun(K,\mathcal{C})$ remains stable [100, 1.1.3.1].

Also important is that any stable $(\infty,1)$-category $\mathcal{C}$ comes with a natural *enrichment over spectra.* More precisely the mapping spaces $Map_{\mathcal{C}}(X,Y)$ have a natural structure of an ∞-loop space. To see this we can use the fact the suspension and loop functors in $\mathcal{C}$ are equivalences, so that we can find a new object X' with $X \simeq \Sigma(X')$ so that $Map_{\mathcal{C}}(X,Y) \simeq Map_{\mathcal{C}}(\Sigma(X'),Y) \simeq \Omega Map_{\mathcal{C}}(X,Y)$. Another way to make this precise is to use a universal property of the stabilization which tells us that the composition with $\Omega^\infty : Sp \to \mathcal{S}$ induces an equivalence of $(\infty,1)$-categories $Exc_*(\mathcal{C}, Sp) \simeq Exc_*(\mathcal{C},\mathcal{S})$ (see [100, 1.4.2.22]). In particular, this provides for any object X an essentially unique factorization of the functor $Map_{\mathcal{C}}(X,-) : \mathcal{C} \to \mathcal{S}$ as

$$\begin{tikzcd} \mathcal{C} \arrow[r, "Map_{\mathcal{C}}(X{,}-)"] \arrow[d, dashed, "Map_{\mathcal{C}}^{Sp}(X{,}-)"'] & \mathcal{S} \\ Sp \arrow[ur, "\Omega^\infty"'] & \end{tikzcd} \tag{2.1.16}$$

such that for any object Y, the spectra $Map_{\mathcal{C}}^{Sp}(X,Y)$ can be identified with the collection of spaces $\{Map_{\mathcal{C}}(X,\Sigma^n Y)\}_{n\in\mathbb{Z}}$. Moreover, and since Ω is an equivalence, it is equivalent to the family $\{Map_{\mathcal{C}}(\Omega^n X,Y)\}_{n\in\mathbb{Z}}$. The *Ext groups* $Ext^i_{\mathcal{C}}(X,Y)$ are defined as $\pi_0(Map_{\mathcal{C}}(\Omega^n X,Y))$. If $i \leq 0$ these groups correspond to the homotopy groups of the mapping space $Map_{\mathcal{C}}(X,Y)$.

We can now isolate the full subcategory $\mathcal{P}r^L_{Stb}$ of $\mathcal{P}r^L$ spanned by those presentable $(\infty,1)$-categories which are stable (every morphism of presentable $(\infty,1)$-categories which are stable is exact). Again by [100, 1.1.4.4] and the results in the presentable setting, $\mathcal{P}r^L_{Stb}$ has all small limits and the inclusion $\mathcal{P}r^L_{Stb} \subseteq \mathcal{P}r^L$ preserves them.

We discuss now an adapted version of the Proposition [100, 1.4.4.2] that provides a very helpful characterization of presentable stable $(\infty,1)$-categories. First we introduce some terminology. Let $\mathcal{C}$ be an $(\infty,1)$-category with a zero object. We say that a collection $\mathcal{E}$ of objects in $\mathcal{C}$ *generates* $\mathcal{C}$ if the full subcategory $\mathcal{E}^\perp \subseteq \mathcal{C}$ of all objects A in $\mathcal{C}$ such that $Map_{\mathcal{C}}(E,A) = *$ for all $E \in \mathcal{E}$, consists only of zero objects in $\mathcal{C}$. Let now κ be a regular cardinal. We say that $\mathcal{E}$ is a *family of κ-compact generators of* $\mathcal{C}$ if $\mathcal{E}$ generates $\mathcal{C}$ in the previous sense and each object $E \in \mathcal{E}$ is κ-compact. In particular, we will say that an object X in $\mathcal{C}$ is a *κ-compact generator of* $\mathcal{C}$ if the family $\mathcal{E} = \{X\}$ is a family of κ-compact generators of $\mathcal{C}$.

Proposition 2.1.2. *Let $\mathcal{C}$ be a stable $(\infty,1)$-category. Then, $\mathcal{C}$ is presentable if and only if the following conditions are satisfied:*

(i) $\mathcal{C}$ has arbitrary small coproducts [9];

(ii) the triangulated category $h(\mathcal{C})$ is locally small;

(iii) there exists a regular cardinal κ and a small family $\mathcal{E}$ of κ-compact generators in $\mathcal{C}$. In this case $\mathcal{C}$ is presentable κ-compactly generated in the sense of 2.1.20.

[9] Since $\mathcal{C}$ is stable this is equivalent to ask for all small colimits

Proof. We follow essentially the same arguments of [100, 1.4.4.2]. For the "only if" part, by definition, there is a small $(\infty, 1)$-category $\mathcal{D}$ and a regular cardinal τ, together with an equivalence $\mathcal{C} \simeq Ind_\tau(\mathcal{D})$. The formal completion of $\mathcal{D}$ with τ-small colimits is given by $\mathcal{D} \to \mathcal{D}' = \mathcal{P}(\mathcal{D})^\tau$. Passing to the ind-completions we obtain a map

$$\mathcal{C} \simeq Ind_\kappa(\mathcal{D}) \to Ind_\kappa(\mathcal{P}(\mathcal{D})^\kappa) \simeq \mathcal{P}(\mathcal{D}) \tag{2.1.17}$$

commuting with τ-filtered colimits. From the proof of [99, 5.5.1.1] we know that this map has a left adjoint L that establishes $\mathcal{C}$ as τ-accessible reflexive localization of $\mathcal{P}(\mathcal{D})$. The items (i) and (ii) follow immediately from this. Moreover, the composition functor

$$\mathcal{P}(\mathcal{D}) \to \mathcal{C} \subseteq \mathcal{P}(\mathcal{D}) \tag{2.1.18}$$

preserves τ-filtered colimits. To prove (iii) we consider the family $\mathcal{E}$ of all objects of the form $L(j(d))$ in $\mathcal{C}$ with j the Yoneda embedding $j : \mathcal{D} \to \mathcal{P}(\mathcal{D})$ and $d \in \mathcal{D}$. It follows immediately from the Yoneda lemma, from the fact that the composition (2.1.18) is τ-accessible and from the fact that the right adjoint of L is fully-faithful that $\mathcal{E}$ is a family of τ-compact generators in $\mathcal{C}$. The family is indexed by a small set because $\mathcal{D}$ is small.

For the "if" part, we consider the full subcategory $\mathcal{C}_\mathcal{E}$ of $\mathcal{C}$ spanned by the objects in $\mathcal{E}$, their suspensions and loopings. Inductively, we consider the successive closures under κ-small colimits. As a result we find a full subcategory $\mathcal{C}_\mathcal{E}(\kappa)$ of $\mathcal{C}$ closed under κ-small colimits, suspensions and loopings. Since each $E \in \mathcal{E}$ is κ-compact and the suspensions of κ-compact objects are again κ-compact [10] and κ-compact objects are closed under κ-small colimits, we find that $\mathcal{C}_\mathcal{E}(\kappa)$ is made of κ-compact objects and closed under κ-small colimits. It follows that the inclusion $\mathcal{C}_\mathcal{E}(\kappa) \subseteq \mathcal{C}$ extends to a fully-faithfull functor $F : Ind_\kappa(\mathcal{C}_\mathcal{E}(\kappa)) \to \mathcal{C}$ that commutes with κ-filtrant colimits. Since $Ind_\kappa(\mathcal{C}_\mathcal{E}(\kappa))$ has all κ-small colimits and all κ-filtrant colimits, it has all colimits and F commutes with all colimits. By the hypothesis (ii) and the Remark [99, 5.5.2.10] F has a right adjoint G and the fully-faithfulness implies $G \circ F \simeq Id$. We are reduce to showing that for every $Y \in \mathcal{C}$, the adjunction map $F \circ G(Y) \to Y$ is an equivalence. For that, we consider its fiber Z. Since F is fully-faithful, and G preserves limits, we have $G(Z) \simeq *$ and by adjunction we find that for every object $D \in Ind_\kappa(\mathcal{C}_\mathcal{E}(\kappa))$ we have have $Map(F(D), Z) \simeq Map(D, G(Z)) \simeq *$. In particular, the formula holds for any $D = E \in \mathcal{E}$ and by the definition of generating family we find that Z is a zero object in $\mathcal{C}$ so that the counit map is an equivalence. In particular, $\mathcal{C}$ is a stable κ-compactly generated $(\infty, 1)$-category. In the case $\mathcal{E}$ is a family of ω-compact generators, $\mathcal{C}$ is compactly generated and its full subcategory of compact objects is equivalent to $Idem(\mathcal{C}_\mathcal{E}(\kappa))$. □

Remark 2.1.3. The condition (iii) in the Proposition 2.1.2 is equivalent to the existence of an α-compact generator for some regular cardinal α, not necessarily the same as κ. Indeed, by definition, if $\mathcal{C}$ has a κ-compact generator, then it provides a κ-generating family with a single element. Conversely, if $\mathcal{E} = \{E_i\}_{i \in I}$ is a κ-generating family with multiple objects, by the hypothesis (i), the sum $\coprod_{i \in I} E_i$ exists in $\mathcal{C}$ and is an α-compact generator of $\mathcal{C}$ for some α a regular cardinal (let $\gamma = max\{\kappa, card(I)\}$ and choose α satifying the condition described in 2.1.17).

Remark 2.1.4. The statement given in [100, 1.4.4.2] is somewhat different from ours, namely because the notion of compact generator therein is stronger. More precisely, an object X there is said to be a κ-compact generator if it is κ-compact and such that for any $Y \in \mathcal{C}$, if $\pi_0(Map(X, Y)) = 0$ then Y is a zero object. Of course, if X verifies this condition, the family $\mathcal{E} = \{X\}$ verifies our condition (iii). However, the converse is not necessarily true for the same X and the same cardinal. If X is a κ-generator in our sense, then the infinite coproduct $\coprod_{n \in \mathbb{Z}} \Sigma^n X$ is a generator in the sense of [100, 1.4.4.2] but a priori it will only be κ'-compact for some cardinal $\kappa' \geq \kappa$.

Our presentation is needed to match the familiar results coming from the classical theory of compact generators in triangulated categories. Following Neeman [108], we recall that an object X in

[10] If I is a κ-filtered simplicial set and $d : I \to \mathcal{C}$ is a diagram, we have $Map(\Sigma X, colim_I d_i) \simeq Map(X, \Omega(colim_k d))$ and since $\mathcal{C}$ is stable (which implies that Ω is an equivalence and therefore commutes with colimits) and X is κ-compact, we find that the last space is homotopy equivalent to $Map(X, colim_I(\Omega(d_i)) \simeq colim_I Map(X, \Omega(d_i)) \simeq colim_I Map(\Sigma X, d_i)$

a triangulated category T is said to be *compact* if it commutes with infinite coproducts. Moreover, a collection of objects $\mathcal{E}$ in T is said to *generate* T if its right-orthogonal complement $\mathcal{E}^\perp := \{A \in Ob(T) : Hom_T(E[n], A) = 0, \forall n \in \mathbb{Z}, \forall E \in \mathcal{E}\}$ consists only of zero objects in T. Now, $\mathcal{E}$ is said to be a *family of compact generators of* T *in the sense of Neeman* if it generates T and each $E \in \mathcal{E}$ is compact in the sense of triangulated categories. Finally, an object X is said to be a *compact generator of* T if it is compact and the set $\mathcal{E} = \{X\}$ generates T.

Let now $\mathcal{C}$ be a stable $(\infty,1)$-category and let $\mathcal{E}$ be a collection of objects in $\mathcal{C}$. It follows that $\mathcal{E}$ is a family of compact generators of $h(\mathcal{C})$ in the sense of Neeman if and only it satisfies the condition (iii) for $\kappa = \omega$. Indeed, the two notions of generator agree because $\pi_0 Map_{\mathcal{C}}(\Sigma^n E, A) \simeq \pi_n Map_{\mathcal{C}}(E, A)$. Therefore, it is enough to see that an object X is compact in the triangulated category $h(\mathcal{C})$ if and only it is ω-compact in $\mathcal{C}$. This follows from [100, 1.4.4.1-(3)] and from the fact that coproducts in $\mathcal{C}$ are the same as coproducts in $h(\mathcal{C})$: if $\{X_i\}_{i\in I}$ is a collection of objects in $\mathcal{C}$, its coproduct $\coprod_i X_i$ in $\mathcal{C}$ is a coproduct in $h(\mathcal{C})$ because the functor π_0 commutes with homotopy products; conversely, if $\coprod_i X_i$ is a coproduct in $\mathcal{C}$, by definition, this means that $\pi_0 Map_{\mathcal{C}}(\coprod_i X_i, Z) \simeq \oplus_i \pi_0 Map_{\mathcal{C}}(X_i, Z)$ holds for any $Z \in h(\mathcal{C})$. In particular, this holds for all the loopings $\Omega^n Z$ so that the formula holds for all π_n and $\coprod_i X_i$ is a coproduct in $\mathcal{C}$.

This characterization allows us to detect the property of a stable $(\infty,1)$-category being compactly generated simply by studying its homotopy category. The following example is crucial to algebraic geometry and will play a fundamental role later in this work:

Example 2.1.5. Let X be a quasi-compact and separated scheme. Its underlying $(\infty,1)$-category $\mathcal{D}(X)$ (see below) is stable and its homotopy category $h(\mathcal{D}(X))$ recovers the classical derived category of X. As proved in the Corollary 5.5 of [19], when X is quasi-compact and separated, $h(\mathcal{D}(X))$ is equivalent to the derived category of $\mathcal{O}_X$-modules with quasi-coherent cohomology sheaves. Together with the Theorem 3.1.1 of [23], this tells us that $\mathcal{D}(X)$ has a compact generator in the sense of Neeman and that the compact objects are the perfect complexes. Thus, by the previous discussion, $\mathcal{D}(X)$ is a stable compactly generated $(\infty,1)$-category.

We will write $\mathcal{P}r^L_{\omega,Stb}$ to denote the full subcategory of $\mathcal{P}r^L_\omega$ spanned by the stable $(\infty,1)$-categories that are compactly generated, together with the compact preserving morphisms. The equivalence $\mathcal{P}r^L_\omega \to Cat_\infty(\omega)^{idem}$ of 2.1.20 restricts to an equivalence $\mathcal{P}r^L_{\omega,Stb} \to Cat_\infty^{\mathcal{E}x,idem}$ where the last denotes the (non full) subcategory of Cat_∞^{idem} spanned by the small stable ∞-categories which are idempotent complete, together with the exact functors. This follows from the fact that the idempotent completion of a stable $(\infty,1)$-category remains stable [100, 1.1.3.7], together with the observation that stable $(\infty,1)$-categories have all finite colimits and that exact functors preserve them.

Remark 2.1.6. In [18] the authors identify the subject of *Topological Morita theory* with the study of the $(\infty,1)$-category $\mathcal{P}r^L_{\omega,Stb}$. We will come back to this in 6.2.

To conclude this section, we give a useful result that will be necessary for many of the future applications we have in mind:

Proposition 2.1.7. *Let $f : \mathcal{C} \to \mathcal{D}$ be colimit preserving functor between stable presentable $(\infty,1)$-categories. Assume that*

(i) The $(\infty,1)$-category $\mathcal{C}$ has a family of ω-compact generators $\mathcal{E}$ in the sense of the Proposition 2.1.2 (here we assume, without loss of generality, that $\mathcal{E}$ is closed under suspensions and loopings[11]) and f is fully-faithful when restricted to the objects in this collection;

(ii) for any object $E \in \mathcal{E}$, the object $f(E)$ is ω-compact in $\mathcal{D}$;

[11] We can always assume this because, as discussed in the previous footnote, suspensions of compact objects are compact.

Then, f is fully-faithful. Moreover, if the image of the collection $\mathcal{E}$ in $\mathcal{D}$ is a family of ω-compact generators, then f is an equivalence.

Proof. To start with, we observe that to assume $\mathcal{E}$ to be closed under suspensions and loopings and f to be fully-faithful when restricted to the objects in $\mathcal{E}$ produces the same effects as dropping the condition of stability under suspensions and loopings and asking for the naturally induced maps of spectra

$$Map_{\mathcal{C}}^{Sp}(X,Y) \to Map_{\mathcal{D}}^{Sp}(f(X), f(Y)) \tag{2.1.19}$$

to be equivalences in Sp for any X and Y in $\mathcal{E}$.

Let us now explain the proof. Using the same notations of the Proposition 2.1.2, we have $\mathcal{C} \simeq Ind(\mathcal{C}_{\mathcal{E}}(\omega))$. To deduce fully-faithfulness we prove that the restriction of f to $\mathcal{C}_{\mathcal{E}}(\omega)$ is fully-faithful so that, by the hypothesis (ii) together with [99, 5.3.5.11] we conclude that f is fully-faithful. To see this, it is enough to check that f is fully-faithful when restricted to each one of the subcategories in the inductive construction of $\mathcal{C}_{\mathcal{E}}(\omega)$ (see the proof in [100, 1.4.4.2] for the precise inductive step). Using induction, and since $\mathcal{C}$ is stable, it is enough to check that f is fully-faithful when restricted to finite direct sums and cofibers of objects in the collection $\mathcal{E}$. For direct sums this is immediate. Suppose now we have a cofiber sequence $X \to Y \to Z$ in $\mathcal{C}$ with X and Y in $\mathcal{E}$ and let A be another object in $\mathcal{E}$. In this case, and since the functors $Map_{\mathcal{C}}^{Sp}(A,-)$ are exact by construction, we obtain a cofiber sequence in Sp

$$Map_{\mathcal{C}}^{Sp}(A,X) \to Map_{\mathcal{C}}^{Sp}(A,Y) \to Map_{\mathcal{C}}^{Sp}(A,Z) \tag{2.1.20}$$

Since f commutes with small colimits, the induced sequence $f(X) \to f(Y) \to f(Z)$ is a cofiber sequence and we get a canonical diagram of cofiber sequences in Sp

$$\begin{tikzcd} Map_{\mathcal{C}}^{Sp}(A,X) \arrow[r] \arrow[d, "\sim"] & Map_{\mathcal{C}}^{Sp}(A,Y) \arrow[r] \arrow[d, "\sim"] & Map_{\mathcal{C}}^{Sp}(A,Z) \arrow[d] \\ Map_{\mathcal{D}}^{Sp}(f(A),f(X)) \arrow[r] & Map_{\mathcal{D}}^{Sp}(f(A),f(Y)) \arrow[r] & Map_{\mathcal{D}}^{Sp}(f(A),f(Z)) \end{tikzcd} \tag{2.1.21}$$

where the two first vertical maps are equivalences by hypothesis. We conclude the vertical map on the right is also an equivalence. Finally, for any other cofiber sequence $U \to V \to W$ in $\mathcal{C}$, we conclude using the universal property of the cofiber that $Map_{\mathcal{C}}^{Sp}(f(W), f(Z)) \simeq Map_{\mathcal{C}}^{Sp}(W,Z)$.

To conclude the additional statement we use the definition of generating familiy and the consequences of the Prop. 2.1.2 to reduce everything to prove that the induced restriction $\mathcal{C}_{\mathcal{E}}(\omega) \to \mathcal{D}_{f(\mathcal{E})}(\omega)$ is an equivalence. This follows because f commutes with colimits.

□

2.1.24 Localizations of Stable $(\infty,1)$-categories and Exact Sequences

Our goal in this section is to prove the Proposition 2.1.10 below. Let us start by reviewing some standard terminology for triangulated categories. Let C be triangulated category and let A be a triangulated subcategory. We say that A is thick in C (also said *epaisse*), if it is closed under direct summands. Moreover generally, we say that a triangulated functor $A \to C$ is cofinal if the image of A is thick in C. Recall also that a sequence of triangulated categories $A \to C \to D$ is said to be exact if the composition is zero, the first map is fully-faithful and the inclusion from the Verdier quotient $C/A \hookrightarrow D$ is cofinal, meaning that every object in D is a direct summand of an object in B/A.

Following [18], we say that a sequence in $\mathcal{P}r^L_{Stb}$

$$\mathcal{A} \to \mathcal{C} \to \mathcal{D} \tag{2.1.22}$$

is exact if the composition is zero, the first map is fully-faithful and the diagram

$$\begin{tikzcd} \mathcal{A} \arrow[r, hook] \arrow[d] & \mathcal{C} \arrow[d] \\ * \arrow[r] & \mathcal{D} \end{tikzcd} \tag{2.1.23}$$

is a pushout. Here we denote by $*$ the final object in $\mathcal{P}r^L_{Stb}$. As proved in [18, Prop. 4.5, Prop. 4.6], this notion of exact sequence can be reformulated using the language of localizations: if $\phi : \mathcal{A} \hookrightarrow \mathcal{C}$ is a fully-faithful functor, the cofiber of ϕ can be identified with the accessible reflexive localization

$$\begin{tikzcd} \mathcal{D} \arrow[r, hook] & \mathcal{C} \arrow[l, bend right] \end{tikzcd} \tag{2.1.24}$$

with local equivalences given by the class of edges f in $\mathcal{C}$ with cofiber in the essential image of ϕ. In particular, an object $x \in \mathcal{C}$ is in $\mathcal{D}$ if and only if for every object $a \in \mathcal{A}$ we have $Map_{\mathcal{C}}(a, x) \simeq *$.

Remark 2.1.8. Let $\mathcal{A} \hookrightarrow \mathcal{C} \to \mathcal{D}$ be an exact sequence of presentable stable $(\infty, 1)$-categories as above. If the homotopy category $h(\mathcal{A})$ has a compact generator in the sense of Neeman, say $k \in \mathcal{A}$, then for an object $x \in \mathcal{C}$ to be in $\mathcal{D}$ it is enough to have $Map_{\mathcal{C}}(k, x) \simeq *$. This follows from the arguments in the proof of tne Proposition 2.1.2: every object in $\mathcal{A}$ can be obtained as a colimit of suspensions of k.

Thanks to [18, Prop. 5.9] and to the arguments in the proof of [18, Prop. 5.13], this notion of exact sequence extends the notion given by Verdier in [148]: a sequence $\mathcal{A} \hookrightarrow \mathcal{C} \to \mathcal{D}$ in $\mathcal{P}r^L$ is exact if and only if the sequence of triangulated functors $h(\mathcal{A}) \hookrightarrow h(\mathcal{C}) \to h(\mathcal{D})$ is exact sequence in the classical sense and the inclusion $h(\mathcal{C})/h(\mathcal{A}) \hookrightarrow h(\mathcal{D})$ is an equivalence of triangulated categories. In the compactly generated case we have the following

Proposition 2.1.9. *Let $\mathcal{A} \hookrightarrow \mathcal{C} \to \mathcal{D}$ be a sequence in $\mathcal{P}r^L_{\omega,Stb}$. The following are equivalent:*

1. *the sequence is exact;*

2. *the induced sequence of triangulated functors $h(\mathcal{A}) \hookrightarrow h(\mathcal{C}) \to h(\mathcal{D})$ is exact in the classical sense and the inclusion $h(\mathcal{C})/h(\mathcal{A}) \hookrightarrow h(\mathcal{D})$ is an equivalence;*

3. *the sequence of triangulated functors induced between the homotopy categories of the associated stable subcategories of compact objects $h(\mathcal{A}^\omega) \hookrightarrow h(\mathcal{C}^\omega) \to h(\mathcal{D}^\omega)$ is exact in the classical sense.*

Proof. The equivalence between 1) and 2) follows from the results of [18] discussed above. The equivalence between 2) and 3) follows from the results of B.Keller [82, Section 4.12, Corollary] and the fact that for any compactly generated stable $(\infty, 1)$-category $\mathcal{C}$ we can identify $h(\mathcal{C}^\omega)$ with the full subcategory of compact objects (in the sense of Neeman) in $h(\mathcal{C})$ (see 2.1.4).

□

The following result will become important in the last section of our work:

Proposition 2.1.10. *Let*

$$\begin{tikzcd} & \mathcal{D} \arrow[d, "f"] \\ \mathcal{C} \arrow[r, "L"] & \mathcal{C}_0 \end{tikzcd} \tag{2.1.25}$$

be a diagram in $\mathcal{P}r^L_{\omega,Stb}$ such that

- *The homotopy triangulated category $h(\mathcal{D})$ has a compact generator in the sense of Neeman;*

- *The map $L : \mathcal{C} \to \mathcal{C}_0$ is an accessible reflexive localization of $\mathcal{C}$ obtained by killing a stable subcategory $\mathcal{A} \subseteq \mathcal{C}$ such that $h(\mathcal{A})$ has a compact generator (in the sense of Neeman) and the inclusion $\mathcal{A} \subseteq \mathcal{C}$ is a map in $\mathcal{P}r^L_{\omega,Stb}$.*

Then:

1. *the diagram admits a limit $\sigma =$*

$$\begin{tikzcd} \mathcal{T} \arrow[r] \arrow[d] & \mathcal{D} \arrow[d, "f"] \\ \mathcal{C} \arrow[r, "L"] & \mathcal{C}_0 \end{tikzcd} \tag{2.1.26}$$

   in $\mathcal{P}r^L_{\omega,Stb}$;

2. *the diagram σ remains a pullback after the (non-full) inclusion $\mathcal{P}r^L_{\omega,Stb} \subseteq \mathcal{P}r^L_{Stb}$;*

3. *the homotopy category $h(\mathcal{T})$ has a compact generator in the sense of Neeman.*

Proof. We start by noticing that $\mathcal{P}r^L_{\omega,Stb} \subseteq \mathcal{P}r^L_{Stb}$ preserves colimits (combine [99, 5.5.3.18,5.5.7.6, 5.5.7.7] or see [100, 5.3.2.9]). Therefore, the map $L : \mathcal{C} \to \mathcal{C}_0$ remains a Bousfield localization in the sense discussed above. We recall also that all pullbacks exists in $\mathcal{P}r^L_{Stb}$ and thanks to [100, 1.1.4.4] and to [99, 5.5.3.13] they can be computed in Cat^{big}_∞. In this case, let

$$\begin{tikzcd} \mathcal{V} \arrow[r, "p"] \arrow[d, "q"] & \mathcal{D} \arrow[d, "f"] \\ \mathcal{C} \arrow[r, "L"] & \mathcal{C}_0 \end{tikzcd} \tag{2.1.27}$$

be a pullback for the diagram given by (f, L) in $\mathcal{P}r^L$. Of course, we can assume that f is a categorical fibration and nothing will change up to categorical equivalence (see our discussion about homotopy pullbacks in 4.2). With this, we can actually describe $\mathcal{V}$ as the strict pullback $\mathcal{D} \times_{\mathcal{C}_0} \mathcal{C}$. It follows from the description of compact-objects in the pullback [99, 5.4.5.7] that both maps p and q preserve compact objects. Therefore, to achieve the proof we are reduced to showing that $\mathcal{V}$ is ω-accessible. Indeed, if this is the case, it follows from the universal property of the pullback in $\mathcal{P}r^L_{\omega,Stb}$ and in $\mathcal{P}r^L_{Stb}$ that $\mathcal{V}$ is canonically equivalent to $\mathcal{T}$. To prove that $\mathcal{V}$ is ω-accessible we can make use of the Proposition 2.1.2: it suffices to show that the homotopy category of $\mathcal{V}$ has a compact generator in the sense of Neeman. We can construct one using exactly the same arguments of [140, Prop. 3.9], itself inspired by the arguments of the famous theorem of Bondal - Van den Bergh [23, Thm 3.1.1]:

Let d be a compact generator in $\mathcal{D}$, which exists as part of our assumptions. As f and L are functors in $\mathcal{P}r^L_{\omega,Stb}$, they preserve compact objects and therefore $f(d)$ is compact. As L is a Bousfield localization of compactly generated stable $(\infty, 1)$-categories we can use the famous result of Neeman-Thomason [107, Theorem 2.1] to deduce the existence of a compact object $c \in \mathcal{C}$ whose image in $\mathcal{C}_0$ is $f(d) \oplus (f(d)[1])$. The new object $d' = d \oplus (d[1])$ is again a compact generator in $h(\mathcal{D})$ and since f preserves colimits we conclude the existence of an object $v \in \mathcal{V}$ such that $p(v) = d'$ and $q(v) = c$. Then, [99, Lemma 5.4.5.7] implies that v is a compact object in $\mathcal{V}$.

At the same time, we use our second assumption that $\mathcal{A}$ has a compact generator k. Since k is in $\mathcal{A}$, $L(k)$ is a zero object in $\mathcal{C}_0$. Therefore, it lifts to an object $\tilde{k} \in \mathcal{V}$ with $q(\tilde{k}) = k$ and $p(\tilde{k}) = 0 \in \mathcal{D}$. To deduce that $\tilde{k}$ is a compact object in $\mathcal{V}$ we observe that for any z in $\mathcal{V}$, the mapping space in the pullback is given by the formula

$$Map_{\mathcal{V}}(\tilde{k}, z) \simeq Map_{\mathcal{C}}(k, q(z)) \times_{Map_{\mathcal{C}_0}(L(k),L(q(z)))} Map_{\mathcal{D}}(p(\tilde{k}), p(z)) \tag{2.1.28}$$
$$\simeq Map_{\mathcal{C}}(k, q(z)) \times_{Map_{\mathcal{C}_0}(0,L(q(z)))} Map_{\mathcal{D}}(0, p(z)) \tag{2.1.29}$$
$$\simeq Map_{\mathcal{C}}(k, q(z)) \times_{*} * \simeq Map_{\mathcal{C}}(k, q(z)) \tag{2.1.30}$$

so that, since q commutes with colimits, $\tilde{k}$ is compact in $\mathcal{V}$ if and only if k is compact in $\mathcal{C}$. The last is true because of our hypothesis that the inclusion $\mathcal{A} \subseteq \mathcal{C}$ preserves compact objects.

We claim that the sum $v \oplus \tilde{k}$ is a compact generator of $\mathcal{V}$. Obviously, as a finite sum of compacts, it is compact. We are left to check that it is a generator of $h(\mathcal{V})$. In other words, we have to prove that for an arbitrary object z in $\mathcal{V}$, if z is right-orthogonal to the sum $v \oplus \tilde{k}$ in $h(\mathcal{V})$, then it is a zero object. Notice that z is right-orthogonal to the sum if and only if it is right-orthogonal to v and $\tilde{k}$ at the same time. In particular, the formula (2.1.28) implies that z is right-orthogonal to $\tilde{k} \in h(\mathcal{V})$ if and only if $q(z)$ is right-orthogonal to k in $\mathcal{C}$. Since k is a compact generator of $h(\mathcal{A})$ (by assumption), it follows from the Remark 2.1.8 that $q(z)$ is right-orthogonal to k if and only if $q(z)$ is L-local, meaning that it is in $\mathcal{C}_0$ and we have $i \circ L(q(z)) \simeq q(z)$, where i is the fully faithfull right adjoint of L. Let us assume that z is right-orthogonal to $\tilde{k}$. Then, this discussion implies that

$$Map_{\mathcal{V}}(v, z) \simeq Map_{\mathcal{C}}(c, q(z)) \times_{Map_{\mathcal{C}_0}(f(d'),f(q(z))} Map_{\mathcal{D}}(d', p(z)) \tag{2.1.31}$$

and using the fact that $q(z) \simeq i \circ L(q(z))$, it becomes

$$\simeq Map_{\mathcal{C}}(L(c), L(q(z)) \times_{Map_{\mathcal{C}_0}(f(d'),f(q(z))} Map_{\mathcal{D}}(d', p(z)) \simeq Map_{\mathcal{D}}(d', p(z)) \tag{2.1.32}$$

We conclude that if z is orthogonal to $\tilde{k}$ and v at the same time, then $p(z)$ is orthogonal to d'. However, by construction, d$'$ is again a compact generator of $h(\mathcal{D})$ so that $p(z)$ is zero in $\mathcal{D}$. Since we have $q(z) \simeq i \circ L(q(z)) \simeq i \circ f \circ p(z)$, this implies that $q(z)$ is also zero in $\mathcal{C}$. Using [99, Lemma 5.4.5.5], we find that z is a zero object in $\mathcal{D}$. This concludes the proof.

□

Remark 2.1.11. The proof of the Proposition 2.1.10 works mutatis-mutandis if we replace the hypothesis of single compact generators in $\mathcal{A}$ and $\mathcal{D}$ by the existence of compact generating families. More precisely, and using the same arguments and notations, if $\mathcal{E}_{\mathcal{D}} = \{d_i\}_{i \in I}$ and $\mathcal{E}_{\mathcal{A}} = \{k_j\}_{j \in J}$ are families of compact generators respectively in $\mathcal{D}$ and in $\mathcal{A}$, we can prove that the family $\{\tilde{k}_j \oplus v_i\}_{(i,j) \in I \times J)}$ is a family of compact generators in $\mathcal{T}$.

In particular, we have the following immediate corollary:

Corollary 2.1.12. *Let* $\sigma =$

$$\begin{tikzcd} \mathcal{A} \arrow[r, hook] \arrow[d] & \mathcal{C} \arrow[d] \\ * \arrow[r] & \mathcal{C}_0 \end{tikzcd} \tag{2.1.33}$$

be an exact sequence in $\mathcal{P}r^L_{\omega,Stb}$ *such that* $h(\mathcal{A})$ *admits a family of compact generators in the sense of Neeman. Then, the diagram* σ *is a pullback in* $\mathcal{P}r^L_{Stb}$

Proof. This is the degenerated case of 2.1.10 (together with the Remark 2.1.11) where $\mathcal{D} = 0$. The inclusion $\mathcal{A} \subseteq \mathcal{C}$ admits a canonical factorization through the pullback, which, by the arguments in 2.1.10 and 2.1.11, sends the generating family of $\mathcal{A}$ to a generating family. The conclusion now follows from the Proposition 2.1.7. □

2.1.25 t-structures

t-structures are an important tool in the study of triangulated categories. Following [100, Section 1.2.1] they extend in a natural way to the setting of stable $(\infty,1)$-categories: A t-structure in a stable $(\infty,1)$-category $\mathcal{C}$ is the data of a t-structure in the homotopy category $h(\mathcal{C})$. Given a t-structure $(h(\mathcal{C})_{\leq 0}, h(\mathcal{C})_{\geq 0})$ in $h(\mathcal{C})$, we denote by $\mathcal{C}_{\leq 0}$ (resp. $\mathcal{C}_{\geq 0}$) the full subcategory of $\mathcal{C}$ spanned by the objects in $h(\mathcal{C})_{\leq 0}$ (resp. $h(\mathcal{C})_{\geq 0}$). Moreover, we will write $\mathcal{C}_{\leq n}$ (resp. $\mathcal{C}_{\geq n}$) to denote the image of $\mathcal{C}_{\leq 0}$ (resp. $\mathcal{C}_{\geq 0}$) under the functor Σ^n. Recall also that an object $X \in \mathcal{C}$ is said to be *connective with respect to the t-structure* if it belongs to $\mathcal{C}_{\geq 0}$.

It follows from the axioms for a t-structure that for every $n \in \mathbb{Z}$ the inclusion $\mathcal{C}_{\leq n} \subseteq \mathcal{C}$ admits a left adjoint $\tau_{\leq n}$ [100, 1.2.1.5] and the inclusion $\mathcal{C}_{\geq n} \subseteq \mathcal{C}$ admits a right adjoint $\tau_{\geq n}$ and these two adjoints are related by the existence of a cofiber/fiber sequence

$$\begin{tikzcd} \tau_{\geq n}(X) \arrow[r] \arrow[d] & X \arrow[d] \\ * \arrow[r] & \tau_{\leq n-1}(X) \end{tikzcd} \tag{2.1.34}$$

Moreover, for every $m, n \in \mathbb{Z}$ they are related by a natural equivalence

$$\tau_{\leq m} \circ \tau_{\geq n} \simeq \tau_{\geq n} \circ \tau_{\leq m} \tag{2.1.35}$$

(see [100, 1.2.1.10]).

Remark 2.1.13. The two notations $\tau_{\leq n} : \mathcal{C} \to \tau_{\leq n}\mathcal{C}$ (after 2.1.22) and $\tau_{\leq n} : \mathcal{C} \to \mathcal{C}_{\leq n}$ are not compatible. However, they are compatible when restricted to $\mathcal{C}_{\geq 0}$ and we have $\tau_{\leq n}(\mathcal{C}_{\geq 0}) \simeq \mathcal{C}_{\geq 0} \cap \mathcal{C}_{\leq n}$ (See [100, 1.2.1.9]).

By definition, the *heart* of the t-structure is the full subcategory $\mathcal{C}^{\heartsuit}$ spanned by the objects in the interesection $\mathcal{C}_{\leq 0} \cap \mathcal{C}_{\geq 0}$. It follows from the axioms that $\mathcal{C}^{\heartsuit}$ is equivalent to the nerve of $h(\mathcal{C}^{\heartsuit})$. Given an object $X \in \mathcal{C}$ we denote by $\mathbb{H}_n(X)$ the object of $\mathcal{C}^{\heartsuit}$ obtained by shifting the object $\tau_{\leq n}\tau_{\geq n}(X) \in \mathcal{C}_{\leq n} \cap \mathcal{C}_{\geq n}$.

Remark 2.1.14. The cofiber/fiber sequence (2.1.34) implies that if X is already in $\mathcal{C}_{\leq n}$ (which means that $X \simeq \tau_{\leq n}(X)$) we a have pushout diagram

$$\begin{tikzcd} \Sigma^n\mathbb{H}_n(X) = \tau_{\geq n}\tau_{\leq n}(X) \arrow[r] \arrow[d] & X \arrow[d] \\ * \arrow[r] & \tau_{\leq n-1}(X) \end{tikzcd} \tag{2.1.36}$$

The data of a t-structure in a stable $(\infty,1)$-category $\mathcal{C}$ is completely characterized by the data of the reflexive localization $\mathcal{C}_{\leq 0} \subseteq \mathcal{C}$ [100, 1.2.1.16]. Following this, if $\mathcal{C}$ is an accessible $(\infty,1)$-category we say that the t-structure is *accessible* if this localization is accessible. Moreover, we say that the t-structure is compatible with filtered colimits if the inclusion $\mathcal{C}_{\leq 0} \subseteq \mathcal{C}$ also commutes with filtered colimits.

If $\mathcal{C}$ and $\mathcal{C}'$ are stable $(\infty,1)$-categories carrying t-structures, we say that a functor $f : \mathcal{C} \to \mathcal{C}'$ is *right t-exact* if it is exact and carries $\mathcal{C}_{\leq 0}$ to $(\mathcal{C}')_{\leq 0}$. Respectively, we say that f is *left t-exact* if it is exact and carries $\mathcal{C}_{\geq 0}$ to $(\mathcal{C}')_{\geq 0}$.

To conclude this section we recall the notions of left and right completeness. A t-structure in $\mathcal{C}$ is said to be *left-complete* if the canonical map from $\mathcal{C}$ to the homotopy limit $\widehat{C} := lim_n \mathcal{C}_{\leq n}$ of the diagram

$$... \longrightarrow \mathcal{C}_{\leq 2} \xrightarrow{\tau_{\leq 1}} \mathcal{C}_{\leq 1} \xrightarrow{\tau_{\leq 0}} \mathcal{C}_{\leq 0} \xrightarrow{\tau_{\leq -1}} ... \tag{2.1.37}$$

is an equivalence. A dual definition gives the notion of *right-completeness*. In general this limit is again a stable $(\infty, 1)$-category and its objects can be identified with families $X = \{X_i\}_{i \in \mathbb{Z}}$ such that $X_i \in \mathcal{C}_{\leq i}$ and $\tau_{\leq n} X_i \simeq X_n$ for every $n \leq i$. It admits a natural t-structure where X is in the positive subcategory if each X_i is in $\mathcal{C}_{\geq 0}$. This t-structure makes the canonical map $\mathcal{C} \to lim_n \mathcal{C}_{\leq n}$ both left and right t-exact. Moreover, the restriction $\mathcal{C}_{\leq 0} \to (\widehat{\mathcal{C}})_{\leq 0}$ is an equivalence [100, 1.2.1.17]. In general the difference between $\mathcal{C}$ and $\widehat{\mathcal{C}}$ lays exactly in the connective part. This difference disappears if the t-structure is left-complete: the restriction $\mathcal{C}_{\geq 0} \to \widehat{\mathcal{C}}_{\geq 0} \simeq lim_n(\mathcal{C}_{\leq n} \cap \mathcal{C}_{\geq 0})$ is an equivalence. Thanks to [100, 1.2.1.19], a t-structure is known to be left-complete if and only if the subcategory $\cap_n \mathcal{C}_{\geq n} \subseteq \mathcal{C}$ consists only of zero objects.

Remark 2.1.15. If $\mathcal{C}$ is a stable $(\infty, 1)$-category with a left-complete t-structure then Postnikov towers converge in $\mathcal{C}_{\geq 0}$. This follows from the definition of left-completeness and from the Remark 2.1.13.

Again a classical example of a stable $(\infty, 1)$-category with a t-structure is the $(\infty, 1)$-category of spectra Sp [100, 1.4.3.4, 1.4.3.5, 1.4.3.6] where $Sp_{\geq 0}$ is the full subcategory spanned by the spectrum objects X such that $\pi_n(X) = 0, \forall n \leq 0$. It is both right and left complete and its heart is equivalent to the nerve of the category of abelian groups.

2.1.26 Homological Algebra

The subject of homological algebra can be properly formulated using the language of stable $(\infty, 1)$-categories. If A is a Grothendieck abelian category, we can obtain the classical unbounded derived category of A as the homotopy category of an $(\infty, 1)$-category $\mathcal{D}(A)$. By the main result of [70] the category of unbouded chain complexes $Ch(A)$ admits a model structure for which the weak-equivalences are the quasi-isomorphisms of complexes and the cofibrations are the monomorphisms (this is usually called the *injective model structure*). We define $\mathcal{D}(A)$ as the $(\infty, 1)$-category *underlying* this model structure (see Section 2.2 below). It is stable [100, Prop. 1.3.5.9] and the pair of full subcategories $(\mathcal{D}(A)_{\leq 0}, \mathcal{D}(A)_{\geq 0})$ respectively spanned by the objects whose homology groups vanish in positive degree (resp. negative), determines a right-complete t-structure [100, 1.3.5.21]. This t-structure is not left-complete in general.

If X is a scheme, we know from [65] that $A = Qcoh(X)$ is Grothendieck abelian. The $(\infty, 1)$-category of the Example 2.1.5 is $\mathcal{D}(A)$.

In [100, Section 1.3] the author describes several alternative approaches to access this $(\infty, 1)$-category and its subcategory spanned by the right-bounded complexes. We will not review these results here.

2.2 From Model Categories to $(\infty, 1)$-categories

2.2.0.1 Model categories and ∞-categories

Model categories were invented (see [113]) as axiomatic structures suitable to perform the classical notions of homotopy theory. They have been extensively used and developed (see [69, 68] for an introduction) and still form the canonical way to introduce/present homotopical studies. A typical example is the homotopy theory of schemes which provides the motivation for this work. The primitive ultimate object associated to a model category $\mathcal{M}$ is its homotopy category $h(\mathcal{M})$ which can be obtained as a localization of $\mathcal{M}$ with respect to the class W of weak-equivalences. This localization should be taken in the world of categories. The problems start when we understand that $h(\mathcal{M})$ lacks some of the interesting homotopical information contained in $\mathcal{M}$ up to such a point that it is possible to have two model categories which are not equivalent but their homotopy categories are equivalent (see [37]).

This tells us that $h(\mathcal{M})$ is not an ultimate invariant and that in order to do homotopy theory we should not abandon the setting of model categories. But this brings some troubles. To start with, the theory of model categories is not "closed" meaning that, in general, the collection of morphisms between two model categories does not provide a new model category. Moreover, the theory is not adapted to consider homotopy theories with monoidal structures, their associated theories of homotopy algebra-objects and modules over them.

The quest to solve these problems is one of the possible motivations for the subject of $(\infty,1)$-categories. Every model category should have an associated $(\infty,1)$-category which should work as an ultimate container for the homotopical information in $\mathcal{M}$. In particular, the information about the homotopy category. The original motivation for the subject had its origins in the famous manuscript of A. Grothendieck [64]. In the last few years there were amazing developments and the reader has now available many good references for the different directions [16, 127, 99, 5, 114].
Back to our discussion, the key idea is that every model category $\mathcal{M}$ hides an $(\infty,1)$-category and this $(\infty,1)$-category encodes all the "'homotopical information"' contained in $\mathcal{M}$. The key idea dates to the works of Dwyer-Kan [48, 47] who found out that by performing the "'simplicial localization of $\mathcal{M}$"' - meaning a localization in the world of simplicial categories - instead of the usual localization in the setting of ordinary 1-categories, the resulting object would contain all the interesting homotopical information and, in particular, the classical homotopy category of $\mathcal{M}$ appearing in the "ground" level of this localization. The meaning of the preceding technique became clear once it was understood that simplicial categories are simply one amongst many other possible models for the theory of $(\infty,1)$-categories. Another possible model is provided by the theory of Joyal's quasi-categories, which was extensively developed in the recent years [99]. The method to assign an $(\infty,1)$-category to a model category $\mathcal{M}$ reproduces the original idea of Dwyer and Kan - Start from $\mathcal{M}$, see $\mathcal{M}$ as a trivial $(\infty,1)$-category and perform the localization of $\mathcal{M}$ with respect to the weak-equivalences - not in the world of usual categories - but in the world of $(\infty,1)$-categories. The resulting object will be refer to as the *underlying ∞-category of the model category* $\mathcal{M}$. For a more detailed exposition on this subject we redirect the reader to the exposition in [145].

For our purposes we need to understand that the nerve functor $N : Cat \to \hat{\Delta}$ provides the way to see a category as a trivial quasi-category. By definition, if $\mathcal{M}$ is model category with a class of weak-equivalences W, the underlying $(\infty,1)$-category of $\mathcal{M}$ is the localization $N(\mathcal{M})[W^{-1}]$ obtained in the setting of $(\infty,1)$-categories using the process described in 2.1.11. Moreover, the universal property of this new object implies that its associated homotopy category $h(N(\mathcal{M})[W^{-1}])$ recovers the classical localization. In particular, $N(\mathcal{M})[W^{-1}]$ and $N(\mathcal{M})$ have essentially the same objects. The main technical result which was originally discovered by Dwyer and Kan is the following:

Proposition 2.2.1. *([100, Prop. 1.3.4.20])*
Let $\mathcal{M}$ be a simplicial model category[12]. Then there exists an equivalence of $(\infty,1)$-categories between the underlying ∞-category of $\mathcal{M}$ and the $(\infty,1)$-category $N_\Delta(\mathcal{M}^\circ)$ where N_Δ is the simplicial nerve construction (see [99, Def. 1.1.5.5]) and $\mathcal{M}^\circ$ denotes the full simplicial subcategory of $\mathcal{M}$ of cofibrant-fibrant objects. In other words we have an isomorphism

$$N(\mathcal{M})[W^{-1}] \simeq N_\Delta(\mathcal{M}^\circ) \tag{2.2.1}$$

in the homotopy category of simplicial sets with the Joyal model structure [74].

This statement provides an ∞-generalization of the fundamental result by Quillen (see [113]) telling us that the localization $Ho(\mathcal{M})$ is equivalent to the naive homotopy theory of cofibrant-fibrant objects. By combining this result with [99, Thm. 4.2.4.1], we find a dictionary between the classical notions of homotopy limits and colimit in $\mathcal{M}$ (with $\mathcal{M}$ simplicial) and limits and colimits in the underlying $(\infty,1)$-category of $\mathcal{M}$.

[12] Assume the existence of functorial factorizations

2.2.1 Combinatorial Model Categories

The theory of combinatorial model categories and that of presentable $(\infty, 1)$-categories are equivalent. Morever, this equivalence is compatible with left Bousfield localizations:

Proposition 2.2.2. *([99, A.3.7.4, A.3.7.6]) Let $\mathcal{C}$ be a big $(\infty, 1)$-category. Then, $\mathcal{C}$ is presentable if and only if there exists a big $\mathbb{U}$-combinatorial simplicial model category $\mathcal{M}$ such that $\mathcal{C}$ is the underlying $(\infty, 1)$-category of $\mathcal{M}$. Moreover, if $\mathcal{M}$ is left-proper, left Bousfield localizations of $\mathcal{M}$ [13] correspond bijectively to accessible reflexive localizations of $\mathcal{C}$ (see our Notations).*

This has many important consequences. To start with, if $\mathcal{M} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{N}$ is a Quillen adjunction between combinatorial model categories then it induces an adjunction between the underlying $(\infty, 1)$-categories. To see this, remember from our preliminaries that the localization $N(\mathcal{M})[W^{-1}]$ can be obtained as a fibrant-replacement for the pair $(N(\mathcal{M}), W)$ in the model category of marked simplicial sets. Under the combinatorial hypothesis, $\mathcal{M}$ admits cofibrant and fibrant replacement functors and of course, they preserve weak-equivalences. If we let $\mathcal{M}^c$ denote the full subcategory of $\mathcal{M}$ spanned by the cofibrant objects and W_c the weak-equivalences between them, we will have an inclusion of marked simplicial sets $(N(\mathcal{M}^c), W_c) \subseteq (N(\mathcal{M}), W)$ together with a map in the inverse direction provided by the cofibrant-replacement functor (the same applies for the subcategories of fibrant, resp. cofibrant-fibrant, objects). By the universal property of the localization, these two maps provide an equivalence of $(\infty, 1)$-categories $N(\mathcal{M}^c)[W_c^{-1}] \simeq N(\mathcal{M})[W^{-1}]$. Back to the Quillen adjunction (F, G), Ken Brown's lemma provides a well-defined map of marked simplicial sets

$$(N(\mathcal{M}^c), W_c) \to (N(\mathcal{N}^c), W'_c) \tag{2.2.2}$$

and therefore, a new one

$$N(\mathcal{M}^c)[W_c^{-1}] \to N(\mathcal{N}_c)[W_c'^{-1}] \tag{2.2.3}$$

through the choice of fibrant-replacements in the model category of marked simplicial sets. It is the content of [100, 1.3.4.21] that if the original Quillen adjunction is an equivalence then this map is will also be.

Thanks to the results of [45] we know that every combinatorial model category is Quillen equivalent (by a zig-zag) to a simplicial combinatorial model category. The proposition implies that the underlying $(\infty, 1)$-category of a combinatorial model category is always presentable. In particular, it admits all limits and colimits which, again by the results of [45] together with the [99, Thm. 4.2.4.1], can be computed as homotopy limits and homotopy colimits in $\mathcal{M}$, namely, an object $X \in \mathcal{M}$ is an homotopy limit (resp. colimit) of a diagram $I \to \mathcal{M}$ if and only if it is a limit (resp. colimit) in $N(\mathcal{M})[W^{-1}]$ of the composition $N(I) \to N(\mathcal{M}) \to N(\mathcal{M})[W^{-1}]$, now in the sense of $(\infty, 1)$-categories (see [100, 1.3.4.23 and 1.3.4.24]).

Moreover, combining [99, Thm. 4.2.4.4] again with the main result of [45] we find that for any combinatorial model category $\mathcal{M}$ and small category I, there is an equivalence

$$N(\mathcal{M}^I)[W_{levelwise}^{-1}] \simeq Fun(N(I), N(\mathcal{M})[W^{-1}]) \tag{2.2.4}$$

In particular, for a left Quillen map between combinatorial model categories, the map induced between the underlying $(\infty, 1)$-categories (as above) commutes with colimits. The presentability, together with the adjoint functor theorem ([100, Cor. 5.5.2.9]) implies the existence of a right adjoint $N(\mathcal{M}^c)[W_c^{-1}] \leftarrow N(\mathcal{N}_c)[W_c'^{-1}]$ which we can describe explicitly as the composition

$$N(\mathcal{N}^c) \xrightarrow{P} N(\mathcal{N}^{cf}) \xrightarrow{G} N(\mathcal{M}^f) \xrightarrow{Q} N(\mathcal{M}^{cf}) \hookrightarrow N(\mathcal{M}^c) \tag{2.2.5}$$

where P is a fibrant replacement functor in $\mathcal{N}$ and Q is a cofibrant replacement functor in $\mathcal{M}$.

[13] with respect to a class of morphisms of small generation

In the simplicial case the underlying adjunction can be obtained with simpler technology (see [99, Prop. 5.2.4.6] defining $\bar{F}(X)$ as a fibrant replacement of $F(X)$ and $\bar{G}(Y)$ via a cofibrant replacement of $G(Y)$.

2.2.2 Compactly Generated Model Categories

The following discussion will be usefull in the last part of this work. Let $\mathcal{M}$ be a model category. Recall that an object X in $\mathcal{M}$ is said to be *homotopically finitely presented* if the mapping space functor $Map(X,-)$ commutes with filtered homotopy colimits. Recall also that if $\mathcal{M}$ is cofibrantly generated with I a set of generating cofibrations, then X is said to be a strict finite I-cell if there exists a finite sequence of morphisms in $\mathcal{M}$

$$X_0 = \emptyset \to X_1 \to ... \to X_n = X \tag{2.2.6}$$

such that for any i, we have a pushout square

$$\begin{tikzcd} X_i \arrow[r] & X_{i+1} \\ A \arrow[u] \arrow[r, "s"] & B \arrow[u] \end{tikzcd} \tag{2.2.7}$$

with $s \in I$. Recall also that $\mathcal{M}$ is said to be *compactly generated* if it is cellular and there is a set of generating cofibrations (resp. trivial cofibrations) I (resp. J) whose domains and codomains are cofibrant and (strictly) ω-compact and (strictly) ω-small with respect to the whole category $\mathcal{M}$. We have the following result

Proposition 2.2.3. *([141] Prop. 2.2) Let $\mathcal{M}$ be a compactly generated model category. Then any object is equivalent to a filtered colimit of strict finite I-cell objects. Moreover, if the (strict) filtered colimits in $\mathcal{M}$ are exact, an object X is homotopically finitely presented if and only if it is a retract of a strict finite I-cells object.*

This proposition, together with the results of [100] described in the last section, implies that if $\mathcal{M}$ is a combinatorial compactly generated model category where (strict) filtered colimits are exact, then the compact objects in the presentable $(\infty,1)$-category $N(\mathcal{M})[W^{-1}]$ are exactly the homotopicaly finitely presented objects in $\mathcal{M}$. Moreover, we have a canonical equivalence $N(\mathcal{M})[W^{-1}] \simeq Ind((N(\mathcal{M})[W^{-1}])^{\omega})$ (consult our Notations).

Chapter **3**

Preliminaries II - A World Map of Higher Algebra

Our goal in this section is to review the fundaments of the subject of *higher algebra* as developed in the works of J. Lurie in [100]. We collect the main notions and results and provide some new observations and results needed in the later sections of this work. The reader familiar with the subject can skip this section and consult these results later on.

3.1 ∞-Operads and Symmetric Monoidal $(\infty, 1)$-categories

The (technical) starting point of higher algebra is the definition of a symmetric monoidal structure on a $(\infty, 1)$-category (see Section 3.9 for the philosophical motivations). The guiding principle is that a symmetric monoidal $(\infty, 1)$-category is the data of an $(\infty, 1)$-category together with an operation $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$, a unit object $\Delta[0] \to \mathcal{C}$ and a collection of commutative diagrams providing associative and commutative restraints. There are three main reasons why a precise definition is difficult using brute force: (i) we don't know how to make explicit he whole list of diagrams; (ii) these diagrams are expected to be interrelated; (iii) in higher category theory the data of a commutative diagram is not a mere collection of vertices and edges: commutativity is defined by the existence of higher cells. The first and second problem exist already in the classical setting. The third problem makes the higher setting even more complicated for now to give (i) is to specify higher cells in Cat_∞ and (ii) is to write down explicit relations between them.

In this work we will follow the approach of [100] where a symmetric monoidal $(\infty, 1)$-category is a particular instance of the notion of ∞-operad. In order to understand the idea, we recall that both classical operads and classical symmetric monoidal categories can be seen as particular instances of the more general notion of colored operad (also known as "'multicategory"'). At the same time, classical symmetric monoidal categories can be understood as certain types of diagrams of categories indexed by the category Fin_* of pointed finite sets. Using the *Grothendieck-Construction*, we can encode this diagram-style definition of a symmetric monoidal category in the form of a category cofibered over Fin_* with an additional property - the fiber over a finite set $\langle n \rangle$ is equivalent to the n-th power of the fiber over $\langle 1 \rangle$ (follow the notations below). Moreover, by weakening this form, it is possible to reproduce the notion of a coloured operad in this context. This way - operads, symmetric monoidal structures and coloured operads - are brought to the same setting: everything can be written in the world of "things over Fin_*". The book [92] provides a good introduction to these ideas.

In [103, 33, 32, 31] the authors explore another approach to the theory of $(\infty, 1)$-operads. The key observation is that the theory of simplicial sets admits a natural extension - the theory of dendroidal sets - that allows us to naturally capture the structure of a multicategory. Similarly to simplicial sets, these admit an appropriate homotopy theory, which more recently in [67] was proved to be equivalent to the theory developed by J. Lurie in [100]. We will give a brief overview below.

3.1.1 ∞-operads

In order to provide the formal definitions we need to recall some of the terminology introduced in [100]. We write $\langle n\rangle \in N(Fin_*)$ to denote the finite set $\{0,1,...,n\}$ with 0 as the base point and $\langle n\rangle^+$ to denote its subset of non-zero elements. A morphism $f:\langle n\rangle \to \langle m\rangle$ will be called *inert* if for each $i\in\langle m\rangle^+$, $f^{-1}(\{i\})$ has exactly one element. Alternatively, a map f is inert iff it is surjective and the induced map $\langle n\rangle - f^{-1}(\{0\}) \to \langle m\rangle^+$ is a bijection. Notice that the canonical maps $\langle n\rangle \to \langle 0\rangle$ are inert. Moreover, for each $i\in\langle n\rangle^+$, we write $\rho^i:\langle n\rangle\to\langle 1\rangle$ for the inert map sending i to 1 and everything else to 0. We say that f is *active* if $f^{-1}(\{0\})=\{0\}$.

Definition 3.1.1. *([100]- Definition 2.1.1.10)*
An ∞-operad is an ∞-category $\mathcal{O}^\otimes$ together with a map $p:\mathcal{O}^\otimes \to N(Fin_*)$ satisfying the following list of properties:

1. *For every inert morphism $f:\langle m\rangle\to\langle n\rangle$ and every object C in the fiber $\mathcal{O}^\otimes_{\langle m\rangle} := p^{-1}(\{\langle m\rangle\})$, there exists a p-coCartesian morphism (see [99, Def. 2.4.1.1]) $\bar{f}:C\to\bar{C}$ lifting f. In particular, f induces a functor $f_!:\mathcal{O}^\otimes_{\langle m\rangle}\to\mathcal{O}^\otimes_{\langle n\rangle}$;*
2. *Given objects C in $\mathcal{O}^\otimes_{\langle m\rangle}$ and C' in $\mathcal{O}^\otimes_{\langle n\rangle}$ and a morphism $f:\langle m\rangle\to\langle n\rangle$ in Fin_*, we write $Map^f_{\mathcal{O}^\otimes}(C,C')$ for the disjoint union of those connected components of $Map_{\mathcal{O}^\otimes}(C,C')$ which lie over $f\in Map_{N(Fin_*)}(\langle m\rangle,\langle n\rangle) := Hom_{Fin}(\langle m\rangle,\langle n\rangle)$.*

   We demand the following condition: whenever we choose p-coCartesian morphisms $C'\to C'_i$ lifting the inert morphisms $\rho^i:\langle n\rangle\to\langle 1\rangle$ for $1\le i\le n$ (these liftings exists by (1)), the induced map

   $$Map^f_{\mathcal{O}^\otimes}(C,C')\to\prod Map^{\rho^i\circ f}_{\mathcal{O}^\otimes}(C,C'_i) \tag{3.1.1}$$

   is an homotopy equivalence of spaces;
3. *For each $n\ge 0$, the functors $\rho^i_!:\mathcal{O}^\otimes_{\langle n\rangle}\to\mathcal{O}$ (where $\mathcal{O}^\otimes$ denotes the fiber over $\langle 1\rangle$) induced by the inert maps ρ^i through condition (1), induce an equivalence of $(\infty,1)$-categories $\mathcal{O}^\otimes_{\langle n\rangle}\to\mathcal{O}^n$. In particular, for $n=0$ we have $\mathcal{O}^\otimes_{\langle 0\rangle}\simeq\Delta[0]$.*

Notice that with this definition any ∞-operad $\mathcal{O}^\otimes\to N(Fin_*)$ is a categorical fibration. From now, we will make an abuse of notation and write $\mathcal{O}^\otimes$ for an ∞-operad $p:\mathcal{O}^\otimes\to N(Fin_*)$, ommiting the structure map to $N(Fin_*)$. We denote the fiber over $\langle 1\rangle$ by $\mathcal{O}$ and refer to it as the *underlying ∞-category* of $\mathcal{O}^\otimes$. The objects of $\mathcal{O}$ are called the *colours* or *objects* of the ∞-operad $\mathcal{O}^\otimes$. To illustrate the definition, condition (3) tells us that any object $C\in\mathcal{O}^\otimes$ living over $\langle n\rangle$ can be identified with a unique (up to equivalence) collection $(X_1,X_2,...,X_n)$ where each X_i is an object in $\mathcal{O}$. Moreover, if $C\to C'$ is a coCartesian morphism in $\mathcal{O}^\otimes$ lifting an inert morphism $\langle n\rangle\to\langle m\rangle$ and if $C=(X_1,X_2,...,X_n)$ then C' corresponds (up to equivalence) to the collection $(X_{f^{-1}(\{1\})},...,X_{f^{-1}(\{m\})})$. In other words, coCartesian liftings of inert morphisms $C=(X_1,X_2,...,X_n)\to C'$ in $\mathcal{O}^\otimes$ correspond to the selection of m colours (without repetition) out of the n presented in C. Finally, if $C=(X_i)_{1\le i\le n}$ and $C'=(X'_j)_{1\le i\le m}$ are objects in $\mathcal{O}^\otimes$, condition (2) tells us that

$$Map_{\mathcal{O}^\otimes}((X_i)_{1\le i\le n},(X'_j)_{1\le i\le m})\simeq\prod_j Map_{\mathcal{O}^\otimes}((X_i)_{1\le i\le n},X'_j) \tag{3.1.2}$$

Let $p:\mathcal{O}^\otimes\to N(Fin_*)$ be an ∞-operad. We say that a morphism f in $\mathcal{O}^\otimes$ is *inert* if its image in $N(Fin_*)$ is inert and f is p-coCartesian. We say that f is active if $p(f)$ is active. By [100, 2.1.2.4], the collections ({inert morphism}, {active morphisms}) form a (*strong*) *factorization system* in $\mathcal{O}^\otimes$ ([99, Def. 5.2.8.8]).

The simplest example of an ∞-operad is the identity map $N(Fin_*)\to N(Fin_*)$. Its underlying $(\infty,1)$-category corresponds to $\Delta[0]$. It is called the *commutative ∞-operad* and we use the notation

$\mathcal{C}omm^{\otimes} = N(Fin_*)$. Another simple example is the *trivial* ∞-operad $\mathcal{T}riv^{\otimes}$. By definition, it is given by the subcategory of $N(Fin_*)$ of all objects $\langle n\rangle$ together with the inert morphisms.

More generally, there is a mechanism - the so-called *operadic nerve* $N^{\otimes}(-)$ - to produce an ∞-operad out of a *simplicial coloured operad whose mapping spaces are Kan-complexes.*

Construction 3.1.2. If $\mathcal{A}$ is a simplicial coloured operad, we construct a new simplicial category $\tilde{\mathcal{A}}$ as follows: the objects of $\tilde{\mathcal{A}}$ are pairs $(\langle n\rangle, (X_1, ..., X_n))$ where $\langle n\rangle$ is an object in Fin_* and $(X_1, ..., X_n)$ is a sequence of colours in $\mathcal{A}$. The mapping spaces are defined by the formula

$$Map_{\tilde{\mathcal{A}}}((X_1, ..., X_n), (Y_1, ..., Y_m)) := \coprod_{f:\langle n\rangle\to\langle m\rangle} \prod_{i=1}^{m} Map_{\mathcal{A}}((X_\alpha)_{\alpha\in f^{-1}(\{i\})}, Y_i) \tag{3.1.3}$$

If $\mathcal{A}$ is enriched over Kan complexes, it is immediate that $\tilde{\mathcal{A}}$ is a fibrant simplicial category. Following [100, 2.1.1.23] we set $N^{\otimes}(\mathcal{A}) := N_{\Delta}(\tilde{\mathcal{A}})$. In this case (see [100, 2.1.1.27]) the canonical projection $\pi : N^{\otimes}(\mathcal{A}) \to N(Fin_*)$ is an $(\infty,1)$-operad. In particular, this mechanism works using a classical operad as input.

Example 3.1.3. This mechanism can be used to construct the *associative operad* $\mathcal{A}ss^{\otimes}$. Following the Definition 4.1.1.3 of [100]), we let **Ass** be the multicategory with one color **a** and having as set of operations $Hom(\{\mathbf{a}\}_I, \mathbf{a})$ the set of total order relations on I. In other words, an operation

$$\underbrace{(\mathbf{a}, ..., \mathbf{a})}_{n} \to \mathbf{a} \tag{3.1.4}$$

consists of a choice of a permutation of the n-factors. We can now understand **Ass** as enriched over constant Kan-complexes and applying the Construction 3.1.2 we find a fibrant simplicial category $\widetilde{\mathbf{Ass}}$ whose simplicial nerve is by definition, the associative ∞-operad $\mathcal{A}ss^{\otimes}$.

Explicitly, the objects of $\mathcal{A}ss^{\otimes}$ can be identified with the objects of $N(Fin_*)$. Morphisms $f : \langle n\rangle \to \langle m\rangle$ are given by the choice of a morphisms in $N(Fin_*)$, $\langle n\rangle \to \langle m\rangle$ together with the choice of a total order on each $f^{-1}(\{j\})$ for each $j \in \langle m\rangle^{\circ}$. With this description, it is obvious that $\mathcal{A}ss^{\otimes}$ comes equipped with a map towards $N(Fin_*)$ obtained by forgetting the total orderings.

Example 3.1.4. The associative operad represents the first element in a distinguished family of ∞-operads: for any natural number $n \in \mathbb{N}$, we can construct a fibrant simplicial colored operad [100, Def. 5.1.0.2] whose simplicial nerve $\mathbb{E}_n^{\otimes}$ is called the ∞-operad of *little* n*-cubes*. For every $n \geq 0$ the objects of $\mathbb{E}_n^{\otimes}$ are the same objects of $N(Fin_*)$ and in particular it only has one color. When $n = 1$, there is an equivalence $\mathbb{E}_1^{\otimes} \simeq \mathcal{A}ss$. Moreover, there are natural maps of ∞-operads $\mathbb{E}_n^{\otimes} \to \mathbb{E}_{n+1}^{\otimes}$ and by [100, Cor. 5.1.1.5], the colimit of the sequence (in the $(\infty,1)$-category of operads described in the next section)

$$\mathbb{E}_0^{\otimes} \to \mathbb{E}_1^{\otimes} \to \mathbb{E}_2^{\otimes} \to ... \tag{3.1.5}$$

is the commutative operad $\mathcal{C}omm^{\otimes}$.

3.1.2 The $(\infty,1)$-category of ∞-operads

By definition, a map of ∞-operads is a map of simplicial sets $\mathcal{O}^{\otimes} \to (\mathcal{O}')^{\otimes}$ over $N(Fin_*)$, sending inert morphisms to inert morphisms. Following [100], we write $Alg_{\mathcal{O}}(\mathcal{O}')$ to denote the full subcategory of $Fun_{N(Fin_*)}(\mathcal{O}^{\otimes}, (\mathcal{O}')^{\otimes})$ spanned by the maps of ∞-operads.

The collection of ∞-operads can be organized in a new $(\infty,1)$-category Op_{∞} which can be obtained as the simplicial nerve of the fibrant simplicial category whose objects are the $(\infty,1)$-operads and the mapping spaces are the maximal Kan-complexes inside $Alg_{\mathcal{O}}(\mathcal{O}')$. According [100, Prop. 2.1.4.6], there is a model structure in the category of marked simplicial sets over $N(Fin_*)$ which has Op_{∞} as its underlying $(\infty, 1)$-category.

Let us now briefly present some alternative approaches to the theory ∞-operads:

- As already mentioned in the introduction, there is the approach using dendroidal sets. The language of dendroidal sets was introduced in [103] with the intention of generalizing the interplay between simplicial sets and categories to englobe also colored operads. There the authors construct a category of (rooted) trees Ω which includes Δ as a full subcategory and define dendroidal sets as presheaves of sets over Ω. Any coloured operad admits a nerve encoded by a dendroidal set. More recently in [33] the authors proved the existence of a model structure on dendroidal sets that extends the Joyal's model structure on simplicial sets. A ∞-operad is then defined to be a fibrant object for this model structure.

- There is the approach of dendroidal spaces studied in [31]. By definition, a dendroidal space is a simplicial object in the category of dendroidal sets, or, equivalently, a functor from Ω^{op} to simplicial sets. In loc. cit., the authors introduce a Segal type condition on the objects of this category and a completeness condition like the one for the complete Segal spaces of Rezk. Moreover, they prove the existence of a model structure on dendroidal spaces for which these Segal-like complete objects are exactly the fibrant objects;

- Also in [31] the authors introduce the notion of a Segal operad. By definition these are also dendroidal spaces this time under the requirement that η (by definition, the image of $[0] \in \Delta$ along the inclusion $\Delta \subseteq \Omega$) must be sent to a discrete simplicial set, together with the requirement for a certain Segal-like condition to hold. The authors prove the existence of a model structure on the full subcategory of dendroidal spaces spanned by those objects $X : \Omega^{op} \to \widehat{\Delta}$ that send η to a final object (so-called pre-Segal Operads) and check that the fibrant objects for this model structure are exactly the Segal operads [31, Thm 8.13 and 8.17]. The strictification result [99, 4.2.4.4] provides then an equivalence of $(\infty, 1)$-categories between the underlying $(\infty, 1)$-category of the model structure on pre-Segal Operads and $Fun^{Segal,\eta}(N(\Omega)^{op}, \mathcal{S})$ - the full subcategory of $Fun(N(\Omega)^{op}, \mathcal{S})$ spanned by those dendroidal spaces that satisfy the Segal condition for trees and send η to a discrete space.

- Finally, there is the approach using Simplicial colored Operads. By definition, these are colored operads enriched in simplicial sets. Again, this category admits a model structure for which the fibrant objects are exactly the colored operads enriched in Kan-complexes. See [32, Thm 1.4].

The first, second and third approaches are known to be Quillen equivalent. See, respectively, the Corollary 6.7 and Theorem 8.15 in [31]. The third and fourth approaches are also known to be Quillen equivalent (see [32, Thm 8.4]). More recently in [67], they were all shown to be Quillen equivalent to the approach of J. Lurie used in this overview. In particular their associated underlying $(\infty, 1)$-categories are equivalent and we have

$$\mathcal{O}p_{\infty} \simeq Fun^{Segal,\eta}(N(\Omega)^{op}, \mathcal{S}) \tag{3.1.6}$$

This presentation of the theory of ∞-operads in $\mathcal{S}$ provides some intuition on how to define ∞-operad objects in any cartesian symmetric monoidal $(\infty, 1)$-category $\mathcal{C}^{\otimes}$ (see Section 3.1.8 below). This is not at all immediate using the framework of J. Lurie in [100]. We won't pursue this in this work.

3.1.3 Monoidal and Symmetric Monoidal $(\infty, 1)$-categories

We say that a map of ∞-operads $q : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ is a *fibration of ∞-operads* (respectively *coCartesian fibration of ∞-operads*) if it is a categorical fibration (resp. coCartesian fibration) of simplicial sets (see [99, Def. 2.4.2.1]).

Definition 3.1.5. *Let $\mathcal{O}^{\otimes}$ be an ∞-operad. An* $\mathcal{O}$-monoidal $(\infty,1)$-category *is the data of an ∞-operad $\mathcal{C}^{\otimes}\to N(Fin_*)$ together with a coCartesian fibration of ∞-operads $\mathcal{C}^{\otimes}\to\mathcal{O}^{\otimes}$. A* symmetric monoidal $(\infty,1)$-category *is a $\mathcal{C}omm$-monoidal $(\infty,1)$-category. A* monoidal $(\infty,1)$*-category is an $\mathcal{A}ss$-monoidal $(\infty,1)$-category.*

Let $p:\mathcal{C}^{\otimes}\to N(Fin_*)$ be a symmetric monoidal $(\infty,1)$-category. As for general ∞-operads, we denote by $\mathcal{C}$ the fiber of p over $\langle 1\rangle$ and refer to it as the *underlying* $(\infty,1)$*-category* of $\mathcal{C}^{\otimes}$. To understand how this definition encodes the usual way to see the monoidal operation, we observe that if $f:\langle n\rangle\to\langle m\rangle$ is an active morphism in $N(Fin_*)$ and $C=(X_1,...,X_n)$ is an object in the fiber over $\langle n\rangle$ (notation: $\mathcal{C}^{\otimes}_{\langle n\rangle}$), by the definition of a coCartesian fibration, there exists a p-coCartesian lift of f, $\tilde{f}:C\to C'$ where we can identify C' with a collection $(Y_1,...,Y_m)$, with each Y_i an object in $\mathcal{C}$. The coCartesian property motivates the identification

$$Y_i=\bigotimes_{\alpha\in f^{-1}(\{i\})}X_\alpha \tag{3.1.7}$$

where the equality should be understood only in the philosophical sense. When applied to the active morphisms $\langle 0\rangle\to\langle 1\rangle$ and $\langle 2\rangle\to\langle 1\rangle$ we obtain functors $1:\Delta[0]\to\mathcal{C}$ and $\otimes:\mathcal{C}\times\mathcal{C}\to\mathcal{C}$. We will refer to the first as the *unit* of the monoidal structure. The second recovers the usual multiplication. By playing with the other active morphisms we recover the usual data defining a symmetric monoidal structure. The coherences will appear out of the properties characterizing the coCartesian lifts.

It is an easy but important observation that these operations endow the homotopy category of $\mathcal{C}$ with a symmetric monoidal structure in the classical sense.

Example 3.1.6. Let $\mathcal{C}$ be a classical symmetric monoidal category. By regarding it as a trivial simplicial coloured operad and using the Construction 3.1.2 we obtain an ∞-operad $N^{\otimes}(\mathcal{C})\to N(Fin_*)$ which is a symmetric monoidal $(\infty,1)$-category whose underlying $(\infty,1)$-category is equivalent to the nerve $N(\mathcal{C})$.

Remark 3.1.7. Let $\mathcal{C}^{\otimes}\to N(Fin_*)$ be a symmetric monoidal $(\infty,1)$-category. Then, $\mathcal{C}^{\otimes}$ has a naturally associated monoidal structure obtained as the pullback $\mathcal{C}^{\otimes}\times_{N(Fin_*)}\mathcal{A}ss^{\otimes}\to\mathcal{A}ss^{\otimes}$ performed along the natural map $\mathcal{A}ss^{\otimes}\to N(Fin_*)$. It follows from the definitions that both these monoidal structures have the same underlying $(\infty,1)$-category.

3.1.4 Monoidal Functors

Let $p:\mathcal{C}^{\otimes}\to\mathcal{O}^{\otimes}$ and $q:\mathcal{D}^{\otimes}\to\mathcal{O}^{\otimes}$ be $\mathcal{O}$-monoidal $(\infty,1)-$ *categories* and let $F:\mathcal{C}^{\otimes}\to\mathcal{D}^{\otimes}$ be a map of ∞-operads over $\mathcal{O}^{\otimes}$. Let us first consider $\mathcal{O}=\mathcal{C}omm$. For any object object $C=(X,Y)$ in the fiber over $\langle 2\rangle$, the definitions allow us to extract a natural morphism in $\mathcal{D}$

$$F(X)\otimes F(Y)\to F(X\otimes Y) \tag{3.1.8}$$

which in general doesn not have to be an equivalence. In other words, operadic maps correspond to *lax* monoidal functors.

Back to the general case, the full compatibility between the monoidal structures is equivalent to ask for F to send p-coCartesian morphisms in $\mathcal{C}^{\otimes}$ to q-coCartesian morphisms in $\mathcal{D}^{\otimes}$. These are called $\mathcal{O}$-*monoidal functors* and we write $Fun^{\otimes}_{\mathcal{O}}(\mathcal{C},\mathcal{D})$ to denote their full subcategory inside $Fun_{\mathcal{O}^{\otimes}}(\mathcal{C}^{\otimes},\mathcal{D}^{\otimes})$. It follows from the definitions that an $\mathcal{O}$-monoidal functor $\mathcal{C}^{\otimes}\to\mathcal{D}^{\otimes}$ is an equivalence of ∞-categories if and only if the map induced between the underlying ∞-categories $\mathcal{C}\to\mathcal{D}$ is an equivalence.

3.1.5 Objectwise product on diagram categories

Let $p : \mathcal{C}^{\otimes} \to N(Fin_*)$ be a symmetric monoidal $(\infty,1)$-category. Given an arbitrary simplicial set K, and similarly to the classical case, we can hope for the existence of a monoidal structure in $Fun(K,\mathcal{C})$ defined objectwise, meaning that the product of two functors f, g at an object $k \in K$ should be given by the product of f and g at k, in $\mathcal{C}^{\otimes}$. Indeed, there exists such a structure $Fun(K,\mathcal{C})^{\otimes}$, defined as the homotopy pullback of the diagram of ∞-categories

$$\begin{tikzcd} & Fun(K,\mathcal{C}^{\otimes}) \arrow[d] \\ N(Fin_*) \arrow[r, "\delta"] & Fun(K,N(Fin_*)) \end{tikzcd} \tag{3.1.9}$$

where the vertical map corresponds to the composition with p and the map δ sends an object $\langle n\rangle$ to the constant diagram in $N(Fin_*)$ with value $\langle n\rangle$. By [99, 3.1.2.1], the composition map $Fun(K,\mathcal{C}^{\otimes}) \to Fun(K,N(Fin_*))$ is also a coCartesian fibration and therefore since every object in the diagram 3.1.9 is fibrant, the homotopy pullback is given by the strict pullback. Moreover, the natural map $Fun(K,\mathcal{C})^{\otimes} \to N(Fin_*)$ is a cocartesian fibration because it is the pull-back of a cocartesian fibration. Notice also that the underlying $(\infty,1)$-category of $Fun(K,\mathcal{C})^{\otimes}$ is equivalent to $Fun(K,\mathcal{C})$ by the formulas

$$Fun(K,\mathcal{C})^{\otimes} \times^h_{N(Fin_*)} \Delta[0] \simeq Fun(K,\mathcal{C}^{\otimes}) \times^h_{Fun(K,N(Fin_*))} N(Fin_*) \times^h_{N(Fin_*)} \Delta[0] \tag{3.1.10}$$

$$\simeq Fun(K,\mathcal{C}^{\otimes}) \times^h_{Fun(K,N(Fin_*))} \Delta[0] \simeq Fun(K,\mathcal{C}) \tag{3.1.11}$$

In fact these constructions hold if we consider $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ any coCartesian fibration of operads (see [100, 2.1.3.4]).

3.1.6 Subcategories closed under the monoidal product

If $p : \mathcal{C}^{\otimes} \to N(Fin_*)$ is a symmetric monoidal $(\infty,1)$-category with underlying category $\mathcal{C}$, whenever we have $\mathcal{C}_0 \subseteq \mathcal{C}$ a full subcategory of $\mathcal{C}$ we can ask if the monoidal structure $\mathcal{C}^{\otimes}$ can be restricted to a new one $(\mathcal{C}_0)^{\otimes}$ in $\mathcal{C}_0$. As explained in [100, 2.2.1.1, 2.2.1.2], if $\mathcal{C}_0$ is stable under equivalences (meaning that if X is an object in $\mathcal{C}_0$ and $X \to Y$ (or $Y \to X$) is an equivalence in $\mathcal{C}$, then Y is in $\mathcal{C}_0$) and if $\mathcal{C}_0$ is closed under the tensor product $\mathcal{C}\times\mathcal{C} \to \mathcal{C}$ and contains the unit object, then the restriction of p to the full subcategory $(\mathcal{C}_0)^{\otimes} \subseteq \mathcal{C}^{\otimes}$ spanned by the objects $X = (X_1,...,X_n)$ in $\mathcal{C}^{\otimes}$ where each X_i is in $\mathcal{C}_0$, is again a coCartesian fibration and the inclusion $\mathcal{C}_0^{\otimes} \subseteq \mathcal{C}^{\otimes}$ is a monoidal functor. Moreover, if the inclusion $\mathcal{C}_0 \subseteq \mathcal{C}$ admits a right adjoint τ, it can be naturally extended to a map of ∞-operads $\tau^{\otimes} : \mathcal{C}^{\otimes} \to \mathcal{C}_0^{\otimes}$. In particular, for any ∞-operad $\mathcal{O}^{\otimes}$, $\tau^{\otimes}$ gives a right adjoint to the canonical inclusion

$$Alg_{\mathcal{O}}(\mathcal{C}_0) \hookrightarrow Alg_{\mathcal{O}}(\mathcal{C}) \tag{3.1.12}$$

(see 3.2 below for the theory of algebras).

3.1.7 Monoidal Reflexive Localizations

Let again $p : \mathcal{C}^{\otimes} \to N(Fin_*)$ be a symmetric monoidal $(\infty,1)$-category. In the sequence of the previous topic, we can find situations in which a full subcategory $\mathcal{C}_0 \subseteq \mathcal{C}$ is not stable under the product in $\mathcal{C}$ but we can still define a monoidal structure in $\mathcal{C}_0$. We say that a subcategory $\mathcal{D}^{\otimes} \subseteq \mathcal{C}^{\otimes}$ is a *monoidal reflexive localization of* $\mathcal{C}^{\otimes}$ if the inclusion $\mathcal{D}^{\otimes} \subseteq \mathcal{C}^{\otimes}$ admits a left adjoint map of ∞-operads $L^{\otimes} : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ with $L^{\otimes}$ a monoidal map. By [100, 2.2.1.9], if $\mathcal{C}_0$ is a reflexive localization of $\mathcal{C}$ and the localization satisfies the condition:

$(*)$ for every L-equivalence $f : X \to Y$ in $\mathcal{C}$ (meaning that $L(f)$ is an equivalence) and every object Z in $\mathcal{C}$, the induced map $X \otimes Z \to Y \otimes Z$ is again a L-equivalence (see [100, 2.2.1.6, 2.2.1.7]).

then the full subcategory $\mathcal{C}_0^{\otimes}$ of $\mathcal{C}^{\otimes}$ defined in the previous topic, becomes a monoidal reflexive localization of $\mathcal{C}^{\otimes}$. However, and contrary to the previous situation, the inclusion $\mathcal{C}_0^{\otimes} \subseteq \mathcal{C}^{\otimes}$ will only be lax monoidal.

Remark 3.1.8. If $\mathcal{C}_0^{\otimes} \subseteq \mathcal{C}^{\otimes}$ is a monoidal reflexive localization, then for any ∞-operad $\mathcal{O}^{\otimes}$ the category of algebras $Alg_{\mathcal{O}}(\mathcal{C}_0)$ is a reflexive localization of $Alg_{\mathcal{O}}(\mathcal{C})$. (see 3.2 below for the theory of algebras).

3.1.8 Cartesian and Cocartesian Symmetric Monoidal Structures

We recall the analogues of two classical situations. If $\mathcal{C}$ is a category with finite products and a final object then the operation (-×-) gives birth to a symmetric monoidal structure in $\mathcal{C}$. As explained in [100, Section 2.4.1] there is a mechanism that allows us to extend this classical situation to the ∞-setting. For any $(\infty,1)$-category $\mathcal{C}$ we can construct a new $(\infty,1)$-category $\mathcal{C}^{\times}$ equipped with a map to $N(Fin_*)$ [100, 2.4.1.4] this map being a symmetric monoidal structure if and only if $\mathcal{C}$ admits finite products [100, 2.4.1.5]. More generally, and thanks to the results of [100, 2.4.1.6, 2.4.1.7 and 2.4.1.8] a symmetric monoidal $(\infty,1)$-category $\mathcal{C}^{\otimes}$ is said to be *Cartesian* if its underlying ∞-category $\mathcal{C}$ admits finite products and we have a monoidal equivalence $\mathcal{C}^{\otimes} \simeq \mathcal{C}^{\times}$ which is the identity on $\mathcal{C}$. Moreover, the construction $\mathcal{C}^{\times}$ is characterized by a universal property related to the preservation of products: if $\mathcal{C}$ and $\mathcal{D}$ are $(\infty,1)$-categories with finite products then the space of monoidal maps $\mathcal{C}^{\times} \to \mathcal{D}^{\times}$ is homotopy equivalent to the space of functors $\mathcal{C} \to \mathcal{D}$ that preserve products.

The second classical situation is that of a category with finite sums and an initial object. In the ∞-categorical setting we can apply the preceding argument to the opposite category of $\mathcal{C}$ to deduce the existence of a monoidal structure induced by the disjoint sums in $\mathcal{C}$. In [100, Section 2.4.3] the author provides an independent description of this monoidal structure. Again, from any $(\infty,1)$-category $\mathcal{C}$ we can construct (see [100, 2.4.3.1]) a simplicial set $\mathcal{C}^{\coprod}$ together with a map to $N(Fin_*)$ which we can prove to be always an ∞-operad [100, 2.4.3.3]. Finally, and as explained in the Remark [100, 2.4.3.4] this ∞-operad is a symmetric monoidal $(\infty,1)$-category if and only if and $\mathcal{C}$ has finite sums and an initial object. With this, we say that an ∞-operad is *cocartesian* if it is equivalent to one of the form $\mathcal{C}^{\coprod}$ for some $(\infty,1)$-category $\mathcal{C}$. The assignemt $\mathcal{C} \mapsto \mathcal{C}^{\coprod}$ has a universal property [100, Thm 2.4.3.18]: for any symmetric monoidal $(\infty,1)$-category $\mathcal{D}^{\otimes}$ any map $\mathcal{C} \to CAlg(\mathcal{D})$ can be lifted in a essentially unique way to a lax monoidal functor $\mathcal{C}^{\coprod} \to \mathcal{D}^{\otimes}$.

Finally, if $\mathcal{C}$ is an $(\infty,1)$-category with direct sums and a zero object, the cartesian and cocartesian symmetric monoidal structures are canonically equivalent by means of the description in [100, 2.4.3.19].

In the next section (Remarks 3.2.2 and 3.2.3) we will review how the theory of algebras in a cartesian/ cocartesian structure admits a much more simpler description than in the general case.

3.1.9 Monoid objects

Following [100, Section 2.4.2], if $\mathcal{C}^{\times} \to N(Fin_*)$ is a cartesian symmetric monoidal $(\infty,1)$-category in the sense of the previous subsection, we define for each ∞-operad $p : \mathcal{O}^{\otimes} \to N(Fin_*)$ an *$\mathcal{O}$-monoid object in* $\mathcal{C}$ as being a functor $F : \mathcal{O}^{\otimes} \to \mathcal{C}$ satisfying the usual Segal condition: for each object $C = (x_1, ..., x_n)$ in $\mathcal{O}^{\otimes}$ with $x_1, ..., x_n$ in $\mathcal{O}$ and given p-coCartesian liftings $\tilde{\rho}^i : (x_1, ..., x_n) \to x_i$ for the inert morphisms ρ^i in $N(Fin_*)$, the induced product map $F(C) \to \prod_i F(X_i)$ should an equivalence in $\mathcal{C}$. The collection of $\mathcal{O}$-monoid objects in $\mathcal{C}$ can be organized in a new $(\infty,1)$-category $Mon_{\mathcal{O}}(\mathcal{C})$.

3.2 Algebra Objects

3.2.1 Algebras over an $(\infty, 1)$-operad

Let

$$\begin{tikzcd} & \mathcal{C}^{\otimes} \arrow[d, "p"] \\ \mathcal{O}'^{\otimes} \arrow[r, "f"] & \mathcal{O}^{\otimes} \end{tikzcd} \tag{3.2.1}$$

be a diagram of ∞-operads with p a fibration of ∞-operads. We denote by $Fun_{\mathcal{O}^{\otimes}}(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes})$ the strict pullback

$$\begin{tikzcd} & Fun(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes}) \arrow[d] \\ \Delta[0] \arrow[r, "f"] & Fun(\mathcal{O}'^{\otimes}, \mathcal{O}^{\otimes}) \end{tikzcd} \tag{3.2.2}$$

whose vertices correspond to the dotted maps rendering the diagram commutative

$$\begin{tikzcd} & \mathcal{C}^{\otimes} \arrow[d, "p"] \\ \mathcal{O}'^{\otimes} \arrow[r, "f"] \arrow[ur, dashed] & \mathcal{O}^{\otimes} \end{tikzcd} \tag{3.2.3}$$

By construction, it is an $(\infty, 1)$-category and following [100, Def. 2.1.3.1], we denote by $Alg_{\mathcal{O}'/\mathcal{O}}(\mathcal{C})$ its full subcategory spanned by the maps of ∞-operads defined over $\mathcal{O}^{\otimes}$. We refer to it as the *∞-category of $\mathcal{O}'$-algebras of* $\mathcal{C}$. We consider some special cases:

- If $f = Id$, we will simply write $Alg_{/\mathcal{O}}(\mathcal{C})$ to denote this construction;
- In the particular case $\mathcal{O}^{\otimes} = N(Fin_*)$, this construction recovers the ∞-category of maps of ∞-operads $Alg_{\mathcal{O}'}(\mathcal{C})$ defined earlier in this survey.
- If both $\mathcal{O}'^{\otimes} = \mathcal{O}^{\otimes} = N(Fin_*)$, the ∞-category $Alg_{\mathcal{C}omm}(\mathcal{C})$ can be identified with the ∞-category of sections $s : N(Fin) \to \mathcal{C}^{\otimes}$ of the structure map $p : \mathcal{C}^{\otimes} \to N(Fin_*)$, which send inert morphisms to inert morphisms. This condition forces every $s(\langle n \rangle)$ to be of the form $(X, X, ..., X)$ for some object X in $\mathcal{C}$. Moreover, the image of the active morphisms in $N(Fin_*)$ will produce maps $X \otimes X \to X$ and $1 \to X$ endowing X in $\mathcal{C}$ with the structure of a commutative algebra. The cocartesian property is the machine that produces coherence diagrams. As an example, to extract the first associative restraint we consider the image through s of the diagram

$$\begin{tikzcd} \langle 3 \rangle \arrow[r] \arrow[d] & \langle 2 \rangle \arrow[d] \\ \langle 2 \rangle \arrow[r] & \langle 1 \rangle \end{tikzcd} \tag{3.2.4}$$

of active maps in $N(Fin_*)$. Since $s(\langle 1 \rangle)$ lives in the fiber over $\langle 1 \rangle$, the cocartesian property will ensure the existence of a uniquely determined (up to homotopy) new commutative square in $\mathcal{C}$

$$\begin{tikzcd} X \otimes X \otimes X \arrow[r, dashed] \arrow[d, dashed] \arrow[dr, dashed] & X \otimes X \arrow[d, dashed] \\ X \otimes X \arrow[r, dashed] \arrow[ur, Rightarrow] & X \end{tikzcd} \tag{3.2.5}$$

The commutativity restrain follows from the commutativity of the diagram

$$\begin{tikzcd} \langle 2\rangle \arrow[r] \arrow[d] & \langle 1\rangle \\ \langle 2\rangle \arrow[ur] & \end{tikzcd} \tag{3.2.6}$$

in $N(Fin_*)$, where the vertical map is permutation.

These are called *commutative algebra objects of* $\mathcal{C}$ and we write $CAlg(\mathcal{C}) := Alg_{/\mathcal{C}omm}(\mathcal{C})$. In particular, it follows from the description $\mathcal{C}omm^{\otimes} \simeq colim_k E_k^{\otimes}$ that $CAlg(\mathcal{C})$ is equivalent to $lim_k Alg_{E_k}(\mathcal{C})$.

- If $\mathcal{O}'^{\otimes} = \mathcal{A}ss^{\otimes} \to \mathcal{O}^{\otimes} = N(Fin_*)$ is the associative operad, the associated algebra-objects in $\mathcal{C}^{\otimes}$ can be identified with the data of an object X in $\mathcal{C}$ together with a unit and a multiplication satisfying the usual associative coherences which are extracted as explained in the previous discussion. The main difference is that the permutation of factors is no longer a map in $\mathcal{A}ss^{\otimes}$ so that the commutativity restraint disappears. It follows that the composition with $\mathcal{A}ss^{\otimes} \to N(Fin_*)$ produces a forgetful map $CAlg(C) \to Alg_{\mathcal{A}ss}(\mathcal{C})$.

Example 3.2.1. Let $\mathcal{C}$ be a classical symmetric monoidal category. As explained in the Example 3.1.6, the nerve $N(\mathcal{C})$ adquires the structure of a symmetric monoidal $(\infty, 1)$-category. It follows that $CALg(N(\mathcal{C}))$ and $Alg_{\mathcal{A}ss}(N(\mathcal{C}))$ can be identified, respectively, with the nerves of the classical categories of strictly commutative (resp. associative) algebra objects in $\mathcal{C}$ in the classical sense.

- Another important situation is the case when $\mathcal{O}'^{\otimes} = \mathcal{T}riv^{\otimes} \to \mathcal{O}^{\otimes} = N(Fin_*)$ is the trivial operad for which, as expected, we have a canonical equivalence $Alg_{\mathcal{T}riv}(\mathcal{C}) \simeq \mathcal{C}$ (see [100, 2.1.3.5 and 2.1.3.6]).

- Following [100, 4.1.1.6], if both $\mathcal{O}'^{\otimes} = \mathcal{O}^{\otimes} = \mathcal{A}ss^{\otimes}$ we use the notation $Alg(\mathcal{C}) := Alg_{/\mathcal{A}ss}(\mathcal{C})$. It follows from this notation that if $\mathcal{C}^{\otimes}$ is a symmetric monoidal $(\infty, 1)$-category, we have a canonical equivalence between $Alg_{\mathcal{A}ss}(\mathcal{C})$ and $Alg(\mathcal{C}^{\otimes} \times_{N(Fin_*)} \mathcal{A}ss^{\otimes})$ where $\mathcal{C}^{\otimes} \times_{N(Fin_*)} \mathcal{A}ss^{\otimes}$ is the $\mathcal{A}ss$-monoidal $(\infty, 1)$-category obtained as explained in the Remark 3.1.7.

The theory of algebras becomes much simpler in the case of cartesian and cocartesian monoidal structures. The following two remarks collect some of these aspects:

Remark 3.2.2. If $\mathcal{C}^{\otimes}$ is cartesian symmetric monoidal $(\infty, 1)$-category, we have a canonical map relating the theory of algebras with the theory of monoids described in the previous section

$$Alg_{\mathcal{O}}(\mathcal{C}) \to Mon_{\mathcal{O}}(\mathcal{C}) \tag{3.2.7}$$

By [100, 2.4.2.5] this map is an equivalence. We will use this in the next section.

Remark 3.2.3. In the classical situation if $\mathcal{C}$ is a category with finite sums $\coprod$ then every object X in $\mathcal{C}$ carries admits a unique structure of commutative algebra, where the codiagonal map $X \coprod X \to X$ is the multiplication. In the ∞-setting this situation has its analoge for any Cocartesian ∞-operad as a consequence of the fact that Cocartesian ∞-operads are determined by their underlying $(\infty, 1)$-categories in a very strong sense. More precisely (see [100, 2.4.3.16]), for any unital generalized ∞-operad $\mathcal{O}^{\otimes}$ and any Cocartesian ∞-operad $\mathcal{C}^{\coprod}$ the restriction map $Alg_{\mathcal{O}}(\mathcal{C}) \to Fun(\mathcal{O}, \mathcal{C})$ is an equivalence of $(\infty, 1)$-categories. In particular, when $\mathcal{O}^{\otimes}$ is the commutative or the associative operad[1], the evaluation functors $Alg_{\mathcal{A}ss}(\mathcal{C}) \to \mathcal{C}$ and $CAlg(\mathcal{C}) \to \mathcal{C}$ are equivalences so that the forgetful map

$$\begin{tikzcd} CAlg(\mathcal{C}) \arrow[rr] \arrow[dr, "\sim"] & & Alg_{\mathcal{A}ss}(\mathcal{C}) \arrow[dl, "\sim"'] \\ & \mathcal{C} & \end{tikzcd} \tag{3.2.8}$$

is also an equivalence. In particular, by choosing an inverse to $CAlg(\mathcal{C}) \to \mathcal{C}$ we find a precise way to reproduce the classical situation.

3.2.2 Symmetric Monoidal $(\infty, 1)$-categories as commutative algebras in Cat_{∞} and Monoidal Localizations

Let us consider the $(\infty, 1)$-category of small $(\infty, 1)$-categories Cat_{∞} (see [99, Chapter 3]). The cartesian product endows Cat_{∞} with a symmetric monoidal structure $Cat_{\infty}^{\otimes}$ which can be obtained as the operadic nerve of the combinatorial simplicial model category of marked simplicial sets with the cartesian model structure. The objects of $Cat_{\infty}^{\otimes}$ are the finite sequences of $(\infty, 1)$-categories $(\mathcal{C}_1, ..., \mathcal{C}_n)$ and the morphisms $(\mathcal{C}_1, ..., \mathcal{C}_n) \to (\mathcal{D}_1, ..., \mathcal{D}_m)$ over a map $f : \langle n \rangle \to \langle m \rangle$ are given by families of maps

$$\prod_{j \in f^{-1}(\{i\})} C_j \to D_i \tag{3.2.9}$$

with $1 \leq i \leq m$. Using the Grothendieck-construction of [99, Thm. 3.2.0.1], the objects of $CAlg(Cat_{\infty}) \simeq Mon_{\mathcal{C}omm}(Cat_{\infty})$ can be identified with small symmetric monoidal $(\infty, 1)$-categories and the maps of algebras are identified with the monoidal functors (see [99, 2.4.2.6]). The same idea works if we replace Cat_{∞} by Cat_{∞}^{big}. These examples will play a vital role throughout this work.

Remark 3.2.4. Thanks to [100, 2.2.4.9] the forgetful functor $CAlg(Cat_{\infty}) \to Op_{\infty}$ admits left adjoint $Env^{\otimes}$. Given an ∞-operad $\mathcal{O}^{\otimes}$, the symmetric monoidal $(\infty, 1)$-category $Env^{\otimes}(\mathcal{O}^{\otimes})$ is called the *monoidal envelope of* $\mathcal{O}^{\otimes}$.

The theory in 3.1.7 can be extended to localizations which are not necessarily reflexive. Recall from our preliminaries on higher category theory that the formula $(\mathcal{C}, W) \mapsto \mathcal{C}[W^{-1}]$ provides a left adjoint to the fully-faithful map $Cat_{\infty} \subseteq \mathcal{W}Cat_{\infty}$. This makes Cat_{∞} a reflexive localization of $\mathcal{W}Cat_{\infty}$. The last carries a natural monoidal structure given by the cartesian product of pairs which extends the cartesian product in Cat_{∞}. By [100, 4.1.3.2] the formula $(\mathcal{C}, W) \mapsto \mathcal{C}[W^{-1}]$ commutes with products so that, as explained in 3.1.8 it extends to a monoidal functor

$$\mathcal{W}Cat_{\infty}^{\times} \to Cat_{\infty}^{\times} \tag{3.2.10}$$

inducing a left adjoint to the inclusion

$$CAlg(Cat_{\infty}) \subseteq CAlg(\mathcal{W}Cat_{\infty}) \tag{3.2.11}$$

We can identify the objects in $CAlg(\mathcal{W}Cat_{\infty})$ with the pairs $(\mathcal{C}^{\otimes}, W)$ where $\mathcal{C}^{\otimes}$ is a symmetric monoidal $(\infty, 1)$-category and W is collection of edges in the underlying ∞-category $\mathcal{C}$, together with the condition that the operations $\otimes^n : \mathcal{C}^n \to \mathcal{C}$ send sequences of edges in W to a new edge in W. The previous adjunction is telling us that every time we have a symmetric monoidal $(\infty, 1)$-category $\mathcal{C}^{\otimes}$ together with a collection of edges W which is compatible with the operations, then there is natural symmetric monoidal structure $\mathcal{C}^{\otimes}[W^{-1}]^{\otimes}$ in the localization $\mathcal{C}[W^{-1}]$. Plus, the localization functor is monoidal and has the obvious universal property (see [100, 4.1.3.4]).

Notice that the condition $(*)$ in 3.1.7 is exactly asking for the edges $W = L - equivalences$ to satisfy the compatibility described in the present discussion.

[1]These are both unital operads

3.2.3 Change of Operad and Free Algebras

Let us consider now a diagram of ∞-operads.

$$\begin{tikzcd} & & \mathcal{C}^{\otimes} \arrow[d, "p"] \\ \mathcal{O}_2^{\otimes} \arrow[r, "\alpha"] & \mathcal{O}_1^{\otimes} \arrow[r, "f"] & \mathcal{O}^{\otimes} \end{tikzcd} \tag{3.2.12}$$

with p a fibration. Composition with α produces a forgetful functor

$$Alg_{\mathcal{O}_1/\mathcal{O}}(\mathcal{C}) \to Alg_{\mathcal{O}_2/\mathcal{O}}(\mathcal{C}) \tag{3.2.13}$$

The main result of [100, Section 3.1.3] is that, under some mild hypothesis on $\mathcal{C}^{\otimes}$[2], we can use the theory of *operadic Kan extensions* (see [100, Section 3.1.2]) to ensure the existence of a left adjoint F to this functor [100, Cor. 3.1.3.5]. For each algebra $X \in Alg_{\mathcal{O}_2/\mathcal{O}}(\mathcal{C})$, $F(X)$ can be understood as the *free* $\mathcal{O}_1$*-Algebra generated by the* $\mathcal{O}_2$ *algebra* X [100, Def. 3.1.3.1]: for each color $b \in \mathcal{O}_2$, the value of $F(X)$ at b is given by the operadic p-colimit of the diagram consisting of all active morphisms over $b \in \mathcal{O}_2^{\otimes}$, whose source is in the image of α.

As explained in [100, 3.1.3.6, 3.1.3.9, 3.1.3.13], this left adjoint admits a very precise description in the case where $\mathcal{O}_2$ is the trivial operad and $\mathcal{O}_1 = \mathcal{O}$ is the associative or the commutative operad. Given a trivial algebra X in $\mathcal{C}$, for the first (see [100, Prop. 4.1.1.14]) we obtain

$$F(X)(\langle 1\rangle) = \coprod_{n\geq 0} X^{\otimes n} \tag{3.2.14}$$

while in the commutative case (see [100, Ex. 3.1.3.13]) we obtain the previous formula mod out by the action of the permutation groups

$$F(X)(\langle 1\rangle) = \coprod_{n\geq 0} (X^{\otimes n}/\Sigma_n) \tag{3.2.15}$$

3.2.4 Limits and Colimits of algebras

Another important feature of the ∞-categories $Alg_{/\mathcal{O}}(\mathcal{C})$ is the existence of limits and colimits. The first exist whenever they exist in $\mathcal{C}$ and can be computed using the forgetful functor (see [100, Prop. 3.2.2.1 and Cor. 3.2.2.5]). The existence of colimits needs a more careful discussion. In order to make the colimit of algebras an algebra we need to ask for a certain compatibility of the monoidal structure with colimits in $\mathcal{C}$. This observation motivates the notion of an $\mathcal{O}$*-monoidal* $(\infty,1)$*-category compatible with* $\mathcal{K}$*-indexed colimits*, with $\mathcal{K}$ a given collection of simplicial sets (see [100, 3.1.1.18, 3.1.1.19] and our review in 3.2.8 below). The definition demands the existence of $\mathcal{K}$-colimits on each $\mathcal{C}_x$ (for each $x \in \mathcal{O}$), and also that the multiplication maps associated to the monoidal structure preserve all colimits indexed by the simplicial sets $K \in \mathcal{K}$, separately in each variable. The main result ([100, Cor. 3.2.3.2, 3.2.3.3]) is that if $\mathcal{C}^{\otimes}$ is an $\mathcal{O}$-monoidal $(\infty,1)$-category compatible with $\mathcal{K} = \{\kappa - \text{small simplicial sets}\}$-colimits and if $\mathcal{O}^{\otimes}$ is an essentially κ-small ∞-operad, then $Alg_{/\mathcal{O}}(\mathcal{C})$ admits κ-small colimits. However and in general, contrary to limits, colimits cannot be computed using the forgetful functor to $\mathcal{C}_x$ for each color $x \in \mathcal{O}$.

Remark 3.2.5. In particular, if $\mathcal{C}^{\otimes}$ is a symmetric monoidal $(\infty,1)$-category compatible with colimits, one can easily check that its natural $\mathcal{A}ss$-monoidal structure $\mathcal{D}^{\otimes} := \mathcal{C}^{\otimes} \times_{N(Fin_*)} \mathcal{A}ss^{\otimes}$ is also compatible with colimits so that $Alg_{\mathcal{A}ss}(\mathcal{C}) \simeq Alg(\mathcal{D})$ has all small colimits.

[2]These mild conditions hold for any symmetric monoidal $(\infty,1)$-category *compatible with all small colimits* (see below) and this will be enough for our present purposes.

3.2.5 Transport of Algebras via Monoidal functors

Let

$$\begin{tikzcd} \mathcal{C}^{\otimes} \arrow[rr, "f"] \arrow[dr, "p"'] & & \mathcal{D}^{\otimes} \arrow[dl, "q"] \\ & \mathcal{O}^{\otimes} & \end{tikzcd} \tag{3.2.16}$$

be a morphism of $(\infty,1)$-operads (not necessarily monoidal) with both p and q given by fibrations of $(\infty,1)$-operads. In this case, as the composition of maps of ∞-operads is again a map of ∞-operads f induces a composition map

$$f_* : Alg_{/\mathcal{O}}(\mathcal{C}) \to Alg_{/\mathcal{O}}(\mathcal{D}) \tag{3.2.17}$$

which we can easily see makes the diagrams

$$\begin{tikzcd} Alg_{/\mathcal{O}}(\mathcal{C}) \arrow[r, "f_*"] \arrow[d, "ev_x"] & Alg_{/\mathcal{O}}(\mathcal{D}) \arrow[d, "ev_x"] \\ \mathcal{C}_x \arrow[r, "f_x"] & \mathcal{D}_x \end{tikzcd} \tag{3.2.18}$$

commute for any color $x \in \mathcal{O}$.

3.2.6 Tensor product of Algebras

Let $q : \mathcal{C}^{\otimes} \to N(Fin_*)$ be an ∞-operad. Thanks to [100, 3.2.4.1, 3.2.4.3], for any ∞-operad $\mathcal{O}^{\otimes}$ the category of algebras $Alg_{\mathcal{O}}(\mathcal{C})$ can be endowed again with the structure of ∞-operad $p : Alg_{\mathcal{O}}(\mathcal{C})^{\otimes} \to N(Fin_*)$. Moreover, a morphism α in $Alg_{\mathcal{O}}(\mathcal{C})^{\otimes}$ is p-cocartesian if and only if for each color $x \in \mathcal{O}$ its image through the evaluation functor $e_x : Alg_{\mathcal{O}}(\mathcal{C})^{\otimes} \to \mathcal{C}^{\otimes}$ is q-cocartersian. In particular, $Alg_{\mathcal{O}}(\mathcal{C})^{\otimes}$ is a symmetric monoidal $(\infty,1)$-category if and only if $\mathcal{C}^{\otimes}$ is, and in this case the evaluation functors e_x are symmetric monoidal. In other words, the category of algebras inherits a tensor product given by the tensor operation in the underlying category $\mathcal{C}$. In particular, for any morphism of ∞-operads $\mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}$, since the forgetful functor $f : Alg_{\mathcal{O}}(\mathcal{C}) \to Alg_{\mathcal{O}'}(\mathcal{C})$ is defined over the evaluation functors e_x, it extends to a monoidal map $Alg_{\mathcal{O}}(\mathcal{C})^{\otimes} \to Alg_{\mathcal{O}'}(\mathcal{C})^{\otimes}$.

Remark 3.2.6. Extending the discussion in 3.2.5, by the universal property of the simplicial set $Alg_{\mathcal{O}}(\mathcal{C})^{\otimes}$ (see [100, Const. 3.2.4.1]), any monoidal functor $\mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ between symmetric monoidal $(\infty,1)$-categories, extends to a monoidal functor between the symmetric monoidal categories of algebras $Alg_{\mathcal{O}}(\mathcal{C})^{\otimes} \to Alg_{\mathcal{O}}(\mathcal{D})^{\otimes}$. By the properties of this monoidal structure, for every color $x \in \mathcal{O}$, the evaluation maps e_x provide a commutative diagram of monoidal functors

$$\begin{tikzcd} Alg_{\mathcal{O}}(\mathcal{C})^{\otimes} \arrow[r] \arrow[d, "e_x"] & Alg_{\mathcal{O}}(\mathcal{D})^{\otimes} \arrow[d, "e_x"] \\ \mathcal{C}^{\otimes} \arrow[r] & \mathcal{D}^{\otimes} \end{tikzcd} \tag{3.2.19}$$

As in the classical case, if $\mathcal{O} = \mathcal{C}omm$, this monoidal structure is coCartesian (Prop. 3.2.4.7 of [100]). More generally, for $\mathcal{O} = \mathbb{E}_k$ there is a formula relating this monoidal structure with coproducts in $\mathcal{C}$ [100, Theorem 5.3.3.3].

3.2.7 Tensor Product of ∞-operads

The $(\infty,1)$-category of ∞-operads admits a symmetric monoidal structure where the tensor product of two operads $\mathcal{O}^{\otimes}$ and $(\mathcal{O}')^{\otimes}$ is characterized ([100, 2.2.5.3]) by the data of a map of simplicial sets $f: \mathcal{O}^{\otimes}\times(\mathcal{O}')^{\otimes} \to \mathcal{O}^{\otimes}\otimes(\mathcal{O}')^{\otimes}$ with the following universal property: for any ∞-operad $\mathcal{C}^{\otimes}$, composition with f induces an equivalence

$$Alg_{(\mathcal{O}^{\otimes}\otimes(\mathcal{O}')^{\otimes})_{\langle 1\rangle}}(\mathcal{C}) \simeq Alg_{\mathcal{O}}(Alg_{\mathcal{O}'}(C)) \tag{3.2.20}$$

where $Alg_{\mathcal{O}'}(C)$ is viewed with the operadic structure of the previous section. In particular, the unit is the trivial operad.

This monoidal structure can be defined at the level of marked simplicial sets over $N(Fin_*)$ and can be seen to be compatible with the model structure therein [100, 2.2.5.7, 2.2.5.13]. Moreover, it is compatible with the natural inclusion

$$Cat_{\infty} \hookrightarrow Op_{\infty} \tag{3.2.21}$$

so that the cartesian product of $(\infty,1)$-categories is sent to this new product of operads [100, Prop. 2.2.5.15].

An important application is the description of the ∞-operad $\mathbb{E}_{i+j}^{\otimes}$ as the tensor product of $\mathbb{E}_i^{\otimes}$ with $\mathbb{E}_j^{\otimes}$ [100, 5.1.2.2]. In particular, this characterizes an $\mathbb{E}_n^{\otimes}$-algebra X in a symmetric monoidal $(\infty,1)$-category $\mathcal{C}^{\otimes}$ has a collection of n different associative multiplications on X, monoidal with respect to each other.

3.2.8 Symmetric Monoidal Structures and Compatibility with Colimits

As mentioned in 3.2.2 the objects of $CAlg(Cat_{\infty})$ can be identified with the (small) symmetric monoidal $(\infty,1)$-categories. We have an analogue for the (small) symmetric monoidal $(\infty,1)$-categories compatible with $\mathcal{K}$-indexed colimits: as indicated in 2.1, given an arbitrary $(\infty,1)$-category $\mathcal{C}$ together with a collection $\mathcal{K}$ of arbitrary simplicial sets and a collection $\mathcal{R}$ of diagrams indexed by simplicial sets in $\mathcal{K}$, we can fabricate a new $(\infty,1)$-category $\mathcal{P}_{\mathcal{R}}^{\mathcal{K}}(\mathcal{C})$ with the universal property described by the formula (2.1.11). We can now use this mechanism to fabricate a monoidal structure in $Cat_{\infty}(\mathcal{K})$ induced by the cartesian structure of Cat_{∞}. Given two small $(\infty,1)$-categories $\mathcal{C}$ and $\mathcal{C}'$ having all the $\mathcal{K}$-indexed colimits, we consider the collection $\mathcal{R}=\mathcal{K}\boxtimes\mathcal{K}$ of all diagrams $p: K \to \mathcal{C}\times\mathcal{C}'$ such that $K\in\mathcal{K}$ and p is constant in one of the product components, and define a new $(\infty,1)$-category $\mathcal{C}\otimes\mathcal{C}' := \mathcal{P}_{\mathcal{K}\boxtimes\mathcal{K}}^{\mathcal{K}}(\mathcal{C}\times\mathcal{C}')$. By construction it admits all the $\mathcal{K}$-indexed colimits and comes equipped with a map $\mathcal{C}\times\mathcal{C}' \to \mathcal{C}\otimes\mathcal{C}'$ endowed with the following universal property: for any $(\infty,1)$-category $\mathcal{D}$ having all the $\mathcal{K}$-indexed colimits, the composition map

$$Fun_{\mathcal{K}}(\mathcal{C}\otimes\mathcal{C}',\mathcal{D}) \to Fun_{\mathcal{K}\boxtimes\mathcal{K}}(\mathcal{C}\times\mathcal{C}',\mathcal{D}) \tag{3.2.22}$$

is an equivalence. The right-side denotes the category of all $\mathcal{K}$-colimit preserving functors and the left-side denotes the category spanned by the functors preserving $\mathcal{K}-$colimits separately in each variable.

We can now use this operation to define a symmetric monoidal structure in $Cat_{\infty}(\mathcal{K})$. For that, we start with $Cat_{\infty}^{\otimes} \to N(Fin_*)$ the cartesian monoidal structure in Cat_{∞}, and we consider the (non-full) subcategory $Cat_{\infty}(\mathcal{K})^{\otimes}$ whose objects are finite sequences $(\mathcal{C}_1,...,\mathcal{C}_n)$ where each $\mathcal{C}_i$ admits all $\mathcal{K}$-indexed colimits, together with those morphisms $(\mathcal{C}_1,..,\mathcal{C}_n)\to(\mathcal{D}_1,...,\mathcal{D}_m)$ in $Cat_{\infty}^{\otimes}$ over some $f:\langle n\rangle\to\langle m\rangle$, which correspond to a family of maps

$$\prod_{j\in f^{-1}(\{i\})} \mathcal{C}_j \to \mathcal{D}_i \tag{3.2.23}$$

given by functors commuting with $\mathcal{K}$-indexed colimits separately in each variable. We can now use the universal property described in the previous paragraph to prove that $Cat_{\infty}(\mathcal{K})^{\otimes}$ is a cocartesian fibration: given a morphism $f:\langle n\rangle\to\langle m\rangle$ and a sequence of $(\infty,1)$-categories $X=(\mathcal{C}_1,...,\mathcal{C}_n)$ having all the $\mathcal{K}$-indexed colimits, a locally cocartesian lifting for f at X in $Cat_\infty(\mathcal{K})^{\otimes}$ is given by the family of universal maps

$$\prod_{j\in f^{-1}(\{i\})} \mathcal{C}_j \to \mathcal{D}_i := \mathcal{P}^{\mathcal{K}}_{\boxtimes_{j\in f^{-1}(\{i\}}\mathcal{K}}(\prod_{j\in f^{-1}(\{i\}} \mathcal{C}_j) \tag{3.2.24}$$

which we know commutes with $\mathcal{K}$-indexed colimits separately in each variable. The fact that the composition of locally cocartesian morphisms is locally cocartesian follows from [99, 5.3.6.11] Moreover, it follows from this formula that the canonical inclusion $Cat_\infty(\mathcal{K})^{\otimes} \to Cat_\infty^{\otimes}$ is a lax-monoidal functor (see [100, 4.8.1.3, 4.8.1.4] for the full details).

Finally, the objects of $CAlg(Cat_\infty(\mathcal{K}))$ can be naturally identified with the symmetric monoidal $(\infty,1)$-categories compatible with $\mathcal{K}$-colimits (see [100, Remark 4.8.1.9])

More generally, given two arbitrary collections of simplicial sets $\mathcal{K} \subseteq \mathcal{K}'$, it results from the universal properties defining the monoidal structures that the inclusion

$$Cat_\infty^{big}(\mathcal{K}') \subseteq Cat_\infty^{big}(\mathcal{K}) \tag{3.2.25}$$

is lax monoidal and its (informal) left adjoint $\mathcal{C} \mapsto \mathcal{P}^{\mathcal{K}'}_{\mathcal{K}}(\mathcal{C})$ (see our review in 2.1.15) is monoidal. In other words, for every inclusion $\mathcal{K} \subseteq \mathcal{K}^{'}$ of collections of simplicial sets, if $\mathcal{C}^{\otimes}$ is a symmetric monoidal $(\infty,1)$-category compatible with all $\mathcal{K}$-colimits, the $(\infty,1)$-category $\mathcal{P}^{\mathcal{K}'}_{\mathcal{K}}(\mathcal{C})$ inherits a canonical symmetric monoidal structure $\mathcal{P}^{\mathcal{K}'}_{\mathcal{K}}(\mathcal{C})^{\otimes}$ compatible with all the $\mathcal{K}'$-indexed colimits. Moreover, the canonical functor $\mathcal{C} \to \mathcal{P}^{\mathcal{K}'}_{\mathcal{K}}(\mathcal{C})$ extends to a monoidal functor $\mathcal{C}^{\otimes} \to \mathcal{P}^{\mathcal{K}'}_{\mathcal{K}}(\mathcal{C})^{\otimes}$ and, again by ignoring the set-theoretical aspects, the previous adjunction extends to a new one (see [100, 4.8.1.10])

$$CAlg(Cat_\infty^{big}(\mathcal{K}')) \underset{i}{\overset{\longleftarrow}{\longrightarrow}} CAlg(Cat_\infty^{big}(\mathcal{K})) \tag{3.2.26}$$

Example 3.2.7. In the particular case when $\mathcal{K}$ is empty and $\mathcal{K}'$ is the collection of all small simplicial sets, this tells us that if $\mathcal{C}^{\otimes}$ is a small symmetric monoidal $(\infty,1)$-category, the ∞-category of ∞-presheaves on $\mathcal{C}$ inherits a natural symmetric monoidal structure $\mathcal{P}^{\otimes}(\mathcal{C})$, commonly called the *convolution product*. Moreover, the Yoneda map is monoidal and satisfies the following universal property: for any symmetric monoidal $(\infty,1)$-category $\mathcal{D}^{\otimes}$ with all small colimits, the natural map given by composition

$$Fun^{\otimes,L}(\mathcal{P}(\mathcal{C}),\mathcal{D}) \to Fun^{\otimes}(\mathcal{C},\mathcal{D}) \tag{3.2.27}$$

is an equivalence.

Example 3.2.8. The same mechanism tell us that the Ind-completion of a small symmetric monoidal $(\infty,1)$-category also acquires a symmetric monoidal structure. In fact, Ind is a symmetric monoidal equivalence (see [100, 5.3.2.11]).

Example 3.2.9. Following the discussion in 2.1.18, $Cat_\infty(\{Idem\})$ can be identified with Cat_∞^{idem}. In this case, the previous discussion endows Cat_∞^{idem} with a symmetric monoidal structure and the idempotent-completion $Idem(-)$ is a monoidal left adjoint to the inclusion $Cat_\infty^{idem} \subseteq Cat_\infty$.

Example 3.2.10. Let $\mathcal{C}^{\otimes}$ be a symmetric monoidal $(\infty,1)$-category. We say that $\mathcal{C}$ is *closed* if for each object $X \in \mathcal{C}$ the map $(-\otimes X) : \mathcal{C} \to \mathcal{C}$ has a right adjoint. In other words, $\mathcal{C}^{\otimes}$ is closed if and only if for any objects X and Y in $\mathcal{C}$ there is an object X^Y and a map $Y^X \otimes X \to Y$ inducing an homotopy equivalence $Map_{\mathcal{C}}(Z\otimes X, Y) \simeq Map_{\mathcal{C}}(Z, Y^X)$. If $\mathcal{C}^{\otimes}$ is closed symmetric monoidal $(\infty,1)$-category and its underlying ∞-category $\mathcal{C}$ has all small colimits, then $\mathcal{C}^{\otimes}$ is a symmetric monoidal $(\infty,1)$-category compatible with all small colimits. An important example is the cartesian symmetric monoidal structure on Cat_∞, where the right adjoint to $(-\times X)$ is provided by the construction $Fun(X,-)$. All small colimits exist in Cat_∞ because it is presentable.

3.3 Modules over an Algebra

We now recall the theory of module-objects over an algebra-object. We mimic the classical situation: in a symmetric monoidal category $\mathcal{C}$, each algebra-object A has an associated theory of modules $Mod_A(\mathcal{C})$ and under some nice assumptions on $\mathcal{C}$, this new category has a natural monoidal product. This provides an example of a more general object - a collection of ∞-operads indexed by the objects of a small $(\infty,1)$-category.

3.3.1 Generalized $(\infty,1)$-operads and Operadic families

We start with a review of the appropriate language to formulate the notion of a family of ∞-operads indexed by the collection of objects of an $(\infty,1)$-category $\mathcal{B}$. The theory of modules provides an example, with $\mathcal{B}=CAlg(\mathcal{C})$ for a symmetric monoidal $(\infty,1)$-category $\mathcal{C}$.

In [100], the author develops two equivalent ways to formulate this idea of family. The first is the notion of a $\mathcal{B}$*-operadic family* (see [100, 2.3.2.10]). It consists of a categorical fibration $p:\mathcal{O}^{\otimes}\to \mathcal{B}\times N(Fin_*)$ such that

1. for each object $b\in\mathcal{B}$, the fiber $\mathcal{O}^{\otimes}\times_{\mathcal{B}\times N(Fin_*)}\{b\}\to N(Fin_*)$ is an ∞-operad. In particular, we can identify an object X in the fiber of $(b,\langle n\rangle)$ as a sequence $(b;(X_1,...,X_n))$ by choosing p-cocartesian liftings $X\to X_i$ of the canonical morphisms $(id_b,\rho_i:\langle n\rangle\to\langle 1\rangle)$;
2. For any $Z=(b';(Z_1,...,Z_m))$ and $X=(b;(X_1,...,X_n))$ in $\mathcal{O}^{\otimes}$ and every morphism $(u,f):(b',\langle m\rangle)\to(b,\langle n\rangle)$ in $\mathcal{B}\times N(Fin_*)$, we ask for the canonical map
$$Map^{u,f}_{\mathcal{O}^{\otimes}}(Z,X)\to\prod_{i=1}^{n} Map^{u,\rho_i\circ f}_{\mathcal{O}^{\otimes}}(Z,X_i) \tag{3.3.1}$$
to be an equivalence. In this notation, $Map^{u,f}_{\mathcal{O}^{\otimes}}$ denotes the connected component of $Map_{\mathcal{O}^{\otimes}}$ of all morphisms lying over (u,f).

The second notion is that of a *generalized* ∞*-operad* ([100, Defn 2.3.2.1]). It is given by the data of an $(\infty,1)$-category $\mathcal{O}^{\otimes}$ equipped with a map $q:\mathcal{O}^{\otimes}\to N(Fin_*)$ such that:

1. For any object X over $\langle n\rangle$ and any inert morphism $f:\langle n\rangle\to\langle m\rangle$, there is a q-cocartesian lifting $\tilde{f}:X\to X'$ of f. In particular, these induce functors $f_!:\mathcal{O}^{\otimes}_{\langle n\rangle}\to\mathcal{O}^{\otimes}_{\langle m\rangle}$ and if $\langle m\rangle=\langle 0\rangle$ we find a canonical map $\mathcal{O}^{\otimes}\to\mathcal{O}^{\otimes}_{\langle 0\rangle}$. Let X be a object over $\langle n\rangle$. Choose $X\to X_i$ a p-cocartesian lifting for each ρ_i. Moreover, choose a p-cocartesian lifting $X\to X_0$ for the canonical map $\langle n\rangle\to\langle 0\rangle$ and p-cocartesian liftings $X_i\to X_{i,0}$ for the null map $\langle 1\rangle\to\langle 0\rangle$. Because of the cocartesian property, the diagram
$$\begin{tikzcd}
 & X_1 \arrow[ddr] & \\
 & X_2 \arrow[dr] & \\
X \arrow[uur] \arrow[ur] \arrow[dr] & \vdots & X_0\simeq X_{i,0} \\
 & X_n \arrow[ur] &
\end{tikzcd} \tag{3.3.2}$$
commutes in $\mathcal{O}^{\otimes}$ (up to equivalence).

2. For each $\langle n\rangle$, the natural map

$$\mathcal{O}^{\otimes}_{\langle n\rangle} \to \underbrace{\mathcal{O}^{\otimes}_{\langle 1\rangle} \times_{\mathcal{O}^{\otimes}_{\langle 0\rangle}} .... \times_{\mathcal{O}^{\otimes}_{\langle 0\rangle}} \mathcal{O}^{\otimes}_{\langle 1\rangle}}_{n} \tag{3.3.3}$$

induced by the morphisms $\rho_i : \langle n\rangle \to \langle 1\rangle$, is an equivalence. This condition is weaker than the condition in the definition of an ∞-operad for it does not force $\mathcal{O}^{\otimes}_{\langle 0\rangle}$ to be contractible. This second axiom allows us to identify an object X over $\langle n\rangle$ with a sequence of objects $(X_1, ..., X_n)$ in $\mathcal{O}^{\otimes}_{\langle 1\rangle}$ living over the same (up to equivalence) object $X_0 \in \mathcal{O}^{\otimes}_{\langle 0\rangle}$ and motivates the notation $X = (X_0; X_1, ..., X_n)$.

3. Let $X = (X_0; X_1, ..., X_n)$ and $Z = (Z_0; Z_1, ..., Z_m)$ be objects in $\mathcal{O}^{\otimes}$. For any $f : \langle m\rangle \to \langle n\rangle$, we ask for the canonical map

$$Map^{f}_{\mathcal{O}^{\otimes}}(Z, X) \to (\prod_{I=1}^{n} Map^{\rho_i \circ f}_{\mathcal{O}^{\otimes}} Map_{\mathcal{O}^{\otimes}}(Z, X_i)) \times_{Map^{0:\langle n\rangle \to \langle 0\rangle}_{\mathcal{O}^{\otimes}}(Z, X_0)} Map_{\mathcal{O}^{\otimes}_{\langle 0\rangle}}(Z_0, X_0) \tag{3.3.4}$$

to be an equivalence.

According to [100, 2.3.2.11], the two notions are equivalent: if $q : \mathcal{O}^{\otimes} \to \mathcal{B} \times N(Fin_*)$ is a $\mathcal{B}$-operadic family, the composition with the projection towards $N(Fin_*)$ is a generalized ∞-operad and the canonical projection $\mathcal{O}^{\otimes}_{\langle 0\rangle} \to \mathcal{B}$ is a categorical equivalence. Conversely, if $p : \mathcal{O}^{\otimes} \to N(Fin_*)$ is a generalized ∞-operad, the product of p with the canonical projection $\mathcal{O}^{\otimes} \to \mathcal{O}^{\otimes}_{\langle 0\rangle}$ is a $\mathcal{O}^{\otimes}_{\langle 0\rangle}$-operadic family. These two constructions are mutually inverse. Notice also that if $\mathcal{B} = \Delta[0]$, we recover the notion of ∞-operad.

Let $p : \mathcal{O}^{\otimes} \to N(Fin_*)$ be a generalized ∞-operad. We say that a morphism in $\mathcal{O}^{\otimes}$ is *inert* if it is p-cocartesian and its image in $N(Fin_*)$ is inert. If $\mathcal{O}^{\otimes}$ and $(\mathcal{O}')^{\otimes}$ are generalized ∞-operads, we say that a map of simplicial sets $f : \mathcal{O}^{\otimes} \to (\mathcal{O}')^{\otimes}$ is a map of generalized ∞-operads if it is defined over $N(Fin_*)$ and sends inert morphisms to inert morphisms. According to [100, 2.3.2.4], there is a left proper, combinatorial simplicial model structure in the category of marked simplicial sets over $N(Fin_*)$ having the generalized ∞-operads as cofibrant-fibrant objects. We denote by Op^{gn}_{∞} its underlying $(\infty, 1)$-category. According to [100, 2.3.2.6], the model structure for ∞-operads is a Bousfield localization of this model structure for generalized ∞-operads. At the level of the underlying $(\infty, 1)$-categories this is the same as saying that Op_{∞} is a reflexive localization of Op^{gn}_{∞}. The inclusion understands an ∞-operad as generalized ∞-operad whose indexing category is $\Delta[0]$.

In the language of $(\infty, 1)$-categories, the relation between the two notions of operadic families can now be understood by using an adjunction: the assignment $\mathcal{O}^{\otimes} \mapsto \mathcal{O}^{\otimes}_{\langle 0\rangle}$ sending a generalized ∞-operad to its fiber over $\langle 0\rangle$ can be understood as a functor $F : Op^{gn}_{\infty} \to Cat_{\infty}$ and as explained in [100, 2.3.2.9], the map sending an $(\infty, 1)$-category $\mathcal{B}$ to the generalized ∞-operad $\mathcal{B} \times N(Fin_*)$ is a fully-faithful right adjoint of F. In this language, $\mathcal{O}^{\otimes} \to \mathcal{B} \times N(Fin_*)$ is an operadic family if and only it is a fibration of ∞-operads and its adjoint morphism $\mathcal{O}^{\otimes}_{\langle 0\rangle} \to \mathcal{B}$ is a trivial Kan fibration.

3.3.2 The $(\infty, 1)$-category of modules over an algebra-object

Let $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a fibration of ∞-operads. Following the results of [100, Section 3.3], by assuming a *coherence* condition on the ∞-operad $\mathcal{O}^{\otimes}$ (see [100, 3.3.1.9]), it is possible to construct a $Alg_{/\mathcal{O}}(\mathcal{C})$-operadic family $Mod^{\mathcal{O}}(\mathcal{C})^{\otimes} \to Alg_{/\mathcal{O}}(\mathcal{C}) \times \mathcal{O}^{\otimes}$ whose fiber over an algebra A

$$Mod^{\mathcal{O}}_{A}(\mathcal{C})^{\otimes} \to \mathcal{O}^{\otimes} \tag{3.3.5}$$

can be understood as a theory of A-modules. We will not reproduce here the details of this construction. Let us just say that if $\mathcal{O}^{\otimes}$ is *unital* (see [100, Def. 2.3.1.1]) [3], the category $Alg_{/\mathcal{O}}(\mathcal{C}) \times \mathcal{O}^{\otimes}$ can be described by means of a nice universal property in the homotopy theory of simplicial sets over $\mathcal{O}^{\otimes}$ with the Joyal model structure (see [100, 3.3.3.5, 3.3.3.6, 3.3.3.7]). Also by means of a universal property, we can then define a new simplicial set $Mod^{\mathcal{O}}(\mathcal{C})^{\otimes}$ over $\mathcal{O}^{\otimes}$ (see [100, Const. 3.3.3.1, Def. 3.3.3.8]), together with a canonical map

$$Mod^{\mathcal{O}}(\mathcal{C})^{\otimes} \to Alg_{/\mathcal{O}}(\mathcal{C}) \times \mathcal{O}^{\otimes} \tag{3.3.6}$$

which by [100, 3.3.3.16] is a fibration of generalized ∞-operads. Moreover, by [100, Thm. 3.3.3.9], if $\mathcal{O}^{\otimes}$ is coherent, then for each algebra A, the fiber $Mod_A^{\mathcal{O}}(\mathcal{C})^{\otimes} := Mod^{\mathcal{O}}(\mathcal{C})^{\otimes} \times_{Alg_{/\mathcal{O}}(\mathcal{C})} \{A\} \to \mathcal{O}^{\otimes}$ is a fibration of ∞-operads.

3.3.3 Monoidal Structures in Categories of Modules

Under some extra conditions on $\mathcal{C}^{\otimes}$, it is possible to prove that $Mod_A^{\mathcal{O}}(\mathcal{C})^{\otimes}$ is actually an $\mathcal{O}$-monoidal structure, with A as a unit object. These sufficient conditions are already visible in the classical situation: If $\mathcal{C}$ is the category of abelian groups with the usual tensor product, and A is a (classical) commutative ring (a commutative algebra object in $\mathcal{C}$), then the tensor product of A-modules M and N is by definition, the colimit of

$$M \otimes A \otimes N \rightrightarrows M \otimes N \tag{3.3.7}$$

where $\otimes$ denotes the tensor product of abelian groups and the two different arrows correspond, respectively, to the multiplication on M and N. In order for this pushout to be a new A-module we need to assume that $\otimes$ commutes with certain colimits. By [100, 3.4.4.6], if $\mathcal{C}^{\otimes}$ is an $\mathcal{O}$-monoidal $(\infty, 1)$-category compatible with κ-small colimits (for κ an uncountable regular cardinal) and if $\mathcal{O}^{\otimes}$ is a κ-small coherent ∞-operad, then the map $Mod_A^{\mathcal{O}}(\mathcal{C})^{\otimes} \to \mathcal{O}^{\otimes}$ is a coCartesian fibration of ∞-operads which is again compatible with κ-small colimits. In particular this result is valid for algebras over the ∞-operad $\mathbb{E}_k^{\otimes}$, for any $k \geq 0$, because it is known to be coherent (see [100, Thm. 5.1.1.1] for the general case and [100, 3.3.1.12] for the commutative operad).

3.3.4 Limits and Colimits of Modules

Another important feature of the module-categories $Mod_A^{\mathcal{O}}(\mathcal{C})_x^{\otimes}$ (for each $x \in \mathcal{O}$) is the existence of limits, which can be computed directly on each $\mathcal{C}_x$ using the forgetful functor (Thm 3.4.3.1 of [100]). The existence of colimits requires again the compatibility of the monoidal structure with colimits on each $\mathcal{C}_x$. If $\mathcal{C}^{\otimes}$ is compatible with κ-small colimits, then again by [100, 3.4.4.6], colimits in $Mod_A^{\mathcal{O}}(\mathcal{C})^{\otimes}$ can be computed in the underlying categories $\mathcal{C}_x$ by means of the forgetful functors $Mod_A^{\mathcal{O}}(\mathcal{C})_x^{\otimes} \to \mathcal{C}_x$, for each color $x \in \mathcal{O}$.

3.3.5 Algebra-objects in the category of modules

We also recall another important result relating algebra objects in $Mod_A^{\mathcal{O}}(\mathcal{C})^{\otimes}$ and algebras B in $\mathcal{C}$ equipped with a map of algebras $A \to B$.

Proposition 3.3.1. *([100, Cor. 3.4.1.7]) Let $\mathcal{O}^{\otimes}$ be a coherent operad and let $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a fibration of ∞-operads. Let $A \in Alg_{/\mathcal{O}}(\mathcal{C})$ be an $\mathcal{O}$-algebra object of $\mathcal{C}$. Then the obvious map*

$$Alg_{/\mathcal{O}}(Mod_A^{\mathcal{O}}(\mathcal{C})) \to Alg_{/\mathcal{O}}(\mathcal{C})_{A/} \tag{3.3.8}$$

is a categorical equivalence (where $Alg_{/\mathcal{O}}(\mathcal{C})_{A/}$ denotes the $(\infty, 1)$-category of objects B in $Alg_{\mathcal{O}}(\mathcal{C})$ equipped with a map of algebras $A \to B$ - see [99, 1.2.9.2]).

In particular, if $\mathcal{C}^{\otimes}$ is coCartesian fibration compatible with all small colimits, $Alg_{/\mathcal{O}}(\mathcal{C})_{A/}$ inherits a monoidal structure from $Mod_A^{\mathcal{O}}(\mathcal{C})^{\otimes}$ and by the discussion above this structure is cocartesian.

[3] In particular coherent operads are unital

Under the same conditions, it is also true that for any algebra $\widetilde{B} \in Alg_{/\mathcal{O}}(Mod_A^{\mathcal{O}}(\mathcal{C}))$ the canonical map

$$Mod_{\widetilde{B}}^{\mathcal{O}}(Mod_A^{\mathcal{O}}(\mathcal{C}))^{\otimes} \to Mod_B^{\mathcal{O}}(\mathcal{C})^{\otimes}. \tag{3.3.9}$$

is an equivalence of ∞-operads, with B the image of $\widetilde{B}$ through (3.3.8). See [100, Cor. 3.4.1.9].

3.3.6 Modules over associative algebras

We now review the particular situation over the ∞-operad $\mathcal{A}ss^{\otimes}$. Let $\mathcal{C}^{\otimes}$ be a monoidal $(\infty,1)$-category. Given an associative algebra A in $\mathcal{C}^{\otimes}$, it is possible to introduce two new constructions $LMod_A(\mathcal{C})$, $RMod_A(\mathcal{C})$ encoding, respectively, the theories of left and right modules over A. Their objects are pairs (A, M) where A is an object in $Alg(\mathcal{C})$ and M is an object in $\mathcal{C}$ equipped with a left (resp. right) action of A. This idea can be made precise with the construction of two new ∞-operads $\mathcal{LM}^{\otimes}$ and $\mathcal{RM}^{\otimes}$ fabricated to shape left-modules (see [100, Definitions 4.2.1.7 and 4.2.1.13]), respectively, right-modules. Let us overview the mechanism for left-modules. Grosso modo, $\mathcal{LM}^{\otimes}$ is the operadic nerve of a classical operad LM constructed to have two colours $\mathbf{a}$ and $\mathbf{m}$ and a unique morphism

$$(\underbrace{\mathbf{a}, \mathbf{a}, ..., \mathbf{a}}_{n}, \mathbf{m}) \longrightarrow \mathbf{m}. \tag{3.3.10}$$

for each $n \in \mathbb{N}$, these being the only morphisms where the color $\mathbf{m}$ appears. Moreover, the full subcategory spanned by the color $\mathbf{a}$ recovers the associative operad. At the same time, the projection sending the two colors $(\mathbf{a}, \mathbf{m})$ in $\mathcal{LM}^{\otimes}$ to the unique color in $\mathcal{A}ss^{\otimes}$, determines a fibration of ∞-operads.

Following [100, Def. 4.2.1.13] we set $LMod(\mathcal{C}) := Alg_{\mathcal{LM}/\mathcal{A}ss}(\mathcal{C})$. From an object $s \in LMod(\mathcal{C})$ we can extract an associative algebra-object in $\mathcal{C}$, $s|_{\mathcal{A}ss^{\otimes}} \in Alg(\mathcal{C})$, an object $s(\mathbf{m}) = M$ in $\mathcal{C}$ and a multiplication map $(A \otimes M \to M) := s((\mathbf{a}, \mathbf{m}) \to \mathbf{a})$ which with the help of the cocartesian property of $\mathcal{C}^{\otimes} \to \mathcal{A}ss^{\otimes}$ satisfies all the coherences defining the usual module-structure (see the Example 4.2.1.18 of [100]). If we fix A an associative algebra object in $\mathcal{C}$, we obtain $LMod_A(\mathcal{C})$ - the left-modules in $\mathcal{C}$ over the algebra A - as the fiber over A of the canonical map $LMod(\mathcal{C}) \to Alg(\mathcal{C})$ induced by the composition with the inclusion $\mathcal{A}ss^{\otimes} \subseteq \mathcal{LM}^{\otimes}$.

Given a pair of associative algebras A and B, it is also possible to perform a third construction ${}_ABMod_B(\mathcal{C})$ encoding the theory of bimodules over the pair (A, B) (left-modules over A and right-modules over B). Again, the construction starts with the fabrication of an ∞-operad $\mathcal{BM}^{\otimes}$ whose algebras in $\mathcal{C}$ are identified with bimodules (see Definitions 4.3.1.6 and 4.3.1.12 of [100]). By construction there are inclusions of ∞-operads

$$\mathcal{A}ss^{\otimes} \xhookrightarrow{()^+} \mathcal{LM}^{\otimes} \hookrightarrow \mathcal{BM}^{\otimes} \hookleftarrow \mathcal{RM}^{\otimes} \xhookleftarrow{()_-} \mathcal{A}ss^{\otimes} \tag{3.3.11}$$

which implies the existence of forgetful functors

$$LMod_A(\mathcal{C}) \longleftarrow {}_ABMod_B(\mathcal{C}) \longrightarrow RMod_B(\mathcal{C}) \tag{3.3.12}$$

which, in general, are not equivalences.

As explained by [100, Thm. 4.4.1.28] this new theory of modules is related to the general theory by means of a canonical equivalence

$$Mod_A^{\mathcal{A}ss}(\mathcal{C}) \xrightarrow{\sim} {}_ABMod_A(\mathcal{C}) \tag{3.3.13}$$

where $Mod_A^{\mathcal{A}ss}(\mathcal{C})$ is by definition, the underlying ∞-category of $Mod_A^{\mathcal{A}ss}(\mathcal{C})^{\otimes}$ (the general construction). Under some general conditions, for any triple (A, B, C) of associative algebras in $\mathcal{C}$ it is possible to fabricate a map of $(\infty, 1)$-categories

$$ {}_A BMod_B(\mathcal{C}) \times {}_B BMod_C(\mathcal{C}) \to {}_A Mod_C(\mathcal{C}) \tag{3.3.14} $$

encoding a *relative tensor product* (see [100, Def. 4.4.2.10, Eg. 4.4.2.11]). It can be understood as a generalization of the formula (3.3.7), replacing it by the geometric realization of a whole simplicial object $Bar_B(M,N)_\bullet$ given informally by the formula

$$ Bar_B(M,N)_n = M \otimes B^n \otimes N \tag{3.3.15} $$

(see [100, Construction 4.4.2.7, Theorem 4.4.2.8]). By [100, 4.4.2.15, 4.4.3.12], if $\mathcal{C}$

$(***)$ admits geometric realizations of simplicial objects and the tensor product preserves geometric realizations of simplicial objects, separately in each variable;

then, the fibration of ∞-operads $Mod_A^{\mathcal{A}ss}(\mathcal{C})^\otimes \to \mathcal{A}ss^\otimes$ (obtained by the general methods) is an $\mathcal{A}ss$-monoidal $(\infty,1)$-category with the monoidal structure given by the relative tensor product. Moreover, if $\mathcal{C}$ admits all small colimits and the tensor product is compatible with them in each variable, the equivalence $Mod_A^{\mathcal{A}ss}(\mathcal{C}) \simeq {}_A BMod_A(\mathcal{C})$ will send the existing abstract-nonsense-monoidal structure on $Mod_A^{\mathcal{A}ss}(\mathcal{C})$ provided by the general discussion in 3.3.3 to this relative tensor product.

Remark 3.3.2. As mentioned before, the theory of left-modules, resp. right-modules, does not have to be equivalent to the general theory (as we will see in the next section, this is true in the commutative case). For this reason it is convenient to have a theory of limits and colimits specific for left, resp. right, modules. See the Corollaries [100, 4.2.3.3 and 4.2.3.5].

3.3.7 Modules over commutative algebras

Finally, if A is a commutative algebra in a symmetric monoidal $(\infty,1)$-category $\mathcal{C}^\otimes$, the forgetful map $Mod_A(\mathcal{C}) := Mod_A^{\mathcal{C}omm}(\mathcal{C}) \to Mod_A^{\mathcal{A}ss}(\mathcal{C})$ fits in a commutative diagram

$$ \begin{tikzcd} & Mod_A(\mathcal{C}) \arrow[dl] \arrow[d] \arrow[dr] & \\ & Mod_A^{\mathcal{A}ss}(\mathcal{C}) \arrow[d, "\sim"] & \\ LMod_A(\mathcal{C}) & {}_A BMod_A(\mathcal{C}) \arrow[l] \arrow[r] & RMod_A(\mathcal{C}) \end{tikzcd} \tag{3.3.16} $$

and by [100, 4.4.1.4, 4.4.1.6], the diagonal arrows are equivalences. Moreover, by [100, Thm. 4.5.2.1], if $\mathcal{C}$ satisfies (***), then the ∞-operad $Mod_A(\mathcal{C})^\otimes := Mod_A^{Comm}(\mathcal{C})^\otimes$ is a symmetric monoidal $(\infty,1)$-category and its tensor product can be identified with the composition

$$ \begin{tikzcd} Mod_A(\mathcal{C}) \times Mod_A(\mathcal{C}) \arrow[r] & {}_A BMod_A(\mathcal{C}) \times {}_A BMod_A(\mathcal{C}) \arrow[r, "\otimes_A"] & {}_A BMod_A(\mathcal{C}) \arrow[d] \\ & & LMod_A(\mathcal{C}) \simeq Mod_A(\mathcal{C}) \end{tikzcd} \tag{3.3.17} $$

Moreover, if $\mathcal{C}$ admits all small colimits, this monoidal structure agrees with the therefore existing abstract-nonsense structure provided by the discussion in 3.3.3.

3.3.8 Base Change

We now review the procedure of base change. If $\mathcal{C}^{\otimes}$ is a symmetric monoidal $(\infty,1)$-category and $f : A \to B$ is a morphism of commutative algebras, the pre-composition with f produces a forgetful functor

$$f_* : Mod_B(\mathcal{C}) \to Mod_A(\mathcal{C}) \tag{3.3.18}$$

which in general is not a monoidal functor. Assuming $\mathcal{C}$ satisfies the condition $(***)$, the relative tensor product discussed in the previous section provides monoidal structures in $Mod_A(\mathcal{C})$ and $Mod_B(\mathcal{C})$. The [100, Thm. 4.5.3.1] enhances this result with the additional fact that $p : Mod(\mathcal{C})^{\otimes} \to CAlg(\mathcal{C}) \times N(Fin_*)$ is a cocartesian fibration. The construction of p-cocartesian liftings is achieved using the relative tensor product construction: every morphism of algebras $f : A \to B$ admits p-cocartesian liftings which we can informally describe with the formula

$$M \mapsto L_{A\to B}(M) := M \otimes_A B \tag{3.3.19}$$

Using the Grothendieck construction, this formula assembles to a left adjoint to the forgetful functor f_*. Moreover, the properties of the relative tensor product in [100, 4.4.2.9] imply that this left adjoint is monoidal.

It is also evident by the nature of the construction (obtained via cocartesian liftings) that for any composition $A \to B \to C$ and for any $\mathcal{A}$-module $\mathcal{M}$, there are natural equivalences $(M \otimes_A C) \simeq (M \otimes_A B) \otimes_B C$.

3.3.9 Transport of Modules via a monoidal functor

Our goal in this section is to explain how given $f : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ a monoidal functor between symmetric monoidal $(\infty,1)$-categories, we can associate to every commutative algebra $A \in CAlg(\mathcal{C})$ a natural map

$$Mod_A(\mathcal{C}) \xrightarrow{f_A} Mod_{f(A)}(\mathcal{D}) \tag{3.3.20}$$

and under some nice hypothesis on f, $\mathcal{C}^{\otimes}$ and $\mathcal{D}^{\otimes}$ this new map will again be monoidal with respect to the monoidal structure on modules described in 3.3.3 and 3.3.7. Moreover, we want to see that if $A \to B$ is map of commutative algebras, then the diagram

$$\begin{tikzcd} Mod_A(\mathcal{C}) \arrow[r, "f_A"] \arrow[d, "-\otimes_A B"] & Mod_{f(A)}(\mathcal{D}) \arrow[d, "-\otimes_{f(A)} f(B)"] \\ Mod_B(\mathcal{C}) \arrow[r, "f_B"] & Mod_{f(B)}(\mathcal{D}) \end{tikzcd} \tag{3.3.21}$$

commutes. Here the vertical maps are the base change maps of 3.3.8.

We start with the construction of the maps f_A. For that, we recall that the generalized operads $Mod(\mathcal{C})^{\otimes}$ and $Mod(\mathcal{D})^{\otimes}$ are defined in terms of a universal property as simplicial sets over $N(Fin_*)$ (See [100, Construction 3.3.3.1, Definition 3.3.3.8]). Using this universal property we can deduce that f induces a map $F : Mod(\mathcal{C})^{\otimes} \to Mod(\mathcal{D})^{\otimes}$ sending inert morphims to inert morphisms, and fitting in a commutative diagram

$$\begin{tikzcd} Mod(\mathcal{C})^{\otimes} \arrow[r, "F"] \arrow[d, "p"] & Mod(\mathcal{D})^{\otimes} \arrow[d, "q"] \\ CAlg(\mathcal{C}) \times N(Fin_*) \arrow[r, "f_* \times Id"] & CAlg(\mathcal{D}) \times N(Fin_*) \end{tikzcd} \tag{3.3.22}$$

where the map f_* is the transport of algebras explained in 3.2.5. We obtain the maps f_A as the restriction of F to the fiber over A. In the commutative case, the [100, Theorem 4.5.3.1] explained in the previous section tells us that if $\mathcal{C}$ and $\mathcal{D}$ both satisfy $(***)$, then both maps p and q are cocartesian fibrations. Our goal follows immediately from the following property

Proposition 3.3.3. *Let $\mathcal{C}^{\otimes}$ and $\mathcal{D}^{\otimes}$ be symmetric monoidal $(\infty,1)$-categories such that both $\mathcal{C}$ and $\mathcal{D}$ both admit geometric realizations of simplicial objects and the tensor product preserves them on each variable. Let $f : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ be a monoidal functor commuting with geometric realizations of simplicial objects. Then, the induced map F in the diagram (3.3.22) sends p-cocartesian morphisms to q-cocartesian morphisms.*

Proof. As F is a functor, because of [99, Lemma 2.4.2.8] we are reduced to showing that F sends locally p-cocartesian morphisms in $Mod(\mathcal{C})^{\otimes}$ to locally q-cartesian morphisms in $Mod(\mathcal{D})^{\otimes}$. One can now easily check that both p and q satisfy the requirements of the Lemma [100, 4.5.3.4] so that, as F preserves inert morphisms, we are reduced to showing that the induced maps

$$\begin{tikzcd} Mod_A(\mathcal{C})^{\otimes} \arrow[rr] \arrow[dr, "p_A"] & & Mod_{f(A)}(\mathcal{D})^{\otimes} \arrow[dl, "q_A"] \\ & N(Fin_*) & \end{tikzcd} \tag{3.3.23}$$

and

$$\begin{tikzcd} Mod(\mathcal{C}) \arrow[r] \arrow[d] & Mod(\mathcal{D}) \arrow[d] \\ CAlg(\mathcal{C}) \arrow[r] & CAlg(\mathcal{D}) \end{tikzcd} \tag{3.3.24}$$

both send locally cocartesian morphisms to locally cocartesian morphisms. By the inspection of the proofs of [100, 4.5.2.1] for the first case and [100, 4.5.3.6, 4.6.2.17] for the second, we conclude that everything is reduced to showing that f sends the relative tensor product in $\mathcal{C}$ to the relative tensor product in $\mathcal{D}$. By inspection of [100, Construction 4.4.2.7 and Theorem 4.4.2.8] we conclude that this follows immediately from our assumptions on f.

□

3.4 Endomorphisms Objects

In this section we review the theory of endomorphism objects as developed in [100, Sections 6.1 and 6.2]. Let $\mathcal{C}^{\otimes}$ be a monoidal $(\infty,1)$-category. As reviewed in the section 3.3.6, to every object $A \in Alg(\mathcal{C})$ we can associate a new $(\infty,1)$-category $LMod_A(\mathcal{C})$ whose objects consist of objects m in $\mathcal{C}$ equipped with a multiplication $A \otimes m \to m$ satisfying the usual coherences for modules. We can now generalize the notion of an A-module to include objects m belonging to any $(\infty,1)$-category $\mathcal{M}$ where $\mathcal{C}$ acts. More precisely, recall that $\mathcal{C}$ can be understood as an object in $Alg_{\mathcal{A}ss}(Cat_{\infty})$ and therefore $\mathcal{C}$ itself admits a theory of left-modules $LMod_{\mathcal{C}}(Cat_{\infty})$ - the objects of this $(\infty,1)$-category can be understood as $(\infty,1)$-categories $\mathcal{M}$ equipped with an action $\bullet : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$ satisfying the standard coherences for module-objects in Cat_{∞}. We generalize the notion of an A-module to include objects $m \in M$ endowed with a multiplication $A \bullet m \to m$ satisfying the standard coherences for modules. This can be made precise as follows. Let $\overline{\mathcal{M}}$ be an object in $LMod_{\mathcal{C}}(Cat_{\infty})$. Explicitly, $\overline{\mathcal{M}}$ is a map of ∞-operads

$$\begin{tikzcd} & Cat_{\infty}^{\times} \arrow[d] \\ \mathcal{LM}^{\otimes} \arrow[ur, "\overline{\mathcal{M}}"] \arrow[r] & N(Fin_*) \end{tikzcd} \tag{3.4.1}$$

whose restriction to $\mathcal{A}ss^{\otimes} \subseteq \mathcal{LM}^{\otimes}$ is the monoidal $(\infty,1)$-category $\mathcal{C}^{\otimes}$ and whose evaluation at the color $\mathbf{m}$ is another $(\infty,1)$-category $\mathcal{M}$. Since $Cat_{\infty}^{\times}$ is cartesian, we have an equivalence $Alg_{\mathcal{LM}}(Cat_{\infty}) \simeq Mon_{\mathcal{LM}}(Cat_{\infty})$ and therefore we can use the Grothendieck construction to present the diagram $\overline{\mathcal{M}}$ in the form of a cocartesian fibration of ∞-operads $\mathcal{O}^{\otimes}_{\overline{\mathcal{M}}} \to \mathcal{LM}^{\otimes}$ where we recover $\mathcal{O}^{\otimes}_{\overline{\mathcal{M}}} \times_{\mathcal{LM}^{\otimes}} \mathcal{A}ss^{\otimes} \simeq \mathcal{C}^{\otimes}$ and $\mathcal{O}^{\otimes}_{\overline{\mathcal{M}}} \times_{\mathcal{LM}^{\otimes}} \{\mathbf{m}\} \simeq \mathcal{M}$ and the action $\bullet : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$ is again extracted from the cocartesian property.

In this setting, an object $\overline{m} \in LMod(\mathcal{C}, \mathcal{M}) := Alg_{/\mathcal{LM}}(\mathcal{O}_{\overline{M\}}})$ provides exactly the concept we seek: the restriction $\overline{m}|_{\mathcal{A}ss^{\otimes}}$ is an algebra-object in $\mathcal{O}^{\otimes}_{\overline{M}} \times_{\mathcal{LM}^{\otimes}} \mathcal{A}ss^{\otimes} \simeq \mathcal{C}^{\otimes}$; the value at $\mathbf{m}$ is an object m in $\mathcal{O}^{\otimes}_{\overline{M}} \times_{\mathcal{LM}^{\otimes}} \{\mathbf{m}\} \simeq \mathcal{M}$ and the image of canonical morphism $(\mathbf{a}, \mathbf{m}) \to \mathbf{m}$ provides, via the cocartesian property of $\mathcal{O}^{\otimes}_{\overline{M}} \to \mathcal{LM}^{\otimes}$, a map $A \bullet m \to m$ which, also because of the cocartesian property, will satisfies all the standard coherences we seek.

There are canonical projections $LMod(\mathcal{C}, \mathcal{M}) \to Alg(\mathcal{C})$ and $LMod(\mathcal{C}, \mathcal{M}) \to \mathcal{M}$ induced, respectively, by the composition with the inclusion $\mathcal{A}ss^{\otimes} \subseteq \mathcal{LM}^{\otimes}$ and the inclusion $\{\mathbf{m}\} \subseteq \mathcal{LM}^{\otimes}$ (see [100, Def. 4.2.1.13]). For each associative algebra A in $\mathcal{C}$, the fiber $LMod_A(\mathcal{C}, \mathcal{M}) := LMod(\mathcal{C}, \mathcal{M}) \times_{Alg(\mathcal{C})} \{A\}$ gathers the collection of objects m in $\mathcal{M}$ endowed with a left action of A satisfying the standard coherences of being a module-object. Similarly, for each object $m \in \mathcal{M}$, the fiber $LMod(\mathcal{C}, \mathcal{M}) \times_{\mathcal{M}} \{m\}$ codifies all the different ways in which the object m can be endowed the structure of an A-module, for some associative algebra A in $\mathcal{C}$.

Remark 3.4.1. If $\mathcal{C}^{\otimes} \to \mathcal{A}ss^{\otimes}$ is a monoidal $(\infty,1)$-category, the tensor operation provides $\mathcal{C}$ with the structure of a $\mathcal{C}$-module and we recover $LMod(\mathcal{C}) \simeq LMod(\mathcal{C}, \mathcal{M} = \mathcal{C})$.

Remark 3.4.2. This construction uses the data of symmetric monoidal $(\infty,1)$-category $\mathcal{C}^{\otimes}$ and a module $\mathcal{M}$ over it as initial ingredients. However, the defining ingredient is the data of the cartesian fibration $\mathcal{O}^{\otimes}_{\overline{M}} \to \mathcal{LM}^{\otimes}$. Dropping the cocartesian condition we can reproduce the situation with the data of fibration of ∞-operads $p : \mathcal{O}^{\otimes} \to \mathcal{LM}^{\otimes}$. This gives rise to what in [100]-Definition 4.2.1.12 is called a *weak enrichment* of $\mathcal{M} := \mathcal{O}^{\otimes} \times_{\mathcal{LM}^{\otimes}} \{\mathbf{m}\}$ over $\mathcal{C}^{\otimes} := \mathcal{O}^{\otimes} \times_{\mathcal{LM}^{\otimes}} \mathcal{A}ss^{\otimes}$.

We now have the following important result:

Proposition 3.4.3. *(see [100, Cor. 4.7.2.42]) Let $\mathcal{C}$ be a monoidal $(\infty,1)$-category and $\overline{\mathcal{M}}$ be an object in $LMod_{\mathcal{C}}(Cat_{\infty})$. Let m be an object in $\mathcal{M} = \overline{\mathcal{M}}(\boldsymbol{m})$. Then, the canonical map $p : LMod(\mathcal{C}, \mathcal{M}) \times_{\mathcal{M}} \{m\} \to Alg(\mathcal{C})$ is a right fibration (in particular it is a cartesian fibration and its fibers are ∞-groupoids).*

In [100]-Section 4.7.2, the author proves this result by constructing a new monoidal $(\infty,1)$-category $\mathcal{C}^{+}[m]$ whose objects can be identified with pairs (X, η) where X is an object in $\mathcal{C}$ and $\eta : X \bullet m \to m$ is a morphisms in $\mathcal{M}$. The canonical map $Alg(\mathcal{C}^{+}[m]) \to Alg(\mathcal{C})$ is a right fibration ([100, 4.7.2.39]) and the conclusion follows from the existence of an equivalence of right-fibrations $LMod(\mathcal{C}, \mathcal{M}) \times_{\mathcal{M}} \{m\} \simeq Alg(\mathcal{C}^{+}[m])$ (see [100, Cor. 4.7.2.40]).

In the context of the previous result, we say that the object $m \in \mathcal{M}$ admits a *classifying object for endomorphisms* if the right fibration p is representable. Because of [99, Thm. 4.4.4.5], this amounts to the existence an algebra-object $End(m) \in Alg(\mathcal{C})$ and an equivalence of right fibrations $LMod(\mathcal{C}, \mathcal{M}) \times_{\mathcal{M}} \{m\} \simeq Alg(\mathcal{C})_{/End(m)}$ over $Alg(\mathcal{C})$. In this case, for each associative algebra-object A in $\mathcal{C}$ we have a canonical homotopy equivalence

$$Map_{Alg(\mathcal{C})}(\mathcal{A}, End(m)) \simeq \{A\} \times_{\mathcal{A}ss^{\otimes}} LMod(\mathcal{C}, \mathcal{M}) \times_{\mathcal{M}} \{m\} \tag{3.4.2}$$

In other words, morphisms of algebras $A \to End(m)$ correspond to A-module structures on m.

Remark 3.4.4. Following the arguments in the proof of the previous result, and due to [100, Cor. 3.2.2.4], the existence of a classifying object for endomorphisms for m can be deduced from the existence of a final object in $Alg(\mathcal{C}^{+}[m])$.

We will be mostly interested in finding classifying objects for endomorphisms in the case when $\mathcal{C} = \mathcal{M}$ is Cat_∞ with the cartesian product. In other words, we want to have, for any monoidal $(\infty,1)$-category $A \in Alg_{\mathcal{A}ss}(\mathcal{C} = Cat_\infty)$ and any $(\infty,1)$-category $m \in \mathcal{M} = Cat_\infty$, a new monoidal $(\infty,1)$-category $End(m)$ such that the space of categories m left-tensored over A is homotopy equivalent to the space of monoidal functors $A \to End(m)$. As expected, $End(m)$ exists: it can be canonically identified with the $(\infty,1)$-category of endofunctors of m - $Fun(m,m)$ - equipped with the *strict* monoidal structure $End(m)^\otimes \to \mathcal{A}ss^\otimes$ induced by the composition of functors. The fact that $End(m)^\otimes$ has the required universal property follows from the universal property of $Fun(m,m)$ as an internal-hom in Cat_∞, and from [100, 4.7.2.39, 3.2.2.4] (see also the Remark 6.2.0.5 of loc.cit).

3.5 Idempotent Algebras

In this section we review the theory of idempotents as developed in [100, Section 4.8.2]. Let $\mathcal{C}^\otimes$ be a symmetric monoidal $(\infty,1)$-category with unit 1 and let E be an object in $\mathcal{C}$. A morphism $e : 1 \to E$ is said to be *idempotent* if the product morphism $id_E \otimes e : E \otimes 1 \to E \otimes E$ is an equivalence. Since $\mathcal{C}^\otimes$ is symmetric this is equivalent to ask for $e \otimes id_E$ to be an equivalence. We write (E,e) to denote an idempotent. The first important result concerning idempotents is that a pair (E,e) is an idempotent if and only if the product functor $E \otimes - : \mathcal{C} \to \mathcal{C}$ makes $\mathcal{C}_E$ - the essential image of $(E \otimes -)$ - a full reflexive subcategory of $\mathcal{C}$ (see [100, 4.8.2.4]). Notice that $\mathcal{C}_E$ equals the full subcategory of $\mathcal{C}$ spanned by those objects in $\mathcal{C}$ which are stable under products with E. By [100, 4.8.2.7], this localization is monoidal and therefore $\mathcal{C}_E$ inherits a symmetric monoidal structure $\mathcal{C}_E^\otimes$ where the unit object is E and the product map $(E \otimes -)$ extends to a monoidal map $\mathcal{C}^\otimes \to \mathcal{C}_E^\otimes$. Its right adjoint (the inclusion) is lax-monoidal and therefore induces an inclusion

$$CAlg(\mathcal{C}_E) \to CAlg(\mathcal{C}) \tag{3.5.1}$$

and since E is the unit in $\mathcal{C}_E$ we can use this inclusion to equip E with the structure of a commutative algebra in $\mathcal{C}$ for which the multiplication map $E \otimes E \to E$ is an equivalence. In fact, by the [100, 4.8.2.9], there is a perfect matching between idempotents and commutative algebras whose multiplication map is an equivalence (these are called *idempotent-algebras*). More precisely, if we denote by $CAlg^{idem}(\mathcal{C})$ the full subcategory of commutative algebra objects in $\mathcal{C}$ whose multiplication map $A \otimes A \to A$ is an equivalence, the natural composition

$$CAlg^{idem}(\mathcal{C}) \subseteq CAlg(\mathcal{C}) \simeq CAlg(\mathcal{C})_{1/} \to \mathcal{C}_{1/} \tag{3.5.2}$$

sending an commutative algebra object A to its unit $1 \to A$ morphism, is fully-faithful and its image consists exactly of the idempotent objects in $\mathcal{C}$.

The main result for idempotents can be stated as follows:

Proposition 3.5.1. *([100]-Prop.4.8.2.10) Let $\mathcal{C}^\otimes$ be a symmetric monoidal $(\infty,1)$-category and let (E,e) be an idempotent which we now know can be given by the unit of a commutative algebra object A in $\mathcal{C}$ (which is unique up to equivalence). Then, the natural forgetful map $Mod_A(\mathcal{C})^\otimes \to \mathcal{C}^\otimes$ induces an equivalence $Mod_A(\mathcal{C})^\otimes \to \mathcal{C}_E^\otimes$.*

3.6 Presentability

The results in this work depend crucially on the presentability of the closed cartesian symmetric monoidal $(\infty,1)$-category Cat_∞ (see [99, Prop. 3.1.3.9 and Cor. 3.1.4.4]). By [99, Prop. 5.5.4.15], the theory of presentable $(\infty,1)$-categories admits a very friendly theory of localizations: every localization with respect to a set of morphisms admits a description by means of local objects and, conversely, every (small) local theory is a localization. This feature will play a vital role in the proceeding sections where we shall work with presentable symmetric monoidal $(\infty,1)$-categories.

Let $\mathcal{K}$ be the collection of all small simplicial sets. By definition (see [100, 3.4.4.1]) a *presentable $\mathcal{O}$-monoidal $(\infty,1)$-category* is an $\mathcal{O}$-monoidal $(\infty,1)$-category compatible with $\mathcal{K}$-colimits such that for each color $x \in \mathcal{O}$, the fiber $\mathcal{C}_x$ is presentable. In this case, it is a corollary of the *Adjoint Functor Theorem* that $\mathcal{C}^{\otimes}$ is necessary closed.

Remark 3.6.1. Let $\mathcal{C}^{\otimes}$ is a presentable symmetric monoidal $(\infty,1)$-category and $\mathcal{D}$ be a reflexive localization of $\mathcal{C}$ with left-adjoint L satisfying the hypothesis $(*)$ of 3.1.7. Then the induced structure in $\mathcal{D}^{\otimes}$ is again a presentable symmetric monoidal $(\infty,1)$-category. This follows because of the condition $(*)$ and because both colimits and tensor products in $\mathcal{D}^{\otimes}$ are determined by the ones in $\mathcal{C}^{\otimes}$. In particular, and following the discussion in 3.2.2, these same reasons together with [99, 5.5.4.20], imply that the left-adjoint $L^{\otimes}: \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ has the expected universal property of the monoidal localization (along the collection of L-equivalences) within presentable symmetric monoidal $(\infty,1)$-categories.

Remark 3.6.2. Let $\mathcal{O}^{\otimes}$ be a small coherent $(\infty,1)$-operad and let $\mathcal{C}^{\otimes}$ be a presentable $\mathcal{O}$-monoidal $(\infty,1)$-category. By [100, 3.2.3.5] $Alg_{/\mathcal{O}}(\mathcal{C})$ is a presentable $(\infty,1)$-category and by [100, 3.4.4.2] $Mod_A^{\mathcal{O}}(\mathcal{C})^{\otimes}$ is presentable $\mathcal{O}$-monoidal.

There is also monoidal version of the adjoint functor theorem that will be useful to us in the future: by the Corollary [100, 7.3.2.7], if $f^{\otimes}: \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ is a lax-functor between $\mathcal{O}$-monoidal $(\infty,1)$-categories such that for each color $x \in \mathcal{O}$ the $f_x^{\otimes}$ has a right-adjoint, then $f^{\otimes}$ itself admits a right adjoint, which moreover is also map of ∞-operads. In particular, if $\mathcal{C}^{\otimes}$ and $\mathcal{D}^{\otimes}$ are presentable symmetric monoidal $(\infty,1)$-categories and $f^{\otimes}$ is a monoidal functor such that its underlying map $f: \mathcal{C} \to \mathcal{D}$ commutes with colimits, then by the Adjoint Functor Thereom [99, 5.5.2.9] it has a right-adjoint g which by the preceding discussion can be extented to a map of ∞-operads $g^{\otimes}: \mathcal{D}^{\otimes} \to \mathcal{C}^{\otimes}$.

3.6.1 The Monoidal Structure in $\mathcal{P}r^L$

Following the notations from [99] we write $\mathcal{P}r^L$ for full subcategory of $Cat_{\infty}^{big}(\mathcal{K})$ (with $\mathcal{K}$ denoting the collection of all small simplicial sets) spanned by the presentable $(\infty,1)$-categories together with the colimit preserving functors. By [100, 4.8.1.14], $\mathcal{P}r^L$ is closed under the monoidal structure in $Cat_{\infty}^{big}(\mathcal{K})^{\otimes}$ described by the formula (3.2.22) and therefore inherits a symmetric monoidal structure $(\mathcal{P}r^L)^{\otimes}$: if $\mathcal{C}_0$ and $\mathcal{C}_0'$ are two small ∞-categories, the tensor product $\mathcal{P}(\mathcal{C}_0) \otimes \mathcal{P}(\mathcal{C}_0')$ is given by $\mathcal{P}(\mathcal{C}_0 \times \mathcal{C}_0')$. More generally, if $\mathcal{C}$ and $\mathcal{C}'$ are two presentable $(\infty,1)$-categories and S is a small collection of morphism in $\mathcal{C}$, the product $(S^{-1}\mathcal{C}) \otimes \mathcal{C}'$ is the localization $T^{-1}(\mathcal{C} \otimes \mathcal{C}')$ where T is the image of the collection $S \times \{id_X\}_{X \in Obj(\mathcal{C}')}$ via the canonical morphism $\mathcal{C} \times \mathcal{C}' \to \mathcal{C} \otimes \mathcal{C}'$. By [99, Thm. 5.5.1.1] this is enough to describe any product and also to conclude that the unit object is the $(\infty,1)$-category of spaces $\mathcal{S} = \mathcal{P}(*)$.

The objects in $CAlg(\mathcal{P}r^L)$ can now be identified with the presentable symmetric monoidal $(\infty,1)$-categories (see again [100, Remark 4.8.1.9]). Plus, this symmetric monoidal structure is closed: for any pair of presentable $(\infty,1)$-categories $\mathcal{A}$ and $\mathcal{B}$, the $(\infty,1)$-category $Fun^L(\mathcal{A},\mathcal{B})$ of colimit-preserving functors $\mathcal{A} \to \mathcal{B}$ is again presentable and provides an internal-hom object in $\mathcal{P}r^{L,\otimes}$ (see [100, Remark 4.8.1.17]). Since $\mathcal{P}r^L$ admits all small colimits (by the combination of [99, Corollary 5.5.3.4 and Theorem 5.5.3.18]), we conclude that $(\mathcal{P}r^L)^{\otimes}$ is a symmetric monoidal structure compatible with all small colimits.

The following result will also be important to us:

Proposition 3.6.3. *The symmetric monoidal $(\infty,1)$-category $\mathcal{P}r^{L,\otimes}$ admits classifying objects for endomorphisms: for each presentable $(\infty,1)$-category M, the $(\infty,1)$-category $End^L(\mathcal{M})$ of colimit-preserving endomorphisms of M is the underlying $(\infty,1)$-category of a presentable monoidal $(\infty,1)$-category $End^L(M)^{\otimes} \to \mathcal{A}ss^{\otimes}$ whose monoidal operation is determined the composition of functors. Moreover, for any presentable symmetric monoidal $(\infty,1)$-category, we have a canonical homotopy equivalence*

$$Maps_{Alg_{\mathcal{A}ss}(\mathcal{P}r^L)}(\mathcal{C}^{\otimes}, End^L(M)^{\otimes}) \simeq \{\mathcal{C}^{\otimes}\} \times_{\mathcal{A}ss^{\otimes}} LMod(\mathcal{P}r^L, \mathcal{P}r^L) \times_{\mathcal{P}r^L} \{M\} \tag{3.6.1}$$

Proof. For the part that concerns the monoidal structure on $End^L(M)$, we know that $End(M)$ admits a monoidal structure $End(M)^{\otimes} \to \mathcal{A}ss^{\otimes}$ where the fiber over $\langle n\rangle$ is isomorphic to $\prod_{\langle n\rangle} Fun(M,M)$. We take $End^L(M)^{\otimes}$, by definition, the full subcategory of $End(M)^{\otimes}$ spanned by those sequences $(f_1,...,f_n)$ where each f_i is a colimit-preserving endofunctor of M. The fact that the composition $q: End^L(M)^{\otimes} \subseteq End(M)^{\otimes} \to \mathcal{A}ss^{\otimes}$ is still a cocartesian fibration follows immediately from the fact that the composition of colimit-preserving functors is colimit-preserving. It follows also that this monoidal structure is strictly associative because this holds for $End(M)^{\otimes}$.

To prove that this monoidal structure is presentable (see [100, Def. 3.4.4.1]) it suffices to observe that: (i) since M is presentable, $End^L(M)$ is also presentable (See [99, Prop. 5.5.3.8]); (ii) since colimits in $End^L(M)$ are computed objectwise in M ([99, Cor. 5.1.2.3]) and the objects in $End^L(M)$ are, by definition, colimit-preserving functors, the cocartesian fibration q is compatible with small colimits (See [100, Def. 3.1.1.18]).

To conclude, the fact that $End^L(M)$ provides a classifying object for endomorphisms results from the same arguments as in [100, Remark 6.2.0.5]: since $End^L(M)$ has the property of internal-hom object in $\mathcal{P}r^L$, it provides a final object in $(\mathcal{P}r^L)^+[\mathcal{P}r^L]$. The Corollary [100, 3.2.2.4] applied to $End^L(M)^{\otimes}$ concludes the proof.

□

3.6.2 The Monoidal Structure in $\mathcal{P}r^L_\kappa$

Let κ be a regular cardinal. Following [100, 5.3.2.9 and 5.3.2.11]), the monoidal structure in $\mathcal{P}r^L$ restricts to a monoidal structure in the (non-full) subcategory $\mathcal{P}r^L_\kappa \subset \mathcal{P}r^L$. Moreover, if $\mathcal{K}$ denotes the collection of κ-small simplicial sets together with the simplicial set $Idem$, the equivalence

$$Ind_\kappa : Cat_\infty(\mathcal{K}) \to \mathcal{P}r^L_\kappa \tag{3.6.2}$$

of the discussion in 2.1.20 is compatible with the monoidal structures (where on the left side we consider the monoidal structure described in 3.2.8).

To see this we use the fact the monoidal structure in $\mathcal{P}r^L$ is the restriction of the monoidal structure described in 3.2.8 for the $(\infty,1)$-category of big $(\infty,1)$-categories with all colimits together with colimit preserving functors. The discussion in the same section implies also that Ind_κ is monoidal, so that the product of κ-compactly generated in $\mathcal{P}r^L$ is again compactly generated. Moreover, if x is a κ-compact object in $\mathcal{C}$ and y is a κ-compact object in $\mathcal{C}'$, their product $x\otimes y$ is a κ-compact object in the product $\mathcal{C}\otimes\mathcal{C}'$ and the collection of κ-compact objects in $\mathcal{C}\otimes\mathcal{C}'$ is generated by the objects of this form under κ-small colimits. This implies that if $\mathcal{C}$ and $\mathcal{C}'$ and $\mathcal{D}$ are κ-compactly generated, the equivalence in (3.2.22) restricts to an equivalence between the full subcategory of $Fun_{\mathcal{K}}(\mathcal{C}\otimes\mathcal{C}',\mathcal{D})$ spanned by those functors which preserve κ-compact objects and the full subcategory of $Fun_{\mathcal{K}\boxtimes\mathcal{K}}(\mathcal{C}\times\mathcal{C}',\mathcal{D})$ spanned by the functors which preserve κ-compact objects.

Let now $\mathcal{P}r^{L,\otimes}_\kappa$ denote the (non-full) subcategory of $\mathcal{P}r^{L,\otimes}$ spanned by the objects $(\mathcal{C}_1,...,\mathcal{C}_n)$ where each $\mathcal{C}_i$ is a κ-compactly generated $(\infty,1)$-category, together with the maps $(\mathcal{C}_1,..,\mathcal{C}_n)\to(\mathcal{D}_1,...,\mathcal{D}_m)$ over some $f:\langle n\rangle\to\langle m\rangle$ corresponding to those families of functors

$$\{u_i : \prod_{j\in f^{-1}(\{i\})} \mathcal{C}_j \to \mathcal{D}_i\}_{i\in\{1,...,m\}} \tag{3.6.3}$$

in Cat_∞^{big} where each functor commutes with colimits separately in each variable and sends κ-compact objects to κ-compact objects. It follows that if $f:\langle n\rangle\to\langle m\rangle$ is a map in $N(Fin_*)$ and $X=(\mathcal{C}_1,...,\mathcal{C}_n)$ is a sequence of κ-compactly generated $(\infty,1)$-categories, then the map in $\mathcal{P}r^L$ corresponding to the family of universal functors

$$\prod_{j\in f^{-1}(\{i\})} \mathcal{C}_j \to \mathcal{D}_i := \otimes_{j\in f^{-1}(\{i\})}\mathcal{C}_j \simeq \mathcal{P}^{\mathcal{K}}_{\boxtimes_{j\in f^{-1}(\{i\}}\mathcal{K}}(\prod_{j\in f^{-1}(\{i\}} \mathcal{C}_j) \tag{3.6.4}$$

is in $\mathcal{P}r^L_\kappa$ (because it commutes with colimits separately in each variable and preserves compact objects because of the discussion above) and provides a cocartesian lift to f at X. It follows that the non-full inclusion $\mathcal{P}r^L_\kappa \subseteq \mathcal{P}r^L$ is monoidal.

3.7 Dualizable Objects

We recall a notion of duality. If $\mathcal{C}^{\otimes}$ is a monoidal $(\infty,1)$-category with a unit 1, we say that an object X is right-dualizable, or that it has a right-dual, if there exist an object $\check{X}$ together with morphisms

$$1 \xrightarrow{\alpha_X} \check{X}\otimes X \qquad\qquad X\otimes \check{X} \xrightarrow{\beta_X} 1 \tag{3.7.1}$$

such that the compositions

$$X \simeq X\otimes 1 \xrightarrow{Id_X\otimes \alpha_X} X\otimes\check{X}\otimes X \xrightarrow{\beta_X\otimes Id_X} 1\otimes X\simeq X \tag{3.7.2}$$

$$\check{X}\simeq 1\otimes \check{X} \xrightarrow{\alpha_X\otimes Id_{\check{X}}} \check{X}\otimes X\otimes \check{X} \xrightarrow{Id_{\check{X}}\otimes\beta_X} \check{X}\otimes 1\simeq \check{X}$$

are homotopic to the identity maps in $\mathcal{C}$. These restraints are equivalent to asking that for any pair of objects Y and Z in $\mathcal{C}$, the multiplication with the dual induces a homotopy equivalence

$$Map_{\mathcal{C}}(X\otimes Y, Z)\simeq Map_{\mathcal{C}}(Y,\check{X}\otimes Z) \tag{3.7.3}$$

There is also an obvious notion of left-dual and of course if $\mathcal{C}^{\otimes}$ is symmetric the two notions coincide.

Remark 3.7.1. In particular, if $\mathcal{C}^{\otimes}$ admits internal-hom objects and X has a dual, then we have for every object Y in $\mathcal{C}$, a canonical equivalence $Y^X\simeq \check{X}\otimes Y$.

It follows also from the definition that any monoidal functor preserves dualizable objects.

3.8 Stability

3.8.1 Stable Monoidal $(\infty,1)$-categories

Let $Cat_\infty^{\mathcal{E}x}$ denote the $(\infty,1)$-category of small stable ∞-categories together with the exact functors. The inclusion $Cat_\infty^{\mathcal{E}x}\subseteq Cat_\infty$ preserves finite products (as a result of [100, Thm. 1.1.1.4]) and therefore $Cat_\infty^{\mathcal{E}x}$ inherits a symmetric monoidal structure $(Cat_\infty^{\mathcal{E}x})^{\otimes}$ induced from the cartesian structure in Cat_∞. By definition (see Def. 8.3.4.1 of [100]) a *stable* $\mathcal{O}$*-monoidal* ∞*-category* is an $\mathcal{O}$-monoidal ∞-category $q:\mathcal{C}^{\otimes}\to\mathcal{O}^{\otimes}$ such that for each color $X\in\mathcal{O}$, the fiber $\mathcal{C}_X$ is a stable ∞-category and the monoidal operations are exact separately in each variable. In particular, the monoidal structure commutes with finite colimits. The small stable symmetric monoidal ∞-categories can be identified with commutative algebra objects in $(Cat_\infty^{\mathcal{E}x})^{\otimes}$.

If $\mathcal{C}^{\otimes}\to\mathcal{O}^{\otimes}$ is an $\mathcal{O}$-monoidal ∞-category compatible with all colimits, then it is stable $\mathcal{O}$-monoidal if and only if for each color $x\in\mathcal{O}$ the fiber $\mathcal{C}_x$ is stable. This is obvious because, by definition, the monoidal structure preserves colimits on each variable and therefore is exact on each variable. These will be called *stable presentable* $\mathcal{O}$*-monoidal* ∞*-categories*. We know that stable presentable $(\infty,1)$-categories form a full subcategory $\mathcal{P}r^L_{\mathcal{S}tb}$ of $\mathcal{P}r^L$ which by [100, 4.8.2.10, 4.8.2.18] is closed under the tensor structure in $\mathcal{P}r^L$. Moreover, following [100, 4.8.1.17], if $\mathcal{C}$ and $\mathcal{D}$ are stable presentabled $(\infty,1)$-categories, $Fun^L(\mathcal{C},\mathcal{D})$ is again stable presentable so that the monoidal structure in $\mathcal{P}r^L_{\mathcal{S}tb}$ is closed. We can identify stable presentable symmetric monoidal $(\infty,1)$-categories with the objects in $CAlg(\mathcal{P}r^L_{\mathcal{S}tb})$.

Remark 3.8.1. Let $\mathcal{C}$ be a stable $\mathcal{O}$-monoidal $(\infty,1)$-category compatible with all colimits. Then, for any $A \in Alg_{/\mathcal{O}}(\mathcal{C})$ the symmetric monoidal $(\infty,1)$-categories $Mod_A^{\mathcal{O}}(\mathcal{C})^{\otimes}$ is stable. This follows immediately from the fact that for each colour $x \in \mathcal{O}$, pushouts and pullbacks in $Mod_A^{\mathcal{O}}(\mathcal{C})_x$ are computed in $\mathcal{C}_x$ by means of the forgetful functor $Mod_A^{\mathcal{O}}(\mathcal{C})_x^{\otimes} \to \mathcal{C}_x$ (which is conservative). Moreover, since $Mod_A^{\mathcal{O}}(\mathcal{C})^{\otimes}$ is again compatible with colimits, the multiplication maps of the monoidal structure are exact on each variable. Notice however that the same is not true for algebras because colimits are not computed directly as colimits in the underlying category.

Example 3.8.2. The canonical example of a stable symmetric monoidal $(\infty,1)$-category is the $(\infty,1)$-category of spectra *Sp* with the smash product structure. One way to obtain this monoidal structure is to prove that *Sp* is an idempotent object in $\mathcal{P}r^{L,\otimes}$ [100, Prop. 4.8.2.18]. Our results in this work provide an alternative way to obtain this monoidal structure. We will return to this in the Example 4.2.16.

To conclude this section we recall an helpful characterization of compact generators in categories of modules over a stable presentable $(\infty,1)$-category.

Proposition 3.8.3. *Let $\mathcal{C}^{\otimes}$ be a stable presentable symmetric monoidal $(\infty,1)$-category. Suppose that its underlying $(\infty,1)$-category $\mathcal{C}$ admits a family $\mathcal{E} = \{E_i\}_{i\in I}$ of κ-compact generators in the sense of 2.1.23. Then, for any commutative algebra object A in $\mathcal{C}$, the family $\{A \otimes E_i\}_{i\in I}$ is a family of κ-compact generators in the $(\infty,1)$-category $Mod_A(\mathcal{C})$ (this makes sense because by the previous remark the category of modules is stable).*

Proof. By definition, $A\otimes E_i$ is the image of E_i under the base-change monoidal functor $(-\otimes A) : \mathcal{C}^{\otimes} \to Mod_A(\mathcal{C})^{\otimes}$. This functor is a left adjoint to the forgetful functor. The result follows immediately from this adjunction, together with the fact the forgetful functor is conservative and commutes with colimits ([100, 3.4.4.6]). ☐

3.8.2 Compatibility with t-structures

Let now $\mathcal{C}^{\otimes}$ be a stable symmetric monoidal $(\infty,1)$-category and suppose that $\mathcal{C}$ is equipped with a t-structure $((\mathcal{C})_{\leq 0}, (\mathcal{C})_{\geq 0})$. Following [100, 2.2.1.3] we say that the monoidal structure is compatible with the t-structure if the full subcategory $\mathcal{C}_{\geq 0}$ contains the unit object and is closed under the tensor product. In this case, it inherits a symmetric monoidal structure. Moreover, the truncation functors $\tau_{\leq n} : (\mathcal{C})_{\geq 0} \to (\mathcal{C})_{\geq 0}$ are monoidal [100, 2.2.1.8] and in particular, the subcategories $(\mathcal{C}_{\geq 0} \cap \mathcal{C}_{\leq n})$ are monoidal reflexive localizations of $\mathcal{C}_{\geq 0}$ [100, 2.2.1.10]. In particular, the heart $\mathcal{C}^{\heartsuit}$ inherits a symmetric monoidal structure and the zero-homology functor $\mathbb{H}_0 : \mathcal{C}_{\geq 0} \to \mathcal{C}^{\heartsuit}$ is monoidal.

Given an ∞-operad $\mathcal{O}^{\otimes}$, we write $Alg_{\mathcal{O}}(\mathcal{C})^{cn}$ for the full subcategory of $Alg_{\mathcal{O}}(\mathcal{C})$ spanned by the algebra objects whose underlying object in $\mathcal{C}$ is in $\mathcal{C}_{\geq 0}$. Since $\mathcal{C}_{\geq 0}$ inherits a monoidal structure, we have a fully-faithfull map $Alg_{\mathcal{O}}(\mathcal{C}_{\geq 0}) \subseteq Alg_{\mathcal{O}}(\mathcal{C})$ and its image can be identified with $Alg_{\mathcal{O}}(\mathcal{C})^{cn}$. It follows from the discussion in 3.1.6 that the right adjoint $\tau_{\geq 0} : \mathcal{C} \to \mathcal{C}_{\geq 0}$ extends to a right adjoint to the inclusion

$$Alg_{\mathcal{O}}(\mathcal{C})^{cn} \hookrightarrow Alg_{\mathcal{O}}(\mathcal{C}) \tag{3.8.1}$$

Assume now that the t-structure is left complete. In this case we have an equivalence $\mathcal{C}_{\geq 0} \simeq lim_n(\mathcal{C}_{\geq 0} \cap \mathcal{C}_{\leq n})$. In fact this equivalence in Cat_{∞} lifts to an equivalence in $CAlg(Cat_{\infty})$ through the forgetful functor $CAlg(Cat_{\infty}) \to Cat_{\infty}$. Indeed, the functors $\tau_{\leq n} : \mathcal{C}_{\geq 0} \to \mathcal{C}_{\geq 0} \cap \mathcal{C}_{\leq n}$ are monoidal and limits in $CAlg(Cat_{\infty})$ are computed in Cat_{∞} by means of the same forgetful map. In particular, since the forgetful map $CAlg(Cat_{\infty}) \subseteq Op_{\infty}$ has a left adjoint (see 3.2.4), it commutes with limits so that $\mathcal{C}_{\geq 0}^{\otimes}$ is the limit of $(\mathcal{C}_{\geq 0} \cap \mathcal{C}_{\leq n})^{\otimes}$ in Op_{∞}. In particular, for any ∞-operad $\mathcal{O}^{\otimes}$, we have an equivalence

$$Alg_{\mathcal{O}}(\mathcal{C})^{cn} \simeq lim_n Alg_{\mathcal{O}}(\mathcal{C}_{\geq 0} \cap \mathcal{C}_{\leq n}) \tag{3.8.2}$$

If we assume that $\mathcal{C}$ is presentable, then $Alg_{\mathcal{O}}(\mathcal{C})$ will also be presentable and in particular the subcategory of n-truncated objects $\tau_{\leq n}Alg_{\mathcal{O}}(\mathcal{C})$ is a reflexive localization of $Alg_{\mathcal{O}}(\mathcal{C})$. Moreover, since the truncation functor given by the t-structure is monoidal, it exhibits $Alg_{\mathcal{O}}(\mathcal{C}_{\geq 0}\cap\mathcal{C}_{\leq n})$ also a reflexive localization of $Alg_{\mathcal{O}}(\mathcal{C})$ (see 3.1.8) so that the two subcategories are equivalent. Together with the equivalence (3.8.2) this implies that Postnikov towers converge in $Alg_{\mathcal{O}}(\mathcal{C})^{cn}$.

Again, an important example is the $(\infty,1)$-category of spectra $Sp^{\otimes}$ [100, Lemma 7.1.1.7]. More generally, for any connective $\mathbb{E}_{k+1}$-algebra R in Sp, the category of left modules $LMod_R(Sp)$ inherits a natural left-complete t-structure [100, 7.1.1.10, 7.1.1.13] together with a compatible $\mathbb{E}_k$-monoidal structure [100, 7.1.2.5, 7.1.3.15].

3.9 From symmetric monoidal model categories to symmetric monoidal $(\infty,1)$-categories

3.9.1 The (monoidal) link

The link between model categories and $(\infty,1)$-categories described in the Section 2.2 can now be extended to the world of monoidal structures. Recall that a model category $\mathcal{M}$ equipped with a monoidal structure $\otimes$ is said to be a *monoidal model category* if the monoidal structure is closed, the tensor functor is a left-Quillen bifunctor and the unit of the monoidal structure is a cofibrant object in $\mathcal{M}$. The main idea is that

Every symmetric monoidal model category *"presents" a symmetric monoidal* $(\infty,1)$*-category.*

Following the [100, Example 4.1.3.6], if $\mathcal{M}$ is a symmetric monoidal model category (see Definition 4.2.6 of [69]) then the underlying ∞-category of $\mathcal{M}$ inherits a canonical symmetric monoidal structure which we denote here as $N(\mathcal{M})[W^{-1}]^{\otimes}\to N(Fin_*)$. It can be obtained as follows: first recall that in a symmetric monoidal model category, the product of cofibrant objects is again cofibrant and by the Ken Brown's Lemma, the product of weak-equivalences between cofibrant objects is again a weak-equivalence. This implies that the full subcategory of cofibrant objects in $\mathcal{M}$ inherits a monoidal structure and we can regard it as a simplicial coloured operad $(\mathcal{M}^c)^{\otimes}$ enriched over constant simplicial sets. This way, its operadic nerve $N^{\otimes}((\mathcal{M}^c)^{\otimes})\to N(Fin_*)$ provides a trivial ∞-operad which furthermore is a symmetric monoidal $(\infty,1)$-category with underlying ∞-category equivalent to $N(M^c)$ (see the Example 3.1.6). Since the restriction of the monoidal structure to the cofibrant objects preserves weak-equivalences, we can understand the pair $(N^{\otimes}((\mathcal{M}^c)^{\otimes}),W)$ as an object in $CAlg(\mathcal{W}Cat_{\infty})$ and we define the underlying symmetric monoidal $(\infty,1)$-category of $\mathcal{M}$ as the monoidal localization (see 3.2.2)

$$N(\mathcal{M})[W^{-1}]^{\otimes}:=N^{\otimes}((\mathcal{M}^c)^{\otimes})[W_c^{-1}]^{\otimes} \tag{3.9.1}$$

It follows from the definitions that its underlying ∞-category is canonically equivalent to the underlying ∞-category of $\mathcal{M}$. Moreover, it comes canonically equipped with a universal monoidal functor $N^{\otimes}((\mathcal{M}^c)^{\otimes})\to N(\mathcal{M})[W^{-1}]^{\otimes}$.

At the same time, if $\mathcal{M}$ comes equipped with a *compatible simplicial enrichment*, then $\mathcal{M}^{\circ}$, although not a simplicial monoidal category (because the product of fibrant objects is not fibrant in general), can be seen as the underlying category of a simplicial coloured operad $(\mathcal{M}^{\circ})^{\otimes}$ where the colours are the cofibrant-fibrant objects in $\mathcal{M}$ and the operation space is given by

$$Map_{(\mathcal{M}^{\circ})^{\otimes}}(\{X_i\}_{i\in I},Y):=Map_{\mathcal{M}}(\bigotimes_i X_i,Y) \tag{3.9.2}$$

which is a Kan-complex because Y is fibrant and the product of cofibrant objects is cofibrant. With this, we consider the ∞-operad given by the operadic nerve $N^{\otimes}((\mathcal{M}^{\circ})^{\otimes})$. By [100, 4.1.3.10], this ∞-operad is a symmetric monoidal $(\infty,1)$-category and the product of cofibrant-fibrant objects X,Y is given by the choice of a trivial cofibration $X\otimes Y\to Z$ providing a fibrant replacement for the product in $\mathcal{M}$. The key result, proved in [100, Cor. 4.1.3.16], is the existence an ∞-symmetric-monoidal-generalization of the Proposition 2.2.1. Namely, the symmetric monoidal $(\infty,1)$-category $N(M)[W^{-1}]^{\otimes}$ is monoidal equivalent to $N^{\otimes}((\mathcal{M}^{\circ})^{\otimes})$.

A particular instance of this is when $\mathcal{M}$ is a cartesian closed combinatorial simplicial model category with a cofibrant final object. In this case, it is a symmetric monoidal model category with respect to the product and we can consider its operadic nerve $N^{\otimes}((\mathcal{M}^{\circ})^{\times})$. From [100, 2.4.1.10], this is equivalent to a Cartesian structure in the underlying ∞-category of $\mathcal{M}$ - $N_{\Delta}(\mathcal{M}^{\circ})^{\times}$.

A monoidal left-Quillen map ([69]-Def. 4.2.16) between monoidal model categories induces a monoidal functor between the underlying symmetric monoidal $(\infty,1)$-categories. This is because the monoidal localization was constructed as a functor $CAlg(\mathcal{W}Cat_{\infty}) \to CAlg(Cat_{\infty})$. In the simplicial case we can provide a more explicit construction:

Construction 3.9.1. Let $\mathcal{M} \to \mathcal{N}$ be a monoidal left-Quillen functor between two combinatorial simplicial symmetric monoidal model categories. Let G be its right adjoint. We construct a monoidal map between the associated operadic nerves

$$\begin{tikzcd} N^{\otimes}((\mathcal{M}^{\circ})^{\otimes}) \arrow[rr, "F^{\otimes}"] \arrow[dr] & & N^{\otimes}((\mathcal{N}^{\circ})^{\otimes}) \arrow[dl] \\ & N(Fin_*) & \end{tikzcd} \tag{3.9.3}$$

For that, we consider the simplicial category $\mathcal{A}$ whose objects are triples $(i, \langle n\rangle, (X_1, ..., X_n))$ with $i \in \{0,1\}$, $\langle n\rangle \in Fin_*$ and X_1,..., X_n are objects in $\mathcal{M}$ if $i=0$ and in $\mathcal{N}$ if $i=1$. The mapping spaces

$$Map_{\mathcal{A}}((i, \langle n\rangle, (X_1, ..., X_n)), (j, \langle m\rangle, (Y_1, ..., Y_n))) \tag{3.9.4}$$

are defined to be the mapping spaces in $\tilde{M}$ ([4]) (resp. $\tilde{N}$) if $i,j=0$ (resp. $i,j=1$). If $i=1$ and $j=0$, we declare it to be empty and finally, if $i=0$ and $j=1$, we define it as

$$Map_{\tilde{N}}((\langle n\rangle, (F(X)_1, ..., F(X)_n)), (\langle m\rangle, (Y_1, ..., Y_n))) \tag{3.9.5}$$

which by the adjunction (F,G) and the fact that F is strictly monoidal, are the same as

$$Map_{\tilde{M}}((\langle n\rangle, (X_1, ..., X_n)), (\langle m\rangle, (G(Y_1), ..., G(Y_n)))) \tag{3.9.6}$$

The composition is the obvious one induced from $\mathcal{M}$ and $\mathcal{N}$. We consider the full simplicial subcategory $\mathcal{A}^{\circ}$ spanned by the objects $(i, \langle n\rangle, (X_1, ..., X_n))$ where each X_i is cofibrant-fibrant (respectively in $\mathcal{M}$ or $\mathcal{N}$ depending on the value of i). It follows that $\mathcal{A}^{\circ}$ is enriched over Kan-complexes (because $\mathcal{M}$ and $\mathcal{N}$ are simplicial model categories and F is left Quillen) and therefore its simplicial nerve is an $(\infty,1)$-category. Moreover, it admits a canonical projection $p : N_{\Delta}(\mathcal{A}^{\circ}) \to N(Fin_*) \times \Delta[1]$ whose fibers

$$\{0\} \times_{N(Fin_*)\times\Delta[1]} N_{\Delta}(\mathcal{A}^{\circ}) \simeq N^{\otimes}((\mathcal{M}^{\circ})^{\otimes}) \tag{3.9.7}$$

and

$$\{1\} \times_{N(Fin_*)\times\Delta[1]} N_{\Delta}(\mathcal{A}^{\circ}) \simeq N^{\otimes}((\mathcal{N}^{\circ})^{\otimes}) \tag{3.9.8}$$

recover the operadic nerves of $\mathcal{M}$ and $\mathcal{N}$, respectively.

Proposition 3.9.2. *The projection $p : N_{\Delta}(\mathcal{A}^{\circ}) \to N(Fin_*) \times \Delta[1]$ is a cocartesian fibration of $(\infty,1)$-categories.*

[4]consult the notation in the Construction 3.1.2

Proof. We follow the arguments in the proof of [100, 4.1.3.15] using the characterization of p-cocartesian morphisms of [99, 2.4.4.3]. We have to prove that for any edge $u : (i, \langle n\rangle) \to (j, \langle m\rangle)$ in $N(Fin_*)\times\Delta[1]$ and any object $C := (i, \langle n\rangle, (X_1, ..., X_n))$ over $(i, \langle n\rangle)$, there is a p-cocartesian lifting $\tilde{u}$ of u starting at C. In other words, we have to prove that for any morphism $v : (j, \langle m\rangle) \to (k, \langle l\rangle)$ and for any object E lying over $\langle l\rangle$, the induced map

$$Map^{v}_{N_\Delta(\mathcal{A}^\circ)}(C', E) \to Map^{v\circ u}_{N_\Delta(\mathcal{A}^\circ)}(C, E) \tag{3.9.9}$$

is an homotopy equivalence. By definition a morphism u consists of a pair $(i \to i', f : \langle n\rangle \to \langle m\rangle)$ with $i \to i'$ an edge in $\Delta[1]$ and f a morphism in Fin_*. Therefore, we are reduced to showing the following cases:

- Case $i = j = k = 0$: Given an object $C := (0, \langle n\rangle, (X_1, ..., X_n))$ over $(0, \langle n\rangle)$ and $f : \langle n\rangle \to \langle m\rangle$ we define objects $\tilde{X}_\lambda$ by means of taking fibrant-replacements

$$u_\lambda : \otimes_{\alpha\in f^{-1}(\{\lambda\})} X_\alpha \to \tilde{X}_\lambda, \text{ with } \lambda \in \{1, ..., m\} \tag{3.9.10}$$

  where the maps u_α are the trivial cofibrations given by the fibrant replacement functor. Finally, we set $C' := (0, \langle m\rangle, (\tilde{X}_1, ..., \tilde{X}_m))$ and $u : C \to \mathcal{C}'$ the unique map in $N_\Delta(\mathcal{A}^\circ)$ induced by the product of the trivial cofibrations u_λ. Let now $E := (0, \langle l\rangle, (E_1, ..., E_l))$ be an object over $(0, \langle l\rangle)$ and $g : \langle m\rangle \to \langle l\rangle$ a morphism. Using the definitions, the map of (3.9.9) can be idntified with the composition map along the product of the u_λ

$$\prod_{\beta\in\langle l\rangle} Map_{N_\Delta(\mathcal{M}^\circ)}(\bigotimes_{\lambda\in g^{-1}(\{\beta\})} \tilde{X}_\lambda, E_\beta) \to \prod_{\beta\in\langle l\rangle} Map_{N_\Delta(\mathcal{M}^\circ)}(\bigotimes_{\sigma\in(g\circ f)^{-1}(\{\beta\})} X_\sigma, E_\beta) \tag{3.9.11}$$

$$\simeq \prod_{\beta\in\langle l\rangle} Map_{N_\Delta(\mathcal{M}^\circ)}(\bigotimes_{\lambda\in g^{-1}(\{\beta\})} (\bigotimes_{\alpha\in(f)^{-1}(\{\lambda\})} X_\alpha), E_\beta) \tag{3.9.12}$$

  is an homotopy equivalence. Here the last isomorphism follows from the natural identification of the two products $\bigotimes_{\sigma\in(g\circ f)^{-1}(\{\beta\})} X_\sigma$ and $\bigotimes_{\lambda\in g^{-1}(\{\beta\})}(\bigotimes_{\alpha\in(f)^{-1}(\{\lambda\})} X_\alpha)$. This follows because each of the E_β is a fibrant object in $\mathcal{M}$ and the maps u_α are trivial cofibrations so that their product is also a trivial cofibration (since the monoidal structure is assumed to be compatible with the model structure).

- Case $i = j = k = 1$: Follows by the same arguments as in the previous case, this time using the monoidal model structure in $\mathcal{N}$

- Case $i = 0, j = k = 1$: Given an object $C := (0, \langle n\rangle, (X_1, ..., X_n))$ over $(0, \langle n\rangle)$ we have to find a new object $C' := (1, \langle m\rangle, (\tilde{X}_1, ..., \tilde{X}_m))$ together with a p-cocartesian morphism in $N_\Delta(\mathcal{A}^\circ)$

$$\tilde{u} : C \to C' \tag{3.9.13}$$

  defined over u. Recall that by definition, the connected component of the mapping space

$$Map_{N_\Delta(\mathcal{A}^\circ)}((i, \langle n\rangle, (X_0, ..., X_n)), (j, \langle m\rangle, (Y_1, ..., Y_m))) \tag{3.9.14}$$

  spanned by the maps which are defined over u was defined to be the mapping space

$$\prod_{\lambda\in\langle m\rangle} Map_{\mathcal{N}^\circ}(\bigotimes_{\alpha\in f^{-1}\{\lambda\}} F(X_\alpha), \tilde{Y}_\lambda) \tag{3.9.15}$$

  With this in mind, we define $\tilde{X}_\lambda$ to be a fibrant replacement for the product

$$u_\lambda : \bigotimes_{\alpha\in f^{-1}\{\lambda\}} F(X_\alpha) \to \tilde{X}_\lambda \tag{3.9.16}$$

where each u_λ is the trivial cofibration that comes out from the device of the model structure providing the functorial factorizations. Finally, we take $\tilde{u}$ to be the point in $Map_{N_\Delta(\mathcal{A}^\circ)}((0,(X_0,...,X_1)),(1,(\tilde{X}_1,...,\tilde{X}_m))$ corresponding the product of the trivial cofibrations u_λ. Notice that each $\tilde{X}_\lambda$ is cofibrant-fibrant in $\mathcal{N}$ because the product of cofibrants is cofibrant and F preserves cofibrant objects. We are now reduced to the task of proving that $\tilde{u}$ is a p-cocartesian morphism. In our situation, this is equivalent to say that for any morphism $v=(id_1,g):(1,\langle m\rangle)\to(1,\langle l\rangle)$ in $\Delta[1]\times N(Fin_*)$ and any object $E:=(1,\langle l\rangle,(E_1,...,E_l))$ over $(1,\langle l\rangle)$, the composition map with $\tilde{u}$

$$Map^v_{N_\Delta(\mathcal{A}^\circ)}((1,\langle n\rangle,(\tilde{X}_0,...,\tilde{X}_n)),(1,\langle l\rangle,(E_1,...,E_l)))\to \tag{3.9.17}$$

$$Map^{v\circ u}_{N_\Delta(\mathcal{A}^\circ)}((0,\langle n\rangle,(X_0,...,X_1)),(1,\langle l\rangle,(E_1,...,E_l))) \tag{3.9.18}$$

is a weak-equivalence of simplicial sets (here we denote by $Map^v_{N_\Delta(\mathcal{A}^\circ)}(-,-)$ the directed component of $Map_{N_\Delta(\mathcal{A}^\circ)}(-,-)$ of those maps which are defined over v). It is immediate from the definitions that the previous map is the composition map

$$\Pi_{\beta\in\{1,...,l\}}Map_{N_\Delta(\mathcal{N}^\circ)}(\bigotimes_{\lambda\in g^{-1}(\{\beta\})}\tilde{X}_\lambda,E_\beta)\to \tag{3.9.19}$$

$$\Pi_{\beta\in\{1,...,l\}}Map_{N_\Delta(\mathcal{N}^\circ)}(\bigotimes_{\sigma\in(g\circ f)^{-1}(\{\beta\})}F(X_\sigma),E_\beta)\simeq \tag{3.9.20}$$

$$\simeq\Pi_{\beta\in\{1,..,l\}}Map_{N_\Delta(\mathcal{N}^\circ)}(\bigotimes_{\lambda\in g^{-1}(\{\beta\})}(\bigotimes_{\alpha\in(f)^{-1}(\{\lambda\})}F(X_\alpha)),E_\beta) \tag{3.9.21}$$

where the last isomorphism follows from the natural identification of the two products $\bigotimes_{\sigma\in(g\circ f)^{-1}(\{\beta\})}F(X_\sigma)$ and $\bigotimes_{\lambda\in g^{-1}(\{\beta\})}(\bigotimes_{\alpha\in(f)^{-1}(\{\lambda\})}F(X_\alpha))$. Finally, we can see that this previous map is the one obtained by the product of the pos-composition with the trivial cofibrations u_λ. Since the monoidal structure is given by a Quillen bifunctor, the product of trivial cofibrations is a trivial cofibration and therefore as each of the E_β is fibrant, the map between the mapping spaces is a trivial fibration and so a weak-equivalence. To conclude, the product of trivial fibrations is always a trivial fibration.

- Case $i=j=0,k=1$: In this case we define $u:C\to C'$ as in the first case. We need to check that for any object $E:=(1,\langle l\rangle,(E_1,...,E_l))$ over $(1,\langle l\rangle)$ and for any morphism $g:\langle m\rangle\to\langle l\rangle$, the composition map of (3.9.9) is an homotopy equivalence. Again using the definitions, the left side of (3.9.9) is given by

$$\Pi_{\beta\in\{1,...,l\}}Map_{N_\Delta(\mathcal{N}^\circ)}(\bigotimes_{\lambda\in g^{-1}(\{\beta\})}F(\tilde{X}_\lambda),E_\beta) \tag{3.9.22}$$

which as F is monoidal is isomorphic to

$$\Pi_{\beta\in\{1,...,l\}}Map_{N_\Delta(\mathcal{N}^\circ)}(F(\bigotimes_{\lambda\in g^{-1}(\{\beta\})}\tilde{X}_\lambda),E_\beta) \tag{3.9.23}$$

and the right side is given by

$$\Pi_{\beta\in\{1,...,l\}}Map_{N_\Delta(\mathcal{N}^\circ)}(\bigotimes_{\sigma\in(g\circ f)^{-1}(\{\beta\})}F(\tilde{X}_\sigma),E_\beta) \tag{3.9.24}$$

which again because F is monoidal, is isomorphic to

$$\Pi_{\beta\in\{1,...,l\}} Map_{N_\Delta(\mathcal{N}^\circ)}(F(\bigotimes_{\sigma\in(g\circ f)^{-1}(\{\beta\})} \tilde{X}_\sigma), E_\beta) \tag{3.9.25}$$

Finally, using the same change of variables in the tensor product applied in the previous case we conclude that under these identifications the map in (3.9.9) can be identified with the composition map along the tensor product of the image maps $F(u_\lambda)$. As each of the E_β is fibrant and each of the maps u_λ is a trivial cofibration and F is left-Quillen (thus preserving trivial cofibrations), we conclude that the composition map is an homotopy equivalence of fibrant simplicial sets.

This concludes the proof.

□

Finally, we can now extract the monoidal functor $F^\otimes$ using [99, 5.2.1.4]. It is also clear from the proof of the Proposition 3.9.2 that the underlying functor of $F^\otimes$ is the map $\bar{F}$ described in [99, 5.2.4.6] given by the composition of F with a fibrant replacement functor in $\mathcal{N}$.

3.9.2 Strictification of Algebras and Modules

In some very specific cases the theory of algebras can be performed directly within the setting of model categories. In other words, it admits a strictification. An important result of [124] (Theorem 4.1) is that if $\mathcal{M}$ is a combinatorial monoidal model category satisfying the *monoid axiom* (Definition 3.3 of [124]), then the category $Alg(\mathcal{M})$ of strict associative algebra objects in $\mathcal{M}$ admits a new combinatorial model structure where:

- a map in $Alg(\mathcal{M})$ is a weak-equivalence if and only if it is a weak-equivalence in $\mathcal{M}$;
- a map in $Alg(\mathcal{M})$ is a fibration if and only if it is a fibration in $\mathcal{M}$;
- the forgetful functor $Alg(\mathcal{M}) \to \mathcal{M}$ is a right-Quillen map that preserves cofibrant objects.
- this model structure in $Alg(\mathcal{M})$ is simplicial if the model structure in $\mathcal{M}$ is.

Using this results, we can create a comparison map between the underlying $(\infty,1)$-category of $Alg(\mathcal{M})$ and the $(\infty,1)$-category of algebra-objects in the underlying $(\infty,1)$-category of $\mathcal{M}$. More precisely, using the fact the forgetful functor $Alg(\mathcal{M}) \to \mathcal{M}$ preserves cofibrant objects, we have natural inclusions $Alg(\mathcal{M})^c \subseteq Alg(\mathcal{M}^c) \subseteq Alg(\mathcal{M})$ which preserve weak-equivalences. Passing to the localizations (in the sense of 2.1.11), the cofibrant-replacement functor produces equivalences

$$N(Alg(\mathcal{M})^c)[W_{c_{alg}}^{-1}] \simeq N(Alg(\mathcal{M}^c))[W_c^{-1}] \simeq N(Alg(\mathcal{M}))[W^{-1}] \tag{3.9.26}$$

where $W_{c_{alg}}$ denotes the class of weak-equivalences between cofibrant algebras and W_c is the class of weak-equivalences between algebras whose underlying objects in $\mathcal{M}$ are cofibrant. Finally, using the fact that the localization map $N(\mathcal{M}^c) \to N(\mathcal{M}^c)[W_c^{-1}]$ is monoidal, it provides a map $Alg(N(\mathcal{M}^c)) \to Alg(N(\mathcal{M}^c)[W_c^{-1}])$ which sends weak-equivalences in $\mathcal{M}$ between cofibrant objects to equivalences. The universal property of the localization provides a canonical map

$$\begin{tikzcd}
N(Alg(\mathcal{M})^c)[W_{c_{alg}}^{-1}] \simeq Alg(N(\mathcal{M}^c))[W_c^{-1}] \arrow[r, dashed] & Alg(N(\mathcal{M}^c)[W_c^{-1}]) \\
Alg(N(\mathcal{M}^c)) \arrow[u] \arrow[ur] &
\end{tikzcd} \tag{3.9.27}$$

rendering the diagram homotopy commutative. By the [100, Thm. 4.1.4.4], if $\mathcal{M}$ is a combinatorial monoidal model category and either (a) all objects are cofibrant or (b) $\mathcal{M}$ is left-proper, the cofibrations are generated by the cofibrations between cofibrant objects and $\mathcal{M}$ is symmetric and satisfies the monoid axiom, then, this canonical map is an equivalence of $(\infty,1)$-categories. In the next section we will see this result applied to the theory of differential graded algebras.

Remark 3.9.3. This strictification result can be extended to a monoidal functor. More precisely, recall from 3.2.6 that the category of algebras inherits a monoidal structure induced from the one in the base monoidal category. As in the Remark 3.2.6, the functor $Alg(N(\mathcal{M}^c)) \to Alg(N(\mathcal{M}^c)[W_c^{-1}])$ extends to a monoidal functor and using the monoidal localization of 3.2.2 we can also promote the map in (3.9.27) to a monoidal functor.

There is also a strictification result for bimodules over associative algebras. Given two strictly associative algebra objects A and B in a combinatorial monoidal model category $\mathcal{M}$, we can set a model structure in the classical category of bimodules in $\mathcal{M}$, $BiMod(A,B)(\mathcal{M})$, for which the weak-equivalences W_{Mod} are given by the weak-equivalences of $\mathcal{M}$ [100, Prop. 4.3.3.15] and by [100, 4.3.3.17] we have

$$N(BiMod(A,B)(\mathcal{M}))[W_{Mod}^{-1}] \simeq_A BMod_B(N(\mathcal{M})[W^{-1}]) \tag{3.9.28}$$

A similar result holds for commutative algebras ([100, Thm 4.5.4.7]) whenever the strict theory admits an appropriate model structure (as in [100, 4.5.4.6]). In particular, it works also for modules over commutative algebras.

Remark 3.9.4. In the general situation, there are no model structures for algebras or modules. This is exactly one of the main motivations to develop a theory of algebras and modules within the more fundamental setting of $(\infty,1)$-categories. The theory of motives is one of those important cases where model category theory does not work.

Remark 3.9.5. Recall that an $(\infty,1)$-category is presentable iff there exists a combinatorial simplicial model category $\mathcal{M}$ such that $\mathcal{C}$ is the underlying ∞-category of $\mathcal{M}$ (which means, $\mathcal{C} \simeq N_\Delta(\mathcal{M}^\circ)$) (see [99, A.3.7.6]). There is a similar statement for presentable monoidal $(\infty,1)$-categories, replacing the simplicial nerve by the operadic nerve (see [100, 4.1.4.9] for a sketch of proof).

3.10 Higher Algebra over a classical commutative ring k

The discussion in this section will be important later. Let k be a (small) commutative ring and denote by $Mod(k)$ the 1-category of small sets endowed with the structure of module over k. We will write $Ch(k)$ to denote the big category of (unbounded) chain complexes of small k-modules. This is a Grothendieck abelian category. Recall also the existence of a symmetric tensor product of complexes given by the formula

$$(E \otimes E')_n := \bigoplus_{i+j=n} (E_i \otimes_k E_j) \tag{3.10.1}$$

where $\otimes_k$ denotes the tensor product of k-modules. This monoidal structure is closed, with internal-hom $\underline{Hom}_{Ch(k)}(E,E')$ given by the formula

$$\underline{Hom}_{Ch(k)}(E,E')_n := \prod_i Hom_k(E_i, E_{i+n}) \tag{3.10.2}$$

where the differential $d_n : \underline{Hom}_{Ch(k)}(E,E')_n \to \underline{Hom}_{Ch(k)}(E,E')_{n+1}$ sends a family $\{f_i\}$ to the family $\{d \circ f_i - (-1)^n f_{i+1}\}$.

The category $Ch(k)$ carries a left proper combinatorial model structure [69, Theorem 2.3.11] where the weak-equivalences are the quasi-isomorphisms of complexes, the fibrations are the surjections (and so every object is fibrant). We will call it the projective model structure on complexes. The cofibrant complexes (see the Lemma 2.3.6 and the Remark 2.3.7 of [69]) are the *DG-projective complexes*. In particular, every cofibrant complex is a complex of projective (and therefore flat) modules and any bounded below complexes of projective k-modules is cofibrant. Moreover, by the Proposition 4.2.13 of loc.cit, this model structure is compatible with the tensor product of complexes and so $Ch(k)$ is a closed symmetric monoidal model category. Following 3.9.1, the proper way to encode the study of complexes of k-modules up to quasi-equivalences is the underlying $(\infty,1)$-category $\mathcal{D}(k)$ of the model category $Ch(k)$. This is a particular case of the Example 2.1.5 with $X = Spec(k)$. In particular, $\mathcal{D}(k)$ is stable with a compact generator k and with compact objects the perfect complexes. $\mathcal{D}(k)$ acquires a symmetric monoidal structure $\mathcal{D}(k)^{\otimes}$ (as explained in 3.9.1).

Remark 3.10.1. This method to obtain $\mathcal{D}(k)$ is not the one described in 2.1. This is because the projective model structure does not agree with the injective one. However, since the weak-equivalences are the same, the resulting $(\infty,1)$-categories obtained by localization are equivalent.

We now review the theory of algebra objects over k. By definition, a strict *dg-algebra* over k is a strictly associative algebra-object in $Ch(k)$ with respect to the tensor product of complexes. We will denote the category of dg-algebras as $Alg_{\mathcal{A}ss}(Ch(k))$. As explained in the Example 3.2.1, the nerve $N(Alg_{\mathcal{A}ss}(Ch(k)))$ is equivalent to $Alg_{\mathcal{A}ss}(N(Ch(k))$ - the theory of algebras described in the previous section - so that the notations are coherent. Thanks to [124, Thm 4.1] the model structure in $Ch(k)$ extends to a model structure in $Alg_{\mathcal{A}ss}(Ch(k))$ with fibrations and weak-equivalences given by the underlying fibrations and quasi-isomorphisms of complexes[5]. This model structure satisfies the condition (b) of the previous section (see [100, 7.1.4.6]). In this case, denoting its underlying $(\infty,1)$-category by $N(Alg_{\mathcal{A}ss}(Ch(k))^c)[W_c^{-1}]$, the strictification result provides an equivalence

$$N(Alg_{\mathcal{A}ss}(Ch(k))^c)[W_c^{-1}] \xrightarrow{\sim} Alg_{\mathcal{A}ss}(\mathcal{D}(k)) \tag{3.10.3}$$

.

Remark 3.10.2. The situation for commutative algebras is not so satisfatory. In general the model structure on complexes does not extend to the category of strictly commutative algebra objects in $Ch(k)$. However, if k contains the field of rational numbers $\mathbb{Q}$, the model structure extends [100, Prop. 7.1.4.11] and the strictification result holds [100, 7.1.4.7]. Writing $CDGA_k$ to denote its underlying $(\infty,1)$-category, the canonical map given by the universal property of the localization

$$CDGA_k \to CAlg(\mathcal{D}(k)) \tag{3.10.4}$$

is an equivalence.

The $(\infty,1)$-category $\mathcal{D}(k)$ carries a natural right-complete t-structure where $\mathcal{D}(k)_{\geq 0}$ is the full subcategory spanned by the complexes with zero homology in negative degree. Its heart is the category of modules over k and the functor $\mathbb{H}_n : \mathcal{C} \to \mathcal{C}^{\heartsuit}$ corresponds to the classical nth-homology functor H_n. This t-structure is also left-complete. Indeed, this follows because k is a generator in $\mathcal{D}(k)$ and using the formula $H_i(X) \simeq \pi_i(Map_{\mathcal{D}(k)}(k,X)), \forall i \geq 0$ we see that if all the homology groups of an object X are zero so is X. Moreover, the monoidal structure in $\mathcal{D}(k)$ is compatible with this t-structure (as we can suppose that our complexes are cofibrant, this follows directly from the Kunneth spectral sequence, or alternatively, using the same methods as in [100, 7.1.1.7]). Following the discusion in 3.8.2, the left-completness implies that for any ∞-operad $\mathcal{O}^{\otimes}$, we have $\tau_{\leq n} Alg_{\mathcal{O}}(\mathcal{D}(k))^{cn} \simeq Alg_{\mathcal{O}}(\mathcal{D}(k)_{\geq 0} \cap \mathcal{D}(k)_{\leq n})$ and that Postnikov towers converge

$$Alg_{\mathcal{O}}(\mathcal{D}(k))^{cn} \simeq lim_n Alg_{\mathcal{O}}(\mathcal{D}(k)_{\geq 0} \cap \mathcal{D}(k)_{\leq n}) \tag{3.10.5}$$

[5]This model structure is left proper if k is a field

In particular, the heart $\mathcal{D}(k)^{\heartsuit} = \mathcal{D}(k)_{\geq 0} \cap \mathcal{D}(k)_{\leq 0}$ inherits a symmetric monoidal structure which we can identify with the classical tensor product of k-modules. In the associative (resp. commutative) case the category of algebras $\tau_{\leq 0} Alg_{\mathcal{A}ss}(\mathcal{D}(k))^{cn}$ (resp. $\tau_{\leq 0} CAlg(\mathcal{D}(k))^{cn}$) can be identified with the nerve of the classical category of associative (resp. commutative k-algebras). Moreover, since the map $\mathbb{H}_0 : \mathcal{C}_{\geq 0} \to \mathcal{C}^{\heartsuit}$ is monoidal, it extends to a map of algebras $\mathbb{H}_0 : Alg_{\mathcal{A}ss}(\mathcal{C})^{cn} \to Alg_{\mathcal{A}ss}(\mathcal{C}^{\heartsuit})$ so that if A is a connective associative (resp. commutative) algebra object in $\mathcal{D}(k)$, $\mathbb{H}_0(A)$ is an associative (resp. commutative) algebra in the classical sense.

As in the non-connective case, the theory of connective algebras admits a strictification result. More precisely, $Alg_{\mathcal{A}ss}(\mathcal{D}(k))^{cn}$ is equivalent to the underlying $(\infty,1)$-category SR_k of a simplicial model structure in the category of simplicial associative algebras over k, where the weak equivalences are the maps of simplicial algebras inducing a weak-equivalence between the underlying simplicial sets [100, 7.1.4.18].

Remark 3.10.3. As in 3.10.2, if k contains the field of rational numbers, $CAlg(\mathcal{D}(k))^{cn}$ is equivalent to the underlying $(\infty,1)$-category SCR_k of a simplicial model structure in the category of simplicial commutative k-algebras, with weak-equivalences given by the weak-equivalences between the underlying simplicial sets [100, 7.1.4.20]. In fact, the model structure for simplicial commutative algebras exists for any ring k and it can be proved that SCR_k is equivalent to the completion of the 1-category of commutative k-algebras of the form $k[X_1, ..., X_n]$, $n \geq 0$, under sifted colimits [97, 4.1.9].

Remark 3.10.4. The study of higher algebra over a commutative ring k can be understood as a small part of the much vaster subject of higher algebra in the $(\infty,1)$-category of spectra *Sp*. Indeed, we can understand a commutative ring k as 0-truncated connective commutative algebra object in $Sp^{\otimes}$ and using the same ideas as in [123] we can deduce an equivalence of stable presentable symmetric monoidal $(\infty,1)$-categories $Mod_k(Sp)^{\otimes} \simeq \mathcal{D}(k)^{\otimes}$ defined by sending a complex E to the mapping spectrum subjacent to $Map_{\mathcal{D}(k)}(k, E)$ (see [100, 7.1.2.6, 7.1.2.7, 7.1.2.13]). Moreover, the category of modules $Mod_k(Sp)$ inherits a left-complete t-structure induced from the one in *Sp* (see [100, 7.1.1.13]) and we can easily check that the formula $E \mapsto Map_{\mathcal{D}(k)}(k, E)$ is compatible with the t-structures. In particular, this implies that for any ∞-operad $\mathcal{O}^{\otimes}$, we have an equivalences $Alg_{\mathcal{O}}(\mathcal{D}(k)) \simeq Alg_{\mathcal{O}}(Sp)_{k/}$ and $Alg_{\mathcal{O}}(\mathcal{D}(k))^{cn} \simeq Alg_{\mathcal{O}}(Sp)^{cn}_{k/}$.

3.11 Cotangent Complexes and Square-Zero Extensions

Later on in Chapter 6 we construct a functor L_{pe} connecting the classical theory of schemes to the noncommutative world. One of the steps (see Prop. 6.3.8) requires the following noncommutative analogue of [144, Prop. 2.2.2.4] and [100, 7.4.3.18]:

Lemma 3.11.1. *Let A be an object in $Alg_{\mathcal{A}ss}(\mathcal{D}(k))^{cn}$. The following are equivalent:*

1) A is a ω-compact object in $Alg_{\mathcal{A}ss}(\mathcal{D}(k))$;

2) $\mathbb{H}_0(A)$ is a finitely presented associative algebra over k and the cotangent complex $\mathbb{L}_A$ is a compact object in $Mod_A^{\mathcal{A}ss}(\mathcal{D}(k))$;

In order to prove this we need to say what is the cotangent complex of a connective dg-algebra. This is a particular instance of a more general notion. Following [52] we recall how to define the *cotangent complex* of an $\mathcal{O}$-algebra-object in a stable symmetric monoidal $(\infty,1)$-category $\mathcal{C}^{\otimes}$ compatible with colimits.

Let $\mathcal{C}^{\otimes}$ be a stable symmetric monoidal $(\infty,1)$-category compatible with colimits. Let $\mathcal{O}^{\otimes}$ be a κ-small coherent ∞-operad and let $A \in Alg_{\mathcal{O}}(\mathcal{C})$ be an algebra-object in $\mathcal{C}$. Given a module-object $M \in Mod_A^{\mathcal{O}}(\mathcal{C})$ and using the hypothesis that the monoidal structure is compatible with colimits we expect that the direct sum $A \oplus M$ comes naturally equipped with the structure an $\mathcal{O}$-algebra-object in $\mathcal{C}$ where the multiplication is determined by

$$(A \oplus M) \otimes (A \oplus M) \simeq (A \otimes A) \oplus (A \otimes M) \oplus (A \otimes M) \oplus (M \otimes M) \to A \oplus M \tag{3.11.1}$$

where in the last step we use the multiplication $A \otimes A \to A$, the module action $A \otimes M \to M$ and the zero map $M \otimes M \to M$. This new $\mathcal{O}$-algebra-object comes naturally equipped with a morphism of $\mathcal{O}$-algebras $A \oplus M \to A$ which we can informally describe via the formula $(a, m) \to a$. Its fiber can be naturally identified with the module M. This construction should give rise to a functor

$$Mod_A^{\mathcal{O}}(\mathcal{C}) \to Alg_{\mathcal{O}}(\mathcal{C})_{./A} \tag{3.11.2}$$

In [52, Thm. 3.4.2] the author provides a precise way to perform this construction, proving that for any stable symmetric monoidal $(\infty, 1)$-category $\mathcal{C}^{\otimes}$ compatible with colimits and any coherent ∞-operad $\mathcal{O}^{\otimes}$, there is a canonical equivalence

$$Stab(Alg_{\mathcal{O}}(\mathcal{C})_{./A}) \simeq Fun_{\mathcal{O}}(\mathcal{O}, Mod_A^{\mathcal{O}}(\mathcal{C})) \tag{3.11.3}$$

for any $\mathcal{O}$-algebra A in $\mathcal{C}$ (see also [100, 7.3.4.13]). In particular, if the operad only has one color, we have an equivalence between the category of modules and the stabilization of algebras. Also in this case, this equivalence recovers the functor in (3.11.2) as the delooping functor Ω^{∞} (See Section 4.2.2 for an explanation of the notations).

By definition, a derivation of A into M is the data of a morphism of $\mathcal{O}$-algebras $A \to A \oplus M$ over A. It is an easy exercice to see that this notion recovers the classical definition using the Leibniz rule. We set $Der(A, M) := Map_{Alg_{\mathcal{O}}(\mathcal{C})_{./A}}(A, A \oplus M)$ to denote the space of derivations with values in M. The formula $M \mapsto Der(A, M)$ provides a functor $(Mod_A^{\mathcal{O}}(\mathcal{C}))^{op} \to \mathcal{S}$ which, through the Grothendieck construction, corresponds to a left fibration over $Mod_A^{\mathcal{O}}(\mathcal{C})$. By definition, the (absolute) *cotangent complex of* A is an object $\mathbb{L}_A \in Mod_A^{\mathcal{O}}(\mathcal{C})$ which makes this left fibration representable. In other words, it has the universal property

$$Map_{Mod_A^{\mathcal{O}}(\mathcal{C})}(\mathbb{L}_A, M) \simeq Map_{Alg_{\mathcal{O}}(\mathcal{C})_{./A}}(A, A \oplus M) \tag{3.11.4}$$

which allows us to understand the formula $A \mapsto \mathbb{L}_A$ as a left adjoint L_A to the functor in (3.11.2), evaluated in A. In particular, if $\mathcal{C}$ is presentable this left adjoint exists because of the adjoint functor theorem together with the fact that (3.11.2) commutes with limits [52, Lemma 3.1.3]. Moreover, under the equivalence between modules and the stabilization of algebras, L_A can be identified with the suspension functor Σ^{∞}.

Example 3.11.2. When applied to the example $\mathcal{C}^{\otimes} = \mathcal{D}(k)^{\otimes}$ and for $\mathbb{E}_1 \simeq \mathcal{A}ss$, this definition recovers the classical associative cotangent complex introduced by Quillen and studied in [91], given by the kernel of the multiplication map $A \otimes_k A^{op} \to A$ in the $(\infty, 1)$-category $Mod_A^{\mathcal{A}ss}(\mathcal{D}(k))$. Recall also that $Mod_A^{\mathcal{A}ss}(\mathcal{D}(k))$ is equivalent to ${}_ABMod_A(\mathcal{D}(k))$ which by the discussion in 3.9.2 is equivalent to the underlying $(\infty, 1)$-category of the model category of strict A-bimodules in the model category of complexes $Ch(k)$. This example will play an important role later on.

Remark 3.11.3. The notion of cotangent complex is well-behaved with respect to base-change. If $f : A \to A'$ is a morphism of $\mathcal{O}$-algebras we can put together the functors $A \oplus -$ and $A' \oplus -$ in a diagram

$$\begin{tikzcd} Mod_A^{\mathcal{O}}(\mathcal{C}) \arrow[r, "A\oplus -"] & Alg_{\mathcal{O}}(\mathcal{C})_{./A} \\ Mod_{A'}^{\mathcal{O}}(\mathcal{C}) \arrow[u, "For"] \arrow[r, "A'\oplus -"] & Alg_{\mathcal{O}}(\mathcal{C})_{./A'} \arrow[u, "(- \times_{A'} A)"'] \end{tikzcd} \tag{3.11.5}$$

where For is the map that considers an A'-module as an A-modules via f and the map $(- \times'_A A)$ is obtained by computing the fiber product of a morphism $C \to A'$ with respect to f. The fact that this diagram commutes follows from the equivalence relating modules and the stabilization of algebras and from the definition of *tangent bundle* studied in [100, Section 8.3.1]. Moreover, the commutativity of this diagram implies the commutativity of the diagram associated to the left adjoints

$$\begin{tikzcd} Mod_A^{\mathcal{O}}(\mathcal{C}) \arrow[d, "A'\otimes_A -"] & Alg_{\mathcal{O}}(\mathcal{C})_{./A} \arrow[l, "L_A"'] \arrow[d, "f\circ -"] \\ Mod_{A'}^{\mathcal{O}}(\mathcal{C}) & Alg_{\mathcal{O}}(\mathcal{C})_{./A'} \arrow[l, "L_{A'}"] \end{tikzcd} \tag{3.11.6}$$

where now $A' \otimes_A -$ is the base change with respect to f and the $(f \circ -)$ is the map obtained by composing with f. In particular, we find that $A' \otimes_A \mathbb{L}_A$ is equivalent to $L_{A'}$ evaluated at $f : A \to A'$.

Remark 3.11.4. The notion of cotangent complex has a relative version. For any $\mathcal{O}$-algebra R in $\mathcal{C}$, the $(\infty,1)$-category $Mod_R^{\mathcal{O}}(\mathcal{C})$ is again a stable symmetric monoidal $(\infty,1)$-category compatible with colimits. In particular, under the equivalence $Alg_{\mathcal{O}}(Mod_R^{\mathcal{O}}(\mathcal{C})) \simeq Alg_{\mathcal{O}}(\mathcal{C})_{R/.}$, for any R-algebra $f : R \to A$ the previous discussion provides a functor

$$Mod_A^{\mathcal{O}}(\mathcal{C}) \simeq Mod_A^{\mathcal{O}}(Mod_R^{\mathcal{O}}(\mathcal{C})) \to Alg_{\mathcal{O}}(Mod_R^{\mathcal{O}}(\mathcal{C}))_{./A} \simeq (Alg_{\mathcal{O}}(\mathcal{C})_{R/.})_{./A} \tag{3.11.7}$$

sending a A-module M to the R-algebra $A \oplus M$ defined over A. The *relative cotangent complex* of $f : R \to A$ is by definition the absolute cotangent complex of f as an algebra-object in $Alg_{\mathcal{O}}(Mod_R^{\mathcal{O}}(\mathcal{C})) \simeq (Alg_{\mathcal{O}}(\mathcal{C})_{R/.})_{./A}$. This definition recovers the absolute version when R is the unit object. In what follows we will only need the absolute case.

Remark 3.11.5. In [52, Theorem 3.1.10] the author provides a characterization of $\mathbb{L}_A$ for any $\mathbb{E}_n$-algebra A in a stable presentable symmetric monoidal $(\infty,1)$-category $\mathcal{C}^{\otimes}$ such that $\mathcal{C}$ is generated under small colimits by the unit: $\Sigma^n(\mathbb{L}_A)$ is the cofiber of the canonical map $Free(1) \to A$ in $Mod_A^{\mathbb{E}_n}(\mathcal{C})$, where 1 is the unit of the monoidal structure and $Free : \mathcal{C} \to Mod_A^{\mathbb{E}_n}(\mathcal{C})$ is the left adjoint to the forgetful functor $Mod_A^{\mathbb{E}_n}(\mathcal{C}) \to \mathcal{C}$. This adjoint exists because colimits of modules are computed in $\mathcal{C}$ (See also [100, Theorem 7.3.5.1]).

The notion of derivation can also be presented using the idea of a *square-zero extension.* If $d : A \to A \oplus M$ is a derivation, we fabricate a new $\mathcal{O}$-algebra $\tilde{A}$ as the pullback in $Alg_{\mathcal{O}}(\mathcal{C})$

$$\begin{tikzcd} \tilde{A} \arrow[r, "f"] \arrow[d] & A \arrow[d, "d"] \\ A \arrow[r, "d_0"] & A \oplus M \end{tikzcd} \tag{3.11.8}$$

where $d_0 : A \to A \oplus M$ is the zero derivation $a \mapsto (a, 0)$. Since the functor $Alg_{\mathcal{O}}(\mathcal{C}) \to \mathcal{C}$ preserves limits, the diagram (3.11.8) provides a pullback diagram in $\mathcal{C}$ and given a morphism $* \to A$ in $\mathcal{C}$, we can identify the fiber $\tilde{A} \times_A *$ in $\mathcal{C}$ with the loop $\Omega(M)$. Indeed, we have a pullback in $\mathcal{C}$

$$\begin{tikzcd} \tilde{A} \times_A * \arrow[r, "f"] \arrow[d] & (A \times_A *) \simeq * \arrow[d, "d"] \\ * \simeq (A \times_A *) \arrow[r, "d_0"] & (A \oplus M) \times_A * \end{tikzcd} \tag{3.11.9}$$

and since the fiber of the canonical map $A \oplus M \to A$ can be identified with M, we find $\tilde{A} \times_A * \simeq \Omega(M)$.

A morphism of algebras $B \to A$ is said to be a *square-zero extension of A by $\Omega(M)$* if there is a derivation d of A with values in $M \simeq \Sigma(\Omega(M))$ such that $B \simeq \tilde{A}$. Thanks to [100, Theorem 7.4.1.26] if $\mathcal{C}^{\otimes}$ is a stable presentable $\mathbb{E}_k$-monoidal $(\infty, 1)$-category with a compatible t-structutre, then the formula $(A \to A \oplus M) \mapsto (f : \tilde{A} \to A)$ establishes an equivalence between the theory of derivations and the subcategory of $Fun(\Delta[1], Alg_{\mathbb{E}_k}(\mathcal{C}))$ spanned by the square-zero extensions (see [100, Section 7.4.1] for a precise formulation).

Remark 3.11.6. In the presence of a square-zero extension (3.11.8), every $\mathcal{O}$-algebra B induces a pullback diagram of spaces.

$$\begin{tikzcd} Map_{Alg_{\mathcal{O}}(\mathcal{C})}(B, \tilde{A}) \arrow[r] \arrow[d] & Map_{Alg_{\mathcal{O}}(\mathcal{C})}(B, A) \arrow[d] \\ Map_{Alg_{\mathcal{O}}(\mathcal{C})}(B, A) \arrow[r] & Map_{Alg_{\mathcal{O}}(\mathcal{C})}(B, A \oplus M) \end{tikzcd} \tag{3.11.10}$$

Let $\phi : B \to A$ be a morphism of algebras. It follows that we can describe the fiber of the morphism $Map_{Alg_{\mathcal{O}}(\mathcal{C})}(B, \tilde{A}) \to Map_{Alg_{\mathcal{O}}(\mathcal{C})}(B, A)$ over the point corresponding to ϕ with the help of the cotangent complex of B. More precisely, we observe first that the mapping space $Map_{Alg_{\mathcal{O}}(\mathcal{C})_{./A}}(B, A \oplus M)$ (where B is defined over A via ϕ) fits in a pullback diagram

$$\begin{tikzcd} Map_{Alg_{\mathcal{O}}(\mathcal{C})_{./A}}(B, A \oplus M) \arrow[r] \arrow[d] & Map_{Alg_{\mathcal{O}}(\mathcal{C})}(B, A \oplus M) \arrow[d] \\ \Delta[0] \arrow[r, "\phi"] & Map_{Alg_{\mathcal{O}}(\mathcal{C})}(B, A) \end{tikzcd} \tag{3.11.11}$$

where the right vertical map is the composition with the canonical map $A \oplus M \to A$. By tensoring with $(- \times_{Map_{Alg_{\mathcal{O}}(\mathcal{C})}(B,A)} \Delta[0])$ the diagram (3.11.10) produces a new pullback diagram

$$\begin{tikzcd} Map_{Alg_{\mathcal{O}}(\mathcal{C})}(B, \tilde{A}) \times_{Map_{Alg_{\mathcal{O}}(\mathcal{C})}(B,A)} \Delta[0] \arrow[r] \arrow[d] & Map_{Alg_{\mathcal{O}}(\mathcal{C})}(B, A) \times_{Map_{Alg_{\mathcal{O}}(\mathcal{C})}(B,A)} \Delta[0] \arrow[d] \\ Map_{Alg_{\mathcal{O}}(\mathcal{C})}(B, A) \times_{Map_{Alg_{\mathcal{O}}(\mathcal{C})}(B,A)} \Delta[0] \arrow[r] & Map_{Alg_{\mathcal{O}}(\mathcal{C})_{./A}}(B, A \oplus M) \end{tikzcd} \tag{3.11.12}$$

with

$$Map_{Alg_{\mathcal{O}}(\mathcal{C})}(B, A) \times_{Map_{Alg_{\mathcal{O}}(\mathcal{C})}(B,A)} \Delta[0] \simeq \Delta[0] \tag{3.11.13}$$

so that the fiber $Map_{Alg_{\mathcal{O}}(\mathcal{C})}(B, \tilde{A}) \times_{Map_{Alg_{\mathcal{O}}(\mathcal{C})}(B,A)} \Delta[0]$ becomes the space of paths in

$$Map_{Alg_{\mathcal{O}}(\mathcal{C})_{./A}}(B, A \oplus M) \tag{3.11.14}$$

between the point $B \xrightarrow{\phi} A \xrightarrow{d} A \oplus M$ and the point $B \xrightarrow{\phi} A \xrightarrow{d_0} A \oplus M$. To conclude, we can use the adjunctions of the Remark 3.11.3 to find equivalences

$$Map_{Alg_{\mathcal{O}}(\mathcal{C})_{./A}}(B, A \oplus M) \simeq Map_{Mod_A^{\mathcal{O}}(\mathcal{C})}(L_A(\phi), M) \simeq Map_{Mod_A^{\mathcal{O}}(\mathcal{C})}(A \otimes_B \mathbb{L}_B, M) \tag{3.11.15}$$

$$\simeq Map_{Mod_B^{\mathcal{O}}(\mathcal{C})}(\mathbb{L}_B, For(M))$$

which combined, provide

$$Map_{Alg_{\mathcal{O}}(\mathcal{C})}(B,\tilde{A}) \times_{Map_{Alg_{\mathcal{O}}(\mathcal{C})}(B,A)} \Delta[0] \simeq \Omega_{0,d\circ\phi} Map_{Mod_B^{\mathcal{O}}(\mathcal{C})}(\mathbb{L}_B, For(M)) \tag{3.11.16}$$

In particular we identify an obstruction to the existence of liftings: if the point corresponding to $d\circ\phi \in Map_{Mod_B^{\mathcal{O}}(\mathcal{C})}(\mathbb{L}_B, For(M))$ is not in the same connected component of the point corresponding to the zero derivation, there wont't be any liftings.

We now collect the last ingredient to prove the Lemma 3.11.1:

Theorem 3.11.7. *(Lurie [100, Corollary 7.4.1.28]) Let $\mathcal{C}^{\otimes}$ be a stable presentable symmetric monoidal $(\infty,1)$-category equipped with a compatible t-structure (in the sense of 3.8.2). Then for every $k \geq 0$ and any algebra $A \in Alg_{\mathbb{E}_k}(\mathcal{C})^{cn}$ the morphisms in the Postnikov tower*

$$... \to \tau_{\leq 2}A \to \tau_{\leq 1}A \to \tau_{\leq 0}A \tag{3.11.17}$$

are square-zero extensions. More precisely, and following the Remark 2.1.14, for every $n \geq 0$ the truncation map $\tau_{\leq n}A \to \tau_{\leq n-1}A$ is a square-zero extension of $\tau_{\leq n-1}A$ by a module-structure in $\mathbb{H}_n(A)[n]$. This is equivalent to the existence of a derivation $d_n : \tau_{\leq n-1}A \to \tau_{\leq n-1}A \oplus \mathbb{H}_n(A)[n+1]$ and a pullback diagram of algebras

$$\begin{tikzcd} \tau_{\leq n}A \arrow[r] \arrow[d] & \tau_{\leq n-1}A \arrow[d, "d_n"] \\ \tau_{\leq n-1}A \arrow[r] & \tau_{\leq n-1}A \oplus \mathbb{H}_n(A)[n+1] \end{tikzcd} \tag{3.11.18}$$

We have now all the ingredients to prove the lemma.

Proof of the Lemma 3.11.1:
We follow the same methods as in [144, Prop. 2.2.2.4]. We first prove that 1) implies 2).

The fact that $\mathbb{H}_0(A)$ is finitely presented as an associative algebra follows from the fact that $\mathbb{H}_0$ commutes with colimits (it is a left adjoint), together with the fact that π_0 commutes with colimits in the $(\infty,1)$-category of spaces. The fact that $\mathbb{L}_A$ is compact follows from the universal property of the cotangent complex together with the following facts:

i) As explained before, the functor $(A \oplus -)$ of (3.11.2) can be identified with a delooping functor Ω^{∞}. Therefore it commutes with filtered colimits;

ii) by assumption, A is compact.

We now prove that 2) implies 1). To start with, we observe that since A is by assumption connective, it is enough to check that A is compact in the full subcategory $Alg_{\mathcal{A}ss}(\mathcal{D}(k))^{cn}$ spanned by the connective objects. Indeed, recall from 3.8.2 that the truncation functor $\tau_{\leq 0}$ is a right adjoint to the inclusion $Alg_{\mathcal{A}ss}(\mathcal{D}(k))^{cn} \subseteq Alg_{\mathcal{A}ss}(\mathcal{D}(k))$. We can easily check that $\tau_{\leq 0}$ commutes with filtered colimits (because the homology groups commute with filtered colimits) so that for any filtered system $\{\mathcal{C}_i\}_{i\in I}$ in $Alg_{\mathcal{A}ss}(\mathcal{D}(k))$ we have

$$Map_{Alg_{\mathcal{A}ss}(\mathcal{D}(k))}(A, colim_I C_i) \simeq Map_{Alg_{\mathcal{A}ss}(\mathcal{D}(k))^{cn}}(A, \tau_{\leq 0} colim_I C_i) \tag{3.11.19}$$

$$\simeq Map_{Alg_{\mathcal{A}ss}(\mathcal{D}(k))^{cn}}(A, colim_I \tau_{\leq 0} C_i)$$

so that A is compact in $Alg_{\mathcal{A}ss}(\mathcal{D}(k))^{cn}$ if and only if it is compact in $Alg_{\mathcal{A}ss}(\mathcal{D}(k))$.

We start now by proving that A is almost compact, meaning that A is compact with respect to any filtered system in $Alg_{\mathcal{A}ss}(\mathcal{D}(k))^{cn}_{\leq n}$, for every $n \geq 0$. We proceed by induction. The case $n = 0$ follows by the hypothesis. Let us suppose we know this is true for $n-1$ and prove it for n. Let $\{C_i\}_{i\in I}$ be a filtered system in $Alg_{\mathcal{A}ss}(\mathcal{D}(k))^{cn}_{\leq n}$. The discussion in 3.10 together with the Theorem 3.11.7 implies that for each i, C_i admits a Postnikov decomposition

$$C_i = \tau_{\leq n}(C_i) \to \tau_{\leq n-1}(C_i) \to ... \to \tau_{\leq 0}(C_i) \tag{3.11.20}$$

where each morphism is a square-zero extension providing a pullback diagram

$$\begin{tikzcd} C_i = (C_i)_{\leq n} \arrow[r] \arrow[d] & (C_i)_{\leq n-1} \arrow[d, "d_n"] \\ (C_i)_{\leq n-1} \arrow[r] & (C_i)_{\leq n-1} \oplus \mathbb{H}_n(C_i)[n+1] \end{tikzcd} \tag{3.11.21}$$

in $Alg_{\mathcal{A}ss}(\mathcal{D}(k))^{cn}$ where the lower horizontal map is the zero map and right vertical map corresponds to the canonical derivation d_n associated to the square-zero extension $C_i \to \tau_{\leq n-1}C_i$. This diagram induces a pullback diagram of spaces

$$\begin{tikzcd} Map_{Alg_{\mathcal{A}ss}(\mathcal{D}(k))}(A, C_i) \arrow[r] \arrow[d] & Map_{Alg_{\mathcal{A}ss}(\mathcal{D}(k))}(A, \tau_{\leq n-1}(C_i)) \arrow[d] \\ Map_{Alg_{\mathcal{A}ss}(\mathcal{D}(k))}(A, (C_i)_{\leq n-1}) \arrow[r] & Map_{Alg_{\mathcal{A}ss}(\mathcal{D}(k))}(A, \tau_{\leq n-1}(C_i) \oplus \mathbb{H}_n(C_i)[n+1]) \end{tikzcd} \tag{3.11.22}$$

and the Remark 3.11.6 implies that the fiber of the map

$$Map_{Alg_{\mathcal{A}ss}(\mathcal{D}(k))}(A, C_i) \longrightarrow Map_{Alg_{\mathcal{A}ss}(\mathcal{D}(k))}(A, \tau_{\leq n-1}(C_i)) \tag{3.11.23}$$

over a map $u : A \to \tau_{\leq n-1}(C_i)$ is given by the space of paths in $Map_{Mod_A^{\mathcal{A}ss}}(\mathbb{L}_A, \mathbb{H}_n(C_i)[n+1])$ between the zero derivation and the point corresponding to the composition $d_n \circ u$. This reduces everything to the analysis of the diagram

$$\begin{tikzcd} colim_I \Omega_{0,d_n\circ u} Map_{Mod_A^{\mathcal{A}ss}(\mathcal{D}(k))}(\mathbb{L}_A, \mathbb{H}_n(C_i)[n+1]) \arrow[r] \arrow[d] & \Omega_{0,d_n\circ u} Map_{Mod_A^{\mathcal{A}ss}(\mathcal{D}(k))}(\mathbb{L}_A, \mathbb{H}_n(colim_I C_i)[n+1]) \arrow[d] \\ colim_I Map_{Alg_{\mathcal{A}ss}(\mathcal{D}(k))}(A, C_i) \arrow[r] \arrow[d] & Map_{Alg_{\mathcal{A}ss}(\mathcal{D}(k))}(A, colim_I C_i) \arrow[d] \\ colim_I Map_{Alg_{\mathcal{A}ss}(\mathcal{D}(k))}(A, \tau_{\leq n-1}(C_i)) \arrow[r] & Map_{Alg_{\mathcal{A}ss}(\mathcal{D}(k))}(A, \tau_{\leq n-1}(colim_I C_i)) \end{tikzcd} \tag{3.11.24}$$

We observe that

a) The left column is a fiber sequence because filtered colimits are exact in the $(\infty,1)$-category of spaces. For the same reason, there is an equivalence between the top left entry in the diagram and

$$\Omega_{0,d_n\circ u} colim_I Map_{Mod_A^{\mathcal{A}ss}(\mathcal{D}(k))}(\mathbb{L}_A, \mathbb{H}_n(C_i)[n+1]) \tag{3.11.25}$$

b) The right column is also a fiber sequence. This follows from the result of 3.11.7 and the Remark 3.11.6 applied to the colimit algebra $colim_I C_i$;

c) The top entry on the right is equivalent to

$$\Omega_{0,d_n\circ u} Map_{Mod_A^{\mathcal{A}ss}(\mathcal{D}(k))}(\mathbb{L}_A, colim_I \mathbb{H}_n(C_i)[n+1]) \tag{3.11.26}$$

.

This is because the functor $\mathbb{H}_n$ is equivalent to the classical nth-homology functor and therefore commutes with filtered colimits.

Finally, the induction hypothesis together with the fact that $(-)_{\leq n}$ is a left adjoint (and therefore commutes with colimits), implies that the lower horizontal arrow is an equivalence. The assumption that $\mathbb{L}_A$ is compact implies that the top horizontal map is also an equivalence. It follows that the middle one is also an equivalence. This proves that A is almost compact in $Alg_{\mathcal{A}ss}(\mathcal{D}(k))^{cn}$.

We now complete the proof by showing that A is compact. Since the $(\infty,1)$-category $Mod_A^{\mathcal{A}ss}((\mathcal{D}(k)))$ is equivalent to the underlying $(\infty,1)$-category of the model structure on strict A-bimodules in $Ch(k)$ (see 3.11.2) and the last is compactly generated in the sense of 2.2.2, the Proposition 2.2.3 implies that $\mathbb{L}_A$ is a compact object in $Mod_A^{\mathcal{A}ss}((\mathcal{D}(k)))$ if and only if it is given by a finite strict cell object in the model category of bimodules. In this case, with our hypothesis that $\mathbb{L}_A$ is compact, we can find a natural number $n_0 \geq 0$ such that for any object $M \in Mod_A^{\mathcal{A}ss}(\mathcal{D}(k))$ concentrated in degrees strictly bigger than n_0 we have $\pi_0 Map_{Mod_A^{\mathcal{A}ss}(\mathcal{D}(k))}(\mathbb{L}_A, M) \simeq 0$. In particular, for any connective algebra C, the combination of the fiber sequence of the Remark 3.11.6 and the Theorem 3.11.7 implies that homotopy classes of maps $A \to C$ are in bijection with homotopy classes of maps $A \to \tau_{\leq n_0}(C)$, In other words, we have

$$\pi_0 Map_{Alg_{\mathcal{A}ss}(\mathcal{D}(k))}(A, C) \simeq \pi_0 Map_{Alg_{\mathcal{A}ss}(\mathcal{D}(k))}(A, \tau_{\leq n_0}(C)) \tag{3.11.27}$$

We now use this to showing that A is compact. Let $\{C_i\}_{i\in I}$ be a filtered system in $Alg_{\mathcal{A}ss}(\mathcal{D}(k))^{cn}$. Using the fact that π_n commutes with filtered homotopy colimits of spaces and that $Alg_{\mathcal{A}ss}(\mathcal{D}(k))^{cn}$ admits all limits (it is a co-reflexive localization of $Alg_{\mathcal{A}ss}(\mathcal{D}(k))$), we are reduced to showing that the natural map

$$colim_I \pi_0 Map_{Alg_{\mathcal{A}ss}(\mathcal{D}(k))}(A, \Omega^n C_i) \to \pi_0 Map_{Alg_{\mathcal{A}ss}(\mathcal{D}(k))}(A, colim_I \Omega^n C_i) \tag{3.11.28}$$

is an equivalence. We show that the formula is true for any filtered system of algebras $\{U_i\}_{i\in I}$, because we have a commutative diagram

$$\begin{tikzcd} colim_I \pi_0 Map_{Alg_{\mathcal{A}ss}(\mathcal{D}(k))}(A, U_i) \arrow[r] \arrow[d, "\sim"] & \pi_0 Map_{Alg_{\mathcal{A}ss}(\mathcal{D}(k))}(A, colim_I U_i) \arrow[d, "\sim"] \\ colim_I \pi_0 Map_{Alg_{\mathcal{A}ss}(\mathcal{D}(k))}(A, \tau_{\leq n_0}(U_i)) \arrow[r, "\sim"] & \pi_0 Map_{Alg_{\mathcal{A}ss}(\mathcal{D}(k))}(A, colim_I \tau_{\leq n_0}(U_i)) \end{tikzcd} \tag{3.11.29}$$

where the vertical arrows are equivalences because of (3.11.27) together with fact that $\tau_{\leq n_0}$ is a left adjoint, and the lower horizontal map is an equivalence because A is almost compact. This concludes the proof.

□

This completes our preliminaries

Part I

- Universal Property of the Motivic Stable Homotopy Theory of Schemes

Chapter **4**

Inversion of an Object in a Symmetric Monoidal $(\infty, 1)$-category and the Relation with Symmetric Spectrum Objects

In 4.1 and following some ideas of [144], we deal with the formal inversion of an object in a symmetric monoidal $(\infty, 1)$-category. First we deal with the situation for small $(\infty, 1)$-categories (Propositions 4.1.1 and 4.1.2) and then we extend the result to the presentable setting (Prop. 4.1.11). This method allow us to invert any object and the result is endowed with the expected universal property. In 4.2 we deal with the notion of spectrum-objects. These can be defined either via a limit kind of construction or via a colimit. When applied to a presentable $(\infty, 1)$-category both methods coincide. Still in this section, we recall a classical theorem (see [150]) which says that, under a certain symmetric condition on X, the formal inversion of an object in a symmetric monoidal category is equivalent to the standard 1-category of spectra with respect to X. In the Corollary 4.2.13 we prove that this results also holds in the ∞-setting: if the object we want to invert satisfies the symmetry condition then the underlying $(\infty, 1)$-category of the formal inversion is nothing but the stabilization with respect to the chosen object.

Finally, in 4.3 we use the results of [71] to compare our formal inversion to the more familiar notion of symmetric spectra. Our main result (Theorem 4.3.1) ensures that the construction of symmetric spectrum objects with respect to a given symmetric object X together with the convolution product, is the "model category" incarnation of our ∞-categorical phenomenon of inverting X.

4.1 Formal inversion of an object in a Symmetric Monoidal $(\infty, 1)$-category

Let $\mathcal{C}^{\otimes}$ be a symmetric monoidal $(\infty, 1)$-category and let X be an object in $\mathcal{C}$. We will say that X is invertible with respect to the monoidal structure if there is an object X^* such that $X \otimes X^*$ and $X^* \otimes X$ are both equivalent to the unit object. Since the monoidal structure is symmetric, it is enough to have one of these conditions. It is an easy observation that this condition depends only on the monoidal structure induced on the homotopy category $h(\mathcal{C})$, because equivalences are exactly the isomorphisms in $h(\mathcal{C})$. Alternatively, we can see that an object X in $\mathcal{C}$ is invertible if and only if the map "multiplication by X" $= (X \otimes -) : \mathcal{C} \to \mathcal{C}$ is an equivalence of $(\infty, 1)$-categories. Indeed, if X has an inverse X^* then the maps $(X \otimes -)$ and $(X^* \otimes -)$ are inverses since the coherences of the monoidal structure can be used to fabricate the homotopies. Conversely, if $(X \otimes -)$ is an equivalence, the essential surjectivity provides an object X^* such that $X \otimes X^* \simeq \mathbf{1}_{\mathcal{C}}$. The symmetry provides an equivalence $\mathbf{1}_{\mathcal{C}} \simeq X^* \otimes X$.

Our main goal is to produce from the data of $\mathcal{C}^{\otimes}$ and $X \in \mathcal{C}$, a new symmetric monoidal $(\infty, 1)$-category $\mathcal{C}^{\otimes}[X^{-1}]$ together with a monoidal map $\mathcal{C}^{\otimes} \to \mathcal{C}^{\otimes}[X^{-1}]$ sending X to an invertible object and universal with respect to this property. In addition, we would like this construction to hold within the world of presentable symmetric monoidal $(\infty, 1)$-categories. Our steps follow the original ideas of [144], where the authors studied the inversion of an element in a strictly commutative algebra object in a symmetric monoidal model category.

We start by analyzing the theory for a small symmetric monoidal $(\infty, 1)$-category $\mathcal{C}^{\otimes}$. In this case, and following the discussion in 3.2.2, $\mathcal{C}^{\otimes}$ can be identified with an object in $CAlg(Cat_{\infty})$. The objects of $Mod_{\mathcal{C}}(Cat_{\infty})$ can be identified with $(\infty, 1)$-categories endowed with an "action" of $\mathcal{C}$ and we will refer to them simply as $\mathcal{C}^{\otimes}$-*Modules*. By the Proposition 3.3.1, $CAlg(Mod_{\mathcal{C}^{\otimes}}(Cat_{\infty}))$ is equivalent to $CAlg(Cat_{\infty})_{\mathcal{C}^{\otimes}/.}$ where the objects are small symmetric monoidal $(\infty, 1)$-categories $\mathcal{D}^{\otimes}$ equipped with a monoidal map $\mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$. We denote by $CAlg(Cat_{\infty})^{X}_{\mathcal{C}^{\otimes}/.}$ the full subcategory of $CAlg(Cat_{\infty})_{\mathcal{C}^{\otimes}/}$ spanned by the algebras $\mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ whose structure map sends X to an invertible object. The main observation is that the objects in $CAlg(Cat_{\infty})^{X}_{\mathcal{C}^{\otimes}/.}$ can be understood as local objects in $CAlg(Cat_{\infty})_{\mathcal{C}^{\otimes}/.}$ with respect to a certain set of morphisms: there is a forgetful functor

$$CAlg(Cat_{\infty})_{\mathcal{C}^{\otimes}/.} \simeq CAlg(Mod_{\mathcal{C}^{\otimes}}(Cat_{\infty})) \to Mod_{\mathcal{C}^{\otimes}}(Cat_{\infty}) \tag{4.1.1}$$

and since $Cat_{\infty}^{\times}$ is a presentable symmetric monoidal $(\infty, 1)$-category, this functor admits a left adjoint $Free_{\mathcal{C}^{\otimes}}(-)$ assigning to each $\mathcal{C}^{\otimes}$-module $\mathcal{D}$ the free commutative $\mathcal{C}^{\otimes}$-algebra generated by $\mathcal{D}$ (see [100, 3.1.3.5] and our survey in the previous chapter). We will denote by $\mathcal{S}_X$ the collection of morphisms in $CAlg(Cat_{\infty})_{\mathcal{C}^{\otimes}/.}$ consisting of the single morphism

$$Free_{\mathcal{C}^{\otimes}}(\mathcal{C}) \xrightarrow{Free_{\mathcal{C}^{\otimes}}(X \otimes -)} Free_{\mathcal{C}^{\otimes}}(\mathcal{C}) \tag{4.1.2}$$

where $\mathcal{C}$ is understood as a $\mathcal{C}^{\otimes}$-module in the obvious way using the monoidal structure. We prove the following

Proposition 4.1.1. *Let $\mathcal{C}^{\otimes}$ be a symmetric monoidal $(\infty, 1)$-category. Then the full subcategory $CAlg(Cat_{\infty})^{X}_{\mathcal{C}^{\otimes}/.}$ coincides with the full subcategory of $CAlg(Cat_{\infty})_{\mathcal{C}^{\otimes}/.}$ spanned by the $\mathcal{S}_X$-local objects. Moreover, since $Cat_{\infty}^{\times}$ is a presentable symmetric monoidal $(\infty, 1)$-category, the $(\infty, 1)$-categories $CAlg(Cat_{\infty})$ and $CAlg(Cat_{\infty})_{\mathcal{C}^{\otimes}/.}$ are also presentable (see Corollary 3.2.3.5 of [100]) and the results of the Proposition 5.5.4.15 in [99] follow. We deduce the existence a left adjoint $\mathcal{L}^{\otimes}_{(\mathcal{C}^{\otimes}, X)}$*

$$\begin{tikzcd} CAlg(Cat_{\infty})^{\mathcal{S}_X-local}_{\mathcal{C}^{\otimes}/} = CAlg(Cat_{\infty})^{X}_{\mathcal{C}^{\otimes}/} \arrow[r, hook] & CAlg(Cat_{\infty})_{\mathcal{C}^{\otimes}/} \arrow[l, bend right, "\mathcal{L}^{\otimes}_{(\mathcal{C}^{\otimes}, X)}"'] \end{tikzcd} \tag{4.1.3}$$

In particular, the data of this adjunction provides the existence of a symmetric monoidal $(\infty, 1)$-category $\mathcal{L}^{\otimes}_{(\mathcal{C}^{\otimes}, X)}(\mathcal{C}^{\otimes})$ equipped with a canonical monoidal map $f : \mathcal{C}^{\otimes} \to \mathcal{L}^{\otimes}_{(\mathcal{C}^{\otimes}, X)}(\mathcal{C}^{\otimes})$ sending X to an invertible object.

Proof. The only thing to check is that both subcategories coincide. Let $\phi : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ be a $\mathcal{C}$-algebra where X is sent to an invertible object. By the definition of the functor $Free_{\mathcal{C}^{\otimes}}(\mathcal{C})$ we have a commutative diagram

$$\begin{tikzcd} Map_{CAlg(Cat_{\infty})_{\mathcal{C}^{\otimes}/}}(Free_{\mathcal{C}^{\otimes}}(\mathcal{C}), \mathcal{D}^{\otimes}) \arrow[r] \arrow[d, "\sim"] & Map_{CAlg(Cat_{\infty})_{\mathcal{C}^{\otimes}/}}(Free_{\mathcal{C}^{\otimes}}(\mathcal{C}), \mathcal{D}^{\otimes}) \arrow[d, "\sim"] \\ Map_{Mod_{\mathcal{C}^{\otimes}}(Cat_{\infty})}(\mathcal{C}, \mathcal{D}) \arrow[r] & Map_{Mod_{\mathcal{C}^{\otimes}}(Cat_{\infty})}(\mathcal{C}, \mathcal{D}) \end{tikzcd} \tag{4.1.4}$$

where the lower horizontal map is described by the formula $\alpha \mapsto \alpha \circ (X \otimes -)$. Since ϕ is monoidal, the diagram commutes

$$\begin{tikzcd} \mathcal{C} \arrow[r, "(X\otimes -)"] \arrow[d, "\phi"] & \mathcal{C} \arrow[d, "\phi"] \\ \mathcal{D} \arrow[r, "(\phi(X)\otimes -)"'] & \mathcal{D} \end{tikzcd} \tag{4.1.5}$$

and the lower map is in fact homotopic to the one given by the formula $\alpha \mapsto (\phi(X) \otimes -) \circ \alpha$. Since $\phi(X)$ is invertible in $\mathcal{D}^{\otimes}$, there exists an object λ in $\mathcal{D}$ such that the maps $\phi(X) \otimes -)$ and $(\lambda \otimes -)$ are inverses and therefore the lower map in (4.1.4), and as a consequence the top map, are isomorphisms of homotopy types.

Let now $\mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ be a $\mathcal{C}^{\otimes}$-algebra, local with respect to $\mathcal{S}_X$. In particular, the map

$$Map_{Mod_{\mathcal{C}^{\otimes}}(Cat_{\infty})}(\mathcal{C}, \mathcal{D}) \to Map_{Mod_{\mathcal{C}^{\otimes}}(Cat_{\infty})}(\mathcal{C}, \mathcal{D}) \tag{4.1.6}$$

induced by the composition with $(X \otimes -)$ is an isomorphism of homotopy types and in particular we have $\pi_0(Map_{Mod_{\mathcal{C}^{\otimes}}(Cat_{\infty})}(\mathcal{C}, \mathcal{D})) \simeq \pi_0(Map_{Mod_{\mathcal{C}^{\otimes}}(Cat_{\infty})}(\mathcal{C}, \mathcal{D}))$. We deduce the existence of a dotted arrow

$$\begin{tikzcd} \mathcal{D} \arrow[r, "X\otimes -"] \arrow[d, "\phi"] & \mathcal{D} \arrow[dl, dotted, "\alpha"] \\ \mathcal{D} & \end{tikzcd} \tag{4.1.7}$$

rendering the diagram of modules commutative and since α is a map of $\mathcal{C}^{\otimes}$-modules and ϕ is monoidal we find $\phi(1) \simeq \alpha(X \otimes 1) \simeq \phi(X) \otimes \alpha(1)$. Using the symmetry we find that $\alpha(1 \otimes X) \simeq \alpha(1) \otimes \phi(X) \simeq 1$ which proves that $\phi(X)$ has an inverse in $\mathcal{D}^{\otimes}$. $\square$

We will now study the properties of the base change along the morphism $\mathcal{C}^{\otimes} \to \mathcal{L}^{\otimes}_{(\mathcal{C}^{\otimes},X)}(\mathcal{C}^{\otimes})$. In order to establish some insight, let us point out that everything fits in a commutative diagram

$$\begin{tikzcd} CAlg(Cat_{\infty})_{\mathcal{L}^{\otimes}_{(\mathcal{C}^{\otimes},X)}(\mathcal{C}^{\otimes})/.} \simeq CAlg(Mod_{\mathcal{L}^{\otimes}_{(\mathcal{C}^{\otimes},X)}(\mathcal{C}^{\otimes})}(Cat_{\infty})) \arrow[r] \arrow[d] & CAlg(Mod_{\mathcal{C}^{\otimes}}(Cat_{\infty})) \simeq CAlg(Cat_{\infty})_{\mathcal{C}^{\otimes}/.} \arrow[d] \\ Mod_{\mathcal{L}^{\otimes}_{(\mathcal{C}^{\otimes},X)}(\mathcal{C}^{\otimes})}(Cat_{\infty}) \arrow[r, "f_*"] & Mod_{\mathcal{C}^{\otimes}}(Cat_{\infty}) \end{tikzcd} \tag{4.1.8}$$

where the horizontal arrows are induced by the forgetful map given by the composition with $\mathcal{C}^{\otimes} \to \mathcal{L}^{\otimes}_{(\mathcal{C}^{\otimes},X)}(\mathcal{C}^{\otimes})$ and the vertical arrows are induced by the forgetful map produced by the change of ∞-operads $\mathcal{T}riv^{\otimes} \to \mathcal{C}omm^{\otimes}$. Since Cat_{∞} with the cartesian product is a presentable symmetric monoidal $(\infty, 1)$-category, there is a base change functor

$$\begin{tikzcd} Mod_{\mathcal{L}^{\otimes}_{(\mathcal{C}^{\otimes},X)}(\mathcal{C}^{\otimes})}(Cat_{\infty}) \arrow[r] & Mod_{\mathcal{C}^{\otimes}}(Cat_{\infty}) \arrow[l, bend right, "\mathcal{L}_{(\mathcal{C}^{\otimes},X)}"'] \end{tikzcd} \tag{4.1.9}$$

and by the general theory we have an identification of $f_*(\mathcal{L}_{(\mathcal{C}^{\otimes},X)}(M)) \simeq M \otimes_{\mathcal{C}^{\otimes}} (\mathcal{L}^{\otimes}_{(\mathcal{C}^{\otimes},X)}(\mathcal{C}^{\otimes}))$ given by the tensor product in $Mod_{\mathcal{C}^{\otimes}}(Cat_{\infty})$. This map is monoidal and therefore induces a left adjoint

$$\begin{tikzcd} CAlg(Cat_\infty)_{\mathcal{L}^{\otimes}_{(\mathcal{C}^{\otimes},X)}(\mathcal{C}^{\otimes})/} \arrow[r] & CAlg(Cat_\infty)_{\mathcal{C}^{\otimes}/} \arrow[l, bend right, "\widetilde{\mathcal{L}}_{(\mathcal{C}^{\otimes},X)}"'] \end{tikzcd} \tag{4.1.10}$$

which by the discussion in 3.2.5 fits in a commutative diagram

$$\begin{tikzcd} CAlg(Cat_\infty)_{\mathcal{L}^{\otimes}_{(\mathcal{C}^{\otimes},X)}(\mathcal{C}^{\otimes})/.} \simeq CAlg(Mod_{\mathcal{L}^{\otimes}_{(\mathcal{C}^{\otimes},X)}(\mathcal{C}^{\otimes})}(Cat_\infty)) \arrow[d, "forget"] & CAlg(Mod_{\mathcal{C}^{\otimes}}(Cat_\infty)) \simeq CAlg(Cat_\infty)_{\mathcal{C}^{\otimes}/.} \arrow[l, "\widetilde{\mathcal{L}}_{(\mathcal{C}^{\otimes},X)}"'] \arrow[d, "forget"] \\ Mod_{\mathcal{L}^{\otimes}_{(\mathcal{C}^{\otimes},X)}(\mathcal{C}^{\otimes})}(Cat_\infty) & Mod_{\mathcal{C}^{\otimes}}(Cat_\infty) \arrow[l, "\mathcal{L}_{(\mathcal{C}^{\otimes},X)}"] \end{tikzcd} \tag{4.1.11}$$

where the vertical maps forget the algebra structure. We now prove the following statement, which was originally proved in [144] in the context of model categories:

Proposition 4.1.2. *Let $\mathcal{C}^{\otimes}$ be a small symmetric monoidal $(\infty,1)$-category and X be an object in $\mathcal{C}$. Let $f : \mathcal{C}^{\otimes} \to \mathcal{L}^{\otimes}_{(\mathcal{C}^{\otimes},X)}(\mathcal{C}^{\otimes})$ be the natural map constructed above. Then*

1. *the composition map*

$$CAlg(Cat_\infty)_{\mathcal{L}^{\otimes}_{(\mathcal{C}^{\otimes},X)}(\mathcal{C}^{\otimes})/.} \to CAlg(Cat_\infty)_{\mathcal{C}^{\otimes}/.} \tag{4.1.12}$$

is fully faithful and its image coincides with $CAlg(Cat_\infty)^{X}_{\mathcal{C}^{\otimes}}$;

2. *the forgetful functor*

$$f_* : Mod_{\mathcal{L}^{\otimes}_{(\mathcal{C}^{\otimes},X)}(\mathcal{C}^{\otimes})}(Cat_\infty) \to Mod_{\mathcal{C}^{\otimes}}(Cat_\infty) \tag{4.1.13}$$

is fully faithful and its image coincides with the full subcategory of $Mod_{\mathcal{C}^{\otimes}}(Cat_\infty)$ spanned by those $\mathcal{C}$-modules where X acts as an equivalence.

A major consequence is that

Corollary 4.1.3. *The left adjoint $\widetilde{\mathcal{L}}_{(\mathcal{C}^{\otimes},X)}$ provided by the base change is naturally equivalent to the left adjoint $\mathcal{L}^{\otimes}_{(\mathcal{C}^{\otimes},X)}$ provided by Proposition 4.1.1.*

Moreover, since the diagram (4.1.11) commutes, we have the formula $\mathcal{L}_{(\mathcal{C}^{\otimes},X)}(\mathcal{D}) \simeq \mathcal{L}^{\otimes}_{(\mathcal{C}^{\otimes},X)}(\mathcal{D}^{\otimes})_{\langle 1\rangle}$ for any $\mathcal{D}^{\otimes} \in CAlg(Cat_\infty)_{\mathcal{C}^{\otimes}/}$.

In order to prove Proposition 4.1.2, we will need some preliminary steps. We start by recalling some notation: Let $\mathcal{E}^{\otimes}$ be a symmetric monoidal $(\infty,1)$-category. A morphism of commutative algebras $A \to B$ in $\mathcal{E}$ is called an epimorphism (see [144]-Definition 1.2.6.1-1) if for any commutative A-algebra C, the mapping space $Map_{CAlg(\mathcal{E})}(B,C)$ is either empty or weakly contractible. In other words, the space of dotted maps of A-algebras

$$\begin{tikzcd} C & \\ A \arrow[u] \arrow[r] & B \arrow[ul, dotted] \end{tikzcd} \tag{4.1.14}$$

rendering the diagram commutative is either empty or consisting of a unique map, up to equivalence. We can rewrite this definition in a different way. As a result of the general theory, if $\mathcal{E}^{\otimes}$ is compatible with all small colimits, the ∞-category $CAlg(\mathcal{E})_{A/}$ inherits a coCartesian tensor product (see [100, 3.2.4.7]) which we denote here as $\otimes_A$. In this case it is immediate the conclusion that a map $A \to B$ is an epimorphism if and only if the canonical map $B \to B \otimes_A B$ is an equivalence. Of course, this is happens if and only if the induced colimit map $B \otimes_A B \to B$ is also an equivalence. We prove the following

Proposition 4.1.4. *Let $\mathcal{E}^{\otimes}$ be a symmetric monoidal $(\infty, 1)$-category compatible with all small colimits and let $f : A \to B$ be a morphism of commutative algebras in $\mathcal{E}$. The following are equivalent:*

1. *f is an epimorphism;*
2. *The natural map $f_* : Mod_B(\mathcal{E}) \to Mod_A(\mathcal{E})$ is fully faithful;*

Moreover, if these equivalent conditions are satisfied, the forgetful map

$$CAlg(\mathcal{E})_{B/.} \to CAlg(\mathcal{E})_{A/.} \tag{4.1.15}$$

is also fully faithful.

Proof. With the hypothesis that the monoidal structure is compatible with colimits, the general theory gives us a base-change functor

$$(- \otimes_A B) : Mod_A(\mathcal{E}) \to Mod_B(\mathcal{E}) \tag{4.1.16}$$

left adjoint to the forgetful map f_*. In this case f_* will be fully faithful if and only if the counit of the adjunction is an equivalence. If the counit is an equivalence in particular we deduce that the canonical map $B \otimes_A B \to B$ is an equivalence and therefore $A \to B$ is an epimorphism. Conversely, if $A \to B$ is an epimorphism, for any B-module M we have

$$M \otimes_A B \simeq ((M \otimes_B B) \otimes_A B) \simeq (M \otimes_B (B \otimes_A B)) \simeq (M \otimes_B B) \simeq M \tag{4.1.17}$$

It remains to prove the additional statement concerning the categories of algebras. Let us consider $u : B \to U$, $v : B \to V$ two algebras over B. We want to prove that the canonical map

$$Map_{CAlg(\mathcal{E})_{B/.}}(U, V) \to Map_{CAlg(\mathcal{E})_{A/.}}(f_*(U), f_*(V)) \tag{4.1.18}$$

is an isomorphism of homotopy types. The points in $Map_{CAlg(\mathcal{E})_{A/.}}(f_*(U), f_*(V)))$ can be identified with commutative diagrams

$$\begin{tikzcd} & & U \arrow[dd] \\ A \arrow[r, "f"] \arrow[urr, bend left, "u\circ f"] \arrow[drr, bend right, "v\circ f"'] & B \arrow[ur, "u"] \arrow[dr, "v"] & \\ & & V \end{tikzcd} \tag{4.1.19}$$

and therefore we can rewrite $Map_{CAlg(\mathcal{E})_{A/.}}(f_*(U), f_*(V))$ as an homotopy pullback diagram

$$Map_{CAlg(\mathcal{E})_{A/}}(B, f_*(V)) \times_{Map_{CAlg(\mathcal{E})_{A/.}}(A, f_*(V))} Map_{CAlg(\mathcal{E})_{B/.}}(U, V) \tag{4.1.20}$$

which by the fact $A \to B$ is an epimorphism and $Map_{CAlg(\mathcal{E})_{A/.}}(A, f_*(V)) \simeq *$, is the same as $Map_{CAlg(\mathcal{E})_{B/.}}(U, V)$. ☐

The following is the main ingredient in the proof of the Proposition 4.1.2.

Proposition 4.1.5. *Let $\mathcal{C}^{\otimes}$ be a small symmetric monoidal ∞-category and let X be an object in $\mathcal{C}$. Then, the canonical map $\mathcal{C}^{\otimes} \to \mathcal{L}^{\otimes}_{(\mathcal{C}^{\otimes}, X)}(\mathcal{C}^{\otimes})$ is an epimorphism.*

Proof. This is a direct result of the characterization of $\mathcal{L}^{\otimes}_{(\mathcal{C}^{\otimes},X)}$ as an adjoint in the Proposition 4.1.1. Indeed, for any algebra $\phi:\mathcal{C}^{\otimes}\to\mathcal{D}^{\otimes}$, either ϕ does not send X to an invertible object and in this case $Map_{CAlg(Cat_{\infty})_{\mathcal{C}^{\otimes}/.}}(\mathcal{L}^{\otimes}_{(\mathcal{C}^{\otimes},X)}(\mathcal{C}^{\otimes}),\mathcal{D}^{\otimes})$ is necessarily empty or, ϕ sends X to an invertible object and we have by the universal properties

$$Map_{CAlg(Cat_{\infty})_{\mathcal{C}^{\otimes}/.}}(\mathcal{L}^{\otimes}_{(\mathcal{C}^{\otimes},X)}(\mathcal{C}^{\otimes}),\mathcal{D}^{\otimes})\simeq Map_{CAlg(Cat_{\infty})_{\mathcal{C}^{\otimes}/.}}(\mathcal{C}^{\otimes},\mathcal{D}^{\otimes})\simeq * \tag{4.1.21}$$

□

Proof of Proposition 4.1.2: By the results above we know that both maps are fully faithful. It suffices now to analyze their images.

1. If $\phi:\mathcal{C}^{\otimes}\to\mathcal{D}^{\otimes}$ is in the image, $\mathcal{D}^{\otimes}$ is an algebra over $\mathcal{L}^{\otimes}_{(\mathcal{C}^{\otimes},X)}(\mathcal{C}^{\otimes})$, there exists a monoidal factorization

$$\begin{tikzcd} \mathcal{C}^{\otimes} \arrow[r, "\phi"] \arrow[d] & \mathcal{D}^{\otimes} \\ \mathcal{L}^{\otimes}_{(\mathcal{C}^{\otimes},X)}(\mathcal{C}^{\otimes}) \arrow[ur, dotted] & \end{tikzcd} \tag{4.1.22}$$

and therefore X is sent to an invertible object. Conversely, if $\phi:\mathcal{C}^{\otimes}\to\mathcal{D}^{\otimes}$ sends X to an invertible object, $\phi:\mathcal{C}^{\otimes}\to\mathcal{D}^{\otimes}$ is local with respect to $Free_{\mathcal{C}^{\otimes}}(X\otimes -):Free_{\mathcal{C}^{\otimes}}(\mathcal{C})\to Free_{\mathcal{C}^{\otimes}}(\mathcal{C})$ and therefore the adjunction morphisms of the Proposition 4.1.1 fit in a commutative diagram

$$\begin{tikzcd} \mathcal{C}^{\otimes} \arrow[r, "\phi"] \arrow[d] & \mathcal{D}^{\otimes} \arrow[d, "\sim"] \\ \mathcal{L}^{\otimes}_{(\mathcal{C}^{\otimes},X)}(\mathcal{C}^{\otimes}) \arrow[r, "\mathcal{L}^{\otimes}_{(\mathcal{C}^{\otimes},X)}(\phi)"] & \mathcal{L}^{\otimes}_{(\mathcal{C}^{\otimes},X)}(\mathcal{D}^{\otimes}) \end{tikzcd} \tag{4.1.23}$$

where the right vertical map is an equivalence and we deduce the existence of a monoidal map presenting $\mathcal{D}^{\otimes}$ as a $\mathcal{L}^{\otimes}_{(\mathcal{C}^{\otimes},X)}(\mathcal{C}^{\otimes})$-algebra, therefore being in the image of f_*.

2. Again, it remains to prove the assertion about the image. If M is a $\mathcal{C}^{\otimes}$-module in the image, by definition, its module structure is obtained by the composition $\mathcal{C}^{\otimes}\times M\to\mathcal{L}^{\otimes}_{(\mathcal{C}^{\otimes},X)}(\mathcal{C}^{\otimes})\times M\to M$ and therefore the action of X on M is invertible. Conversely, let M be a $\mathcal{C}^{\otimes}$-module where X acts as a equivalence. We want to show that M is in the image of the forgetful functor. Since we know it is fully faithful, this is equivalent to showing that the unit map of the adjunction

$$M\to f_*(\mathcal{L}_{(\mathcal{C}^{\otimes},X)}(M))\simeq M\otimes_{\mathcal{C}^{\otimes}}\mathcal{L}^{\otimes}_{(\mathcal{C}^{\otimes},X)}(\mathcal{C}^{\otimes}) \tag{4.1.24}$$

is an equivalence. To prove this we will need a reasonable description of $Free_{\mathcal{C}^{\otimes}}(M)$ - the free $\mathcal{C}^{\otimes}$ algebra generated by M. Following the Construction 3.1.3.7 and the Example 3.1.3.12 of [100] we know that the underlying $\mathcal{C}^{\otimes}$-module $Free_{\mathcal{C}^{\otimes}}(M)_{\langle 1\rangle}$ can be described as a coprodut

$$\coprod_{n\geq 0} Sym^n(M)_{\mathcal{C}^{\otimes}} \tag{4.1.25}$$

where $Sym^n(M)_{\mathcal{C}^{\otimes}}$ is a colimit diagram in $Mod_{\mathcal{C}^{\otimes}}(Cat_{\infty})$ which can be informally described as $M^{\otimes_{\mathcal{C}^{\otimes}}}/\Sigma_n$ where $\otimes_{\mathcal{C}^{\otimes}}$ refers to the natural symmetric monoidal structure in $Mod_{\mathcal{C}^{\otimes}}(Cat_{\infty})$. Let us proceed.

- The general machinery tells us that $Free_{\mathcal{C}^{\otimes}}(M)$ exists in our case and by construction it comes naturally equipped with a canonical monoidal map $\phi : \mathcal{C}^{\otimes} \to Free_{\mathcal{C}^{\otimes}}(M)$. We remark that the multiplication map $(\phi(X)\otimes -) : Free_{\mathcal{C}^{\otimes}}(M)_{\langle 1\rangle} \to Free_{\mathcal{C}^{\otimes}}(M)_{\langle 1\rangle}$ can be identified with the image $Free_{\mathcal{C}^{\otimes}}(X\otimes -)_{\langle 1\rangle}$ of the multiplication map $(X\otimes -) : M \to M$. Since this last one is an equivalence (by the assumption), we conclude that $Free_{\mathcal{C}^{\otimes}}(M)$ is in fact a $\mathcal{C}^{\otimes}$ algebra where X is sent to an invertible object. This means that it is in fact a $\mathcal{L}^{\otimes}_{(\mathcal{C}^{\otimes},X)}(\mathcal{C}^{\otimes})$-algebra and therefore $Free_{\mathcal{C}^{\otimes}}(M)_{\langle 1\rangle}$ is in fact a $\mathcal{L}^{\otimes}_{(\mathcal{C}^{\otimes},X)}(\mathcal{C}^{\otimes})$-module, which means that the unit map

$$Free_{\mathcal{C}^{\otimes}}(M)_{\langle 1\rangle} \to f_*(\mathcal{L}_{(\mathcal{C}^{\otimes},X)}(Free_{\mathcal{C}^{\otimes}}(M)_{\langle 1\rangle})) \simeq Free_{\mathcal{C}^{\otimes}}(M)_{\langle 1\rangle} \otimes_{\mathcal{C}^{\otimes}} \mathcal{L}^{\otimes}_{(\mathcal{C}^{\otimes},X)}(\mathcal{C}^{\otimes}) \tag{4.1.26}$$

  is an equivalence.

- We observe now that we have a canonical map $M \to Free_{\mathcal{C}^{\otimes}}(M)_{\langle 1\rangle}$ because $Sym^1(M) = M$ and that this map is obviously fully faithful. The unit of the natural transformation associated to the base-change gives us a commutative diagram

$$\begin{tikzcd} M \arrow[r] \arrow[d] & M \otimes_{\mathcal{C}^{\otimes}} \mathcal{L}^{\otimes}_{(\mathcal{C}^{\otimes},X)}(\mathcal{C}^{\otimes}) \arrow[d] \\ Free_{\mathcal{C}^{\otimes}}(M)_{\langle 1\rangle} \arrow[r, "\sim"] & Free_{\mathcal{C}^{\otimes}}(M)_{\langle 1\rangle} \otimes_{\mathcal{C}^{\otimes}} \mathcal{L}^{\otimes}_{(\mathcal{C}^{\otimes},X)}(\mathcal{C}^{\otimes}) \end{tikzcd} \tag{4.1.27}$$

  where the lower arrow is an equivalence from the discussion in the previous item. Since the monoidal structure is compatible with coproducts and using the identification $Sym^n(M)_{\mathcal{C}^{\otimes}} \simeq M^{\otimes^n_{\mathcal{C}^{\otimes}}}/\Sigma_n$, we have

$$Free_{\mathcal{C}^{\otimes}}(M)_{\langle 1\rangle} \otimes_{\mathcal{C}^{\otimes}} \mathcal{L}^{\otimes}_{(\mathcal{C}^{\otimes},X)}(\mathcal{C}^{\otimes}) \simeq \coprod [(M^{\otimes^n_{\mathcal{C}^{\otimes}}})^{\otimes}_{\mathcal{C}^{\otimes}} \mathcal{L}^{\otimes}_{(\mathcal{C}^{\otimes},X)}(\mathcal{C}^{\otimes})]/\Sigma_n \tag{4.1.28}$$

  and finally, using the fact $\mathcal{C}^{\otimes} \to \mathcal{L}^{\otimes}_{(\mathcal{C}^{\otimes},X)}(\mathcal{C}^{\otimes})$ is an epimorphism, we have

$$(\mathcal{L}^{\otimes}_{(\mathcal{C}^{\otimes},X)}(\mathcal{C}^{\otimes}))^{\otimes^n_{\mathcal{C}^{\otimes}}} \simeq \mathcal{L}^{\otimes}_{(\mathcal{C}^{\otimes},X)}(\mathcal{C}^{\otimes}) \tag{4.1.29}$$

  for any $n \geq 0$. We find an equivalence

$$Free_{\mathcal{C}^{\otimes}}(M)_{\langle 1\rangle} \otimes_{\mathcal{C}^{\otimes}} \mathcal{L}^{\otimes}_{(\mathcal{C}^{\otimes},X)}(\mathcal{C}^{\otimes}) \simeq Free_{\mathcal{C}^{\otimes}}(M \otimes_{\mathcal{C}^{\otimes}} \mathcal{L}^{\otimes}_{(\mathcal{C}^{\otimes},X)}(\mathcal{C}^{\otimes}))_{\langle 1\rangle} \tag{4.1.30}$$

  The first diagram becomes

$$\begin{tikzcd} M \arrow[r] \arrow[d] & M \otimes_{\mathcal{C}^{\otimes}} \mathcal{L}^{\otimes}_{(\mathcal{C}^{\otimes},X)}(\mathcal{C}^{\otimes}) \arrow[d] \\ Free_{\mathcal{C}^{\otimes}}(M)_{\langle 1\rangle} = \coprod_{n\geq 0} Sym^n(M)_{\mathcal{C}^{\otimes}} \arrow[r, "\sim"] & \coprod_{n\geq 0} Sym^n(M \otimes_{\mathcal{C}^{\otimes}} \mathcal{L}^{\otimes}_{(\mathcal{C}^{\otimes},X)}(\mathcal{C}^{\otimes}))_{\mathcal{C}^{\otimes}} \end{tikzcd} \tag{4.1.31}$$

  where both vertical maps are now the canonical inclusions in the coproduct. Therefore, since Cat_{∞} has disjoint coproduts (because coproducts can be computed as homotopy coproducts in the combinatorial model category of marked simplicial sets and here coproducts are disjoint), we conclude that the canonical map $M \to M \otimes_{\mathcal{C}^{\otimes}} \mathcal{L}^{\otimes}_{(\mathcal{C}^{\otimes},X)}(\mathcal{C}^{\otimes})$ is also an equivalence.

This concludes the proof.

Remark 4.1.6. Let $\mathcal{C}^{\otimes}$ be a symmetric monoidal $(\infty,1)$-category. Let X and Y be two objects in $\mathcal{C}$ and let $X\otimes Y$ denote their product with respect to the monoidal structure. Since the monoidal structure is symmetric, it is an easy observation that $X\otimes Y$ is an invertible object if and only if X and Y are both invertible. Therefore, we can identify the full subcategory $CAlg(Cat_\infty)^{X\otimes Y}_{\mathcal{C}^{\otimes}/}$ with the full subcategory $CAlg(Cat_\infty)^{X,Y}_{\mathcal{C}^{\otimes}/}$ spanned by the algebra objects $\mathcal{C}^{\otimes}\to\mathcal{D}^{\otimes}$ sending both X and Y to invertible objects. As a consequence, we can provide a relative version of our methods and by the universal properties the diagram

$$\begin{tikzcd}
CAlg(Cat_\infty)_{\mathcal{L}^{\otimes}_{(\mathcal{C}^{\otimes},X)}(\mathcal{C}^{\otimes})/}=CAlg(Cat_\infty)^{X}_{\mathcal{C}^{\otimes}/} \arrow[rr, hookrightarrow] \arrow[d, bend right, "base-change"'] & & CAlg(Cat_\infty)_{\mathcal{C}^{\otimes}/} \arrow[ll, bend right, "base-change"'] \arrow[d, bend left, "base-change"] \\
CAlg(Cat_\infty)_{\mathcal{L}^{\otimes}_{(\mathcal{C}^{\otimes},X\otimes Y)}(\mathcal{C}^{\otimes})/}=CAlg(Cat_\infty)^{X,Y}_{\mathcal{C}^{\otimes}/} \arrow[u, hookrightarrow] \arrow[r, hookrightarrow] & CAlg(Cat_\infty)^{Y}_{\mathcal{C}^{\otimes}/} & =CAlg(Cat_\infty)_{\mathcal{L}^{\otimes}_{(\mathcal{C}^{\otimes},Y)}(\mathcal{C}^{\otimes})/} \arrow[u, hookrightarrow] \arrow[ll, bend left, "base-change"]
\end{tikzcd} \tag{4.1.32}$$

has to commute.

Remark 4.1.7. The results of 4.1.1 and 4.1.2 also hold if we restrict our attention to symmetric monoidal $(\infty,1)$-categories that are ∞-groupoids. More precisely, if $\mathcal{C}^{\otimes}$ is an object in $CAlg(\mathcal{S})$ and X is an object in $\mathcal{C}$, the inclusion

$$CAlg(\mathcal{S})^{X}_{\mathcal{C}^{\otimes}/.}\hookrightarrow CAlg(\mathcal{S})_{\mathcal{C}^{\otimes}/} \tag{4.1.33}$$

admits a left adjoint $\mathcal{L}^{spaces,\otimes}_{\mathcal{C}^{\otimes},X}$. This follows from the same arguments as in 4.1.1, using the fact that $\mathcal{S}$ is presentable. Moreover, as in 4.1.2, we can identify $CAlg(\mathcal{S})^{X}_{\mathcal{C}^{\otimes}/.}$ with the $(\infty,1)$-category of commutative $\mathcal{L}^{spaces,\otimes}_{\mathcal{C}^{\otimes},X}(\mathcal{C}^{\otimes})$-algebras.

Recall now that the existence of a a fully-faithful inclusion $i:\mathcal{S}\subseteq Cat_\infty$. This inclusion is monoidal with respect to the cartesian structures and produces an inclusion $i: CAlg(\mathcal{S})\subseteq CAlg(Cat_\infty)$. Therefore, for every symmetric monoidal ∞-groupoid $\mathcal{C}^{\otimes}$ together with the choice of an object $X\in\mathcal{C}$, we have a commutative diagram

$$\begin{tikzcd}
CAlg(Cat_\infty)^{X}_{i(\mathcal{C}^{\otimes})/.} \arrow[r, hookrightarrow] & CAlg(Cat_\infty)_{i(\mathcal{C}^{\otimes})/.} \\
CAlg(\mathcal{S})^{X}_{\mathcal{C}^{\otimes}/.} \arrow[u, hookrightarrow] \arrow[r, hookrightarrow] & CAlg(\mathcal{S})_{\mathcal{C}^{\otimes}/.} \arrow[u, hookrightarrow]
\end{tikzcd} \tag{4.1.34}$$

from which, using the universal property of the adjuntion in 4.1.1, we can deduce the existence of a canonical monoidal map of symmetric monoidal $(\infty,1)$-categories

$$\mathcal{L}^{\otimes}_{i(\mathcal{C}^{\otimes}),X}(i(\mathcal{C}^{\otimes}))\to i(\mathcal{L}^{spaces,\otimes}_{\mathcal{C}^{\otimes},X}(\mathcal{C}^{\otimes})) \tag{4.1.35}$$

Later on (see the Remark 4.2.15) we will see that under an extra assumption on X this comparison map is an equivalence.

Our goal now is to extend our construction to the setting of presentable symmetric monoidal ∞-categories. The starting observation is that, if $\mathcal{C}^{\otimes}$ is a small symmetric monoidal $(\infty,1)$-category the inversion of an object X can now be rewritten by means of a pushout square in $CAlg(Cat_\infty)$: Since Cat_∞ is a symmetric monoidal $(\infty,1)$-category compatible with all colimits, the forgetful functor

$$CAlg(Cat_\infty) \to Cat_\infty \tag{4.1.36}$$

admits a left adjoint $free^\otimes$ which assigns to an ∞-category $\mathcal{D}$, the *free symmetric monoidal* $(\infty, 1)$-*category generated by* $\mathcal{D}$. An object in $\mathcal{C}$ can be interpreted as a monoidal map $free^\otimes(\Delta[0]) \to \mathcal{C}^\otimes$ where $free^\otimes(\Delta[0])$ is the free symmetric monoidal category generated by one object $*$. By the universal property of $\mathcal{L}^\otimes_{(free^\otimes(\Delta[0]),*)}(free^\otimes(\Delta[0]))$, a monoidal map $\mathcal{C}^\otimes \to \mathcal{D}^\otimes$ sends X to an invertible object if and only if it factors as a commutative diagram

$$\begin{tikzcd} free^\otimes(\Delta[0]) \arrow[r] \arrow[d, "X"] & \mathcal{L}^\otimes_{(free^\otimes(\Delta[0]),*)}(free^\otimes(\Delta[0])) \arrow[d] \\ \mathcal{C}^\otimes \arrow[r] & \mathcal{D}^\otimes \end{tikzcd} \tag{4.1.37}$$

and by the combination of the universal properties, the pushout in $CAlg(Cat_\infty)$

$$\mathcal{C}^\otimes \coprod_{free^\otimes(\Delta[0])} \mathcal{L}^\otimes_{(free^\otimes(\Delta[0]),*)}(free^\otimes(\Delta[0])) \tag{4.1.38}$$

is canonically equivalent to $\mathcal{L}^\otimes_{(\mathcal{C}^\otimes,X)}(\mathcal{C}^\otimes)$. The existence of this pushout is ensured by the fact that $Cat^\times_\infty$ is compatible with all colimits (see [100, 3.2.3.2, 3.2.3.3])

We will use this pushout-version to construct the presentable theory. By the tools described in the section 3.2, if $\mathcal{C}^\otimes$ is a presentable symmetric monoidal $(\infty, 1)$-category (not necessarily small) and X is an object in $\mathcal{C}$, the universal *monoidal* property of presheaves ensures that any diagram like (4.1.37) factors as

$$\begin{tikzcd} free^\otimes(\Delta[0]) \arrow[r] \arrow[d, "j"] & \mathcal{L}^\otimes_{(free^\otimes(\Delta[0]),*)}(free^\otimes(\Delta[0])) \arrow[d, "j'"] \\ \mathcal{P}(free^\otimes(\Delta[0]))^\otimes \arrow[r] \arrow[d, dashed] & \mathcal{P}(\mathcal{L}^\otimes_{(free^\otimes(\Delta[0]),*)}(free^\otimes(\Delta[0])))^\otimes \arrow[d, dashed] \\ \mathcal{C}^\otimes \arrow[r] & \mathcal{D}^\otimes \end{tikzcd} \tag{4.1.39}$$

where $\mathcal{P}^\otimes(-)$ is the natural extension of the symmetric monoidal structure to presheaves, the vertical maps j and j' are the respective Yoneda embeddings (which are monoidal maps) and the dotted arrows are given by colimit-preserving monoidal maps obtained as left Kan extensions.

Definition 4.1.8. *Let $\mathcal{C}^\otimes$ be a presentable symmetric monoidal $(\infty, 1)$-category and let X be an object in $\mathcal{C}$. The* formal inversion *of X in $\mathcal{C}^\otimes$ is the new presentable symmetric monoidal $(\infty, 1)$-category $\mathcal{C}^\otimes[X^{-1}]$ defined by pushout*

$$\mathcal{C}^\otimes[X^{-1}] := \mathcal{C}^\otimes \coprod_{\mathcal{P}(free^\otimes(\Delta[0]))^\otimes} \mathcal{P}(\mathcal{L}^\otimes_{(free^\otimes(\Delta[0]),*)}(free^\otimes(\Delta[0])))^\otimes \tag{4.1.40}$$

in $CAlg(\mathcal{P}r^L)$

Remark 4.1.9. Recall that $\mathcal{P}r^{L,\otimes}$ is compatible with colimits. By [100, 3.2.3.2, 3.2.3.3] the $(\infty, 1)$-category $CAlg(\mathcal{P}r^L)$ has all small colimits so that the previous definition makes sense.

Remark 4.1.10. Let $\mathcal{C}^\otimes$ be a small symmetric monoidal $(\infty, 1)$-category and let X be an object in $\mathcal{C}$. Again by the monoidal universal property of presheaves, the monoidal structure in $\mathcal{C}$ extends to a monoidal structure in $\mathcal{P}(\mathcal{C})$ and it makes it a presentable symmetric monoidal $(\infty, 1)$-category. It is automatic by the universal properties that the inversion $\mathcal{P}(\mathcal{C})^\otimes[X^{-1}]$ in the setting of presentable $(\infty, 1)$-categories is canonically equivalent to $\mathcal{P}(\mathcal{L}^\otimes_{(\mathcal{C}^\otimes,X)}(\mathcal{C}^\otimes))^\otimes$.

As in the *small* context, we analyze the base change with respect to this map. Since $(\mathcal{P}r^L)^{\otimes}$ is compatible with all small colimits, all the machinery related to algebras and modules can be applied. The composition with the canonical map $\mathcal{C}^{\otimes} \to \mathcal{C}^{\otimes}[X^{-1}]$ produces a forgetful functor

$$Mod_{\mathcal{C}^{\otimes}[X^{-1}]}(\mathcal{P}r^L) \to Mod_{\mathcal{C}^{\otimes}}(\mathcal{P}r^L) \tag{4.1.41}$$

and the base-change functor $\mathcal{L}^{Pr}_{(\mathcal{C}^{\otimes},X)} := (-\otimes_{\mathcal{C}^{\otimes}} \mathcal{C}^{\otimes}[X^{-1}])$ exists, is monoidal and therefore induces an adjunction

$$\begin{tikzcd} CAlg(\mathcal{P}r^L)_{\mathcal{C}^{\otimes}[X^{-1}]/.} \arrow[r] & CAlg(\mathcal{P}r^L)_{\mathcal{C}^{\otimes}/.} \arrow[l, bend right, "\mathcal{L}^{Pr,\otimes}_{(\mathcal{C}^{\otimes},X)}"'] \end{tikzcd} \tag{4.1.42}$$

Our main result is the following:

Proposition 4.1.11. *Let $\mathcal{C}^{\otimes}$ be a presentable symmetric monoidal $(\infty,1)$-category. Then*

1. *the canonical map*

$$CAlg(\mathcal{P}r^L)_{\mathcal{C}^{\otimes}[X^{-1}]/.} \to CAlg(\mathcal{P}r^L)_{\mathcal{C}^{\otimes}/.} \tag{4.1.43}$$

is fully faithful and its essential image consists of full subcategory spanned by the algebras $\mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ sending X to an invertible object; In particular we have a canonical equivalence $\mathcal{L}^{Pr,\otimes}_{(\mathcal{C}^{\otimes},X)}(\mathcal{C}^{\otimes}) \simeq \mathcal{C}^{\otimes}[X^{-1}]$

2. *The canonical map*

$$Mod_{\mathcal{C}^{\otimes}[X^{-1}]}(\mathcal{P}r^L) \to Mod_{\mathcal{C}^{\otimes}}(\mathcal{P}r^L) \tag{4.1.44}$$

is fully faithful and its essential image consists of full subcategory spanned by the presentable $(\infty,1)$-categories equipped with an action of $\mathcal{C}$ where X acts as an equivalence.

Proof. Since $(\mathcal{P}r^L)^{\otimes}$ is a closed symmetric monoidal $(\infty,1)$-category (see the discussion in the section 3.6), it is compatible with all colimits and so the results of the Proposition 4.1.4 can be applied. We prove that $\mathcal{C}^{\otimes} \to \mathcal{C}^{\otimes}[X^{-1}]$ is an epimorphism. Indeed, if $\phi: \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ does not send X to an invertible object, by the universal property of the $\mathcal{C}^{\otimes}[X^{-1}]$ as a pushout, the mapping space $Map_{CAlg(Cat_{\infty})_{\mathcal{C}^{\otimes}/}}(\mathcal{C}^{\otimes}[X^{-1}],\mathcal{D}^{\otimes})$ is empty. Otherwise if ϕ sends X to an invertible object, by the universal property of the pushout we have

$$Map_{CAlg(\mathcal{P}r^L)_{\mathcal{C}^{\otimes}/}}(\mathcal{C}^{\otimes}[X^{-1}],\mathcal{D}^{\otimes}) \simeq Map_{CAlg(\mathcal{P}r^L)}(\mathcal{C}^{\otimes}[X^{-1}],\mathcal{D}^{\otimes}) \tag{4.1.45}$$

and the last is given by the homotopy pullback of

$$\begin{tikzcd} & Map_{CAlg(\mathcal{P}r^L)}(\mathcal{P}(\mathcal{L}^{\otimes}_{(free^{\otimes}(\Delta[0]),*)}(free^{\otimes}(\Delta[0])))^{\otimes},\mathcal{D}^{\otimes}) \arrow[d] \\ Map_{CAlg(\mathcal{P}r^L)}(\mathcal{C}^{\otimes},\mathcal{D}^{\otimes}) \arrow[r] & Map_{CAlg(\mathcal{P}r^L)}(\mathcal{P}^{\otimes}(free^{\otimes}(\Delta[0])),\mathcal{D}^{\otimes}) \end{tikzcd} \tag{4.1.46}$$

which, by the universal property of $\mathcal{P}^{\otimes}(-)$ is equivalent to

$$Map_{CAlg(\mathcal{P}r^L)}(\mathcal{C}^{\otimes},\mathcal{D}^{\otimes}) \times_{Map_{CAlg(Cat_{\infty})}(free^{\otimes}(\Delta[0]),\mathcal{D}^{\otimes})} Map_{CAlg(Cat_{\infty})}(\mathcal{L}^{\otimes}_{(free^{\otimes}(\Delta[0]),*)}(free^{\otimes}(\Delta[0])),\mathcal{D}^{\otimes}) \tag{4.1.47}$$

and we use the fact that $free^{\otimes}(\Delta[0]) \to \mathcal{L}^{\otimes}_{(free^{\otimes}(\Delta[0]),*)}(free^{\otimes}(\Delta[0]))$ is an epimorphism to conclude the proof.

It remains now to discuss the images.

1. It is clear by the universal property of the pushout defining $\mathcal{C}^{\otimes}[X^{-1}]$;

2. If M is in the image, the action of X is clearly invertible. Let M be a $\mathcal{C}^{\otimes}$-module with an invertible action of X. By repeating exactly the same arguments as in the proof of Prop. 4.1.11 we get a commutative diagram in $\mathcal{P}r^L$

$$\begin{tikzcd} M \arrow[r] \arrow[d] & M \otimes_{\mathcal{C}^{\otimes}} \mathcal{L}^{\otimes}_{(\mathcal{C}^{\otimes},X)}(\mathcal{C}^{\otimes}) \arrow[d] \\ Free_{\mathcal{C}^{\otimes}}(M)_{\langle 1 \rangle} = \coprod_{n\geq 0} Sym^n(M)_{\mathcal{C}^{\otimes}} \arrow[r, "\sim"] & \coprod_{n\geq 0} Sym^n(M \otimes_{\mathcal{C}^{\otimes}} \mathcal{L}^{\otimes}_{(\mathcal{C}^{\otimes},X)}(\mathcal{C}^{\otimes}))_{\mathcal{C}^{\otimes}} \end{tikzcd} \tag{4.1.48}$$

where the vertical maps are the canonical inclusions in the colimit and $Sym^n(-)_{\mathcal{C}^{\otimes}}$ is now a colimit in $Mod_{\mathcal{C}^{\otimes}}(\mathcal{P}r^L)$. We recall now that coproducts in $\mathcal{P}r^L$ are computed as products in $\mathcal{P}r^R$. Let $u : A \to B$ and $v : X \to Y$ be colimit preserving maps between presentable $(\infty, 1)$-categories and assume the coprodut map $u \coprod v : A \coprod X \to B \coprod Y$ is an equivalence. The coproduct $A \coprod X$ is canonically equivalent to the product $A \times X$ and we have commutative diagrams

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "i"'] & B \arrow[d, "j"] \\ A \coprod X \arrow[r, "\sim", "u \coprod v"'] & B \coprod Y \end{tikzcd} \tag{4.1.49}$$

and

$$\begin{tikzcd} A & & B \arrow[ll, "\bar{u}"'] \\ A \coprod X = A \times X \arrow[u, "p"] & & B \coprod Y = B \times Y \arrow[ll, "\overline{u \coprod v}"] \arrow[u, "q"'] \end{tikzcd} \tag{4.1.50}$$

with i and j the canonical inclusions and p and q the projections. The maps in the second diagram are right adjoints to the maps in the first, with $\overline{u \coprod v} \simeq \bar{u} \times \bar{v}$ and therefore $u \coprod v$ and $\bar{u} \times \bar{v}$ are inverses. Since the projections are essentially surjective, the inclusions i and j are fully faithful and we conclude that u has to be fully faithful and $\bar{u}$ is essentially surjective. To conclude the proof is it enough to check that u is essentially surjective or, equivalently (because u is fully faithful), that $\bar{u}$ is fully-faithful. This is the same as saying that for any diagram as in (4.2.30) with $\bar{u} \times \bar{v}$ fully faithful, $\bar{u}$ is necessarily fully faithful. This is true because Y is presentable and therefore has a final object e and since $\bar{v}$ commutes with limits, for any objects $b_0, b_1 \in Obj(\mathcal{A})$ we have

$$\begin{aligned} Map_B(b_0, b_1) \simeq Map_B(b_0, b_1) \times Map_Y(e, e) &\simeq Map_A(\bar{u}(b_0), \bar{u}(b_1)) \times Map_X(\bar{v}(e), \bar{v}(e)) \quad (4.1.51) \\ &\simeq Map_A(\bar{u}(b_0), \bar{u}(b_1)) \quad (4.1.52) \end{aligned}$$

□

Remark 4.1.12. The considerations in the Remark 4.1.6 work, mutatis mutandis, in the presentable setting.

4.2 Connection with ordinary Spectra and Stabilization

In the previous section we studied the formal inversion of an object X in a symmetric monoidal $(\infty,1)$-category. Our goal for this section is to compare our formal inversion to the more familiar notion of (ordinary) spectrum-objects.

4.2.1 Stabilization

Let $\mathcal{C}$ be an $(\infty,1)$-category and let $G:\mathcal{C}\to\mathcal{C}$ be a functor with a right adjoint $U:\mathcal{C}\to\mathcal{C}$. We define the *stabilization of* $\mathcal{C}$ *with respect to* (G,U) as the limit in Cat_{∞}^{big}

$$Stab_{(G,U)}(\mathcal{C}) := \qquad ... \xrightarrow{U} \mathcal{C} \xrightarrow{U} \mathcal{C} \xrightarrow{U} \mathcal{C} \tag{4.2.1}$$

We will refer to the objects of $Stab_{(G,U)}(\mathcal{C})$ as *spectrum objects in* $\mathcal{C}$ *with respect to* (G,U). As a limit, we have a canonical functor "evaluation at level 0" which we will denote as $\Omega^{\infty}_{\mathcal{C}}: Stab_{(G,U)}(\mathcal{C})\to\mathcal{C}$.

Remark 4.2.1. Let $\mathcal{C}$ is a presentable $(\infty,1)$-category together with a colimit preserving functor $G:\mathcal{C}\to\mathcal{C}$. By the Adjoint Functor Theorem we deduce the existence a right adjoint U to G. Using the equivalence $\mathcal{P}r^L\simeq(\mathcal{P}r^R)^{op}$, and the fact that both inclusions $\mathcal{P}r^L,\mathcal{P}r^R\subseteq Cat_{\infty}^{big}$ preserve limits, we conclude that $Stab_{(G,U)}(\mathcal{C})$ is equivalent to the colimit of

$$\mathcal{C} \xrightarrow{G} \mathcal{C} \xrightarrow{G} \mathcal{C} \xrightarrow{G} ... \tag{4.2.2}$$

Example 4.2.2. The construction of spectrum objects provides a method to stabilize an ∞-category: Let $\mathcal{C}$ be an $(\infty,1)$-category with final object $*$. If $\mathcal{C}$ admits finite limits and colimits we can construct a pair of adjoint functors $\Sigma_{\mathcal{C}}:\mathcal{C}_{*/}\to\mathcal{C}_{*/}$ and $\Omega_{\mathcal{C}}:\mathcal{C}_{*/}\to\mathcal{C}_{*/}$ defined by the formula

$$\Sigma_{\mathcal{C}}(X) := * \coprod_X * \tag{4.2.3}$$

and

$$\Omega_{\mathcal{C}}(X) := * \times_X * \tag{4.2.4}$$

and by [100, Prop. 1.4.2.24] we can define the *stabilization of* $\mathcal{C}$ as the ∞-category

$$Stab(\mathcal{C}) := Stab_{(\Sigma_{\mathcal{C}},\Omega_{\mathcal{C}})}(\mathcal{C}_{*/}) \tag{4.2.5}$$

.

By [100, Cor. 1.4.2.17], $Stab(\mathcal{C})$ is a stable ∞-category and by [100, Corollary 1.4.2.23] the functor $\Omega^{\infty}: Stab(\mathcal{C})\to\mathcal{C}$ has a universal property: for any stable $(\infty,1)$-category $\mathcal{D}$, the composition with Ω^{∞} induces an equivalence

$$Fun'(\mathcal{D},Stab(\mathcal{C}))\to Fun'(\mathcal{D},\mathcal{C}) \tag{4.2.6}$$

between the full subcategories of functors preserving finite limits. Suppose now that $\mathcal{C}$ is presentable. Since $\Omega_{\mathcal{C}}$ by definition commutes with all limits and $\mathcal{P}r^R$ is closed under limits, $Stab(\mathcal{C})$ will also be presentable and Ω^{∞} will also commute with all limits. Therefore, by the Adjoint Functor Theorem it will admit a left adjoint $\Sigma^{\infty}:\mathcal{C}\to Stab(\mathcal{C})$. Using the equivalence $\mathcal{P}r^L\simeq(\mathcal{P}r^R)^{op}$ we find (see [100, Cor. 1.4.4.5]) that Σ^{∞} is characterized by the following universal property: for every stable presentable $(\infty,1)$-category $\mathcal{D}$, the composition with Σ^{∞} induces an equivalence

$$Fun^L(Stab(\mathcal{C}),\mathcal{D})\to Fun^L(\mathcal{C},\mathcal{D}) \tag{4.2.7}$$

Our goal for the rest of this section is to compare this notion of stabilization to something more familiar. Let us start with some precisions about the construction of limits in Cat_∞. By [99, Thm. 4.2.4.1], the stabilization $Stab_{(G,U)}(\mathcal{C})$ can be computed as an homotopy limit for the tower

$$... \xrightarrow{U} \mathcal{C}^\natural \xrightarrow{U} \mathcal{C}^\natural \xrightarrow{U} \mathcal{C}^\natural \tag{4.2.8}$$

in the simplicial model category $\hat{\Delta}_+$ of (big) marked simplicial sets of the [99, Prop. 3.1.3.7] (as a marked simplicial set, $\mathcal{C}^\natural$ is the notation for the pair $(\mathcal{C}, W)$ where W is the collection of all edges in $\mathcal{C}$ which are equivalences). By [99, Thm. 3.1.5.1], the cofibrant-fibrant objects in $\hat{\Delta}_+$ are exactly the objects of the form $\mathcal{C}^\natural$ with $\mathcal{C}$ a quasi-category and, forgetting the marked edges provides a right-Quillen equivalence from $\hat{\Delta}_+$ to $\hat{\Delta}$ with the Joyal model structure. Therefore, to obtain a model for the homotopy limit in $\hat{\Delta}_+$ we can instead compute the homotopy limit in $\hat{\Delta}$ (with the Joyal's structure).

Let now us recall some important results about homotopy limits in model categories. All the following results can be deduced using the Reedy/injective model structures (see [69] or the Appendix section of [99]) to study diagrams in the underlying model category. The first result is that for a pullback diagram

$$\begin{tikzcd} & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & Z \end{tikzcd} \tag{4.2.9}$$

to be an homotopy pullback, it is enough to have Z fibrant and both f and g fibrations. In fact, these conditions can be a bit weakened, and it is enough to have either (i) the three objects are fibrant and one of the maps is a fibration; (ii) if the model category is right-proper, Z is fibrant and one of the maps is a fibration (this last one applies for instance in the model category of simplicial sets with the standard model structure). Secondly, we recall another important fact related to the homotopy limits of towers (again, this can be deduced using the Reedy structure). For the homotopy limit of a tower

$$... \xrightarrow{T_3} X_2 \xrightarrow{T_2} X_1 \xrightarrow{T_1} X_0 \tag{4.2.10}$$

to be given directly by the associated strict limit, it suffices to have the object X_0 fibrant and all the maps T_i given by fibrations. In fact, these towers are exactly the fibrant-objects for the Reedy structure and therefore we can replace any tower by a weak-equivalent one in these good conditions. The following result provides a strict model for the homotopy limit of a tower:

Lemma 4.2.3. *Let $\mathcal{M}$ be a simplicial model category and let $T : \mathbb{N}^{op} \to \mathcal{M}$ be tower in $\mathcal{M}$*

$$... \xrightarrow{T_3} X_2 \xrightarrow{T_2} X_1 \xrightarrow{T_1} X_0 \tag{4.2.11}$$

with each X_n a fibrant object of $\mathcal{M}$. In this case, the homotopy limit $holim_{(\mathbb{N}^{op})}T_n$ is weak-equivalent to the strict pullback of the diagram

$$\begin{tikzcd} & \prod_n X_n^{\Delta[1]} \arrow[d] \\ \prod_n X_n \arrow[r] & \prod_n X_n \times X_n \end{tikzcd} \tag{4.2.12}$$

where the vertical arrow is the fibration[1] *induced by the composition with the cofibration $\partial\Delta[1] \to \Delta[1]$ and the horizontal map is the product of the compositions $\prod_n X_n \to X_n \times X_{n+1} \to X_n \times X_n$ where the last map is the product $Id_{X_n} \times T_n$. Notice that every vertice of the diagram is fibrant.*

Proof. See [61]-VI-Lemma 1.12. □

[1] it is a fibration because of the simplicial assumption

Back to our situation, we conclude that the homotopy limit of

$$... \xrightarrow{U} \mathcal{C}^\natural \xrightarrow{U} \mathcal{C}^\natural \xrightarrow{U} \mathcal{C}^\natural \tag{4.2.13}$$

is given by the explicit strict pullback in $\hat{\Delta}_+$

$$\begin{tikzcd} & \prod_n (\mathcal{C}^\natural)^{\Delta[1]^\sharp} \arrow[d] \\ \prod_n \mathcal{C}^\natural \arrow[r] & \prod_n \mathcal{C}^\natural \times \mathcal{C}^\natural \end{tikzcd} \tag{4.2.14}$$

where $\Delta[1]^\sharp$ is the notation for the simplicial set $\Delta[1]$ with all the edges marked and $(\mathcal{C}^\natural)^{\Delta[1]^\sharp}$ is the coaction of $\Delta[1]^\sharp$ on $\mathcal{C}^\natural$. In fact, it can be identified with the marked simplicial set $Fun'(\Delta[1],\mathcal{C})^\natural$ where $Fun'(\Delta[1],\mathcal{C})$ corresponds to the full-subcategory of $Fun(\Delta[1],\mathcal{C})$ spanned by the maps $\Delta[1] \to \mathcal{C}$ which are equivalences in $\mathcal{C}$.

Let us move further. Consider now a combinatorial simplicial model category $\mathcal{M}$ and let $G : \mathcal{M} \to \mathcal{M}$ be a left simplicial Quillen functor with a right adjoint U. Using the technique described in [99, 5.2.4.6], from the adjunction data we can extract an endo-adjunction of the underlying $(\infty,1)$-category of $\mathcal{M}$

$$\begin{tikzcd} N_\Delta(\mathcal{M}^\circ) \arrow[r, shift left, "\bar{G}"] & N_\Delta(\mathcal{M}^\circ) \arrow[l, shift left, "\bar{U}"] \end{tikzcd} \tag{4.2.15}$$

where $\bar{U}$ can be identified with the composition $Q \circ U$ with Q a simplicial[2] cofibrant-replacement functor in $\mathcal{M}$, which we shall fix once and for all. We can consider the stabilization $Stab_{(\bar{G},\bar{U})}(N_\Delta(\mathcal{M}^\circ))$ given by the homotopy limit

$$... \xrightarrow{\bar{U}} N_\Delta(\mathcal{M}^\circ)^\natural \xrightarrow{\bar{U}} N_\Delta(\mathcal{M}^\circ)^\natural \xrightarrow{\bar{U}} N_\Delta(\mathcal{M}^\circ)^\natural \tag{4.2.16}$$

which we now know, is weak-equivalent to the strict pullback of

$$\begin{tikzcd} & \prod_n Fun'(\Delta[1], N_\Delta(\mathcal{M}^\circ))^\natural \arrow[d] \\ \prod_n N_\Delta(\mathcal{M}^\circ)^\natural \arrow[r] & \prod_n N_\Delta(\mathcal{M}^\circ)^\natural \times N_\Delta(\mathcal{M}^\circ)^\natural \end{tikzcd} \tag{4.2.17}$$

and we know that its underlying simplicial set can be computed as a pullback in $\hat{\Delta}$ by ignoring all the markings. Moreover, by [99, Prop. 4.2.4.4], we have an equivalence of $(\infty,1)$-categories between

$$N_\Delta((\mathcal{M}^I)^\circ) \xrightarrow{\sim} N_\Delta(\mathcal{M}^\circ)^{\Delta[1]} \tag{4.2.18}$$

where I is the categorical interval and $\mathcal{M}^I$ denotes the category of morphisms in $\mathcal{M}$ endowed with the projective model structure (its cofibrant-fibrant objects are the arrows $f : A \to B$ in $\mathcal{M}$ with both A and B cofibrant-fibrant and f a cofibration in $\mathcal{M}$). Moreover, the equivalence above restricts to a new one between the simplicial nerve of $(\mathcal{M}^I)^\circ_{triv}$ (the full simplicial subcategory of $(\mathcal{M}^I)^\circ$ spanned by the arrows $f : A \to B$ which have A and B cofibrant-fibrant and f a trivial cofibration) and $Fun'(\Delta[1], N_\Delta(\mathcal{M}^\circ))$. Using this equivalence, we find an equivalence of diagrams

[2](see for instance the Proposition 6.3 of [115] for the existence of simplicial factorizations in a simplicial cofibrantly generated model category)

$$
\begin{tikzcd}
 & N_\Delta((\mathcal{M}^I)^\circ_{triv}) \arrow[rr, "\sim"] \arrow[d] & & Fun'(\Delta[1], N_\Delta(\mathcal{M}^\circ)) \arrow[d] \\
 & \prod_n N_\Delta(\mathcal{M}^\circ) \times N_\Delta(\mathcal{M}^\circ) \arrow[rr, "id"] & & \prod_n N_\Delta(\mathcal{M}^\circ) \times N_\Delta(\mathcal{M}^\circ) \\
\prod_n N_\Delta(\mathcal{M}^\circ) \arrow[ur] \arrow[rr, "id"] & & \prod_n N_\Delta(\mathcal{M}^\circ) \arrow[ur] &
\end{tikzcd}
\tag{4.2.19}
$$

The homotopy pullbacks of both diagrams are weak-equivalent but since the vertical map on the left diagram is no longer a fibration, the associated strict pullback is no longer a model for the homotopy pullback. We continue: the simplicial nerve functor N_Δ is a right-Quillen functor from the category of simplicial categories with the model structure of [15] to the category of simplicial sets with the Joyal's structure. Therefore, it commutes with homotopy limits and so, the simplicial set underlying the pullback of the previous diagram is in fact given by the simplicial nerve of the homotopy pullback of

$$
\begin{tikzcd}
 & \prod_n (\mathcal{M}^I)^\circ_{triv} \arrow[d] \\
\prod_n \mathcal{M}^\circ \arrow[r] & \prod_n \mathcal{M}^\circ \times \mathcal{M}^\circ
\end{tikzcd}
\tag{4.2.20}
$$

in the model category of simplicial categories.

Let us now progress in another direction. We continue with $\mathcal{M}$ a model category together with $G : \mathcal{M} \to \mathcal{M}$ a Quillen left endofuctor with a right adjoint U. We recall the construction of a category $Sp^{\mathbb{N}}(\mathcal{M}, G)$ of spectrum objects in $\mathcal{M}$ with respect to (G, U): its objects are the sequences $X = (X_0, X_1, ...)$ together with data of morphisms in $\mathcal{M}$, $\sigma_i : G(X_i) \to X_{i+1}$ (by the adjunction, this is equivalent to the data of morphisms $\bar{\sigma}_i : X_i \to U(X_{i+1})$). A morphism $X \to Y$ is a collection of morphisms in $\mathcal{M}$, $f_i : X_i \to Y_i$, compatible with the structure maps σ_i. If $\mathcal{M}$ is a cofibrantly generated model category (see Section 2.1 of [69]) we can equipped $Sp^{\mathbb{N}}(\mathcal{M}, G)$ with a *stable model structure*. First we define the projective model structure: the weak equivalences are the maps $X \to Y$ which are levelwise weak-equivalences in $\mathcal{M}$ and the fibrations are the levelwise fibrations. The cofibrations are defined by obvious left-lifting properties. By the Theorem 1.13 of [71] these form a model structure which is again cofibrantly generated and by the Proposition 1.15 of loc. cit, the cofibrant-fibrant objects are the sequences $(X_0, X_1, ...)$ where every X_i is fibrant-cofibrant in $\mathcal{M}$, and the canonical maps $G(X_i) \to G(X_{i+1})$ are cofibrations. We shall write $Sp^{\mathbb{N}}(\mathcal{M}, G)_{proj}$ to denote this model structure. The stable model structure, denoted as $Sp^{\mathbb{N}}(\mathcal{M}, G)_{stable}$, is obtained as a Bousfield localization of the projective structure so that the new fibrant-cofibrant objects are the *U-spectra*, meaning, the sequences $(X_0, X_1, ...)$ which are fibrant-cofibrant for the projective model structure and such that for every i, the adjoint of the structure map σ_i, $X_i \to U(X_{i+1})$ is a weak-equivalence. (See Theorem 3.4 of [71]).

By [71, Thm 6.3], this construction also works if we assume $\mathcal{M}$ to be a combinatorial simplicial model category and G to be a left simplicial Quillen functor [3]. In this case, $Sp^{\mathbb{N}}(\mathcal{M}, G)$ (both with the stable and the projective structures) is again a combinatorial simplicial model category with mapping spaces given by the pullback

[3] The reader is left with the easy exercise of checking that the following conditions are equivalent for a Quillen adjunction (G, U) between simplicial model categories: (i) G is enriched; (ii) G is compatible with the simplicial action, meaning that for any simplicial set K and any object X we have $G(K \otimes X) \simeq K \otimes G(X)$; (iii) U is compatible with the coaction, meaning that any for any simplicial set K and object Y we have $U(Y^K) \simeq U(Y)^K$; (iv) U is enriched.

$$\begin{tikzcd} & \prod_n Map_{\mathcal{M}}(X_i,Y_i) \arrow[d] \\ \prod_n Map_{\mathcal{M}}(X_i,Y_i) \arrow[r] & \prod_n Map_{\mathcal{M}}(X_i,U(Y_{i+1})) \end{tikzcd} \tag{4.2.21}$$

where

- the horizontal map is the product of the maps

$$Map_{\mathcal{M}}(X_i,Y_i) \to Map_{\mathcal{M}}(X_i,U(Y_{i+1})) \tag{4.2.22}$$

  induced by the composition with the adjoint $\bar{\sigma}_i : Y_i \to U(Y_{i+1})$;

- The vertical map is the product of the compositions

$$Map_{\mathcal{M}}(X_{i+1},Y_{i+1}) \to Map_{\mathcal{M}}(U(X_{i+1}),U(Y_{i+1})) \to Map_{\mathcal{M}}(X_i,U(Y_{i+1})) \tag{4.2.23}$$

  where the first map is induced by U and the second map is the composition with $X_i \to U(X_{i+1})$.

Its points correspond to the collections $f = \{f_i\}_{i\in\mathbb{N}}$ for which the diagrams

$$\begin{tikzcd} X_i \arrow[r] \arrow[d, "f_i"] & U(X_{i+1}) \arrow[d, "U(f_{i+1})"] \\ Y_i \arrow[r] & U(Y_{i+1}) \end{tikzcd} \tag{4.2.24}$$

commute.

By the Proposition 2.2.1, the underlying $(\infty,1)$-categories of $Sp^{\mathbb{N}}(\mathcal{M},G)_{proj}$ and $Sp^{\mathbb{N}}(\mathcal{M},G)_{stable}$ are given, respectively by the simplicial nerves $N_\Delta((Sp^{\mathbb{N}}(\mathcal{M},G)_{proj})^\circ)$ and $N_\Delta((Sp^{\mathbb{N}}(\mathcal{M},G)_{stable})^\circ)$ and by construction the last appears as the full reflexive subcategory of the first, spanned by the U-spectrum objects.

Up to this point we have two different notions of spectrum-objects. Of course they are related. To understand the relation we observe first that $Sp^{\mathbb{N}}(\mathcal{M},G)$ fits in a strict pullback diagram of simplicial categories

$$\begin{tikzcd} Sp^{\mathbb{N}}(\mathcal{M},G) \arrow[r] \arrow[d] & \prod_n (\mathcal{M}^I) \arrow[d] \\ \prod_n \mathcal{M} \arrow[r] & \prod_n \mathcal{M}\times\mathcal{M} \end{tikzcd} \tag{4.2.25}$$

where the top horizontal map is the product of all maps of the form $(X_i)_{i\in\mathbb{N}} \mapsto (X_i \to U(X_{i+1}))$ and the vertical-left map sends a spectrum-object to its underlying sequence of objects. The right-vertical map sends a morphism in $\mathcal{M}$ to its respective source and target and the lower-horizontal map is the product of the compositions $(X_i)_{i\in\mathbb{N}} \mapsto (X_i, X_{i+1}) \mapsto (X_1, U(X_{i+1}))$. All the maps in this diagram are compatible with the simplicial enrichment. We fabricate a new diagram which culminates in (4.2.20).

$$\begin{tikzcd}
 & & & & \prod_n (\mathcal{M}^I)^{\circ}_{triv} \arrow[d, hook] \\
 & & & & \prod_n (\mathcal{M}^I)^{\circ} \arrow[d, hook] \\
Sp^{\mathbb{N}}(\mathcal{M},G)^{\circ}_{stable} \arrow[r, hook] \arrow[uurrrr, dashed, "f"] \arrow[dr, dashed, "a'"'] & Sp^{\mathbb{N}}(\mathcal{M},G)^{\circ}_{proj} \arrow[r, hook] \arrow[urrr, dashed, "e"'] \arrow[d, dashed, "a"'] & Sp^{\mathbb{N}}(\mathcal{M},G) \arrow[r, "x"'] \arrow[d, "y"'] & \prod_n (\mathcal{M}^I) \arrow[r, "b"] \arrow[d, "z"'] & \prod_n (\mathcal{M}^I) \arrow[d] \\
 & \prod_n \mathcal{M}^{\circ} \arrow[r, hook] \arrow[drr, dashed, "c"'] & \prod_n \mathcal{M} \arrow[r, "w"'] & \prod_n \mathcal{M}\times\mathcal{M} & \prod_n \mathcal{M}^{\circ}\times\mathcal{M}^{\circ} \\
 & & & \prod_n \mathcal{M}^{\circ}\times\mathcal{M}^{fib} \arrow[u, hook] \arrow[ur, "d"'] &
\end{tikzcd} \tag{4.2.26}$$

where the maps

1. x, y, z, w are the maps in the diagram (4.2.25);
2. a is the restriction of the projection $Sp^{\mathbb{N}}(\mathcal{M},G) \to \prod_n \mathcal{M}$ (it is well-defined because the cofibrant-fibrant objects in $Sp^{\mathbb{N}}(\mathcal{M},G)$ are supported on sequences of cofibrant-fibrant objects in $\mathcal{M}$)
3. a' is the composition of a with the canonical inclusion;
4. b is the product of the compositions

$$\mathcal{M}^I \xrightarrow{Q} \mathcal{M}^I \times \mathcal{M}^I \longrightarrow \mathcal{M}^I \tag{4.2.27}$$

where $\mathcal{Q}$ is the machine associated to our chosen simplicial functorial factorization of the form "(cofibration, trivial fibration)" (sending a morphism $f : A \to B$ in $\mathcal{M}$ to the pair $(u : A \to X, v : X \to Y)$ with u a cofibration and v a trivial fibration) and the second arrow is the projection in the first coordinate.

5. c is induced by composition of w with the canonical inclusion. Given a sequence of cofibrant-fibrant objects $(X_i)_{i\in\mathbb{N}}$, we have $w((X_i)_{i\in\mathbb{N}}) = (X_i, U(X_i + 1))_{i\in\mathbb{N}}$ with X_i fibrant-cofibrant and $U(X_{i+1})$ fibrant (because U is a right-Quillen functor). Therefore, the composition factors through $\prod_n \mathcal{M}^{\circ} \times \mathcal{M}^{fib}$ and c is well-defined;
6. To obtain d, we consider first the composition

$$\mathcal{M}^{\circ}\times\mathcal{M} \longrightarrow \mathcal{M}^{\circ}\times\mathcal{M}^I \xrightarrow{id\times Q} \mathcal{M}^{\circ}\times(\mathcal{M}^I\times\mathcal{M}^I) \longrightarrow \mathcal{M}^{\circ}\times\mathcal{M}^I \longrightarrow \mathcal{M}^{\circ}\times\mathcal{M} \tag{4.2.28}$$

where the first arrow sends $(X,Y) \mapsto (X, \emptyset \to X)$, the third arrow is induced by the projection of $\mathcal{M}^I \times \mathcal{M}^I \to \mathcal{M}^I$ on the first coordinate and the last arrow is induced by taking the source. All together, this composition is sending a pair (X,Y) to the pair $(X, Q(Y))$ with Q a cofibrant-replacement of Y using the same factorization device of the item (4). In particular, if Y is already fibrant, $Q(Y)$ will be cofibrant-fibrant and we have a dotted arrow

$$\begin{tikzcd}
\mathcal{M}^{\circ}\times\mathcal{M} \arrow[r] & \mathcal{M}^{\circ}\times\mathcal{M} \\
\mathcal{M}^{\circ}\times\mathcal{M}^{fib} \arrow[u, hook] \arrow[r, dashed] & \mathcal{M}^{\circ}\times\mathcal{M}^{\circ} \arrow[u, hook]
\end{tikzcd} \tag{4.2.29}$$

rendering the diagram commutative.

By definition, d is the product of all these dotted maps;

7. e is the map induced by composing $b \circ x$ with the canonical inclusion and it is well-defined for the reasons given also in (2);

8. f is deduced from e by restricting to the U-spectra objects: If $(X_i)_{i\in\mathbb{N}}$ is a U-spectra, the canonical maps $X_i \to U(X_{i+1})$ are weak-equivalences and therefore, when we perform the factorization encoded in the composition $b \circ x$, the first map is necessarily a trivial cofibration and therefore f factors through $\prod_n (\mathcal{M}^I)^{\circ}_{triv}$.

Finally, the fact that everything commutes is obvious from the definition of factorization system. All together, we found a commutative diagram

$$\begin{tikzcd} Sp^{\mathbb{N}}(\mathcal{M},G)^{\circ}_{stable} \arrow[r] \arrow[d] & \prod_n (\mathcal{M}^I)^{\circ}_{triv} \arrow[d] \\ \prod_n \mathcal{M}^{\circ} \arrow[r] & \prod_n \mathcal{M}^{\circ} \times \mathcal{M}^{\circ} \end{tikzcd} \tag{4.2.30}$$

In summary, the upper horizontal map sends a U-spectra $X = (X_i)_{i\in\mathbb{N}}$ to the list of trivial cofibrations $(X_i \to Q(U(X_{i+1})))_{i\in\mathbb{N}}$ and the left-vertical map sends X to its underlying sequence of cofibrant-fibrant objects. By considering the simplicial nerve of the diagram above and using the equivalence of diagrams in (4.2.19), we obtain, using the universal property of the strict pullback, a map

$$\phi : N_{\Delta}((Sp^{\mathbb{N}}(\mathcal{M},G)_{stable})^{\circ}) \to Stab_{(\bar{G},\bar{U})}(N_{\Delta}(\mathcal{M}^{\circ})) \tag{4.2.31}$$

where we identify $Stab_{(\bar{G},\bar{U})}(N_{\Delta}(\mathcal{M}^{\circ}))$ with the strict pullback of the diagram (4.2.17).

The following result clarifies this already long story:

Proposition 4.2.4. *Let $\mathcal{M}$ be a combinatorial simplicial model category and let $G : \mathcal{M} \to \mathcal{M}$ be a left simplicial Quillen functor with a right adjoint U. Let $Sp^{\mathbb{N}}(\mathcal{M},G)^{stable}$ denote the combinatorial simplicial model category of [71] equipped the stable model structure. Then, the canonical map induced by the previous commutative diagram*

$$\phi : N_{\Delta}((Sp^{\mathbb{N}}(\mathcal{M},G)_{stable})^{\circ}) \to Stab_{(\bar{G},\bar{U})}(N_{\Delta}(\mathcal{M}^{\circ})) \tag{4.2.32}$$

is an equivalence of $(\infty,1)$-categories.

Proof. We will prove this by checking the map is essentially surjective and fully-faithful. We start with the essential surjectivity. For that we can restrict ourselves to study of the map induced between the maximal ∞-groupoids (Kan-complexes) on both sides .

$$N_{\Delta}((Sp^{\mathbb{N}}(\mathcal{M},G)_{stable})^{\circ})^{\simeq} \to Stab_{(\bar{G},\bar{U})}(N_{\Delta}(\mathcal{M}^{\circ}))^{\simeq} \tag{4.2.33}$$

To conclude the essential surjectivity it suffices to check that the map induced between the π_0's

$$\pi_0(N_{\Delta}((Sp^{\mathbb{N}}(\mathcal{M},G)_{stable})^{\circ})^{\simeq}) \to \pi_0(Stab_{(\bar{G},\bar{U})}(N_{\Delta}(\mathcal{M}^{\circ}))^{\simeq}) \tag{4.2.34}$$

is surjective. We start by analyzing the right-side. First, the operation $(-)^{\simeq}$ commutes with homotopy limits. To see this, notice that both the $(\infty,1)$-category of homotopy types $\mathcal{S}$ and the $(\infty,1)$-category of small $(\infty,1)$-categories Cat_{∞} are presentable. The combinatorial simplicial model category of simplicial sets with the Quillen structure is a strict model for the first and $\hat{\Delta}_+$ models the second. By combining the Theorem 3.1.5.1 and the Proposition 5.2.4.6 of [99], the inclusion $\mathcal{S} \subseteq Cat_{\infty}$ is in fact a Bousfield (a.k.a reflexive) localization and its the left adjoint can be understood (by its universal property) as the process of inverting all the morphisms. By combining the Proposition 3.3.2.5 and the Corollaries 3.3.4.3 and 3.3.4.6 of [99], we deduce that the inclusion $\mathcal{S} \subseteq Cat_\infty$ commutes with colimits. Since $\mathcal{S}$ and Cat_∞ are presentable, by the Adjoint Functor Theorem (see Corollary 5.5.2.9 of [99]), the inclusion $\mathcal{S} \subseteq Cat_\infty$ admits a right adjoint which, by its universal property can be identified with the operation $(-)^{\simeq}$. An immediate application of this fact is that $\pi_0(Stab_{(\bar{G},\bar{U})}(N_\Delta(\mathcal{M}^\circ))^{\simeq})$ is in bijection with the π_0 of the homotopy limit of the tower of Kan-complexes

$$... \xrightarrow{\bar{U}} N_\Delta(\mathcal{M}^\circ)^{\simeq} \xrightarrow{\bar{U}} N_\Delta(\mathcal{M}^\circ)^{\simeq} \xrightarrow{\bar{U}} N_\Delta(\mathcal{M}^\circ)^{\simeq} \tag{4.2.35}$$

Using the Reedy structure (on $\hat{\Delta}$ with the Quillen structure), we can find a morphism of towers

$$\begin{tikzcd}
... \arrow[r, "\bar{U}"] & N_\Delta(\mathcal{M}^\circ)^{\simeq} \arrow[r, "\bar{U}"] \arrow[d] & N_\Delta(\mathcal{M}^\circ)^{\simeq} \arrow[r, "\bar{U}"] \arrow[d] & N_\Delta(\mathcal{M}^\circ)^{\simeq} \arrow[d] \\
... \arrow[r] & T_2 \arrow[r] & T_1 \arrow[r] & T_0
\end{tikzcd} \tag{4.2.36}$$

where the vertical maps are weak-equivalences of simplicial sets for the Quillen structure, every object is again a Kan-complex but this time the maps in the lower tower are fibrations. By the nature of the weak-equivalences, this morphism of diagrams becomes an isomorphism at the level of the π_0's

$$\begin{tikzcd}
... \arrow[r] & \pi_0(N_\Delta(\mathcal{M}^\circ)^{\simeq}) \arrow[r, "\pi_0(\bar{U})"] \arrow[d, "\sim"] & \pi_0(N_\Delta(\mathcal{M}^\circ)^{\simeq}) \arrow[r, "\pi_0(\bar{U})"] \arrow[d, "\sim"] & \pi_0(N_\Delta(\mathcal{M}^\circ)^{\simeq}) \arrow[d, "\sim"] \\
... \arrow[r] & \pi_0(T_2) \arrow[r] & \pi_0(T_1) \arrow[r] & \pi_0(T_0)
\end{tikzcd} \tag{4.2.37}$$

and therefore the limits $lim_{\mathbb{N}^{op}}\pi_0(N_\Delta(\mathcal{M}^\circ)^{\simeq})$ and $lim_{\mathbb{N}^{op}}\pi_0(T_i)$ are isomorphic. Finally, using the Milnor's exact sequence associated to a tower of fibrations together with the fact that fibrations of simplicial sets are surjective (see Proposition VI-2.15 and Proposition VI-2.12-2 in [61]) we deduce an isomorphism

$$\pi_0(lim_{\mathbb{N}^{op}}T_i) \simeq lim_{\mathbb{N}^{op}}\pi_0(T_i) \tag{4.2.38}$$

and by combining everything we have

$$\pi_0(Stab_{(\bar{G},\bar{U})}(N_\Delta(\mathcal{M}^\circ))^{\simeq}) \simeq lim_{\mathbb{N}^{op}}\pi_0(N_\Delta(\mathcal{M}^\circ)^{\simeq}) \tag{4.2.39}$$

where the right hand side can be identified with the strict limit of the tower of sets

$$... \longrightarrow \pi_0(N_\Delta(\mathcal{M}^\circ)^{\simeq}) \xrightarrow{\pi_0(\bar{U})} \pi_0(N_\Delta(\mathcal{M}^\circ)^{\simeq}) \xrightarrow{\pi_0(\bar{U})} \pi_0(N_\Delta(\mathcal{M}^\circ)^{\simeq}) \tag{4.2.40}$$

and since $\bar{U}$ can be identified with $Q \circ U$, the elements of the last can be presented as sequences $([X_i])_{i\in\mathbb{N}}$ with each $[X_i]$ an equivalence class of an object X_i in $N_\Delta(\mathcal{M}^\circ)$, satisfying $[QU(X_{i+1})] = [X_i]$, which is the same as stating the existence of an equivalence in $N_\Delta(\mathcal{M}^\circ)$ between X_i and $QU(X_{i+1})$. Since we are dealing with cofibrant-fibrant objects, we can find an actual homotopy equivalence $X_i \to QU(X_{i+1})$ and by choosing a representative for each $[X_i]$ together with composition maps $X_i \to QU(X_{i+1}) \to U(X_{i+1})$ we retrieve a U-spectra. This proves that the map is essentially surjective.

It remains to prove ϕ is fully-faithful. Given two U-spectrum objects $X = (X_i)_{i\in\mathbb{N}}$ and $Y = (Y_i)_{i\in\mathbb{N}}$, the mapping space in $N_\Delta((Sp^{\mathbb{N}}(\mathcal{M}, G)_{stable})^\circ)$ between X and Y is given by the pullback[4] of the diagram

[4]see the formula (4.2.21)

$$\begin{tikzcd} & \prod_n Map_{\mathcal{M}}(X_i, Y_i) \arrow[d] \\ \prod_n Map_{\mathcal{M}}(X_i, Y_i) \arrow[r] & \prod_n Map_{\mathcal{M}}(X_i, U(Y_{i+1})) \end{tikzcd} \tag{4.2.41}$$

All vertices in this diagram are given by Kan-complexes (because $\mathcal{M}$ is a simplicial model category, each Y_i and X_i is cofibrant-fibrant and U is right-Quillen) and the vertical map is a fibration. Indeed, it can be identified with product of the compositions

$$Map_{\mathcal{M}}(X_{i+1}, Y_{i+1}) \to Map_{\mathcal{M}}(G(X_i), Y_{i+1})) \simeq Map_{\mathcal{M}}(X_i, U(Y_{i+1})) \tag{4.2.42}$$

where the last isomorphism follows from the adjunction data and the first map is the fibration induced by the composition with structure maps $G(X_i) \to X_{i+1}$ of X (which are cofibrations because X is a U-spectra). Therefore, the pullback square is an homotopy pullback.

At the same time, because of the equivalence of diagrams (4.2.19) the mapping spaces in $Stab_{(\bar{G},\bar{U})}(N_\Delta(\mathcal{M}^\circ))$ between the image of X and the image of Y can obtained[5] as the homotopy pullback of

$$\begin{tikzcd} & \prod_n Map_{\mathcal{M}}(X_i, Y_i) \arrow[d, "U"] \\ & \prod_n Map_{\mathcal{M}}(U(X_i), U(Y_i)) \arrow[d, "Q"] \\ & \prod_n Map_{\mathcal{M}}(QU(X_i), QU(Y_i)) \arrow[d] \\ \prod_n Map_{\mathcal{M}}(X_i, Y_i) \arrow[r] & \prod_n Map_{\mathcal{M}}(X_i, QU(Y_{i+1})) \end{tikzcd} \tag{4.2.43}$$

To conclude the proof it suffices to produce a weak-equivalence between the formulas. Indeed, we produce a map from the diagram (4.2.43) to the diagram (4.2.41), using the identity maps in the outer vertices and in the corner we use the product of the maps induced by the composition with the canonical map $QU(Y_{i+1}) \to U(Y_{i+1})$.

$$Map_{\mathcal{M}}(X_i, QU(Y_{i+1})) \to Map_{\mathcal{M}}(X_i, U(Y_{i+1})) \tag{4.2.44}$$

Of course, this map is a trivial fibration: $\mathcal{M}$ is a simplicial model category, X_i is cofibrant and $QU(Y_{i+1}) \to U(Y_{i+1})$ is a trivial fibration.

□

In the situation of the Proposition 4.2.4, with $\mathcal{M}$ a combinatorial simplicial model category and G a left-simplicial Quillen functor, we know that $Sp^{\mathbb{N}}(\mathcal{M}, G)^\circ_{stable}$ is again combinatorial and simplicial and so, both the underlying $(\infty,1)$-categories $N_\Delta(\mathcal{M}^\circ)$ and $N_\Delta(Sp^{\mathbb{N}}(\mathcal{M}, G)^\circ_{stable})$ are presentable (see the Proposition A.3.7.6 of [99]). Finally, using the Remark 4.2.1 we deduce the existence of canonical equivalence between $N_\Delta(Sp^{\mathbb{N}}(\mathcal{M}, G)^\circ_{stable})$ and the colimit of the sequence

$$N_\Delta(\mathcal{M}^\circ) \xrightarrow{\bar{G}} N_\Delta(\mathcal{M}^\circ) \xrightarrow{\bar{G}} ... \tag{4.2.45}$$

[5]The mapping spaces in the homotopy pullback are the homotopy pullback of the mapping spaces

4.2.2 Stabilization and Symmetric Monoidal Structures

Let us proceed. Our goal now is to compare the construction of spectra with the formal inversion $\mathcal{C}[X^{-1}]^{\otimes}$. The idea of a relation between the two comes from the following classical theorem:

Theorem 4.2.5. *(see Theorem 4.3 of [150])*
Let $\mathcal{C}$ be a symmetric monoidal category with tensor product $\otimes$ and unit 1. Let X be an object in $\mathcal{C}$. Let $Stab_X(\mathcal{C})$ denote the colimit of the sequence

$$... \xrightarrow{X\otimes -} \mathcal{C} \xrightarrow{X\otimes -} \mathcal{C} \xrightarrow{X\otimes -} \mathcal{C} \xrightarrow{X\otimes -} ... \tag{4.2.46}$$

in Cat (up to equivalence). Then, if the action of the cyclic permutation on $X\otimes X\otimes X$ becomes an identity map in $\mathcal{C}$ after tensoring with X an appropriate amount of times (which is the same as saying it is the identity map in $Stab_X(\mathcal{C})$) the category $Stab_X(\mathcal{C})$ admits a canonical symmetric monoidal structure and the canonical functor $\mathcal{C}\to Stab_X(\mathcal{C})$ is monoidal, sends X to an invertible object and is universal with respect to this property.

Proof. We can identify the colimit of the sequence with the category of pairs (A,n) where A is an object in $\mathcal{C}$ and n an integer. The hom-sets are given by the formula

$$Hom_{Stab_X(\mathcal{C})}((A,n),(B,m)) = colim_{(k>-n,-m)} Hom_{\mathcal{C}}(X^{n+k}\otimes A, X^{m+k}\otimes B) \tag{4.2.47}$$

The composition is the obvious one. There is a natural wannabe symmetric monoidal structure on $Stab_X(\mathcal{C})$, namely, the one given by the formula $(A,n)\wedge(B,m) := (A\otimes B, n+m)$. When we try to define this operation on the level of morphisms, we find the need for our hypothesis on X: Let $[f]:(Z,n)\to(Y,m)$ and $[g]:(A,a)\to(B,b)$ be two maps in $Stab_X(\mathcal{C})$. Let $f: X^{\alpha+n}(Z)\to X^{\alpha+m}(Y)$ and $g: X^{\gamma+a}(A)\to X^{\gamma+b}(B)$ be representatives for $[f]$ and $[g]$. Their product has to be a map in $Stab_X(\mathcal{C})$ represented by some map in $\mathcal{C}$, $X^{n+a+k}(Z\otimes A)\to X^{m+b+k}(Y\otimes B)$. In order to define this map from the data of f and g we have to make a choice of which copies of X should be kept together with Z and which should be kept with A. These choices will differ by some permutation of the factors of X, namely, for each two choices there will be a commutative diagram

$$\begin{tikzcd}
X^{n+a+\alpha+\gamma}(Z\otimes A) \arrow[r, "\exists\sigma\in\Sigma_{n+a+\alpha+\gamma}"'] \arrow[d, "\text{Use Choice 1}"] & X^{n+a+\alpha+\gamma}(Z\otimes A) \arrow[d, "\text{Use Choice 2}"] \\
X^{m+b+\alpha+\gamma}(Y\otimes B) \arrow[r, "\exists\sigma'\in\Sigma_{m+b+\alpha+\gamma}"'] & X^{m+b+\alpha+\gamma}(Y\otimes B)
\end{tikzcd} \tag{4.2.48}$$

The reason why we cannot adopt one choice once and for all, is because if we choose different representatives for f and g, for instance, $id_X\otimes f$ and $id_X\otimes g$, we will need a permutation of factors to make the second result equivalent to the one given by our first choice. Therefore, in order to have a well-define product map, it is sufficient to ask for the different permutations of the p-fold product X^p to become equal after tensoring with the identity of X an appropriate amount of times. In other words, they should become an identity map. For this, it is sufficient to ask for the action of the cyclic permutation (123) on X^3 to become the identity. This is because any permutation of p-factors can be built from permutations of 3-factors, by composition.

It is now an exercise to check that this operation, together with the object $(1,0)$ and the natural associators and commutators induced from $\mathcal{C}$, endow $Stab_X(\mathcal{C})$ with the structure of a symmetric monoidal category. Moreover, one can also check that the object $(X,0)$ becomes invertible, with inverse given by $(\mathbf{1},-1)$.

The fact that $Stab_X(\mathcal{C})$ when endowed with this symmetric monoidal structure is universal with respect to the inversion of X comes from fact that any monoidal functor $f:\mathcal{C}\to\mathcal{D}$ sending X to an invertible element produces a morphism of diagrams (in the homotopy category of (small) categories)

$$\begin{tikzcd} \ldots \arrow[r, "X\otimes -"] & \mathcal{C} \arrow[r, "X\otimes -"] \arrow[d, "f"] & \mathcal{C} \arrow[r, "X\otimes -"] \arrow[d, "f"] & \mathcal{C} \arrow[r, "X\otimes -"] \arrow[d, "f"] & \ldots \\ \ldots \arrow[r, "f(X)\otimes -"] & \mathcal{D} \arrow[r, "f(X)\otimes -"] & \mathcal{D} \arrow[r, "f(X)\otimes -"] & \mathcal{D} \arrow[r, "f(X)\otimes -"] & \ldots \end{tikzcd} \tag{4.2.49}$$

together with an associated colimit map $Stab_X(\mathcal{C}) \to Stab_X(\mathcal{D})$. To conclude the proof we need two observations: let $\mathcal{D}$ be a symmetric monoidal category and let U be an invertible object in $\mathcal{D}$, then we the following facts:

1. U automatically satisfies the cocyle condition. This follows from a more general fact. If U is an invertible object in $\mathcal{C}$, we can prove that the group of automorphisms of U in $\mathcal{C}$ is necessarily abelian. This follows from the existence of an isomorphism $U \simeq U \otimes U^* \otimes U$ and the fact that any map $f : U \to U$ can either be written as $f \otimes id_X \otimes id_X$ or $id_X \otimes id_X \otimes f$. Given two maps f and g we can write

$$g \circ f = (g \otimes id_X \otimes id_X) \circ (id_X \otimes id_X \otimes f) = (f \otimes id_X \otimes id_X) \circ (id_X \otimes id_X \otimes g) = f \circ g \tag{4.2.50}$$

   The fact that U satisfies the cocycle condition is an immediate consequence, because the actions of the transpositions $(i, i+1)$ and $(i+1, i+2)$ have to commute and we have the identity $((i, i+1) \circ (i+1, i+2))^3 = id$.

2. the functor $U \otimes - : \mathcal{D} \to \mathcal{D}$ is an equivalence of categories with inverse given by multiplication with U^*, the inverse of U in $\mathcal{D}$. In this case, multiplications by the powers of U and U^* make $\mathcal{D}$ a cocone over the stabilizing diagram. It is an easy observation that the canonical colimit $Stab_U(\mathcal{D}) \to \mathcal{D}$ (which can be described by the formula $(A, n) \mapsto (U^*)^n \otimes A$) is an equivalence. Moreover, since U satisfies the cocycle condition (following the previous item), $Stab_U(\mathcal{D})$ comes naturally equipped with a symmetric monoidal structure and we can check that the colimit map is monoidal. Under these circumstances, any monoidal functor $f : \mathcal{C} \to \mathcal{D}$ with $f(X)$ invertible, gives a canonical colimit map $Stab_X(\mathcal{C}) \to Stab_{f(X)}(\mathcal{D}) \simeq \mathcal{D}$. It is an observation that this map is monoidal under our hypothesis on X. This implies the universal property.

□

Remark 4.2.6. The condition on X appearing in the previous result is trivially satisfied if the action of the cyclic permutation $(X \otimes X \otimes X)^{(1,2,3)}$ is already an identity map in $\mathcal{C}$. For instance, this particular situation holds when $\mathcal{C}$ is the pointed $\mathbb{A}^1$-homotopy category and X is $\mathbb{P}^1$ (See Theorem 4.3 and Lemma 4.4 of [150]).

Our goal now is to find an analogue for the previous theorem in the context of symmetric monoidal $(\infty, 1)$-categories.

Definition 4.2.7. *Let $\mathcal{C}^{\otimes}$ be a symmetric monoidal $(\infty, 1)$-category and let X be an object in $\mathcal{C}$. We say that X is* symmetric *if there is a 2-equivalence in $\mathcal{C}$ between the cyclic permutation $\sigma : (X \otimes X \otimes X)^{(1,2,3)}$ and the identity map of $X \otimes X \otimes X$. In other words, we demand the existence of a 2-simplex in $\mathcal{C}$*

$$\begin{tikzcd} X \otimes X \otimes X \arrow[r, "\sigma"] \arrow[d, "id"'] \arrow[dr, phantom, "\Rightarrow"] & X \otimes X \otimes X \\ X \otimes X \otimes X \arrow[ur, "id"'] & \end{tikzcd} \tag{4.2.51}$$

providing an homotopy between the cyclic permutation and the identity. This is equivalent to the condition that σ is the identity of $X \otimes X \otimes X$ in $h(\mathcal{C})$.

This notion of symmetry is well behaved under equivalences. Moreover, it is immediate that monoidal functors map symmetric objects to symmetric objects.

Remark 4.2.8. Let $\mathcal{V}$ be a symmetric monoidal model category with a cofibrant unit 1. Recall that a unit interval I is a cylinder object for the unit of the monoidal structure $I := C(1)$, together with a map $I \otimes I \to I$ such that the diagrams

$$\begin{tikzcd} 1 \otimes I \simeq I \arrow[r, "\pi"] \arrow[d, "\partial_0 \otimes Id_I"'] & 1 \arrow[d, "\partial_0"] \\ I \otimes I \arrow[r] & I \end{tikzcd} \tag{4.2.52}$$

$$\begin{tikzcd} I \otimes 1 \simeq I \arrow[r, "\pi"] \arrow[d, "Id_I \otimes \partial_0"'] & 1 \arrow[d, "\partial_0"] \\ I \otimes I \arrow[r] & I \end{tikzcd} \tag{4.2.53}$$

and

$$\begin{tikzcd} I \otimes 1 \simeq I \arrow[d, "\partial_1 \otimes Id_I"'] \arrow[dr, "Id_I"] & \\ I \otimes I \arrow[r] & I \end{tikzcd} \tag{4.2.54}$$

commute, where $\partial_0, \partial_1 : 1 \to I$ and $\pi : I \to 1$ are the maps providing I with a structure of cylinder object.

Recall also that two maps $f, g : A \to B$ are said to be homotopic with respect to a unit interval I if there is a map $H : A \otimes I \to B$ rendering the diagram commutative

$$\begin{tikzcd} A \simeq A \otimes 1 \arrow[dr, "Id_A \otimes \partial_0"'] \arrow[drr, bend left, "f"] & & \\ & A \otimes I \arrow[r, "H"] & B \\ A \simeq A \otimes 1 \arrow[ur, "id_A \otimes \partial_1"] \arrow[urr, bend right, "g"'] & & \end{tikzcd} \tag{4.2.55}$$

In [71, Defn. 10.2], the author defines an object X of $\mathcal{V}$ to be *symmetric* if it is cofibrant and if there is a *unit interval* I, together with an homotopy

$$H : X \otimes X \otimes X \otimes I \to X \otimes X \otimes X \tag{4.2.56}$$

between the cyclic permutation σ and the identity map. We observe that if an object X is symmetric in the sense of [71] then it is symmetric as an object in the underlying symmetric monoidal $(\infty,1)$-category of $\mathcal{V}$ in the sense of the Definition 4.2.7. Indeed, since $\mathcal{V}$ is a symmetric monoidal model category with a cofibrant unit, the full subcategory $\mathcal{V}^c$ of cofibrant objects is closed under the tensor product and therefore inherits a monoidal structure, which moreover preserves weak-equivalences in each variable. In Section 3.9 we used this fact to define the underlying symmetric monoidal $(\infty,1)$-category of $\mathcal{V}$, $N((\mathcal{V}^c)^{\otimes})[W_c^{-1}]$ (see Section 3.9 for the notations). Its underlying $(\infty,1)$-category is $N(\mathcal{V}^c)[W^{-1}]$ and its homotopy category is the classical localization in Cat. Moreover, it comes canonically equipped with a monoidal functor $L : N^{\otimes}((\mathcal{V}^c)^{\otimes}) \to N^{\otimes}((\mathcal{V}^c)^{\otimes})[W_c^{-1}]$. Now, if X is symmetric in $\mathcal{V}$ in the sense of [71], the homotopy H forces σ to become the identity in $h(N(\mathcal{V}^c)[W^{-1}])$ (because the classical localization functor is monoidal and the map $I \to 1$ is a weak-equivalence). The conclusion now follows from the commutativity of the diagram induced by the unit of the adjunction (h, N)

$$\begin{tikzcd} N(\mathcal{V}^c) \arrow[r] \arrow[d, "\sim"'] & N(\mathcal{V}^c)[W^{-1}] \arrow[d] \\ N(h(N(\mathcal{V}^c))) \arrow[r] & N(h(N(\mathcal{V}^c)[W^{-1}])) \end{tikzcd} \tag{4.2.57}$$

and the fact that the both horizontal arrows are monoidal and therefore send the cyclic permutation of the monoidal structure in $\mathcal{V}$ to the cyclic permutation associated to the monoidal structure in $N((\mathcal{V}^c)^{\otimes})[W_c^{-1}]$.

We now come to the generalization of the Theorem 10.3 of [71]. The following results relate our formal inversion of an object to the construction of spectrum objects.

Remark 4.2.9. Let $\mathcal{C}^{\otimes}$ be a small monoidal $(\infty,1)$-category and let $\overline{M}$ be an object in $Mod_{\mathcal{C}^{\otimes}}(Cat_{\infty})$ (which we will understand as a left-module). Since Cat_{∞} admits classifying objects for endomorphisms given by the categories of endofunctors, the data of $\overline{M}$ is equivalent to the data of an $(\infty,1)$-category $M := \overline{M}(\mathbf{m})$ together with a monoidal functor $T^{\otimes}: \mathcal{C}^{\otimes} \to End(M)^{\otimes}$ where the last is endowed with the associative monoidal structure induced by the composition of maps of simplicial sets (see [100, 6.2.0.2]). If X is an object in $\mathcal{C}$, the endofunctor $T(X): M \to M$ corresponds to the action of X in M by means of the operation $\mathcal{C} \times M \to M$ encoded in the module-structure. We will call it the *multiplication by* X.

Notice that if the monoidal structure $\mathcal{C}^{\otimes}$ is symmetric, the map $T(X)$ acquires the structure of a map of $\mathcal{C}$-modules. Indeed, as $T^{\otimes}$ is monoidal, it will send an object $(Y,X) \in \mathcal{C}^{\otimes}_{\langle 2\rangle}$ to $(T(Y),T(X))$ in $End(M)^{\otimes}_{\langle 2\rangle}$ and the twisting equivalence $\tau_{Y,X}: (Y,X) \simeq (X,Y)$ to an equivalence $(T(Y),T(X)) \simeq (T(X),T(Y))$. By the definition of coCartesian morphisms in $End(M)^{\otimes}$, the last equivalence provides a natural equivalence $T(Y)\circ T(X) \simeq T(X)\circ T(Y)$ that gives the coherence data making $T(X)$ a map of modules. These coherences define commutative diagrams $\Delta[1] \times \Delta[1] \to Cat_{\infty}$ that we can informally describe as

$$\begin{tikzcd} M \arrow[r, "T(Y)"] \arrow[d, "T(X)"'] & M \arrow[d, "T(X)"] \arrow[dl, Rightarrow, "T(\tau_{Y,X})"] \\ M \arrow[r, "T(Y)"'] & M \end{tikzcd} \tag{4.2.58}$$

More generally, the extra coherences that make $T(X)$ a map of modules are given by the higher order cyclic permutations of factors in $\mathcal{C}^{\otimes}$. The importance of this fact will become clear in the next proposition.

The following is our key result:

Proposition 4.2.10. *Let $\mathcal{C}^{\otimes}$ be a small symmetric monoidal $(\infty,1)$-category and X be a symmetric object in $\mathcal{C}$. Then, for any $\mathcal{C}^{\otimes}$-module $\overline{M}$, the colimit of the diagram of $\mathcal{C}^{\otimes}$-modules*

$$\overline{Stab}_X(\overline{M}) := colimit_{Mod_{\mathcal{C}^{\otimes}}(Cat_{\infty})}(... \longrightarrow \overline{M} \xrightarrow{\overline{T}(X)} \overline{M} \xrightarrow{\overline{T}(X)} \overline{M} \xrightarrow{\overline{T}(X)} ...) \tag{4.2.59}$$

is a $\mathcal{C}^{\otimes}$-module where the multiplication by X is an equivalence.

Proof. Let $d: N(\mathbb{Z}) \to Mod_{\mathcal{C}}(Cat_{\infty})$ be the diagram corresponding to the multiplication by X. Since $Cat_{\infty}^{\times}$ is compatible with all small colimits, the Corollary 3.4.4.6 of [100] [6] implies that d can be extended to a colimit diagram $d': N(\mathbb{Z})^{\triangleright} \to Mod_{\mathcal{C}}(Cat_{\infty})$. Moreover, this extension is a colimit diagram if and only if the composition with the forgetful functor to Cat_{∞} is a colimit diagram. Let

[6] Since we are working the commutative setting, we could also refer to the Corollary 4.2.3.5 of [100]

∞ denote the new joint vertice in $N(\mathbb{Z})^{\triangleright}$ and set $\overline{Stab}_X(\overline{M}) := d'(\infty)$. Moreover, let $\phi_i := d'(i \to \infty)$. As a first step we need to understand how an object $Y \in \mathcal{C}$ acts on this new module $\overline{Stab}_X(\overline{M})$. For that purpose we observe that as the ϕ_i are, by definition, maps of modules, we have commutative diagrams

$$\begin{tikzcd} M \arrow[r, "T(Y)"] \arrow[d, "\phi_i"] & M \arrow[d, "\phi_i"] \\ \overline{Stab}_X(\overline{M}) \arrow[r, dashed, "Y"] & \overline{Stab}_X(\overline{M}) \end{tikzcd}$$

and as the $T(X)'s$ are maps of $\mathcal{C}$-modules, this action of Y on $\overline{Stab}_X(\overline{M})$ appears as the canonical map (induced by the universal property of colimits) produced by the morphism of diagrams $D' : N(\mathbb{Z})^{\triangleright} \times \Delta[1] \to Mod_{\mathcal{C}}(Cat_\infty)$ levelwise given by $T(Y)$. This can be obtained as follows: we consider the diagram $D : N(\mathbb{Z}) \times \Delta[1] \to Mod_{\mathcal{C}}(Cat_\infty)$ obtained by composing the commutative diagrams described in the Remark 4.2.9 side by side. This can also be written as $D : N(\mathbb{Z}) \to Fun(\Delta[1], Mod_{\mathcal{C}}(Cat_\infty))$. By [99, 5.1.2.3] this diagram admits a colimit cone $D' : N(\mathbb{Z})^{\triangleright} \to Fun(\Delta[1], Mod_{\mathcal{C}}(Cat_\infty))$ characterized by the fact that both the source and targe of $D'(\infty)$ are colimit cones of the restrictions to 1 and 0. This presents the action of Y on $\overline{Stab}_X(\overline{M})$ as a colimit of the actions of Y on M. More informally, we now can picture the situation as

$$\begin{tikzcd} \overline{Stab}_X(\overline{M}) \arrow[d, dashed, "Y"] & ... \arrow[r] & M \arrow[r, "T(X)"] \arrow[d, "T(Y)"'] & M \arrow[r, "T(X)"] \arrow[d, "T(Y)"] \arrow[dl, Rightarrow, "T(\tau)"] & M \arrow[r] \arrow[d, "T(Y)"] \arrow[dl, Rightarrow, "T(\tau)"] & ... \\ \overline{Stab}_X(\overline{M}) & ... \arrow[r] & M \arrow[r, "T(X)"'] & M \arrow[r, "T(X)"'] & M \arrow[r] & ... \end{tikzcd} \tag{4.2.60}$$

Our goal now is to understand this action when $Y = X$. But before that it, is important to understand the consecutive composition of two commutative diagrams

$$\begin{tikzcd} M \arrow[r, "T(X)"] \arrow[d, "T(Y)"'] & M \arrow[r, "T(Z)"] \arrow[d, "T(Y)"] \arrow[dl, Rightarrow, "T(\tau_{X,Y})"] & M \arrow[d, "T(Y)"] \arrow[dl, Rightarrow, "T(\tau_{Z,Y})"] \\ M \arrow[r, "T(X)"'] & M \arrow[r, "T(Z)"'] & M \end{tikzcd} \tag{4.2.61}$$

This can be informally describe as a new commutative square

$$\begin{tikzcd} M \arrow[r, "T(Z)\circ T(X)"] \arrow[d, "T(Y)"'] & M \arrow[d, "T(Y)"] \arrow[dl, Rightarrow, "T(\tau_{Z,Y})\circ T(\tau_{X,Y})"] \\ M \arrow[r, "T(Z)\circ T(X)"'] & M \end{tikzcd} \tag{4.2.62}$$

and our main observation is that the horizontal composition $T(\tau_{Z,Y}) \circ T(\tau_{Y,X})$ can be identified with the natural transformation $T(\sigma_{Z,X,Y})$ induced by the cyclic permutation $\sigma_{Z,X,Y} : (Z, X, Y) \to (Y, Z, X)$ in $\mathcal{C}^{\otimes}_{\langle 3\rangle}$. Indeed, $T^{\otimes}$ being monoidal, the equivalence $\sigma_{Z,X,Y}$ produces an equivalence $(T(Z), T(X), T(Y)) \simeq (T(Y), T(Z), T(X))$ which by choosing coCartesian morphisms in $End(M)^{\otimes}$ over $\langle 3\rangle \to \langle 1\rangle$, give the commutativity $T(Z) \circ T(X) \circ T(Y) \simeq T(Y) \circ T(Z) \circ T(X)$. The key point to complete the argument is that the permutation $\sigma_{Z,X,Y} : (Z, X, Y) \to (Y, Z, X)$ can be written as a composition of two consecutive twists, namely $\sigma_{Z,X,Y} \simeq (\tau_{Z,Y}, id_X) \circ (id_Z, \tau_{X,Y})$ and as $T^{\otimes}$ is functorial we have $T(\tau_{Z,Y}) \circ T(\tau_{X,Y}) \simeq T(\sigma_{Z,X,Y})$

Let us now go back to the case when Y and Z are X. In this case, since by assumption X is symmetric, there is a 2-simplex in $\mathcal{C}$ providing an homotopy between σ and the identity of $X \otimes X \otimes X$.

In this case $T(\sigma)$ is equivalent to the identity and the 2-simplex rendering the commutativity of the composition (4.2.62) are the identity faces. By confinality, the map $\overline{Stab}_X(M) \to \overline{Stab}_X(M)$ induced by the morphism of diagrams D is equivalent to the map induced by the $\mathbb{Z}$-indexed diagram given by the composition of the commutative squares

$$\begin{tikzcd}
... \arrow[r, "T(X)"] & M \arrow[r, "T(X)"] \arrow[d, "T(X)"'] & M \arrow[r, "T(X)"] \arrow[d, "T(X)"] \arrow[dl, Rightarrow, "Id"] & M \arrow[r, "T(X)"] \arrow[d, "T(X)"] \arrow[dl, Rightarrow, "Id"] & ... \\
... \arrow[r, "T(X)"'] & M \arrow[r, "T(X)"'] & M \arrow[r, "T(X)"'] & M \arrow[r, "T(X)"'] & ...
\end{tikzcd} \tag{4.2.63}$$

and therefore, by definition of colimit cone, it is an equivalence.

□

Remark 4.2.11. A similar argument shows that the same result holds if X is n-symmetric, meaning that, there exists $n \in \mathbb{N}, n \geq 2$ such that τ^n is equal to the identity map in $h(\mathcal{C})$.

Remark 4.2.12. The Remark 4.2.9 and the Proposition 4.2.10 applies mutatis-mutandis in the presentable setting. This is true because of the Proposition 3.6.3 - $\mathcal{P}r^L$ admits classifying objects for endomorphisms. If M is a presentable $(\infty,1)$-category, $End^L(M)$ is a classifying object for endomorphisms of M, with the associative monoidal structure given by the composition of functors.

We can finally establish the connection between the adjoint $\mathcal{L}^{Pr}_{(\mathcal{C}^{\otimes},X)}$ and the notion of spectra.

Corollary 4.2.13. *Let $\mathcal{C}^{\otimes}$ be a presentable symmetric monoidal $(\infty,1)$-category and let X be a symmetric object in $\mathcal{C}$. Given a $\mathcal{C}^{\otimes}$-module M, $Stab_X(M)$ is a $\mathcal{C}^{\otimes}$-module where X acts as an equivalence and therefore the adjunction of Proposition 4.1.11 provides a map of $\mathcal{C}^{\otimes}$-modules*

$$\mathcal{L}^{Pr}_{(\mathcal{C}^{\otimes},X)}(M) \to Stab_X(M) \tag{4.2.64}$$

This map is an equivalence. In particular, the underlying ∞-category of the formal inversion $\mathcal{C}^{\otimes}[X^{-1}]$ is equivalent to the stabilization $Stab_X(\mathcal{C})$.

Proof. The map can be obtained as a composition

$$\mathcal{L}^{Pr}_{(\mathcal{C}^{\otimes},X)}(M) \to \mathcal{L}^{Pr}_{(\mathcal{C}^{\otimes},X)}(Stab_X(M)) \to Stab_X(M) \tag{4.2.65}$$

where the first arrow is the image of the canonical map $M \to Stab_X(M)$ by the adjunction $\mathcal{L}^{Pr}_{(\mathcal{C}^{\otimes},X)}$ and the second arrow is the counit of the adjunction. In fact, with our hypothesis and because of the previous Proposition, the action of X is invertible in $Stab_X(M)$ and therefore, by the Proposition 4.1.11 the second arrow is an equivalence It remains to prove that the first map is an equivalence. But now, since $Stab_X(M)$ is a colimit and $\mathcal{L}^{Pr}_{(\mathcal{C}^{\otimes},X)}$ is a left adjoint and therefore commutes with colimits, we have a commutative diagram

$$\begin{tikzcd}
\mathcal{L}^{Pr}_{(\mathcal{C}^{\otimes},X)}(M) \arrow[r] \arrow[dr] & \mathcal{L}^{Pr}_{(\mathcal{C}^{\otimes},X)}(Stab_X(M)) \\
& Stab_X(\mathcal{L}^{Pr}_{(\mathcal{C}^{\otimes},X)}(M)) \arrow[u, "\sim"]
\end{tikzcd} \tag{4.2.66}$$

where the diagonal arrow is the colimit map induced by the stabilization of $\mathcal{L}^{Pr}_{(\mathcal{C}^{\otimes},X)}(M)$. It is enough now to observe that if M is a $\mathcal{C}^{\otimes}$-module where the action of X is already invertible, then the canonical map $M \to Stab_X(M)$ is an equivalence of modules. The *2 out of 3* argument concludes the proof.

□

In particular

Corollary 4.2.14. *Let $\mathcal{C}^{\otimes}$ be a stable presentable symmetric monoidal $(\infty,1)$-category and let X be a symmetric object in $\mathcal{C}$. Then $\mathcal{C}^{\otimes}[X^{-1}]$ is again a stable presentable symmetric monoidal $(\infty,1)$-category.*

Proof. If $\mathcal{C}^{\otimes}$ is stable presentable, the multiplication by X is an exact functor. Moreover, since X is symmetric, the previous corollary provides an equivalence $\mathcal{C}[X^{-1}] \simeq Stab_X(\mathcal{C})$ where the last is a colimit in $\mathcal{P}r^L$. Moreover, since the whole diagram is in $\mathcal{P}r^L_{Stb}$ and the last has all colimits and the inclusion $\mathcal{P}r^L_{Stb} \subseteq \mathcal{P}r^L$ commutes with them[7], we find that $\mathcal{C}[X^{-1}]$ is stable. Moreover, since by construction $\mathcal{C}^{\otimes}[X^{-1}]$ is a presentable symmetric monoidal $(\infty,1)$-category, we conclude it is a stable presentable symmetric monoidal $(\infty,1)$-category. □

Remark 4.2.15. Let $\mathcal{C}^{\otimes}$ be a small symmetric monoidal ∞-groupoid and let X be a symmetric object in $\mathcal{C}$. Then, using the same arguments as in the proof of the previous corollary together with the fact that the $(\infty,1)$-category of spaces $\mathcal{S}$ admits classifying objects for endomorphisms, we deduce that the underlying $(\infty,1)$-category of the formal inversion $\mathcal{L}^{spaces,\otimes}_{\mathcal{C}^{\otimes},X}(\mathcal{C}^{\otimes})$ of the Remark 4.1.7 is equivalent to the stabilization $Stab^{spaces}_X(\mathcal{C})$ obtained as the colimit in $\mathcal{S}$ of the diagram induced by the multiplication by X. Moreover, since the inclusion $\mathcal{S} \subseteq Cat_\infty$ admits a right adjoint (the "maximal ∞-groupoid"), it preserves colimits and we see that the comparison map of 4.1.7 is an equivalence

$$\mathcal{L}^{\otimes}_{i(\mathcal{C}^{\otimes}),X}(i(\mathcal{C}^{\otimes}))_{\langle 1\rangle} \simeq Stab_X(i(\mathcal{C})) \simeq i(Stab^{spaces}_X(\mathcal{C})) \simeq \mathcal{L}^{spaces,\otimes}_{\mathcal{C}^{\otimes},X}(\mathcal{C}^{\otimes})_{\langle 1\rangle} \tag{4.2.67}$$

where $Stab_X(i(\mathcal{C}))$ is the stabilization in Cat_∞.

Example 4.2.16. In [100] the author introduces the $(\infty,1)$-category of spectra Sp as the stabilization of the $(\infty,1)$-category of spaces. More precisely, following the notations of the Example 4.2.2 it is given by

$$Sp := Sp^{\mathbb{N}}_{(\Sigma_{\mathcal{S}},\Omega_{\mathcal{S}})}(\mathcal{S}_{*/}) \tag{4.2.68}$$

where $\mathcal{S}$ denotes the $(\infty,1)$-category of spaces. By the Propositions and 1.4.3.6 and 1.4.4.4 of [100] this $(\infty,1)$-category is presentable and stable and by the Proposition 4.8.2.18 of [100] it admits a natural presentable stable symmetric monoidal structure $Sp^{\otimes}$ which can be described by means of a universal property: it is an initial object in $CAlg(\mathcal{P}r^L_{Stb})$. The unit of this monoidal structure is the sphere-spectrum.

Our corollary 4.2.13 provides an alternative characterization of this symmetric monoidal structure. We start with $\mathcal{S}_*$ the $(\infty,1)$-category of pointed spaces. Recall that this $(\infty,1)$-category is presentable and admits a monoidal structure given by the so-called *smash product* of pointed spaces. (see the Remark 4.8.2.14 of [100] and the section 5.2 below). We will denote it as $\mathcal{S}^{\wedge}_*$. According to the Proposition 4.8.2.11 of [100], $\mathcal{S}^{\wedge}_*$ has an universal property amongst the presentable pointed symmetric monoidal $(\infty,1)$-categories: it is a initial one. The unit of this monoidal structure is the pointed space $S^0 = * \coprod *$. We will see below (Corollary 5.2.3 and Remark 5.2.4) that $\mathcal{S}^{\wedge}_*$ is the underlying symmetric monoidal $(\infty,1)$-category of the combinatorial simplicial model category of pointed simplicial sets $\hat{\Delta}_*$ equipped with the classical smash product of spaces. Since S^1 is symmetric in $\hat{\Delta}_*$ with respect to this classical smash (see the Lemma 6.6.2 of [69]), by the Remark 4.2.8 it will also be symmetric in $\mathcal{S}^{\wedge}_*$. Our inversion $\mathcal{S}^{\wedge}_*[(S^1)^{-1}]$ provides a new presentable symmetric monoidal $(\infty,1)$-category and because of the symmetry of S^1, the fact that $(S^1 \wedge -)$ can be identified with $\Sigma_{\mathcal{S}}$ and the Corollary 4.2.13, we conclude that the underlying $(\infty,1)$-category of $\mathcal{S}^{\wedge}_*[(S^1)^{-1}]$ is the stabilization defining Sp and therefore that $\mathcal{S}^{\wedge}_*[(S^1)^{-1}]$ is a presentable stable symmetric monoidal $(\infty,1)$-category. By the universal property of $Sp^{\otimes}$ there is a unique (up to a contractible space of choices) monoidal map

$$Sp^{\otimes} \to \mathcal{S}^{\wedge}_*[(S^1)^{-1}] \tag{4.2.69}$$

[7] To see this we can use the equivalence between $\mathcal{P}r^L_{Stb}$ and $Mod_{Sp}(\mathcal{P}r^L)$ [100, 4.8.2.18] and the identification of the inclusion $\mathcal{P}r^L_{Stb} \subseteq \mathcal{P}r^L$ with the forgetful functor $Mod_{Sp}(\mathcal{P}r^L) \to \mathcal{P}r^L$. Now we use the fact that $\mathcal{P}r^{L,\otimes}$ is compatible with colimits (its has internal-hom objects) and therefore colimits of modules are computed in $\mathcal{P}r^L$ using the forgetful functor [100, 3.4.4.6].

At the same time, since every stable presentable $(\infty,1)$-category is pointed, the universal property of $\mathcal{S}_*^{\wedge}$ ensures the existence of a canonical morphism

$$\mathcal{S}_*^{\wedge} \to Sp^{\otimes} \tag{4.2.70}$$

which is also unique up to a contractible space of choices. This morphism is just the canonical stabilization morphism and it sends S^1 to the sphere-spectrum in Sp and therefore the universal property of the localization provides a factorization

$$\mathcal{S}_*^{\wedge}[(S^1)^{-1}] \to Sp^{\otimes} \tag{4.2.71}$$

which is unique up to homotopy. By combining the two universal properties we find that these two maps are in fact inverses up to homotopy

Remark 4.2.17. The technique of inverting an object provides a way to define the monoidal stabilization of a pointed presentable symmetric monoidal $(\infty,1)$-category $\mathcal{C}^{\otimes}$. It follows from the Proposition 4.8.2.11 of [100] that for any such $\mathcal{C}^{\otimes}$, there is an essentially unique (base-point preserving and colimit preserving) monoidal map $f : \mathcal{S}_*^{\wedge} \to \mathcal{C}^{\otimes}$. Let $f(S^1)$ denote the image of the topological circle through this map. The (presentable) universal property of inverting an object provides an homotopy commutative diagram of commutative algebra objects in $\mathcal{P}r^L$

$$\begin{tikzcd} Sp^{\otimes} \simeq \mathcal{S}_*^{\wedge}[(S^1)^{-1}] \arrow[d, dashed] & \mathcal{S}_*^{\wedge} \arrow[l] \arrow[d, "f"] \\ \mathcal{C}^{\otimes}[f(S^1)^{-1}] & \mathcal{C}^{\otimes} \arrow[l] \end{tikzcd} \tag{4.2.72}$$

The monoidal map $\mathcal{S}_*^{\wedge} \to Sp^{\otimes}$ produces a forgetful functor

$$CAlg(\mathcal{P}r^L)_{Sp^{\otimes}/} \to CAlg(\mathcal{P}r^L)_{\mathcal{S}_*^{\wedge}/} \tag{4.2.73}$$

which by the Proposition 4.1.11 is fully faithful and admits a left adjoint given by the base-change formula $\mathcal{C}^{\otimes} \mapsto Sp^{\otimes} \otimes_{\mathcal{S}_*^{\wedge}} \mathcal{C}^{\otimes}$. The combination of the universal property of the adjunction and the universal property of inverting an object ensures the existence of an equivalence of pointed symmetric monoidal $(\infty,1)$-categories

$$\mathcal{C}^{\otimes}[f(S^1)^{-1}] \simeq Sp^{\otimes} \otimes_{\mathcal{S}_*^{\wedge}} \mathcal{C}^{\otimes} \tag{4.2.74}$$

Finally, combining this with the Example 4.8.1.22 of [100] we deduce that the underlying $(\infty,1)$-category of $\mathcal{C}^{\otimes}[f(S^1)^{-1}]$ is the stabilization $Stab(\mathcal{C})$.

Moreover, we deduce also that if $\mathcal{C}^{\otimes}$ is a stable presentable symmetric monoidal $(\infty,1)$-category and X is any object in $\mathcal{C}$, in order to conclude that the inversion $\mathcal{C}^{\otimes}[X^{-1}]$ is stable presentable it is enough to show that $\mathcal{C}[X^{-1}]$ is pointed, thus extending the result 4.2.14. Indeed, by the previous discussion, $\mathcal{C}$ is stable if and only if $f(S^1)$ is invertible. Since the inversion functor $\mathcal{C}^{\otimes} \to \mathcal{C}^{\otimes}[X^{-1}]$ is monoidal, the image of $f(S^1)$ in $\mathcal{C}[X^{-1}]$ will again by invertible. Finally, if $\mathcal{C}[X^{-1}]$ is pointed, the image of $f(S^1)$ will necessarily correspond to the image of S^1 in $\mathcal{C}[X^{-1}]$, which therefore will be invertible, and so, by the previous discussion, $\mathcal{C}[X^{-1}]$ will be stable.

4.3 Connection with the Symmetric Spectrum objects of Hovey

We recall from [71] the construction of symmetric spectrum objects: Let $\mathcal{V}$ be a combinatorial simplicial symmetric monoidal model category and let $\mathcal{M}$ be a combinatorial simplicial $\mathcal{V}$-model category. Following the Theorem 8.11 of [71], for any object X in $\mathcal{V}$ we can produce a new combinatorial simplicial $\mathcal{V}$-model category $Sp^{\Sigma}(\mathcal{M}, X)$ of spectrum objects in $\mathcal{M}$ endowed with the *stable model structure* and where X acts by an equivalence. In particular, by considering $\mathcal{V}$ as a $\mathcal{V}$-model category (using the monoidal structure) the new model category $Sp^{\Sigma}(\mathcal{V},X)$ inherits the structure of a combinatorial simplicial symmetric monoidal model category and there is left simplicial Quillen monoidal map $\mathcal{V}\to Sp^{\Sigma}(\mathcal{V},X)$ sending X to an invertible object.

This general construction sends an arbitrary combinatorial simplicial $\mathcal{V}$-model category to a combinatorial simplicial $\mathcal{V}$-model category where the action of X is invertible. In fact, by the Theorem 8.11 of [71] $Sp^{\Sigma}(\mathcal{M},X)$ is a combinatorial simplicial $Sp^{\Sigma}(\mathcal{V},X)$-model category. This is a first sign of the fundamental role of the construction of symmetric spectrum objects as an adjoint in the spirit of Section 4.1. We have canonical simplicial left Quillen maps

$$Sp^{\Sigma}(\mathcal{V},X)\xrightarrow{\sim} Sp^{\mathbb{N}}(Sp^{\Sigma}(\mathcal{V},X),X)\xrightarrow{\sim} Sp^{\Sigma}(Sp^{\mathbb{N}}(\mathcal{V},X),X)\longleftarrow Sp^{\mathbb{N}}(\mathcal{V},X) \tag{4.3.1}$$

but in general the last map is not an equivalence. By the Theorem 9.1 of [71] for the last map to be an equivalence we only need $Sp^{\mathbb{N}}(\mathcal{V},X)$ to be a $\mathcal{V}$-model category where X acts as an equivalence. This is exactly the functionality of the symmetric condition on X (see Theorems 10.1 and 10.3 in [71]).

We now state our main result

Theorem 4.3.1. *Let $\mathcal{V}$ be a combinatorial simplicial symmetric monoidal model category whose unit is cofibrant and let X be a symmetric object in $\mathcal{V}$ in the sense of the Remark 4.2.8. Let $Sp^{\Sigma}(\mathcal{V},X)$ denote the combinatorial simplicial symmetric monoidal model category provided by the Theorem 8.11 of [71], equipped the convolution product. Let $\mathcal{C}^{\otimes}$ and $Sp^{\Sigma}_X(\mathcal{C})^{\otimes}$ denote their underlying presentable symmetric monoidal $(\infty,1)$-categories.[8]. The left-Quillen monoidal map $\mathcal{V}\to Sp^{\Sigma}(\mathcal{V},X)$ induces a monoidal functor $\mathcal{C}^{\otimes}\to Sp^{\Sigma}_X(\mathcal{C})^{\otimes}$[9] which sends X to an invertible object, endowing $Sp^{\Sigma}_X(\mathcal{C})^{\otimes}$ with the structure of object in $CAlg(\mathcal{P}r^L)^X_{\mathcal{C}^{\otimes}/}$. In this case, the adjunction of the Prop.4.1.11 provides a monoidal map*

$$\mathcal{C}^{\otimes}[X^{-1}]\simeq \mathcal{L}^{Pr,\otimes}_{(\mathcal{C}^{\otimes},X)}(\mathcal{C}^{\otimes})\to Sp^{\Sigma}_X(\mathcal{C})^{\otimes} \tag{4.3.2}$$

We claim that this map is an equivalence of presentable symmetric monoidal $(\infty,1)$-categories.

Proof. By the remark 4.2.8 if X is symmetric in the sense of [71] then it is symmetric in $\mathcal{C}^{\otimes}$ in the sense of the Definition 4.2.7.

By definition, the map is obtained as a composition

$$\mathcal{L}^{Pr,\otimes}_{(\mathcal{C}^{\otimes},X)}(\mathcal{C}^{\otimes})\longrightarrow \mathcal{L}^{Pr,\otimes}_{(\mathcal{C}^{\otimes},X)}(Sp^{\Sigma}_X(\mathcal{C})^{\otimes})\longrightarrow Sp^{\Sigma}_X(\mathcal{C})^{\otimes} \tag{4.3.3}$$

where the last arrow is the counit of the adjunction of Proposition 4.1.11. To prove that this map is an equivalence it is enough to verify that the map between the underlying $(\infty,1)$-categories

$$\mathcal{L}^{Pr}_{(\mathcal{C}^{\otimes},X)}(\mathcal{C})\to Sp^{\Sigma}_X(\mathcal{C}) \tag{4.3.4}$$

is an equivalence. But now, by the combination of the Corollary 4.2.13 with the main result of the Corollary 10.4 in [71], we find a commutative diagram of equivalences

$$\begin{tikzcd}
\mathcal{L}^{Pr}_{(\mathcal{C}^{\otimes},X)}(\mathcal{C}) \arrow[rr] \arrow[d, "\sim"] & & Sp^{\Sigma}_X(\mathcal{C})=N_{\Delta}(Sp^{\Sigma}(\mathcal{V},X)^{\circ}) \arrow[d, "\sim"] \\
Stab_X(\mathcal{C})\simeq N_{\Delta}(Sp^{\mathbb{N}}(\mathcal{V},X)^{\circ}) \arrow[r, "\sim"] & Stab_X(N_{\Delta}(Sp^{\Sigma}(\mathcal{V},X)^{\circ})) \simeq & N_{\Delta}(Sp^{\mathbb{N}}(Sp^{\Sigma}(\mathcal{V},X),X)^{\circ})
\end{tikzcd} \tag{4.3.5}$$

where the left vertical map is an equivalence because X is symmetric in $\mathcal{C}^{\otimes}$; the equivalence $Stab_X(\mathcal{C})\simeq N_{\Delta}(Sp^{\mathbb{N}}(\mathcal{V},X)^{\circ})$ follows from the Proposition 4.2.4 with $G=(X\otimes -)$ (it is a left Quillen functor

[8] By the Corollary 4.1.3.16 of [100] we have monoidal equivalences $\mathcal{C}^{\otimes}\simeq \mathcal{N}^{\otimes}_{\Delta}((\mathcal{V}^{\circ})^{\otimes})$ and $Sp^{\Sigma}_X(\mathcal{C})^{\otimes}\simeq N^{\otimes}_{\Delta}((Sp^{\Sigma}(\mathcal{V},X)^{\circ})^{\otimes})$ and therefore both $\mathcal{C}^{\otimes}$ and $Sp^{\Sigma}_X(\mathcal{C})^{\otimes}$ are presentable symmetric monoidal $(\infty,1)$-categories
[9] see the Prop. 3.9.2

because X is cofibrant), and the fact that $\mathcal{C}$ is presentable; the right vertical map is an equivalence because X is already invertible in $N_\Delta(Sp^\Sigma(\mathcal{V},X)^\circ)$ and because a Quillen equivalence between combinatorial model categories induces an equivalence between the underlying $(\infty,1)$-categories (see Lemma 1.3.4.21 of [100]). This same last argument, together with the Corollary 10.4 of [71], justifies the fact that the lower horizontal map is an equivalence.

□

Remark 4.3.2. In the proof of Theorem 4.3.1, we used the condition on X twice. The first using the result of [71] and the second with the Proposition 4.2.4. We believe the use of this condition is not necessary. Indeed, everything comes down to prove an analogue of Proposition 4.2.4 for the construction of symmetric spectrum objects, replacing the natural numbers by some more complicated partially ordered set. If such a result is possible, then the construction of symmetric spectra in the combinatorial simplicial case can be presented as a colimit of a diagram of simplicial categories. In this case, the Proposition 4.2.10 would follow immediately even without the condition on X. We will not pursue this topic here since it won't be necessary for our goals.

Example 4.3.3. The combination of the Theorem 4.3.1 together with the Remark 4.2.8 and the Example 4.2.16 provides a canonical equivalence of presentable symmetric monoidal presentable $(\infty,1)$-categories $Sp^\otimes \simeq N_\Delta^\otimes(Sp^\Sigma(\hat{\Delta}_*,S^1))$.

4.4 Compact Generators in the Stabilization

To finish this chapter we use the results of the previous sections to prove a technical result concerning the existence of compact generators on the stabilization with respect to a given object. More precisely, let $\mathcal{C}$ be a presentable $(\infty,1)$-category with a zero object $*$, together with a colimit preserving endofunctor $G:\mathcal{C}\to\mathcal{C}$ (with right adjoint U). Then, we prove that if $\mathcal{C}$ admits a family of compact generators $\mathcal{E}$ (in the sense earlier discussed in 2.1.23), then the stabilitzation $Stab_{(G,U)}(\mathcal{C})$ (which in this case we can identify with the colimit in the Remark 4.2.1) also admits a family of compact generators that we can easily describe in terms of $\mathcal{E}$. By the results in the previous section, this result can then be applied to the inversion of a (symmetric) object in a presentable symmetric monoidal $(\infty,1)$-category. In the later chapters of this thesis we will use these results to give a gentle description of a generating family in the motivic stable homotopy theory of schemes.

The crutial point is the following observation

Proposition 4.4.1. *Let $\mathcal{C}$ be a presentable $(\infty,1)$-category with a zero object $*$, together with a colimit preserving endofunctor G. Consider the colimit cone (in $\mathcal{P}r^L$) of the Remark 4.2.1*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "G"] \arrow[dr, "F_0"'] & \mathcal{C} \arrow[r, "G"] \arrow[d, "F_1"] & \mathcal{C} \arrow[r, "G"] \arrow[dl, "F_2"] & \dots \\ & Stab_{(G,U)}(\mathcal{C}) & & \end{tikzcd} \tag{4.4.1}$$

Let $\mathcal{E}=\{E_\alpha\}_{\alpha\in A}$ be a (small) family of compact generators in the sense discussed in Section 2.1.23. Then, if the right adjoint U of G preserves filtered colimits, the family $\{F_n(E_\alpha)\}_{n\in\mathbb{N},\alpha\in A}$ is a family of compact generators in $Stab_{(G,U)}(\mathcal{C})$.

Proof. As discussed earlier in this chapter, this colimit cone in $\mathcal{P}r^L$ is equivalent to the limit cone in Cat_∞^{big} of

$$\begin{tikzcd} & & Stab_{(G,U)}(\mathcal{C}) \arrow[dl, "ev_2"'] \arrow[d, "ev_1"] \arrow[dr, "ev_0"] & \\ \dots \arrow[r, "U"] & \mathcal{C} & \mathcal{C} \arrow[l, "U"'] & \mathcal{C} \arrow[l, "U"'] \end{tikzcd} \tag{4.4.2}$$

where the maps in the diagram are right-adjoints to the maps in (4.4.1). As also discussed before, it can be computed as the homotopy pullback of the diagram (4.2.14). It follows that if U commutes with filtered colimits, as colimits of diagrams are computed objectwise, both arrows in (4.2.14) preserve filtered colimits. By [99, 5.4.5.5, 5.4.5.7] we deduce that the canonical maps ev_n (which are equal to the composition of the projection $Stab_{(G,U)}(\mathcal{C}) \to \prod_{n\in\mathbb{N}} \mathcal{C}$ with the canonical projections) preserve filtered colimits. It follows then from the adjunctions (F_n, ev_n) and from the definitions that the family $\{F_n(E_\alpha)\}_{n\in\mathbb{N},\alpha\in A}$ consists of compact objects in $Stab_{(G,U)}(\mathcal{C})$ and because of the description of the initial object in the pullback (see [99, 5.4.5.5]) is again generating. □

Essentially the same argument allows us to prove the following result:

Proposition 4.4.2. *Let $\mathcal{C}^\otimes$ be a presentable symmetric monoidal $(\infty,1)$-category and let X be a symmetric object in $\mathcal{C}$. Let $\mathcal{E} = \{E_\alpha\}_{\alpha\in A}$ be a (small) family of compact generators in the sense discussed in Section 2.1.23. Then, if the internal-hom functor $\underline{Hom}_{\mathcal{C}}(X,-)$ preserves filtered colimits, the underlying $(\infty,1)$-category of the monoidal inversion $\mathcal{C}^\otimes[X^{-1}]$ admits a family of compact generators given by $\{(X^{-1})^n \otimes E_\alpha\}_{n\in\mathbb{N},\alpha\in A}$. Here, $(X^{-1})^n \otimes E_\alpha$ denotes the tensor product in $\mathcal{C}^\otimes[X^{-1}]$ of the $\otimes$-inverse of X and E_α denotes the image of the E_α along the canonical map $\mathcal{C}^\otimes \to \mathcal{C}^\otimes[X^{-1}]$.*

Proof. In this case, by the Corollary 4.2.13, the stabilization with respect to the multiplication by X is equivalent to the underlying $(\infty,1)$-category of the formal inversion $\mathcal{C}^\otimes[X^{-1}]$ and the important point is that the colimit is taken inside the $(\infty,1)$-category of $\mathcal{C}^\otimes$-modules (see the Proposition 4.2.10)

$$\begin{tikzcd} \overline{\mathcal{C}} \arrow[r, "\overline{T}(X)"] \arrow[dr, "F_0"'] & \overline{\mathcal{C}} \arrow[r, "\overline{T}(X)"] \arrow[d, "F_1"] & \overline{\mathcal{C}} \arrow[r, "\overline{T}(X)"] \arrow[dl, "F_2"] & ... \\ & \overline{Stab}_X(\overline{\mathcal{C}}) & & \end{tikzcd} \tag{4.4.3}$$

As colimits of modules are computed by means of the forgetful functor (see [100, 3.4.4.6]) we can use exactly the same arguments of the previous proposition to prove again that the family $\{F_n(E_\alpha)\}$ is a generating family: as the internal-hom functor $\underline{Hom}_{\mathcal{C}}(X,-)$ preserves filtered colimits, these objects will be compact in the stabilization. The new important thing to this situation is the fact that the F_n are now maps of $\mathcal{C}^\otimes$-modules, where now thanks to the Corollary 4.2.13 we know that the $\mathcal{C}^\otimes$-module structure in the stabilization is given by the monoidal structure in $\mathcal{C}^\otimes[X^{-1}]$. It follows that for any object $C \in \mathcal{C}$, we have structural equivalences $F_n(T(X)(C)) \simeq X \otimes F_n(C)$ where the last term is the tensor product in $\mathcal{C}^\otimes[X^{-1}]$ with the image of X along the canonical map $\mathcal{C}^\otimes \to \mathcal{C}^\otimes[X^{-1}]$ (by definition, canonically equivalent to F_0). This, together with commutativity of the diagram (4.4.3) implies that for any $n \geq 0$ we have $F_0 \simeq X^n \otimes F_n$ or in other words $F_n \simeq (X^{-1})^n \otimes F_0$ thus concluding the proof. □

Example 4.4.3. Let $\mathcal{C} = \mathcal{S}_*$ be the $(\infty,1)$-category of pointed spaces equiped with the smash product. It has a zero object and the image of the point along the pointing map $(-)_+ : \mathcal{S} \to \mathcal{S}_*$ is a compact generator. After the Example 4.2.16 the family of objects $\Omega^n(\Sigma^\infty((*)_+))$ with $n \geq 0$ is a family of compact generators in spectra. Here Σ^∞ denotes the canonical map $\mathcal{S}_* \to Sp$.

Chapter **5**

Universal Characterization of the Motivic Stable Homotopy Theory of Schemes

In this chapter we use the results of Chapter 4, together with the techniques of [99, 100], to completely characterize the $\mathbb{A}^1$-homotopy theory of schemes and its associated motivic stabilization by means of a universal property inside the world of symmetric monoidal $(\infty, 1)$-categories.

Let $\mathbb{U} \in \mathbb{V} \in \mathbb{W}$ be universes. In the following sections, we shall write $Sm^{ft}(S)$ to denote the $\mathbb{V}$-small category of smooth separated $\mathbb{U}$-small schemes of finite type over a Noetherian $\mathbb{U}$-scheme S of finite Krull dimension.

5.1 $\mathbb{A}^1$-Homotopy Theory of Schemes

The main idea in the subject is to "do homotopy theory with schemes" in more or less the same way we do with spaces, by thinking of the affine line $\mathbb{A}^1$ as an "interval". One first difficulty is that the category of schemes does not admit all colimits. In [105], the authors constructed a *place* to realize this idea. The construction proceeds as follows: start from the category of schemes and add formally all the colimits. Then make sure that the following two principles hold:

I) the line $\mathbb{A}^1$ becomes contractible;

II) if X is a scheme and U and V are two open subschemes whose union equals X in the category of schemes then make sure that their union continues to be X in the new place;

The original construction in [105] was performed using the techniques of model category theory and this *place* is the homotopy category of a model category $\mathcal{M}_{\mathbb{A}^1}$. During the last years their methods were revisited and reformulated in many different ways. In [46], the author presents a "universal" characterization of the original construction using the theory of Bousfield localizations for model categories[1] together with a universal characterization of the theory of simplicial presheaves, within model categories. The construction of [46] can be summarized by the expression

$$\mathcal{M}_{\mathbb{A}^1} = L_{\mathbb{A}^1} L_{HyperNis}((SPsh(Sm^{ft}(S)))) \tag{5.1.1}$$

where $SPsh(-)$ stands for simplicial presheaves with the projective model structure, $L_{HyperNis}$ corresponds the Bousfield localization with respect to the collection of the hypercovers associated to the *Nisnevich topology* (see below) and $L_{\mathbb{A}^1}$ corresponds to the Bousfield localization with respect to the collection of all projection maps $X \times \mathbb{A}^1 \to X$.

It is clear today that model categories should not be taken as fundamental objects, but rather, we should focus on their associated $(\infty, 1)$-categories. In this section, we use the insights of [46] to

[1] see [68]

perform the construction of an $(\infty,1)$-category $\mathcal{H}(S)$ directly within the setting of ∞-categories. By the construction, it will have a universal property and using the link described in Section 2.2 and the theory developed by J.Lurie in [99] relating Bousfield localizations to localizations of ∞-categories, we will be able to prove that $\mathcal{H}(S)$ is equivalent to the ∞-category underlying the $\mathbb{A}^1$ model category of Morel-Voevodsky.

The construction of $\mathcal{H}(S)$ proceeds as follows. We start from the category of smooth schemes of finite type over S - $Sm^{ft}(S)$ and consider it as a trivial $\mathbb{V}$-small $(\infty,1)$-category $N(Sm^{ft}(S))$. Together with the *Nisnevich topology* ([109]), it acquires the structure of an ∞-*site* (see Definition 6.2.2.1 of [99]). By definition (see Def. 1.2 of [105]) the Nisnevich topology is the topology generated by the pre-topology whose covering families of an S-scheme X are the collections of étale morphisms $\{f_i : U_i \to X\}_{i\in I}$ such that for any $x \in X$ there exists an $i \in I$ and $u_i \in U_i$ such that f_i induces an isomorphism between the residual fields $k(x) \simeq k(u_i)$. Recall from [105] (Def. 1.3) that an elementary Nisnevich square is a commutative square of schemes

$$\begin{tikzcd} p^{-1}(U) \arrow[r] \arrow[d] & V \arrow[d, "p"] \\ U \arrow[r, "i"] & X \end{tikzcd} \tag{5.1.2}$$

such that

a) $i : U \hookrightarrow X$ is an open immersion of schemes;

b) $p : V \to X$ is an étale map;

c) the square (5.1.2) is a pullback. In particular $p^{-1}(U) \to V$ is also an open immersion.

d) the canonical projection $p^{-1}(X - U) \to X - U$ is an isomorphism where we consider the closed subsets $Z := X - U$ and $p^{-1}Z$ both equipped with the reduced structures of closed subschemes;

and from this we can easily deduce that

e) the square

$$\begin{tikzcd} V \arrow[d, "p"'] & p^{-1}(Z) \arrow[l] \arrow[d] \\ X & Z := X - U \arrow[l] \end{tikzcd} \tag{5.1.3}$$

is a pullback with both Z and $p^{-1}(Z)$ equipped with the reduced structures;

e) the square (5.1.2) is a pushout.

The crucial fact is that each family $(V \to X, U \to X)$ as above forms a Nisnevich covering and the families of this form provide a *basis* for the Nisnevich topology (see the Proposition 1.4 of [105]). We consider the very big $(\infty,1)$-category $\mathcal{P}^{big}(N(Sm^{ft}(S))) := Fun(N(Sm^{ft}(S))^{op}, \widehat{\mathcal{S}})$ of presheaves of (big) homotopy types over $N(Sm^{ft}(S))$ (See Section 5.1 of [99]) which has the expected universal property (Thm. 5.1.5.6 of [99]): it is the free completion of $N(Sm^{ft}(S))$ with $\mathbb{V}$-small colimits (in the sense of ∞-categories). Using the Proposition 4.2.4.4 of [99] we can immediately identify $\mathcal{P}^{big}(N(Sm^{ft}(S)))$ with the underlying ∞-category of the model category of simplicial presheaves on $Sm^{ft}(S)$ endowed with the projective model structure. The results of [99] provide an ∞-analogue for the classical Yoneda embedding, meaning that we have a fully faithful map of ∞-categories $j : N(Sm^{ft}(S)) \to \mathcal{P}^{big}(N(Sm^{ft}(S)))$ and as usual we will identify a scheme X with its image $j(X)$. We now restrict to those objects in $\mathcal{P}^{big}(N(Sm^{ft}(S)))$ which are sheaves with respect to the Nisnevich topology. Because the Nisnevich squares form a basis for the Nisnevich topology, an object $F \in \mathcal{P}^{big}(N(Sm^{ft}(S)))$ is a sheaf iff it maps Nisnevich squares to pullback squares. In particular, every representable $j(X)$ is a sheaf (because Nisnevich squares are pushouts). Following [99, 5.5.4.15], the inclusion of the full subcategory $Sh^{big}_{Nis}(Sm^{ft}(S)) \subseteq \mathcal{P}^{big}(N(Sm^{ft}(S)))$ admits a left adjoint (which is known to be exact - Lemma 6.2.2.7 of [99]) and provides a canonical example of an ∞-topos (See Definition 6.1.0.4 of [99]). More importantly to our needs, this is an example of a presentable localization of a presentable $(\infty,1)$-category and we can make use of the results discussed in Section 3.8.

Remark 5.1.1. When S is Noetherian of finite Krull dimension, the category of smooth schemes $Sm^{ft}(S)$ can be replaced by the category of *affine* smooth schemes of finite type over S, $N(AffSm^{ft})(S)$, and the resulting $(\infty,1)$-categories $Sh^{big}_{Nis}(Sm^{ft}(S))$ and $Sh^{big}_{Nis}(N(AffSm^{ft})(S))$ are equivalent. This follows because we can identify $Sm^{ft}(S)$ with a full subcategory of $\mathcal{P}^{big}(N(AffSm^{ft})(S))$ using the map sending a smooth scheme X to the representable functor $Y \in N(AffSm^{ft}(k)) \mapsto Hom_{Sm^{ft}(S)}(Y,X)$, and this identification is compatible with the Nisnevich topologies. For more details see [104]. See also [96, Section 2].

Next step, we consider the *hypercompletion* of the ∞-topos $Sh^{big}_{Nis}(Sm^{ft}(S))$ (see Section 6.5.2 of [99]). By construction, it is a presentable localization of $Sh^{big}_{Nis}(Sm^{ft}(S))$ and by the Corollary 6.5.3.13 of [99] it coincides with $Sh^{big}_{Nis}(Sm^{ft}(S))^{hyp}$: the localization of $\mathcal{P}^{big}(N(Sm^{ft}(S)))$ spanned by the objects which are local with respect to the class of *Nisnevich hypercovers*.

Finally, we reach the last step: We will restrict ourselves to those sheaves in $Sh^{big}_{Nis}(Sm^{ft}(S))^{hyp}$ satisfying $\mathbb{A}^1$-invariance, meaning those sheaves F such that for any scheme X, the canonical map induced by the projection $F(X) \to F(X \times \mathbb{A}^1)$ is an equivalence. More precisely, we consider the localization of $Sh^{big}_{Nis}(Sm^{ft}(S))^{hyp}$ with respect to the class of all projection maps $\{X \times \mathbb{A}^1 \to X\}_{X \in Obj(Sm^{ft}(S))}$. We will write $\mathcal{H}(S)$ for the result of this localization and write $l_{\mathbb{A}^1} : Sh^{big}_{Nis}(Sm^{ft}(S))^{hyp} \to \mathcal{H}(S)$ for the localization functor. Notice that $\mathcal{H}(S)$ is very big, presentable with respect to the universe $\mathbb{V}$. It is also clear from the construction that $\mathcal{H}(S)$ comes naturally equipped with a universal characterization:

Theorem 5.1.2. *Let $Sm^{ft}(S)$ be the category of smooth schemes of finite type over a base noetherian scheme S and let $L : N(Sm^{ft}(S)) \to \mathcal{H}(S)$ denote the composition of localizations*

$$N(Sm^{ft}(S)) \to \mathcal{P}^{big}(N(Sm^{ft}(S))) \to Sh^{big}_{Nis}(Sm^{ft}(S)) \to Sh^{big}_{Nis}(Sm^{ft}(S))^{hyp} \to \mathcal{H}(S) \tag{5.1.4}$$

Then, for any $(\infty,1)$-category $\mathcal{D}$ with all $\mathbb{V}$-small colimits, the map induced by composition with L

$$Fun^L(\mathcal{H}(S),\mathcal{D}) \to Fun(N(Sm^{ft}(S)),\mathcal{D}) \tag{5.1.5}$$

is fully faithful and its essential image is the full subcategory of $Fun(N(Sm^{ft}(S)),\mathcal{D})$ spanned by those functors satisfying Nisnevich descent *and* $\mathbb{A}^1$-invariance. *The left-side denotes the full subcategory of $Fun(\mathcal{H}(S),\mathcal{D})$ spanned by the colimit preserving maps.*

Proof. The proof follows from the combination of the universal property of presheaves with the universal properties of each localization in the construction and from the fact that for the Nisnevich topology in $Sm^{ft}(S)$, descent is equivalent to hyperdescent (see [152, Prop. 5.9] or [105, 3- 1.16] or [96, Section 1]) and therefore the localization $Sh^{big}_{Nis}(Sm^{ft}(S)) \to Sh^{big}_{Nis}(Sm^{ft}(S))^{hyp}$ is an equivalence. □

Our goal now is to provide the evidence that $\mathcal{H}(S)$ really is the underlying $(\infty,1)$-category of the $\mathbb{A}^1$- model category of Morel-Voevodsky. In fact, we already saw that our first step coincides with the first step in the construction of $\mathcal{M}_{\mathbb{A}^1}$ - simplicial presheaves are a model for ∞-presheaves. It remains to prove that our localizations produce the same results as the Bousfield localizations. But of course, this follows from the results the appendix of [99] applied to the model category $\mathcal{M} := SPsh(Sm^{ft}(S))$. (See our introductory survey 2.2.1).

Remark 5.1.3. It is important to remark that the sequence of functors in the Theorem 5.1.2 can be promoted to a sequence of monoidal functors with respect to the cartesian monoidal structures

$$N(Sm^{ft}(S))^{\times} \to \mathcal{P}^{big}(N(Sm^{ft}(S)))^{\times} \to Sh^{big}_{Nis}(Sm^{ft}(S))^{\times} \simeq (Sh^{big}_{Nis}(Sm^{ft}(S))^{hyp})^{\times} \to \mathcal{H}(S)^{\times} \tag{5.1.6}$$

The first is the Yoneda map which we know commutes with limits. The second map is the sheafification functor which we also know is left exact. The last functor is a monoidal localization because of the definition of the $\mathbb{A}^1$-equivalences. These localized monoidal structures are cartesian because of the existence of fully faithful right adjoints. Furthermore, they are presentable (see the Remark 3.6.1).

5.2 The monoidal structure in $\mathcal{H}(S)_*$

Let $\mathcal{H}(S)$ be the $(\infty,1)$-category introduced in the last section. Since it is presentable it admits a final object $*$ and the $(\infty,1)$-category of pointed objects $\mathcal{H}(S)_*$ is also presentable (see [99, Prop. 5.5.2.10]). In this case, since the forgetful functor $\mathcal{H}(S)_* \to \mathcal{H}(S)$ commutes with limits, by the Adjoint Functor Theorem (see [99, Cor. 5.5.2.9]) it admits a left adjoint $()_+ : \mathcal{H}(S) \to \mathcal{H}(S)_*$ which we can identify with the formula $X \mapsto X_+ := X \coprod *$. In order to follow the stabilization methods of Morel-Voevodsky we need to explain how the cartesian product in $\mathcal{H}(S)$ extends to a symmetric monoidal structure in $\mathcal{H}(S)_*$ and how the pointing morphism becomes monoidal.

This problem fits in a more general setting. Recall that the $(\infty,1)$-category of spaces $\mathcal{S}$ is the unit for the symmetric monoidal structure $\mathcal{P}r^{L,\otimes}$. In [100, Prop. 4.8.2.11] it is proved that the pointing morphism $-\coprod * : \mathcal{S} \to \mathcal{S}_*$ endows $\mathcal{S}_*$ with the structure of an idempotent object in $\mathcal{P}r^{L,\otimes}$ and proves that its associated local objects are exactly the pointed presentable $(\infty,1)$-categories. It follows from the general theory of idempotents that the product functor $\mathcal{C} \mapsto \mathcal{C}\otimes\mathcal{S}_*$ is a left adjoint to the inclusion functor $\mathcal{P}r^L_{Pt} \subseteq \mathcal{P}r^L$. Also from the general theory, this left adjoint is monoidal. The final ingredient is that for any presentable $(\infty,1)$-category $\mathcal{C}$ there is an equivalence of $(\infty,1)$-categories $\mathcal{C}_* \simeq \mathcal{C}\otimes\mathcal{S}_*$ (see the [100, Example 4.8.1.20]) and via this equivalence, the pointing map $\mathcal{C} \to \mathcal{C}_*$ is equivalent to the product map $id_{\mathcal{C}} \otimes ()_+ : \mathcal{C}\otimes\mathcal{S} \to \mathcal{C}\otimes\mathcal{S}_*$ where $()_+$ denotes the pointing map of spaces. Altogether, we have the following result

Corollary 5.2.1. *(Lurie) The formula $\mathcal{C} \mapsto \mathcal{C}_*$ defines a monoidal left adjoint to the inclusion $\mathcal{P}r^L_{Pt} \subseteq \mathcal{P}r^L$ and therefore induces a left adjoint to the inclusion $CAlg(\mathcal{P}r^L_{Pt}) \subseteq CAlg(\mathcal{P}r^L)$. In other words, for any presentable symmetric monoidal $(\infty,1)$-category $\mathcal{C}^{\otimes}$, there exists a pointed presentable symmetric monoidal $(\infty,1)$-category $\mathcal{C}_*^{\wedge(\otimes)}$ whose underlying $(\infty,1)$-category is $\mathcal{C}_*$, together with a monoidal functor $\mathcal{C}^{\otimes} \to \mathcal{C}_*^{\wedge(\otimes)}$ extending the pointing map $\mathcal{C} \to \mathcal{C}_*$, and satisfying the following universal property:*

$(*)$ for any pointed presentable symmetric monoidal $(\infty,1)$-category $\mathcal{D}^{\otimes}$, the composition

$$Fun^{\otimes,L}(\mathcal{C}_*^{\wedge(\otimes)}, \mathcal{D}^{\otimes}) \to Fun^{\otimes,L}(\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes}) \tag{5.2.1}$$

is an equivalence.

Remark 5.2.2. In the situation of the previous corollary, given a functor $F : \mathcal{C} \to \mathcal{D}$ with $\mathcal{D}$ being pointed, its canonical extension $\tilde{F}$

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d] & \mathcal{D} \\ \mathcal{C}_* \arrow[ur, dashed, "\tilde{F}"] & \end{tikzcd} \tag{5.2.2}$$

is naturally identified with the formula $(u : * \to X) \mapsto cofiber F(u) \in \mathcal{D}$.

The symmetric monoidal structure $\mathcal{C}_*^{\wedge(\otimes)}$ will be called the *smash product induced by* $\mathcal{C}^{\otimes}$. Of course, if $\mathcal{C}^{\otimes}$ is already pointed we have an equivalence $\mathcal{C}_*^{\wedge(\otimes)} \simeq \mathcal{C}^{\otimes}$. In the particular case when $\mathcal{C}^{\otimes}$

is Cartesian, we will use the notation $\mathcal{C}_*^{\wedge} := \mathcal{C}_*^{\wedge(\times)}$.

Let us now progress in a different direction. Let $\mathcal{M}$ be a combinatorial simplicial model category. Assume also that $\mathcal{M}$ is cartesian closed and that its final object $*$ is cofibrant. This makes $\mathcal{M}$ a symmetric monoidal model category with respect to the cartesian product and we have an monoidal equivalence

$$N_{\Delta}^{\otimes}((\mathcal{M}^{\circ})^{\times}) \simeq N_{\Delta}(\mathcal{M}^{\circ})^{\times} \tag{5.2.3}$$

Moreover, because the cartesian product provides a Quillen bifunctor, $N_{\Delta}(\mathcal{M}^{\circ})^{\times}$ is a presentable symmetric monoidal $(\infty,1)$-category and therefore, using the Corollary above, we can equip $N_{\Delta}(\mathcal{M}^{\circ})_*$ with a canonical presentable symmetric monoidal structure $N_{\Delta}(\mathcal{M}^{\circ})_*^{\wedge}$ for which the pointing map becomes monoidal

$$N_{\Delta}(\mathcal{M}^{\circ})^{\times} \to N_{\Delta}(\mathcal{M}^{\circ})_*^{\wedge} \tag{5.2.4}$$

Independently of this, we can consider the natural model structure in $\mathcal{M}_*$ (see the Remark 1.1.8 in [69]). Again, it is combinatorial and simplicial and comes canonically equipped with a left-Quillen functor $(-)_+ : \mathcal{M} \to \mathcal{M}_*$ defined by the formula $X \mapsto X \coprod *$. Moreover, it acquires the structure of a symmetric monoidal model category via the usual definition of the smash product, given by the formula

$$(X,x) \wedge (Y,y) := \frac{(X,x) \times (Y,y)}{(X,x) \vee (Y,y)} \tag{5.2.5}$$

It is well-known that this formula defines a symmetric monoidal structure with unit given by $(*)_+$ and by the Proposition 4.2.9 of [69] it is compatible with the model structure in $\mathcal{M}_*$. Let $N_{\Delta}^{\otimes}(((\mathcal{M}_*)^{\circ})^{\wedge_{usual}})$ be its underlying symmetric monoidal $(\infty,1)$-category. The same result also tells us that the left-Quillen map $\mathcal{M} \to \mathcal{M}_*$ is monoidal. By the Proposition 3.9.2, it induces a monoidal map between the underlying symmetric monoidal $(\infty,1)$-categories.

$$f^{\otimes} : N_{\Delta}(\mathcal{M}^{\circ})^{\times} \to N_{\Delta}^{\otimes}(((\mathcal{M}_*)^{\circ})^{\wedge_{usual}}) \tag{5.2.6}$$

Of course, $N_{\Delta}^{\otimes}(((\mathcal{M}_*)^{\circ})^{\wedge_{usual}})$ is a pointed presentable symmetric monoidal $(\infty,1)$-category and by the universal property defining the smash product we obtain a monoidal map

$$N_{\Delta}(\mathcal{M}^{\circ})_*^{\wedge(\otimes)} \to N_{\Delta}^{\otimes}(((\mathcal{M}_*)^{\circ})^{\wedge_{usual}}) \tag{5.2.7}$$

Corollary 5.2.3. *The above map is an equivalence of presentable symmetric monoidal $(\infty,1)$-categories.*

Proof. Since the map is monoidal, the proof is reduced to the observation that the underlying map

$$f = f_{\langle 1\rangle}^{\otimes} : N_{\Delta}(\mathcal{M}^{\circ})_* \to N_{\Delta}(((\mathcal{M}_*)^{\circ}) \tag{5.2.8}$$

is an equivalence. To prove this, we observe first that since $*$ is cofibrant, we have an equality of simplicial sets $N_{\Delta}(\mathcal{M}^{\circ})_* = N_{\Delta}((\mathcal{M}^{\circ})_*)$. Secondly, we observe that the cofibrant-fibrant objects in $(\mathcal{M}_*)$ are exactly the pairs $(X, * \to X)$ with X cofibrant-fibrant in $\mathcal{M}$ and $* \to X$ a cofibration. This means there is a natural inclusion of $(\infty,1)$-categories $i : N_{\Delta}(((\mathcal{M}_*)^{\circ}) \subseteq N_{\Delta}((\mathcal{M}^{\circ})_*)$. It follows from the definition of the model structure in $\mathcal{M}_*$ that this inclusion is essentially surjective: if $(X, * \to X)$ is an object in $N_{\Delta}((\mathcal{M}^{\circ})_*)$, we consider the factorization of $* \to X$ through a cofibration followed by a trivial fibration in $\mathcal{M}$,

$$* \to X' \simeq X \tag{5.2.9}$$

Of course, $(X', * \to X')$ is an object in $N_{\Delta}(((\mathcal{M}_*)^{\circ})$ and it is equivalent to $(X, * \to X)$ in $N_{\Delta}((\mathcal{M}^{\circ})_*)$.

Finally, we notice that the composition $i \circ f : N_{\Delta}(\mathcal{M}^{\circ}) \to N_{\Delta}((\mathcal{M}_*)^{\circ}) \subseteq N_{\Delta}((\mathcal{M}^{\circ})_*)$ yields the result of the canonical pointing map $N_{\Delta}(\mathcal{M}^{\circ}) \to N_{\Delta}(\mathcal{M}^{\circ})_*$. Indeed, the pointing map is characterized by the universal property of the homotopy pushout, and since $* \to X$ in $N_\Delta((\mathcal{M}_*)^\circ)$ is a cofibration and X is also cofibrant, the coproduct $X \coprod *$ is an homotopy coproduct. The result now follows from the universal property of the pointing map. $\square$

Remark 5.2.4. If $\mathcal{M} = \hat{\Delta}$ is the model category of simplicial sets with the cartesian product, it satisfies the above conditions and we find a monoidal equivalence between $\mathcal{S}_*^\wedge$ and the underlying symmetric monoidal $(\infty,1)$-category of $\hat{\Delta}_*$ endowed with the classical smash product of pointed spaces.

Remark 5.2.5. If $\mathcal{C}$ is a simplicial category, the left-Quillen adjunction $\hat{\Delta} \to \hat{\Delta}_*$ extends to a left Quillen adjunction $SPsh(\mathcal{C}) \to SPsh_*(\mathcal{C})$, where $SPsh_*(\mathcal{C})$ corresponds to the category of presheaves of pointed simplicial sets over $\mathcal{C}$, endowed with the projective model structure (see [99]-Appendix). It follows that $SPsh(\mathcal{C})$ has all the good properties which intervene in the proof of the Corollary 5.2.3 and we find a monoidal equivalence $N_\Delta(SPsh(\mathcal{C})^\circ)_*^\wedge \to N_\Delta^\otimes((SPsh_*(\mathcal{C})^\circ)^{\wedge_{usual}})$ where the last is the underlying symmetric monoidal $(\infty,1)$-category associated to the smash product in $SPsh(\mathcal{C})_*$.

The Corollary 5.2.3 implies that

Corollary 5.2.6. *Let $\mathcal{H}(S)^\times$ be the presentable symmetric monoidal $(\infty,1)$-category underlying the model category $\mathcal{M}_{\mathbb{A}^1}$ encoding the $\mathbb{A}^1$-homotopy theory of Morel-Voevodsky together with the cartesian product. Let $(\mathcal{M}_{\mathbb{A}^1})_*$ be its pointed version with the smash product given by the Lemma 2.13 of [105]. Then, the canonical map induced by the universal property of the smash product*

$$\mathcal{H}(S)_*^\wedge \to N_\Delta^\otimes((((\mathcal{M}_{\mathbb{A}^1})_*)^\circ)^\wedge) \tag{5.2.10}$$

is an equivalence of presentable symmetric monoidal $(\infty,1)$-categories.

In other words and as expected, $\mathcal{H}(S)_*^\wedge$ is the underlying symmetric monoidal $(\infty,1)$-category of the classical construction.

5.3 The Stable Motivic Theory

As in the original setting, we may now consider a *stabilized* version of the theory. In fact, two stabilizations are possible - one with respect to the *topological circle* $S^1 := \Delta[1]/\partial\Delta[1]$ (pointed by the image of $\partial\Delta[1]$) and another one with respect to the *algebraic circle* defined as $\mathbb{G}_m := \mathbb{A}^1 - \{0\}$. The *motivic stabilization* of the theory is by definition, the stabilization with respect to the product $\mathbb{G}_m \wedge S^1$ which we know is equivalent to $(\mathbb{P}^1,\infty)$ in $\mathcal{H}(S)_*$: consider the Nisnevich covering of $(\mathbb{P}^1,1)$ given by two copies of $\mathbb{A}^1$ both pointed at 1, together with the maps $\mathbb{A}^1 \to \mathbb{P}^1$ sending $x \mapsto (1:x)$, respectively, $x \mapsto (x:1)$. Their intersection is $\mathbb{A}^1 - \{0\}$ (also pointed at 1). The result follows because this square is a pushout (as a consequence of forcing Nisnevich descent), because $\mathbb{A}^1$ is contractible in $\mathcal{H}(S)_*$ (as a consequence of forcing $\mathbb{A}^1$-invariance and the Remark 5.2.2) and finally, because the suspension $\mathcal{H}(S)_*$ is canonically identified with the smash product with the circle (as explained by the Example 4.2.16). The conclusion follows because $(\mathbb{P}^1,\infty)$ and $(\mathbb{P}^1,1)$ are $\mathbb{A}^1$-homotopic via the map $x \mapsto (1:x)$.

Definition 5.3.1. *([150, Definition 5.7]) Let S be a base scheme. The* stable motivic $\mathbb{A}^1$ ∞-category over S *is the underlying $(\infty,1)$-category of the presentable symmetric monoidal ∞-category $\mathcal{SH}(S)^\otimes$ defined by the formula*

$$\mathcal{SH}(S)^\otimes := \mathcal{H}(S)_*^\wedge[(\mathbb{P}^1,\infty)^{-1}] \tag{5.3.1}$$

as in the Definition 4.1.8.

The standard way to define the stable motivic theory is to consider the combinatorial simplicial symmetric monoidal model category $Sp^\Sigma((\mathcal{M}_{\mathbb{A}^1})_*,(\mathbb{P}^1,\infty))$ where $\mathcal{M}_*$ is equipped with the smash product. By the [150, Lemma 4.4] together with the Remark 4.2.7, we know that $(\mathbb{P}^1,\infty)$ is symmetric and consequently the Theorem 4.3.1 ensures that $\mathcal{SH}(S)^\otimes$ recovers the classical definition. In addition, since we have an equivalence $(\mathbb{P}^1,\infty)\simeq\mathbb{G}_m\wedge S^1$, the universal properties provide canonical monoidal equivalences of presentable symmetric monoidal $(\infty,1)$-categories

$$\mathcal{SH}(S)^{\otimes}\simeq(\mathcal{H}(S)^{\wedge}_{*})[(\mathbb{G}_m\wedge S^1)^{-1}]\simeq(\mathcal{H}(S)^{\wedge}_{*})[((\mathbb{P}^1,\infty)\wedge S^1)^{-1}]\simeq((\mathcal{H}(S)^{\wedge}_{*})[(S^1)^{-1}])[(\mathbb{P}^1,\infty)^{-1}] \tag{5.3.2}$$

Since S^1 is symmetric in $\mathcal{S}^{\wedge}_{*}$ (see [69, Lemma 6.6.2] together with the Remark 4.2.8) it is also symmetric in $\mathcal{H}(S)^{\wedge}_{*}$ (because it is given by the image of the unique colimit preserving monoidal map $\mathcal{S}^{\wedge}_{*}\to\mathcal{H}(S)^{\wedge}_{*}$). In this case, we can use the Proposition 4.2.13 to deduce that the underlying ∞-category of $(\mathcal{H}(S)^{\wedge}_{*})[(S^1)^{-1}]$ is equivalent to the stable ∞-category $Stab(\mathcal{H}(S))$. Plus, since $(\mathcal{H}(S)^{\wedge}_{*})[(S^1)^{-1}]$ is presentable by construction, the monoidal structure is compatible with colimits, thus exact, separately on each variable. We conclude that it is a stable presentable symmetric monoidal $(\infty,1)$-category.

Finally, because $(\mathbb{P}^1,\infty)$ is symmetric, the Corollary 4.2.14 tells us that $\mathcal{SH}(S)^{\otimes}$ is a *stable presentable symmetric monoidal ∞-category.* In particular its homotopy category is triangulated and inherits a canonical symmetric monoidal structure.

Corollary 5.3.2. *Let S be a base scheme and $Sm^{ft}(S)$ denote the category of smooth schemes of finite type over S, together with the cartesian product. The composition of monoidal functors*

$$\theta^{\otimes}:N(Sm^{ft}(S))^{\times}\to\mathcal{P}^{big}(N(Sm^{ft}(S)))^{\times}\to\mathcal{H}(S)^{\times}\to\mathcal{H}(S)^{\wedge}_{*}\to\mathcal{H}(S)^{\wedge}_{*}[(S^1)^{-1}]\to\mathcal{SH}(S)^{\otimes} \tag{5.3.3}$$

satisfies the following universal property: for any pointed presentable symmetric monoidal $(\infty,1)$-category $\mathcal{D}^{\otimes}$, the composition map

$$Fun^{\otimes,L}(\mathcal{SH}(S)^{\otimes},\mathcal{D}^{\otimes})\to Fun^{\otimes}(N(Sm^{ft}(S))^{\times},\mathcal{D}^{\otimes}) \tag{5.3.4}$$

is fully faithful and its image consists of those monoidal functors $N(Sm^{ft}(S))^{\times}\to\mathcal{D}^{\otimes}$ whose underlying functor satisfy Nisnevich descent, $\mathbb{A}^1$-invariance and such that the cofiber of the image of the point at ∞, $S\xrightarrow{\infty}\mathbb{P}^1$ is an invertible object in $\mathcal{D}^{\otimes}$. Moreover, any pointed presentable symmetric monoidal $(\infty,1)$-category $\mathcal{D}^{\otimes}$ admitting such a monoidal functor is necessarily stable.

Proof. Here, $N(Sm^{ft}(S))$ denotes the standard way to interpret an ordinary 1-category as an $(\infty,1)$-category using the nerve. The Yoneda map $j:N(Sm^{ft}(S))\to\mathcal{P}^{big}(N(Sm^{ft}(S)))$ extends to a monoidal map because of the monoidal universal property of presheaves (consult our introductory section on Higher Algebra). By the Proposition 2.15 pg. 74 in [105], the localization functor $\mathcal{P}^{big}(N(Sm^{ft}(S)))\to\mathcal{H}(S)$ is monoidal with respect to the cartesian structure and therefore extends to a monoidal left adjoint to the inclusion $\mathcal{H}(S)^{\times}\subseteq\mathcal{P}^{big}(N(Sm^{ft}(S)))^{\times}$. The result now follows from the discussion above, together with the Corollaries 5.2.1 and 5.2.6, the Corollary 4.2.14, the Theorem 4.3.1 and Remark 5.2.2.

The last assertion follows from the Remark 4.2.17, together with the fact that $\mathbb{P}^1$ mod out by the point at infinity is the tensor product of S^1 and $\mathbb{G}_m$, so that, since we are dealing with monoidal functors, the conditions defining the image of the composition map force the image S^1 to be tensor invertible in $\mathcal{D}^{\otimes}$. $\square$

To conclude this section we provide a useful description of a family of compact generators in $\mathcal{SH}(S)$. This follows almost immediately from the Proposition 4.4.2:

Proposition 5.3.3. *The stable presentable $(\infty,1)$-category $\mathcal{SH}(S)$ admits a family of compact generators in the sense discussed in 2.1.23. In precise terms it consists of the family of objects $(\mathbb{P}^1_S)^{-n}\otimes\theta(V)$ for $\otimes$ the tensor product in $\mathcal{SH}(S)$, $n\geq 0$ and V a smooth scheme of finite type over S.*

Proof. After the Prop. 4.4.2 we are left to check that the collection of all objects in $\mathcal{H}(S)_*$ given by the image of the canonical map $N(Sm^{ft}(S)) \to \mathcal{H}(S)_*$ is a family of compact generators. The fact that they are generators follows from the Yoneda Lemma. The fact that they are compact follows because a filtered colimit in $\mathcal{P}^{big}(N(Sm^{ft}(S)))$ of Nisnevich+$\mathbb{A}^1$ local-objects is again Nisnevich local (because filtered colimits in spaces commute with homotopy pullbacks) and $\mathbb{A}^1$-invariant. □

Remark 5.3.4. Thanks to the results of [116] and to our discussion in the Prop. 2.1.2, if k is a field admitting resolutions of singularities then the results of the previous proposition can be improved: it is enough to take the collection generated by the image of smooth projective varieties. In this case, it corresponds to the dualizable objects in $\mathcal{SH}(k)^{\otimes}$. In particular the $(\infty,1)$-category $\mathcal{SH}(k)$ is compactly generated by dualizable objects.

5.4 Description using Spectral Presheaves

In this section we give an alternative description of the symmetric monoidal $(\infty,1)$-category $\mathcal{SH}(k)^{\otimes}$ using presheaves of spectra.

Remark 5.4.1. (Spectral Yoneda's Lemma) Recall (for instance, see the discussion in 2.1.23) that any stable $(\infty,1)$-category has a natural enrichment over spectra, determined by means of a universal property. In this remark we recall how to use this universal property to deduce an enriched version of Yoneda's lemma for spectral presheaves. More precisely, if $\mathcal{C}$ is a small $(\infty,1)$-category, we consider the composition of the Yoneda's embedding with the pointing map followed by stabilization $\Sigma^\infty_+ \circ j : \mathcal{C} \hookrightarrow \mathcal{P}(\mathcal{C}) \to \mathcal{P}(\mathcal{C})_* \to Stab(\mathcal{P}(\mathcal{C})) \simeq Fun(\mathcal{C}^{op}, Sp)$ (because the stabilization is a limit). Now, given an object X in $\mathcal{C}$, we can use Yoneda's lemma for $\mathcal{P}(\mathcal{C})$ to construct a natural equivalence of functors $Map_{Fun(\mathcal{C}^{op},Sp)}(\Sigma^\infty_+ \circ j(X), -) \to \Omega^\infty \circ ev_X$, where $ev_X : Fun(\mathcal{C}^{op}, Sp) \to Sp$ is the evaluation map at X. This is possible because the delooping of presheaves is computed objectwise. To conclude, since the composition with Ω^∞ induces an equivalence $Exc_*(\mathcal{C}, Sp) \simeq Exc_*(\mathcal{C}, \mathcal{S})$, we can lift the previous natural equivalence to a new one

$$Map^{Sp}_{Fun(\mathcal{C}^{op},Sp)}(\Sigma^\infty_+ \circ j(X), -) \to ev_X \tag{5.4.1}$$

which, when evaluated at F gives us the Yoneda's formula we seek. This holds for any universe: if $\mathcal{C}$ is only $\mathbb{V}$-small for some universe $\mathbb{V}$ we apply the same arguments as above to the $\mathbb{V}$-small $(\infty,1)$-category of spectra obtained from the stabilization of the $\mathbb{V}$-small $(\infty,1)$-category of spaces.

Now, we start from the $(\infty,1)$-category $N(Sm^{ft}(S))$ and consider the very big $(\infty,1)$-category $Fun(N(Sm^{ft}(S))^{op}, \widehat{Sp})$[2] which is canonically equivalent to $Stab(\mathcal{P}^{big}(N(Sm^{ft}(S)))_*)$. Using the Remark 4.2.17 we obtain a canonical monoidal structure $Fun(N(Sm^{ft}(S))^{op}, \widehat{Sp})^{\otimes}$ defined by the inversion $\mathcal{P}^{big}(N(Sm^{ft}(S)))_*^{\wedge(\otimes)}[(S^1)^{-1}]^{\otimes}$ where $\mathcal{P}^{big}(N(Sm^{ft}(S)))_*^{\wedge(\otimes)}$ is the canonical monoidal smash structure given by the Prop. 5.2.1 extending the monoidal structure $\mathcal{P}^{big}(N(Sm^{ft}(S)))^{\otimes}$ of 3.2.7.

We proceed as before and perform the localization with respect to the Nisnevich topology and $\mathbb{A}^1$. Extra care is needed, for the class of maps with respect to which we need to localize is not the same as for presheaves of spaces. In order to describe these two classes we recall first that $Fun(N(Sm^{ft}(S))^{op}, \widehat{Sp})$ is a stable presentable $(\infty,1)$-category and by the discussion in 2.1.23, for any $G \in Fun(N(Sm^{ft}(S))^{op}, \widehat{Sp})$ we have a mapping spectrum functor $Map^{Sp}(G, -) : Fun(N(Sm^{ft}(S))^{op}, \widehat{Sp}) \to \widehat{Sp}$ which when composed with Ω^∞ recovers the mapping space functor in $Fun(N(Sm^{ft}(S))^{op}, \widehat{Sp})$. Moreover, because of the universal property that defines it and because the composition $\Omega^\infty Map^{Sp}(G, -)$ commutes with all limits, we conclude that $Map^{Sp}(G, -)$ also commutes with all limits. In particular, by the Adjoint functor theorem [99, 5.5.2.9], it admits a left adjoint which we shall denote as $\delta_G : \widehat{Sp} \to Fun(N(Sm^{ft}(S))^{op}, \widehat{Sp})$ and for any $K \in \widehat{Sp}$ and $F \in Fun(N(Sm^{ft}(S))^{op}, \widehat{Sp})$ we have

[2]Here $\widehat{Sp}$ denotes the big $(\infty,1)$-category of spectra, obtained from the stabilization of the big $(\infty,1)$-category of spaces $\widehat{\mathcal{S}}$

$$Map_{Sp}(K, Map^{Sp}(G,F)) \simeq Map_{Fun(N(Sm^{ft}(S))^{op},\widehat{Sp})}(\delta_G(K), F) \tag{5.4.2}$$

We can now use this to define the class of maps that generate the Nisnevich localization. Namely, for any Nisnevich square

$$\begin{tikzcd} W \arrow[r] \arrow[d] & V \arrow[d] \\ U \arrow[r] & X \end{tikzcd} \tag{5.4.3}$$

we consider its image through $\Sigma^\infty_+ \circ j$

$$\begin{tikzcd} \Sigma^\infty_+ \circ j(W) \arrow[r] \arrow[d] & \Sigma^\infty_+ \circ j(V) \arrow[d] \\ \Sigma^\infty_+ \circ j(U) \arrow[r] & \Sigma^\infty_+ \circ j(X) \end{tikzcd} \tag{5.4.4}$$

in $Fun(N(Sm^{ft}(S))^{op}, \widehat{Sp})$. Every commutative square like this produces a commutative diagram of natural transformations

$$\begin{tikzcd} \delta_{\Sigma^\infty_+ \circ j(W)} \arrow[r] \arrow[d] & \delta_{\Sigma^\infty_+ \circ j(V)} \arrow[d] \\ \delta_{\Sigma^\infty_+ \circ j(U)} \arrow[r] & \delta_{\Sigma^\infty_+ \circ j(X)} \end{tikzcd} \tag{5.4.5}$$

and in particular for each $K \in \widehat{Sp}$, a commutative diagram in $Fun(N(Sm^{ft}(S))^{op}, \widehat{Sp})$

$$\begin{tikzcd} \delta_{\Sigma^\infty_+ \circ j(W)}(K) \arrow[r] \arrow[d] & \delta_{\Sigma^\infty_+ \circ j(V)}(K) \arrow[d] \\ \delta_{\Sigma^\infty_+ \circ j(U)}(K) \arrow[r] & \delta_{\Sigma^\infty_+ \circ j(X)}(K) \end{tikzcd} \tag{5.4.6}$$

Finally, we localize with respect to the class of maps

$$\delta_{\Sigma^\infty_+ \circ j(U)}(K) \coprod_{\delta_{\Sigma^\infty_+ \circ j(W)}(K)} \delta_{\Sigma^\infty_+ \circ j(V)}(K) \to \delta_{\Sigma^\infty_+ \circ j(X)}(K) \tag{5.4.7}$$

given by the universal property of the pushout, with K in $(\widehat{Sp})^\omega$ [3] and W,V,U and X part of a Nisnevich square. Finally, the fact that this class satisfies the required properties follows directly from the definition of the functors $\delta_{\Sigma^\infty_+ \circ j(-)}$ as left adjoints to Map^{Sp} and from the enriched version of the Yoneda's lemma 5.4.1.

For the $\mathbb{A}^1$ localization, we localize with respect to the class of all induced maps

$$\delta_{\Sigma^\infty_+ \circ j(X \times \mathbb{A}^1)}(K) \to \delta_{\Sigma^\infty_+ \circ j(X)}(K) \tag{5.4.8}$$

with X in $N(Sm^{ft}(S))^{op}$ and $K \in (\widehat{Sp})^\omega$.

We observe that these localizations are monoidal. This follows because for any two objects G and G' in $Fun(N(Sm^{ft}(S))^{op}, \widehat{Sp})$, we have

[3] Here $(\widehat{Sp})^\omega$ denotes the full subcategory of $\widehat{Sp}$ spanned by the compact objects. Recall that $\widehat{Sp} \simeq Ind((\widehat{Sp})^\omega)$.

$$Map_{Fun(N(Sm^{ft}(S))^{op},\widehat{Sp})}(\delta_{G\otimes G'}(K),F) \simeq Map_{\widehat{Sp}}(K, Map^{Sp}(G\otimes G',F)) \simeq \quad (5.4.9)$$
$$\simeq Map_{\widehat{Sp}}(K, Map^{Sp}(G, \underline{Hom}(G',F)) \simeq Map_{Fun(N(Sm^{ft}(S))^{op},\widehat{Sp})}(\delta_G(K), \underline{Hom}(G',F)) \simeq \quad (5.4.10)$$
$$\simeq Map_{Fun(N(Sm^{ft}(S))^{op},\widehat{Sp})}(\delta_G(K)\otimes G',F)) \quad (5.4.11)$$

where $\underline{Hom}$ denotes the internal-hom in $Fun(N(Sm^{ft}(S))^{op},\widehat{Sp})$[4]. In particular, the previous chain of equivalences implies $\delta_{G\otimes G'}(K)\simeq \delta_G(K)\otimes G'$.

Finally, we denote the result of both these localizations as $Fun^{Nis}_{\mathbb{A}^1}(N(Sm^{ft}(S))^{op},\widehat{Sp})^{\otimes}$. To conclude, we invert $\mathbb{G}_m$ and obtain a new stable presentable symmetric monoidal $(\infty,1)$-category which by the universal properties involved, is canonically monoidal equivalent to $\mathcal{SH}(k)^{\otimes}$.

5.5 Stable Presentable Symmetric Monoidal $(\infty, 1)$-Categories of Motives over a Scheme

One important result in the subject of motives (see [119]-Theorem 1.1) tells us that the homotopy category of modules over the *motivic Eilenberg-Maclane spetrum* $M\mathbb{Z}$ in

$$h(\mathcal{SH}(S)) \simeq h(Sp^{\Sigma}((\mathcal{M}_{\mathbb{A}^1})_*, (\mathbb{P}^1,\infty))) \quad (5.5.1)$$

is (monoidal) equivalent to the triangulated category of motives constructed by Voevodsky in [151] (whenever S is field of characteristic zero).

This brings the study of motives to the realm of abstract homotopy theory: it is encoded in the homotopy theory of module objects in $Sp^{\Sigma}((\mathcal{M}_{\mathbb{A}^1})_*, (\mathbb{P}^1,\infty))$. However, this is exactly one of those situations where the theory of strictly commutative algebra-objects and their associated theories of modules do not have satisfactory model structures in the sense of the Section 3.9. Therefore, this is exactly one of those situations where the techniques of higher algebra are crucial: they provide a direct access to the $(\infty,1)$-category of commutative algebra objects $CAlg(\mathcal{SH}(S))$ where we can recognize the K-theory ring spectrum $\mathbb{K}$ (see for instance [60] and [106]) and the motivic Eilenberg-MacLane spectrum $M\mathbb{Z}$. Moreover, we also have direct access to the theory of modules $\mathcal{M}ot(S) := Mod_{H\mathbb{Z}}(\mathcal{SH}(S))$. In addition, since $\mathcal{SH}(S)^{\otimes}$ is a presentable symmetric monoidal $(\infty,1)$-category, $\mathcal{M}ot(S)$ is also presentable and inherits a natural symmetric monoidal structure $\mathcal{M}ot(S)^{\otimes}$. Plus, since $\mathcal{SH}(S)^{\otimes}$ is stable, the $(\infty,1)$-category $\mathcal{M}ot(S)^{\otimes}$ is also stable because an ∞-category of modules-objects over an algebra in a stable symmetric monoidal ∞-category, is stable. Therefore, the homotopy category $h(\mathcal{M}ot(S))$ carries a canonical triangulated structure and by our results and the main result of [119] it is equivalent to the triangulated category of motives of Voevodsky (see also the recent results in [129]).

We can now reproduce the results of [6, 7] and [30] in this new setting. More precisely, the assignment $S\mapsto \mathcal{SH}^{\otimes}(S)$ can be properly understood as an ∞-functor with values in stable presentable symmetric monoidal $(\infty,1)$-categories and we should study its descent properties and verify the *six-operations* (which have recently been well-understood and reformulated in the setting of symmetric monoidal $(\infty,1)$-categories [93, 94]). This is the subject of chapters 9 and 10.

[4] which exists because $Fun(N(Sm^{ft}(S))^{op},\widehat{Sp})$ is a stable presentable symmetric monoidal $(\infty,1)$-category

Part II

- Noncommutative Motives and K-Theory

Chapter **6**

Noncommutative Motives - Motivic Stable Homotopy Theory of Noncommutative Spaces over a Ring

Our goal in this section is to formulate a motivic stable homotopy theory for the noncommutative spaces of Kontsevich [86, 89, 87]. This new construction will be canonically related to the classical theory for schemes by means of the universal property proved in the previous chapter of this work. We start with a small survey of the main notions and results concerning the Morita theory of dg-categories and its relation to the notion of finite type introduced by Toën-Vaquié in [141]. In the second part we review how a classical scheme gives birth to a dg-category $L_{qcoh}(X)$ with a compact generator and compact objects given by the perfect complexes of quasi-coherent sheaves. As it is well-known, $L_{pe}(X)$ - the full sub-dg-category of $L_{qcoh}(X)$ spanned by the compact objects - is enough to recover the whole $L_{qcoh}(X)$. In [87], Kontsevich proposes the dg-categories of the form $L_{pe}(X)$ as the natural objects of noncommutative geometry. We recall his notions of smoothness and properness for dg-categories and how they relate to the notion of finite type. The last has been understood as the appropriate notion of smoothness while the notion of Kontsevich should be understood as "formal smoothness". Following this, we define the $(\infty, 1)$-category of smooth noncommutative spaces $\mathcal{N}cS$ as the opposite of the $(\infty, 1)$-category of dg-categories of finite type and explain how the formula $X \mapsto L_{pe}(X)$ can be arranged as a monoidal functor from schemes to $\mathcal{N}cS$. Finally, we perform the construction of a motivic stable homotopy theory for these new noncommutative spaces. After the work in the previous chapter, our task is reduced to finding the appropriate analogue for the Nisnevich topology in the noncommutative setting.

6.1 Preliminaries on Dg-categories

6.1.1 The Homotopy Theory of dg-categories over a ring

For a first contact with the subject we recommend the highly pedagogical expositions in [11, 83]. In order to make our statements precise, we will need to work with three universes $\mathbb{U} \in \mathbb{V} \in \mathbb{W}$ (which we will assume big enough to fit our purposes). The reader is invited to verify that none of the definitions and constructions depends on the choice of the universes. The $\mathbb{U}$-small objects will be called *small*, the $\mathbb{V}$-small *big* and the $\mathbb{W}$-small, *very big*. We redirect the reader to the Section 2.1 for our notations and conventions.

Let $\mathbb{U}$ be our fixed base universe. We fix a small commutative ring k and following the discussion in 3.10 we denote by $Ch(k)$ the big category of (unbounded) complexes of small k-modules. By definition, a small dg-category T is a small category strictly enriched over $Ch(k)$. In other words, T consists of: a small collection of objects $Ob(T)$, for every pair of objects $x, y \in Ob(T)$ a complex of small k-modules $T(x, y)$ and composition maps $T(x, y) \otimes T(y, z) \to T(x, z)$ satisfying the standard coherences of a composition. To every small dg-category T we can associated a classical small category $[T]$ with the same objects of T and morphisms given by the zero-homology groups $H^0(T(x,y))$. It is called the *homotopy category of* T.

Let $Cat_{Ch(k)}$ the big category of all small dg-categories together with the $Ch(k)$-enriched functors (dg-functors for short).

Remark 6.1.1. We can of course give sense to these notions within any universe. In our context we will denote by $Cat^{big}_{Ch(k)}$ the very big category of big dg-categories. We will allow ourselves not to mention the universes whenever the context applies for any universe.

Let us provide some important simple examples:

Example 6.1.2. Given a k-dg-algebra A, we denote by A_{dg} the dg-category with a single object and A as endomorphisms, with composition given by the multiplicative structure of A. This assignment provides a functor $(-)_{dg} : Alg_{\mathcal{A}ss}(Ch(k)) \hookrightarrow Cat_{Ch(k)}$. With this, we a have a commutative diagram

$$\begin{tikzcd} k-algebras \arrow[r] \arrow[d, hook] & k-additiveCats \arrow[d, hook] \\ Alg_{\mathcal{A}ss}(Ch(k)) \arrow[r, "(-)_{dg}"] & Cat_{Ch(k)} \end{tikzcd} \tag{6.1.1}$$

where the horizontal maps understand an algebra as a category with one object and the vertical maps understand a k-module as a k-complex concentrated in degree zero.

Example 6.1.3. The category of complexes $Ch(k)$ has a natural structure of k-dg-category with the enrichement given by the internal-hom described in 3.10. We will write $Ch_{dg}(k)$ to denote $Ch(k)$ together with this enrichement.

Construction 6.1.4. Every model category $\mathcal{M}$ with a compatible $Ch(k)$-enrichment (see Def. 4.2.6 of [69]) provides a new dg-category $Int(\mathcal{M})$: the full dg-category of $\mathcal{M}$ spanned by the cofibrant-fibrant objects (usually called the *underlying dg-category of* $\mathcal{M}$). Also in this case, the homotopy category $h(Int(\mathcal{M}))$ recovers the usual homotopy category of $\mathcal{M}$. This machine is crucial to the foundational development of the theory. The dg-category $Ch_{dg}(k)$ of the previous item is a canonical example of this situation since its model structure is compatible with the $Ch(k)$-enrichment. In this case, $Int(Ch_{dg}(k))$ is the full dg-subcategory of $Ch_{dg}(k)$ spanned by the cofibrant complexes (all objects are fibrant).

If T and T' are dg-categories, we can form a third dg-category $T \otimes T'$ where the objects are the pairs (x,y) with x an object in T and y in T' and the mapping complex $T \otimes T'((x,y),(x',y'))$ is given by the tensor product of complexes in $Ch(k)$, $T(x,x') \otimes T(y,y')$. This formula endows $Cat_{Ch(k)}$ with a symmetric monoidal structure with unit 1_k given by the dg-category with a single object and k as its complex of endomorphisms. We can use the general arguments of [84] to deduce the existence of an internal-hom functor: given T and T' there is a new dg-category $\underline{Hom}(T,T')$ and a natural isomorphism

$$Hom_{Cat_{Ch(k)}}(T'', \underline{Hom}(T,T')) \simeq Hom_{Cat_{Ch(k)}}(T'' \otimes T, T') \tag{6.1.2}$$

In particular the objects of $\underline{Hom}(T,T')$ are the $Ch(k)$-enriched functors and the morphisms can be identified with the $Ch(k)$-natural transformations.

Construction 6.1.5. If T is a dg-category, the *dg-category of T-dg-modules* is defined by the formula

$$Ch(k)^T := \underline{Hom}(T, Ch_{dg}(k)) \tag{6.1.3}$$

An object $E \in Ch(k)^T$ can be naturally identified with a formula that assigns to each object $x \in T$ a complex $E(x)$ and for each pair of objects x, y in T, a map of complexes $T(x,y) \otimes E(x) \to E(y)$ compatible with the composition in T.

The dg-category of dg-modules carries a natural $Ch(k)$-model structure induced from the one in $Ch(k)$, with weak-equivalences and fibrations determined objectwise. More generally, for any $Ch(k)$-model category $\mathcal{M}$, the *dg-category of T-modules with values in* $\mathcal{M}$ is defined by the formula $\mathcal{M}^T := \underline{Hom}(T, \mathcal{M})$. If $\mathcal{M}$ is a cofibrantly generated model category then we can equip $\mathcal{M}^T$ with the projective model structure and we can check that this is again compatible with the dg-enrichment. This construction can be made functorial in T: if $f : T \to T'$ is a dg-functor, we have a canonical restriction functor $f^* : \mathcal{M}^{T'} \to \mathcal{M}^T$ defined by sending a T'-module F to the composition $F \circ f$. By the adjoint functor theorem, this functor admits a left adjoint $f_!$ and the pair $(f_!, f^*)$ forms a Quillen adjunction compatible with the $Ch(k)$-enrichment.

Remark 6.1.6. If T is a dg-category of the form A_{dg} as in the Example 6.1.2, $Ch(k)^T$ can be naturally identified with the category of left A-modules in $Ch(k)$. In particular, when endowed with the projective model structure, the dg-category $Int(Ch(k)^T)$ is a dg-enhancement of the classical derived category of A, in the sense that the $[Int(Ch(k)^T)] \simeq D(A)$. Another important case is when T is associated to the product of two dg-algebras $A \otimes B^{op}$. In this case the category of T-modules is naturally isomorphic to $BiMod(A, B)(Ch(k))$.

In practice we are not interested in the strict study of dg-categories but rather on what results from the study of complexes up to quasi-isomorphisms. A *Dwyer-Kan* equivalence of dg-categories is an (homotopic) fully faithful $Ch(k)$-enriched functor $f : T \to T'$ (ie, such that the induced maps $T(x,y) \to T'(f(x), f(y))$ are weak-equivalences in $Ch(k)$) such that the functor induced between the homotopy categories $[T] \to [T']$ is essentially surjective. Of course, if $T \to T'$ is a Dwyer-Kan equivalence, the induced functor $[T] \to [T']$ is an equivalence of 1-categories. It is the main content of [132] that $Cat_{Ch(k)}$ admits a (non left-proper) cofibrantly generated model structure to study these weak-equivalences. Moreover, the model structure is combinatorial because $Cat_{Ch(k)}$ is known to be presentable (see [85]). The fibrations are the maps $T \to T'$ such that the induced applications $T(x,y) \to T'(f(x), f(y))$ are surjections (meaning fibrations in $Ch(k)$) and the map induced between the associated categories has the lifting property for isomorphisms. Therefore, every object is fibrant and the cofibrant objects, which are more difficult to describe, are necessarily enriched over cofibrant complexes (see Prop. 2.3 in [139]). We will address to this model structure as the "standard one".

Remark 6.1.7. The theory of modules over dg-categories is well-behaved with respect to the Dwyer-Kan equivalences. By the Proposition 3.2 of [139], if $f : T \to T'$ is a Dwyer-Kan equivalence of dg-categories, then the adjunction $f^* : \mathcal{M}^{T'} \to \mathcal{M}^T$ is a Quillen equivalence if one of the following situations hold: (i) T and T' are locally cofibrant (meaning: enriched over cofibrant complexes); (ii) the product (using the $Ch(k)$-action) of a cofibrant object A in $\mathcal{M}$ with a weak-equivalence of complexes $C \to D$ is a weak-equivalence in $\mathcal{M}$. In particular the second condition holds if $\mathcal{M} = Ch(k)$. Moreover, by the Proposition 3.3 of [139], if T is locally cofibrant then the evaluation functors $ev_x : \mathcal{M}^T \to \mathcal{M}$ sending $F \mapsto F(x)$, preserve fibrations, cofibrations and weak-equivalences. In particular, $Int(\mathcal{M}^T)$ is made of objectwise cofibrant-fibrant objects in $\mathcal{M}$.

The information of this homotopical study is properly encoded in a new big[1] $(\infty, 1)$-category

$$\mathcal{D}g(k) := N(Cat_{Ch(k)})[W_{DK}^{-1}] \tag{6.1.4}$$

where W_{DK} denote the collection of all Dwyer-Kan equivalences. Because the homotopy category $h(\mathcal{D}g(k))$ recovers the ordinary localization in Cat, the objects of $\mathcal{D}g(k)$ can be again identified with the small dg-categories. Notice that in this situation we cannot apply the Proposition 2.2.1 because $Cat_{Ch(k)}$ is not a simplicial model category.

[1]because $Cat_{Ch(k)}$ is big, its nerve is a a big simplicial set and the localization is obtained as a cofibrant-fibrant replacement in the model category of big marked simplicial sets

Remark 6.1.8. It is important to remark that the inclusion of very big categories $Cat_{Ch(k)} \subseteq Cat_{Ch(k)}^{big}$ is compatible with the model structure of [132] and we have a fully-faithful map of very big $(\infty,1)$-categories $\mathcal{D}g(k) \subseteq \mathcal{D}g(k)^{big}$ (where the last is defined by the same formula using the theory for $\mathbb{V}$-small simplicial sets).

Remark 6.1.9. The functor $(-)_{dg} : Alg_{\mathcal{A}ss}(Ch(k)) \to Cat_{Ch(k)}$ sends weak-equivalences of dg-algebras to Dwyer-Kan equivalences. The localization gives us a canonical map of $(\infty,1)$-categories $Alg_{\mathcal{A}ss}(\mathcal{D}(k)) \to \mathcal{D}g(k)$ (see 3.10). As pointed to me by B. Toën, this map is not fully-faithfull. To see this, we observe that the new map $Alg_{\mathcal{A}ss}(\mathcal{D}(k)) \to \mathcal{D}g(k)$ factors as $Alg_{\mathcal{A}ss}(\mathcal{D}(k)) \to \mathcal{D}g(k)_* \to \mathcal{D}g(k)$ where $\mathcal{D}g(k)_*$ denotes the $(\infty,1)$-category of pointed objects in $\mathcal{D}g(k)$. The first map in the factorization is fully-faithful but the second one is not. Indeed, if A and B are two dg-algebras, the mapping space $Map_{\mathcal{D}g(k)_*}(A_{dg}, B_{dg})$ can be obtained as the homotopy quotient of $Map_{\mathcal{D}g(k)}(A,B)$ by the action of the simplicial group of units in B.

Remark 6.1.10. The theory of A_∞*-categories* of [90] provides an equivalent approach to the homotopy theory of dg-categories.

We now collect some fundamental results concerning the inner structure of the $(\infty,1)$-category $\mathcal{D}g(k)$.

1. *Existence of Limits and Colimits*: This results from the fact that the model structure in $Cat_{Ch(k)}$ is combinatorial, together with the Proposition A.3.7.6 [99] and the main result of [45].

2. *Symmetric Monoidal Structure:* Notice that $Cat_{Ch(k)}$ is *not* a symmetric monoidal model category in the sense of the Definition 4.2.6 in [69]. For instance, the product of the two cofibrant objects is not necessarily cofibrant (Exercise 14 of [11]). Therefore, we cannot apply directly the abstract-machinery reviewed in the Section 3.9 to deduce the existence of a monoidal structure in $\mathcal{D}g(k)$. Luckily, we can overcome this problem and extend the monoidal structure to $\mathcal{D}g(k)$ even under these bad circumstances.

   We observe first that the product of dg-categories whose hom-complexes are cofibrant in $Ch(k)$ (also called *locally cofibrant*) is again a dg-category with cofibrant hom-complexes. This follows from the fact that $Ch(k)$ is a symmetric monoidal model category and so the product of cofibrant complexes is again a cofibrant complex. Second, we notice that the product of weak-equivalences between dg-categories with cofibrant hom-complexes is again a weak-equivalence. It is enough to check that for any triple of locally-cofibrant dg-categories T, T' and S and for any weak-equivalence $T \to T'$, the product $T \otimes S \to T' \otimes S$ is again a weak-equivalence. The fact that the map between the homotopy categories is essentially surjective is immediate. Everything comes down to prove that if M is a cofibrant complex of k-modules and $N \to P$ is quasi-isomorphism between cofibrant complexes then $M \otimes N \to M \otimes P$ is also a quasi-isomorphisms of complexes. Again this follows because $Ch(k)$ is a symmetric monoidal model category (or more precisely, because cofibrant complexes are flat[2] - combine [69, 2.3.6] with the Kunneth spectral sequence.

   The first conclusion of this discussion is that the full-subcategory $Cat_{Ch(k)}^{loc-cof}$ of $Cat_{Ch(k)}$ spanned by the locally-cofibrant dg-categories, is closed under tensor products and contains the unit of $Cat_{Ch(k)}$ and therefore inherits a symmetric monoidal structure

   We have inclusions

$$Cat_{Ch(k)}^{cof} \subseteq Cat_{Ch(k)}^{loc-cof} \subseteq Cat_{Ch(k)} \tag{6.1.5}$$

   mapping weak-equivalences to weak-equivalences. Since in $Cat_{Ch(k)}$ we can choose a functorial cofibrant-replacement Q that preserves sets of objects (see [132] for details) and together with the inclusions in the previous diagram, produces equivalences of $(\infty,1)$-categories

[2] Recall that a complex M is flat if for every acylic complex N the tensor product $N \otimes M$is also acyclic

$$N(Cat^c_{Ch(k)})[W_c^{-1}] \simeq \mathcal{D}g^{loc-cof}(k) \simeq \mathcal{D}g(k) \tag{6.1.6}$$

where we set $\mathcal{D}g^{loc-cof}(k) := N(Cat^{loc-cof}_{Ch(k)})[W^{-1}_{loc-cof}]$.

Finally, since the symmetric monoidal structure in $N(Cat^{loc-cof}_{Ch(k)})$ preserves weak-equivalences, by the discussion in 3.9, the localization $\mathcal{D}g^{loc-cof}(k)$ inherits a natural symmetric monoidal structure $\mathcal{D}g^{loc-cof}(k)^{\otimes} \to N(Fin_*)$ obtained as the monoidal localization of $Cat^{loc-cof}_{Ch(k)}$ seen as a trivial simplicial coloured operad (followed by the Grothendieck construction). We can now use the equivalence $\mathcal{D}g^{loc-cof}(k) \simeq \mathcal{D}g(k)$ to give sense to the product of two arbitrary dg-categories $T \otimes^{\mathbb{L}} T' \simeq Q(T) \otimes T'$. This recovers the famous formula for the derived tensor product.

Remark 6.1.11. The category of strict k-dg-algebras inherits a symmetric monoidal structure, obtained by tensoring the underlying complexes. It follows that the functor $(-)_{dg}$: $Alg_{\mathcal{A}ss}(Ch(k)) \to Cat_{Ch(k)}$ is monoidal. Moreover, since the product of cofibrant dg-algebras remains a dg-algebra with a cofibrant underlying complex (as proved in [124]), we can use the Remark 3.9.3 and the discussion in 3.10 to deduce that $(-)_{dg} : Alg_{\mathcal{A}ss}(\mathcal{D}(k)) \to \mathcal{D}g^{loc-cof}(k) \simeq \mathcal{D}g(k)$ is a monoidal functor.

Notation 6.1.12. Following the previous remark, we will sometimes abuse the notation and identify a dg-algebra A with its associated dg-category A_{dg}.

3. *The Mapping spaces in $\mathcal{D}g(k)$*: The first important technical result of [139] is the characterization of the mapping spaces $Map_{\mathcal{D}g(k)}(T, T')$. The description uses the monoidal structure introduced in the previous item: from the input of two dg-categories T and T', we consider the $Ch(k)$-model category of $T \otimes^{\mathbb{L}} (T')^{op} := Q(T) \otimes (T')^{op}$-dg-modules. Again, this homotopy theory is properly encoded in the $(\infty, 1)$-category

$$(T, T')_\infty := N(Ch(k)^{T \otimes^{\mathbb{L}} (T')^{op}})[W^{-1}_{qis}] \tag{6.1.7}$$

inside which we can isolate the full subcategory spanned by the *right quasi-representable objects.* [3] which we denote here as $rrep(T, T')_\infty$. By the Theorem 4.2 of [139], there is an explicit isomorphism of homotopy types

$$Map_{\mathcal{D}g(k)}(T, T') \simeq rrep(T, T')^{\simeq}_\infty \tag{6.1.8}$$

where $rrep(T, T')^{\simeq}_\infty$ denotes the ∞-groupoid of equivalences in $rrep(T, T')_\infty$.

Remark 6.1.13. The original formulation of this theorem in [139] uses another presentation of the Kan-complex $rrep(T, T')^{\simeq}$. Let $\mathcal{M}$ be a model category with weak-equivalences W. In one direction, we can consider the subcategory W of $\mathcal{M}$ consisting of all the objects in $\mathcal{M}$ together with the weak-equivalences between them. The inclusion $W \subseteq \mathcal{M}$ sends weak-equivalences to weak-equivalences and by using the nerve we have a natural homotopy commutative diagram of $(\infty, 1)$-categories

$$\begin{tikzcd} N(W) \arrow[r] \arrow[d] & N(M) \arrow[d] \\ N(W)[W^{-1}] \arrow[r] & N(M)[W^{-1}] \end{tikzcd} \tag{6.1.9}$$

[3] By definition, these are the $T \otimes (T')^{op}$- modules F such that for any $x \in T$, there is an object $f_x \in T'$ and an isomorphism between $F(x, -)$ and $T'(-, f_x)$ in the homotopy category of $(T')^{op}$-modules

The $(\infty, 1)$-category $N(W)[W^{-1}]$ is a Kan-complex (because every arrow is invertible)[4] and therefore, the map $N(W)[W^{-1}] \to N(M)[W^{-1}]$ factors as $N(W)[W^{-1}] \to N(M)[W^{-1}]^{\simeq} \subseteq N(M)[W^{-1}]$. This map is a weak-equivalence of simplicial sets for the Quillen structure because M is a model category. It results from the foundational works of Dwyer and Kan (see [48, Prop. 4.3]) that the canonical map $N(W) \to N(W)[W^{-1}]$ is also weak-equivalences of simplicial sets for the standard model structure.

4. *The monoidal structure in $\mathcal{D}g(k)$ is closed:* More precisely, by the Theorem 6.1 of [139], for any three small dg-categories A, B and C, there exists a new small dg-category $\mathbb{R}\underline{Hom}(B, C)$ (in the same universe of B and C - see Proposition 4.11 of [139]) and functorial isomorphisms of homotopy types

$$Map_{\mathcal{D}g(k)}(A \otimes^{\mathbb{L}} B, C) \simeq Map_{\mathcal{D}g(k)}(A, \mathbb{R}\underline{Hom}(B, C)) \tag{6.1.10}$$

Furthermore, $\mathbb{R}\underline{Hom}(B, C)$ is naturally equivalent in $\mathcal{D}g(k)$ to the full essentially small sub-dg-category $Int(B \otimes^{L} C^{op} - Mod))_{rrep}$ of $Int(B \otimes^{L} C^{op} - Mod))$ spanned by the right quasi-representable modules.

An immediate implication of this result is that the derived tensor product is compatible with colimits on each variable separately.

5. *Existence of dg-localizations*: The description of the mapping spaces in $\mathcal{D}g(k)$ allows us to prove the existence of a localization process inside the dg-world. By the Corollary 8.7 of [139], given a dg-category $T \in \mathcal{D}g(k)$ together with a class of morphisms S in $[T]$ we can formally construct a new dg-category $L_S T$ together with a map $T \to L_S T$ in $\mathcal{D}g(k)$, such that for any dg-category T' the composition map

$$Map_{\mathcal{D}g(k)}(L_S T, T') \to Map^{S}_{\mathcal{D}g(k)}(T, T') \tag{6.1.11}$$

is a weak-equivalence. Here $Map^{S}_{\mathcal{D}g(k)}(T, T')$ denotes the full simplicial set of $Map_{\mathcal{D}g(k)}(T, T')$ given by the union of all connected components corresponding to morphisms in $h(\mathcal{D}g(k))$ sending S to isomorphisms in $[T']$. Another way to formulate this is to say that for each pair (T, S), the functor $\mathcal{D}g(k) \to \mathcal{S}$ sending $T' \mapsto Map^{S}_{\mathcal{D}g(k)}(T, T')$ is co-representable.

Remark 6.1.14. This localization allows us to prove a dg-analogue of a fundamental result of Quillen for model categories: for a $Ch(k)$-model category $\mathcal{M}$, the dg-localization of $\mathcal{M}$ with respect to its weak-equivalences is equivalent to $Int(\mathcal{M})$ (see [11]).

6.1.2 Morita Theory of dg-categories

Let T be a small dg-category. The enriched version of the Yoneda's Lemma allows us understand T as a full sub-dg-category of $Ch(k)^{T^{op}}$.

$$h : T \to Ch(k)^{T^{op}} \tag{6.1.12}$$

This big dg-category admits a compatible model structure induced from the one in $Ch(k)$. It is well known that this model structure is stable so that its homotopy category inherits a canonical triangulated structure where the exact triangles are the image of the homotopy fibration sequences through the localization map. It is an important remark that for each $x \in T$, the representable h_x is a cofibrant T^{op}-module (it follows again from the Yoneda's lemma and the fact that fibrations are defined as levelwise surjections). By setting $\widehat{T} := Int(Ch(k)^{T^{op}})$, h factors as

[4] See [99, Prop. 1.2.5.1]

$$T \to \widehat{T} \subseteq Ch(k)^{T^{op}} \tag{6.1.13}$$

and from now will use the letter h to denote the first map in this factorization. Of course, when $T = 1_k$ we have $(1_k)^{op} - Mod = Ch(k)$ and therefore $\widehat{1_k} = Int(Ch(k))$.

Remark 6.1.15. It is important to notice that if T is a small locally-cofibrant dg-category, then $\widehat{T}$ will not be locally-cofibrant in general. However, we will see in the Remark 6.1.23 that it is possible to provide an alternative construction of the assignement $T \mapsto \widehat{T}$ that preserves the condition of being locally-cofibrant.

It is also important to remark that the passage $T \mapsto \widehat{T}$ is (pseudo) functorial. If $f : T \to T'$ is a dg-functor, we have a natural restriction map

$$Ch(k)^{(T')^{op}} \to Ch(k)^{T^{op}} \tag{6.1.14}$$

induced by the composition with f^{op}. As a limit preserving map between presentable, it admits a left adjoint and this adjunction is compatible with the model structures. Since all objects are fibrant, the left adjoint restricts to a well-define map $\widehat{T} \to \widehat{T'}$.

We finally come to the subject of (derived) Morita theory. Classically, it can described as the study of algebras up to their (derived) categories of modules. The version for small dg-categories implements the same principle: it is the study of small dg-categories T up to their derived dg-categories of modules $\widehat{T}$. It generalizes the classical theory for algebras for when T is a dg-category coming from an algebra A the homotopy category of $\widehat{T}$ recovers the derived category of A. We will see that the following three constructions are equivalent:

a) the localization of $Cat_{Ch(k)}$ with respect to the class of dg-functors $T \to T'$ for which the induced map $\widehat{T} \to \widehat{T'}$ is a weak-equivalence of dg-categories;

b) the full subcategory of $\mathcal{D}g(k)$ spanned by the *idempotent complete* dg-categories;

c) the (non-full) subcategory of $\mathcal{D}g(k)^{loc-cof,big}$ spanned by the dg-categories of the form $\widehat{T}$ (with T small), together with those morphisms that preserve colimits and compact objects.

The link is made by the notion of a compact object. It is well known that the model category of complexes is combinatorial and compactly generated so that we can apply the results of our discussion in 2.2.2. It is immediate that the same will hold for the projective structure in model category $Ch(k)^T$, for any small dg-category T. Following this, we denote by $\widehat{T}_c$ the full sub-dg-category of $\widehat{T}$ spanned by those cofibrant modules which are homotopicaly finitely presented in the model category $Ch(k)^{T^{op}}$. Again by the general machinery described in 2.2.2, they can be constructed as retracts of strict finite cell-objects and correspond to the compact objects in the underlying $(\infty, 1)$-category of $Ch(k)^{T^{op}}$ and with this in mind we will refer to them as compact.

It follows from the definitions and from the enriched version of the Yoneda Lemma that for any object x in T, the representable dg-module $h(x)$ is compact. In particular, h factors as $T \to (\widehat{T})_c \subseteq \widehat{T}$. At the level of the homotopy categories, this produces a sequence of inclusions $[T] \subseteq [(\widehat{T})_c] \subseteq [\widehat{T}]$ and the fact that $Ch(k)^{T^{op}}$ is stable model category implies two important things: (i) the category $[\widehat{T}]$ has a triangulated structure; (ii) because homotopy pushouts are homotopy pullbacks, a dg-module F is compact if and only if $Map(F, -)$ commutes with arbitrary coproducts. With this we identify the subcategory $[(\widehat{T})_c] \subseteq [\widehat{T}]$ with $[\widehat{T}]_c \subseteq [\widehat{T}]$ - the full triangulated subcategory spanned by the compact objects in the sense of Neeman (see 2.1.4). In particular when $T = 1_k$, the objects in $\widehat{T}_c$ are are exactly the perfect complexes of k-modules.

Remark 6.1.16. Because any compact module is a strict finite I-cell (2.2.2) the dg-category $(\widehat{T})_c$ is essentially small and can be considered as an object in $\mathcal{D}g(k)$. For the same reason, $(\widehat{T})_c$ is stable under shifts, retracts and pushouts.

Recall now that a small dg-category is said to be *idempotent complete* (or *triangulated*) if the dg-functor

$$T \to (\widehat{T})_c \tag{6.1.15}$$

is a weak-equivalence of dg-categories. The first reason why idempotent dg-categories are relevant to Morita theory is because for any small dg-category T, the restriction along $T \to (\widehat{T})_c$

$$\widehat{((\widehat{T})_c)} \to \widehat{T} \tag{6.1.16}$$

is a weak-equivalence of dg-categories (Lemma 7.5-(1) in [139]). In other words, at the level of modules we cannot distinguish between T and $(\widehat{T})_c$. It follows that a dg-functor $f : T \to T'$ induces a weak-equivalence $\widehat{T} \to \widehat{T'}$ if and only if its restriction to compact objects [5] $(\widehat{T})_c \to (\widehat{T'})_c$ is a weak-equivalence. We will denote by $\mathcal{D}g(k)^{idem}$ the full subcategory of $\mathcal{D}g(k)$ spanned by those dg-categories which are idempotent complete.

Proposition 6.1.17. *The formula $T \mapsto (\widehat{T})_c$ provides a left adjoint to the inclusion $\mathcal{D}g(k)^{idem} \subseteq \mathcal{D}g(k)$.*

Proof. In order to prove this result we construct a cocartesian fibration $(\infty,1)$-categories $p : \mathcal{M} \to \Delta[1]$ with $p^{-1}(\{0\}) = \mathcal{D}g(k)$ and $p^{-1}(\{1\}) = \mathcal{D}g(k)^{idem}$. We consider the full $(\infty,1)$-category $\mathcal{M}$ of the product $\mathcal{D}g(k) \times \Delta[1]$ spanned by the pairs $(T, 0)$ where T can be any small dg-category and the pairs of the form $(T, 1)$ only accept dg-categories T which are triangulated. By construction, there is a canonical projection $p : \mathcal{M} \to \Delta[1]$ whose fiber over 0 is $\mathcal{D}g(k)$ and over 1 is $\mathcal{D}g(k)^{idem}$. We are reduced to check that p is a cocartesian fibration. Notice a map in $\mathcal{M}$ over the morphism $0 \to 1$ in $\Delta[1]$ consists in the data of a morphism $T \to T'$ in $\mathcal{M}$ where T is any dg-category and T' is a triangulated one. To say that p is cocartesian is equivalent to say that for any dg-category T, there is a new triangulated dg-category T' together with a morphism $T \to T'$ having the following universal property: for any triangulated dg-category D, the composition map

$$Map_{\mathcal{D}g(k)}(T', D) \to Map_{\mathcal{D}g(k)}(T, D) \tag{6.1.17}$$

is a weak-equivalence of spaces. We set $T' := (\widehat{T})_c$ and $T \to T'$ the yoneda's map. Since any triangulated dg-category D is equivalent in $\mathcal{D}g(k)$ to $\widehat{D}_c$, we are reduced to prove the composition map

$$Map_{\mathcal{D}g(k)}((\widehat{T})_c, \widehat{D}_c) \to Map_{\mathcal{D}g(k)}(T, \widehat{D}_c) \tag{6.1.18}$$

is a weak-equivalence. Using the internal-hom, we are reduced to prove that the natural map

$$\mathbb{R}\underline{Hom}((\widehat{T})_c, \widehat{D}_c) \to \mathbb{R}\underline{Hom}(T, \widehat{D}_c) \tag{6.1.19}$$

is an equivalence in $\mathcal{D}g(k)$. This is the content of the Theorem 7.2-(2) in [139].

□

The existence of this left adjoint, which we will denote as $(\widehat{-})_c$, makes $\mathcal{D}g(k)^{idem}$ a reflexive localization of $\mathcal{D}g(k)$. In particular the idempotent dg-categories can be described as local objects with respect to the class of maps in $\mathcal{D}g(k)$ whose image through the composition of the inclusion with $(\widehat{-})_c$ is an equivalence. This establishes $\mathcal{D}g(k)^{idem}$ as the second approach in the list.

[5] This restriction is well-defined because every compact dg-module in $\widehat{T}$ can be constructed from representables using finite colimits and retracts. The conclusion follows because the map $\widehat{T} \to \widehat{T'}$ sends representables to representables, representables are compact and compact objects are stable under finite colimits and retracts. (See [139] for more details)

Remark 6.1.18. The equivalence $\mathcal{D}g^{loc-cof}(k) \simeq \mathcal{D}g(k)$ restricts to an equivalence $\mathcal{D}g(k)^{idem,loc-cof} \subseteq \mathcal{D}g(k)^{idem}$. Using the Remark 6.1.15 we see that the left adjoint of the Prop. 6.1.17 restricts to a left adjoint to the inclusion $\mathcal{D}g(k)^{idem,loc-cof} \subseteq \mathcal{D}g^{loc-cof}(k)$: if T is small and locally-cofibrant, we can find an equivalent way to define the formula $T \mapsto \widehat{T}$, this time preserving the condition of being locally-cofibrant so that the full subcategory $(\widehat{T})_c$ is locally cofibrant.

We now explain the first approach. Recall that dg-functor $T \to T'$ in $Cat_{Ch(k)}$ is called a *Morita equivalence* if the induced map $\widehat{T} \to \widehat{T'}$ is a weak-equivalence of big dg-categories.

Corollary 6.1.19. *Let W_{Mor} denote the collection of Morita equivalences between small dg-categories. Then there is a canonical isomorphism in the homotopy category of $(\infty,1)$-categories*

$$\mathcal{D}g(k)^{idem} \simeq N(Cat_{Ch(k)})[W_{Mor}^{-1}] \tag{6.1.20}$$

Proof. This follows from the universal property of the localization in the setting of $(\infty,1)$-categories, together with the observations that: (i) every weak-equivalence of dg-categories is in W_{Mor} and (ii) a dg-functor f is in W_{Mor} if and only if its image through the composition of localizations $N(Cat_{Ch(k)}) \to \mathcal{D}g(k) \to \mathcal{D}g(k)^{idem}$ is an equivalence. □

Both sides of this equivalence admit natural symmetric monoidal structures and the equivalence preserves them. More precisely, for the first side we have

Proposition 6.1.20. *The reflexive localization $\mathcal{D}g(k)^{idem,loc-cof} \subseteq \mathcal{D}g^{loc-cof}(k)$ is compatible with the monoidal structure in $\mathcal{D}g^{loc-cof}(k)^{\otimes}$. In other words, there is a natural symmetric monoidal structure in $\mathcal{D}g(k)^{idem,loc-cof}$ for which the left adjoint is monoidal. Informally, it is given by the formula $T \otimes^{idem} T' := (\widehat{T \otimes^{\mathbb{L}} T'})_c$. In particular, the unit is the idempotent completion of the dg-category with a single object with k as endomorphisms.*

Proof. It is enough to check that if $f : T \to T'$ is a morphism in $\mathcal{D}g(k)$ such that $(\widehat{T})_c \to \widehat{T'}_c$ is an equivalence, then for any dg-category $C \in \mathcal{D}g(k)$, the product $f \otimes^{\mathbb{L}} Id_C : T \otimes^{\mathbb{L}} C \to T' \otimes^{\mathbb{L}} C$ will also be sent to an equivalence in $\mathcal{D}g(k)^{idem}$. This follows directly from the Lemma 7.5-(1) in [139]. □

Remark 6.1.21. The combination of the argument in the previous proof, with the Theorem 7.2-(2) of [139] implies that for any idempotent complete dg-category Z and any dg-category T, the internal-hom $\mathbb{R}\underline{Hom}(T, Z)$ is again idempotent. In particular, it provides an internal-hom for the monoidal structure in $\mathcal{D}g(k)^{idem}$.

To find a monoidal structure in the second localization $N(Cat_{Ch(k)})[W_{Mor}^{-1}]$ it suffices to verify that the tensor product of Morita equivalences in $Cat_{Ch(k)}$ is again a Morita equivalence. However, and as for the Dwyer-Kan equivalences, this is not true. It happens that everything is well-behaved if we restrict to locally cofibrant dg-categories (see the Proposition 2.22 in [35] together with the observation that any cofibrant complex is flat). The problem is solved by considering the monoidal localization of the trivial simplicial coloured operad associated to the well-defined monoidal structure in $Cat_{Ch(k)}^{loc-cof}$. The fact that the equivalence in the Corollary 6.1.19 is monoidal follows immediately from the universal property of the monoidal localization.

To compare these two approaches with the third one, it is convenient to have a description of the mapping spaces in $\mathcal{D}g(k)^{idem}$. Being a full subcategory of $\mathcal{D}g(k)$ and using again the Theorem 7.2-(2) of [139] we find equivalences

$$Map_{\mathcal{D}g(k)^{idem}}((\widehat{T})_c, (\widehat{T'})_c) \simeq Map_{\mathcal{D}g(k)}((\widehat{T})_c, (\widehat{T'})_c) \simeq Map_{\mathcal{D}g(k)}(1_k, \mathbb{R}\underline{Hom}((\widehat{T})_c, (\widehat{T'})_c)) \tag{6.1.21}$$

$$\simeq Map_{\mathcal{D}g(k)}(1_k, \mathbb{R}\underline{Hom}(T, (\widehat{T'})_c)) \tag{6.1.22}$$

and the internal-hom $\mathbb{R}\underline{Hom}(T, (\widehat{T'})_c)$ is given by the full sub-dg-category of $Int(T \otimes^{\mathbb{L}} ((\widehat{T'})_c)^{op} - modules)$ spanned by the right-representable. In this particular case, the last can be described as the full sub-dg-category of $\widehat{T^{op} \otimes^{\mathbb{L}} T'}$ spanned by those modules E which for every $x \in T$, the module $E(x, -) : (T')^{op} \to Ch(k)$ is compact. These are called *pseudo-perfect* (over T relatively to T'). Following [141] we will write $\widehat{T^{op} \otimes^{\mathbb{L}} T'}_{pspe}$ for the sub-dg-category of $\widehat{T^{op} \otimes^{\mathbb{L}} T'}$ spanned by the pseudo-perfect dg-modules over T relative to T'. In the next section we will review how the interplay between the notion of pseudo-perfect and compact is essential to express the geometrical behavior of dg-categories. Using this description we have

$$Map_{\mathcal{D}g(k)}(1_k, \mathbb{R}\underline{Hom}(T, \widehat{T'}_c)) \simeq Map_{\mathcal{D}g(k)}(1_k, \widehat{T^{op} \otimes^{\mathbb{L}} T'}_{pspe}) \simeq rrep(1_k, \widehat{T^{op} \otimes^{\mathbb{L}} T'}_{pspe}) \tag{6.1.23}$$

where the last is our notation for the maximal ∞-groupoid of the full subcategory spanned by right-representable in the underlying $(\infty, 1)$-category of all $1_k \otimes^{\mathbb{L}} (\widehat{T^{op} \otimes^{\mathbb{L}} T'}_{pspe})^{op}$-modules. We can easily check this is equivalent to the maximal ∞-groupoid of $pspe(T, T)_\infty \subseteq (T, T')_\infty$ - the full subcategory spanned by the pseudo-perfect modules.

We now come to the third approach. Let $\mathcal{D}g^c(k)$ denote the (non full) subcategory of $\mathcal{D}g(k)^{big}$ spanned by the dg-categories of the form $\widehat{T}$ for some small dg-category T, together with those morphisms $\widehat{T} \to \widehat{T'}$ whose map induced between the homotopy categories $[\widehat{T}] \to [\widehat{T'}]$ commutes with arbitrary sums[6]. Notice that each map in $\mathcal{D}g^c(k)$ corresponds to a unique (up to quasi-isomorphism) $(\widehat{T} \otimes \widehat{T'}^{op})$-dg-module. Let $\mathbb{R}\underline{Hom}_c(\widehat{T}, \widehat{T'})$ be the full sub-dg-category of $\mathbb{R}\underline{Hom}_c(\widehat{T}, \widehat{T'})$ spanned by those modules which induce a sum preserving map $[\widehat{T}] \to [\widehat{T'}]$. Then, by the Theorem 7.2-(1) in [139], for any small dg-category T' the composition with the Yoneda's embedding $h : T \to \widehat{T}$

$$\mathbb{R}\underline{Hom}_c(\widehat{T}, \widehat{T'}) \to \mathbb{R}\underline{Hom}(T, \widehat{T'}) \tag{6.1.24}$$

is an isomorphism in the homotopy category of dg-categories. It follows from the description of the internal-hom as right representable modules, that the last is equivalent to the dg-category $\widehat{T^{op} \otimes^{\mathbb{L}} (T')}$. One corollary of this result (see [139]) is the description of the mapping spaces $Map_{\mathcal{D}g^c(k)}(\widehat{T}, \widehat{T'})$ as the maximal $(\infty, 1)$-groupoid in $(T, T')_\infty$. Another corollary is the existence of a functor $\widehat{(-)} : \mathcal{D}g(k) \to \mathcal{D}g^c(k)$ sending a small dg-category to its category of dg-modules. For an explicit description, we consider the canonical projection $\mathcal{D}g(k)^{big} \times \Delta[1] \to \Delta[1]$ and the full subcategory $\mathcal{M}$ spanned by the vertices (i, T) where if $i = 0$, T is small and if $i = 1$, T is of the form $\widehat{T_0}$ for some small dg-category T_0 and the only admissible maps $(1, T) \to (1, T')$ are the ones in $\mathcal{D}g^c(k)$. The fact that this fibration is cocartesian follows again from the theorem.

To formalize the third approach, we will restrict our attention to a subcategory of $\mathcal{D}g^c(k)$. As we just saw, a map $f : \widehat{T} \to \widehat{T'}$ in $\mathcal{D}g^c(k)$ corresponds to the data of a (uniquely determined) $T \otimes^{\mathbb{L}} (T')^{op}$-module E_f. We will say that f preserves compact objects if for every object $x \in T$, the $(T')^{op}$-module $E_f(x, -)$ is compact. According to our terminology, this is the same as saying that E_f is pseudo-perfect over T relatively to $(T')^{op}$. With this, we denote by $\mathcal{D}g^{cc}(k)$ the (non-full) subcategory of $\mathcal{D}g^c(k)$ containing all the objects together with those maps that preserve compact objects. It follows from the definitions that the mapping spaces $Map_{\mathcal{D}g^{cc}(k)}(\widehat{T}, \widehat{T'})$ are given by the maximal ∞-groupoids inside $pspe(T, T')_\infty$. It is now easy to see that the canonical map $\widehat{(-)} : \mathcal{D}g(k) \to \mathcal{D}g^c(k)$ factors through $\mathcal{D}g^{cc}(k)$. The following proposition establishes $\mathcal{D}g^{cc}(k)$ as a third approach to Morita theory

Proposition 6.1.22. *The composition $\mathcal{D}g(k)^{idem} \hookrightarrow \mathcal{D}g(k) \to \mathcal{D}g^{cc}(k)$ is an equivalence of $(\infty, 1)$-categories. An inverse is given by the formula sending a dg-category $\widehat{T}$ to the full subcategory $(\widehat{T})_c$ spanned by the compact objects.*

[6] This notion is well-defined because the map induced between the homotopy categories is unique up to isomorphism of functors.

Proof. By the definition of $\mathcal{D}g^{cc}(k)$ the map is essentially surjective. It is fully-faithful because the mapping spaces in $\mathcal{D}g^{cc}(k)$ are by definition, the same as in $\mathcal{D}g(k)^{idem}$, corresponding both to the ∞-groupoid of pseudo-perfect modules. □

Remark 6.1.23. Notice that if $\widehat{T}$ is a locally-cofibrant dg-category, then so is $\widehat{T}_c$. In this case, the equivalence $(-)_c$ restricts to an equivalence $\mathcal{D}g^{cc,loc-cof}(k) \to \mathcal{D}g(k)^{idem,loc-cof}$. By choosing an inverse to this functor we solve the problem posed in the Remark 6.1.15 of finding a model for the formula $T \mapsto \widehat{T}$ that preserves the hypothesis of being locally-cofibrant.

To complete the comparison between the second and third approaches, we regard the existence of a symmetric monoidal structure in $\mathcal{D}g^{cc,loc-cof}(k)$ which makes $(-)_c : \mathcal{D}g^{cc,loc-cof}(k) \to \mathcal{D}g(k)^{idem,loc-cof}$ a monoidal functor.

Proposition 6.1.24. *The $(\infty,1)$-category $\mathcal{D}g^{cc,loc-cof}(k)$ is the underlying ∞-category of a symmetric monoidal structure $\mathcal{D}g^{cc,loc-cof}(k)^{\otimes}$. Given two objects $\widehat{T}, \widehat{T'} \in \mathcal{D}g^{cc}(k)$, their monoidal product can be informally described by the formula $\widehat{T} \otimes \widehat{T} = \widehat{T \otimes^{\mathbb{L}} T'}$, where $\otimes^{\mathbb{L}}$ denotes the monoidal structure in $\mathcal{D}g^{loc-cof}(k)$.*

Proof. The proof of this proposition requires two steps. The first concerns the construction of an $(\infty,1)$-category $\mathcal{D}g^{cc,loc-cof}(k)^{\otimes}$ equipped with a map to $N(Fin_*)$. The second step is the prove that this map is a cocartesian fibration. For the first, we start with $\mathcal{D}g^{loc-cof,big}(k)^{\otimes} \to N(Fin_*)$ the symmetric monoidal structure in the $(\infty,1)$-category of the big locally-cofibrant dg-categories (as constructed in the section 6.1.1). By construction, its objects can be identified with the pairs $(\langle n\rangle, (T_1, ..., T_n))$ with $\langle n\rangle \in N(Fin_*)$ and $(T_1, ..., T_n)$ a finite sequence of dg-categories. By the cocartesian property, maps $(\langle n\rangle, (T_1, .., T_n)) \to (\langle m\rangle, (Q_1, ..., Q_m))$ over $f : \langle n\rangle \to \langle m\rangle$ corresponds to families of edges in $\mathcal{D}g(k)^{loc-cof,big}$

$$\bigotimes_{j\in f^{-1}(\{i\})} T_j \to Q_i \tag{6.1.25}$$

with $1 \leq i \leq m$, where $\otimes$ denotes the tensor product in $\mathcal{D}g^{loc-cof,big}(k)^{\otimes}$. Given small dg-categories T, T', Q, we will stay that an object in $\mathbb{R}\underline{Hom}(\widehat{T} \otimes^{\mathbb{L}} \widehat{T'}, \widehat{Q})$ is *multi-continuous* if its image through the canonical adjunction is in $\mathbb{R}\underline{Hom}_c(\widehat{T}, \mathbb{R}\underline{Hom}_c(\widehat{T'}, \widehat{Q}))$.

With this, we consider the (non full)subcategory $\mathcal{D}g^{c,loc-cof}(k)^{\otimes} \subseteq \mathcal{D}g^{loc-cof,big}(k)^{\otimes}$ spanned by the pairs $(\langle n\rangle, (T_1, ..., T_n))$ where each T_i is an object in $\mathcal{D}g^{c}(k)$ together with those morphisms $(\langle n\rangle, (T_1, .., T_n)) \to (\langle m\rangle, (Q_1, ..., Q_m))$ corresponding to the edges

$$\bigotimes_{j\in f^{-1}(\{i\})} T_j \to Q_i \tag{6.1.26}$$

which are multi-continuous. It follows that the composition $\mathcal{D}g^{c,loc-cof}(k)^{\otimes} \subseteq \mathcal{D}g^{loc-cof,big}(k)^{\otimes} \to N(Fin_*)$ is a cocartesian fibration: a cocartesian lifting for a morphism $f : \langle n\rangle \to \langle m\rangle$ at a sequence $(\langle n\rangle, (T_1 = \widehat{t_1}, ..., T_n = \widehat{t_n}))$ is given by the edge corresponding to the family of canonical maps

$$u_i : \bigotimes_{j\in f^{-1}(\{i\})} T_j \to Q_i := \widehat{\otimes^{\mathbb{L}}_{j\in f^{-1}(\{i\})} t_j} \tag{6.1.27}$$

obtained from the identity of $\widehat{\otimes^{\mathbb{L}}_{j\in f^{-1}(\{i\})} t_j}$ using the canonical equivalences

$$\mathbb{R}\underline{Hom}_{multi-continuous}(\widehat{T} \otimes \widehat{T'}, \widehat{A}) := \mathbb{R}\underline{Hom}_c(\widehat{T}, \mathbb{R}\underline{Hom}_c(\widehat{T}, \widehat{A})) \simeq \mathbb{R}\underline{Hom}_c(\widehat{T}, \mathbb{R}\underline{Hom}(T, \widehat{A})) \simeq \tag{6.1.28}$$

$$\simeq \mathbb{R}\underline{Hom}(T, \mathbb{R}\underline{Hom}(T, \widehat{A})) \simeq \mathbb{R}\underline{Hom}(T \otimes T', \widehat{A}) \simeq \mathbb{R}\underline{Hom}_c(\widehat{T \otimes T'}, \widehat{A}) \tag{6.1.29}$$

The same equivalences imply the cocartesian property of the family (u_i).

With this, we are reduced to prove that this monoidal structure restricts to the (non-full) subcategory $\mathcal{D}g^{cc,loc-cof}(k) \subset \mathcal{D}g^{c,loc-cof}(k)$ spanned by the same objects and only those morphisms (**??**) that send a product of compact objects to a compact object of Q_i. We claim now that the restriction $\mathcal{D}g^{cc,loc-cof}(k) \subset \mathcal{D}g^{c,loc-cof}(k) \to N(Fin_*)$ is again a cocartesian fibration. For this purpose, it suffices to observe that the same canonical morphisms (6.1.27) send (by definition of Q_i) a product of compact objects to a compact object

□

By inspection of the proof it is obvious that the map $(-)_c : \mathcal{D}g^{cc,loc-cof}(k) \to \mathcal{D}g(k)^{idem,loc-cof}$ is compatible with the monoidal structures.

In summary, we have three equivalent ways to encode Morita theory.

$$N(Cat^{loc-cof}_{Ch(k)})[W^{-1}_{Mor}]^{\otimes} \simeq (\mathcal{D}g(k)^{idem,loc-cof})^{\otimes} \simeq \mathcal{D}g^{cc,loc-cof}(k)^{\otimes} \tag{6.1.30}$$

Convention 6.1.25. *For the future sections, and for the sake of simplicity, we will omit the fact that these monoidal structures are defined for locally-cofibrant dg-categories and that to make sense of this monoidal product for arbitrary dg-categories, we need to perform cofibrant-replacements to fall in the locally-cofibrant context.*

Furthermore, in [131, Thm 5.1] the author proves the existence of a combinatorial compactly generated model structure in $Cat_{Ch(k)}$ with weak-equivalences the Morita equivalences and the same cofibrations as for the Dwyer-Kan model structure. It follows that the three $(\infty,1)$-categories are presentable. In particular, they have all limits and colimits and we can compute them as homotopy limits and homotopy colimits in $Cat_{Ch(k)}$ with respect to this Morita model structure. In particular, we can prove that $\mathcal{D}g(k)^{idem}$ has a zero object $*$ (the dg-category with one object and one morphism) and that, more generally, finite sums are equivalent to finite products (denoted by $\oplus$). The last follows because for any two small dg-categories T and T' we have a canonical equivalence of big dg-categories $\underline{Hom}(T\coprod T', Ch_{dg}(k)) \simeq \underline{Hom}(T, Ch_{dg}(k)) \times \underline{Hom}(T, Ch_{dg}(k))$ compatible with the natural model structures for dg-modules. We can now use this equivalence to find that $\widehat{(T\coprod T')}_c \simeq \widehat{T}_c \times \widehat{T'}_c$.

6.1.3 Dg-categories with a compact generator

A dg-category $\widehat{T} \in \mathcal{D}g^c(k)$ is said to have a compact generator if the triangulated category $[\widehat{T}]$ has a compact generator in the sense of Neeman (see the Remark 2.1.4). We will say that a small dg-category T has a *compact generator* if the triangulated category $[\widehat{T}]$ admits a compact generator in the previous sense. It follows that T has a compact generator if and only if its idempotent completion $\widehat{T}_c$ has a compact generator (of course, this follows from the equivalence $\widehat{(\widehat{T})_c} \simeq \widehat{T}$).

Let $Perf$ be the composition

$$Alg_{\mathcal{A}ss}(\mathcal{D}(k)) \xrightarrow{(-)_{dg}} \mathcal{D}g(k) \xrightarrow{(\widehat{-})_c} \mathcal{D}g(k)^{idem} \tag{6.1.31}$$

Using the same methods as in [123], it can be proved that T has a compact generator if and only if it is in the essential image of $Perf$. For the "only if" direction we consider the dg-algebra B given by the opposite algebra of endomorphisms of the compact generator in $\widehat{T}$. For the "if" direction, if $T \simeq Perf(B)$ then B, seen as a dg-module over itself, is a compact generator.

Remark 6.1.26. Let $T, T' \in \mathcal{D}g(k)^{idem}$ be idempotent complete dg-categories having a compact generator. Then their tensor product in $\mathcal{D}g(k)^{idem}$ has a compact generator. This follows because the functor $Perf$ is monoidal (see 6.1.11 and 6.1.20).

6.1.4 Dg-categories of Finite Type

In this section we discuss the notion of dg-category of finite type studied by Toën-Vaquie in [141]. In the next section they will give body to our noncommutative spaces.

It follows from the results of [131] that the Morita model structure is combinatorial, compactly generated (see [141, Def. 2.1 and Prop. 2.2]) and satisfies the general conditions of the Proposition 2.2 in [141]. Following the discussion in 2.2.2, we can identify the compact objects in $\mathcal{D}g(k)^{idem}$ with the retracts of finite cell objects and we have a canonical equivalence $\mathcal{D}g(k)^{idem} \simeq Ind((\mathcal{D}g(k)^{idem})^{\omega})$. At the same time in [35]-Theorem 4.3, the authors prove that an object $T \in \mathcal{D}g(k)^{idem}$ is compact if and only if its internal-hom functor $\mathbb{R}\underline{Hom}(T,-)$ in $\mathcal{D}g(k)^{idem,\otimes}$ commutes with filtered colimits. An immediate corollary of this is that the product of compact objects in $\mathcal{D}g(k)^{idem,\otimes}$ is again compact so that the subcategory $(\mathcal{D}g(k)^{idem,\omega}$ inherits a symmetric monoidal structure, which we shall denote as $(\mathcal{D}g(k)^{idem,\omega})^{\otimes}$.

Following [141], we say that an idempotent complete dg-category T is of *finite type* if it is equivalent in $\mathcal{D}g(k)^{idem}$ to a dg-category of the form $Perf(B)$ for some dg-algebra B which is compact as an object in the $(\infty,1)$-category $Alg_{\mathcal{A}ss}(\mathcal{D}(k))$ [7]. In particular a dg-category of finite type has a compact generator.

In [141, Lemma 2.11], the authors prove that a dg-category of the form $Perf(B)$ is compact in $\mathcal{D}g(k)^{idem}$ if and only if B is compact in the $(\infty,1)$-category $Alg_{\mathcal{A}ss}(\mathcal{D}(k))$. In fact, an object in $\mathcal{D}g(k)^{idem}$ is compact if and only if it is of finite type:

Proposition 6.1.27. *(Toën-Vaquié) Let $\mathcal{D}g(k)^{ft}$ denote the full subcategory of $\mathcal{D}g(k)^{idem}$ spanned by the dg-categories of finite type. Then, the inclusion $\mathcal{D}g(k)^{ft} \subseteq (\mathcal{D}g(k)^{idem})^{\omega}$ is an equivalence.*

Proof. By the discussion in 6.1.3 it suffices to prove that any compact dg-category $T \in \mathcal{D}g(k)^{idem}$ has a compact generator. Indeed, we can always write T as a filtered colimit of its subcategories generated by compact objects. Since T is compact it is equivalent to one of these subcategories and therefore the triangulated category $[T]$ is compactly generated by a finite family of objects $\{x_1, ..., x_n\}$ (in the sense of Neeman - see the Remark 2.1.4). Since T is idempotent complete, it admits finite sums and therefore the finite direct product $\oplus x_i$ is a compact generator. □

With this, we have a canonical equivalence $\mathcal{D}g(k)^{idem} \simeq Ind(\mathcal{D}g(k)^{ft})$. It follows that $\mathcal{D}g(k)^{ft}$ is closed under finite direct sums, pushouts and contains the zero object.

Remark 6.1.28. It follows from the Yoneda's lemma that the inclusion $\mathcal{D}g(k)^{ft} \subseteq \mathcal{D}g(k)^{idem}$ commutes with arbitrary limits whenever they exist in $\mathcal{D}g(k)^{ft}$. Moreover, by combining the Prop. 6.1.27 and the discussion in the section 3.2.8, it is monoidal.

6.2 Dg-categories vs stable $(\infty,1)$-categories

This section is merely expository and sketches the relation between the theory of dg-categories and the theory of stable $(\infty,1)$-categories. We aim to somehow justify our choice to work with dg-categories. These results have been known to the experts in the field (I learned them from B. Toën) and have recently been established in [36].

For any commutative ring k, $\mathcal{D}(k)^{\otimes}$ is a stable presentable symmetric monoidal $(\infty,1)$-category. In this case, the universal property of $Sp^{\otimes}$ ensures the existence of a (unique up to a contractible space of choices) monoidal colimit preserving map

$$f : Sp^{\otimes} \to \mathcal{D}(k)^{\otimes} \tag{6.2.1}$$

[7] $Alg_{\mathcal{A}ss}(\mathcal{D}(k))$ is the underlying $(\infty,1)$-category of a compactly generated model structure in the category of strictly associative dg-algebras and again by the discussion in 2.2.2 we can identify its compact objects, up to equivalence, with retracts of strict finite cell objects with respect to the generating cofibrations of the model structure for strict dg-algebras.

sending the sphere spectrum to the ring k seen as complex concentrated in degree zero. This is a morphism of commutative algebras in $\mathcal{P}r^{L,\otimes}$ and therefore produces a base-change adjunction

$$\mathcal{P}r^L_{Stb} \simeq Mod_{Sp^{\otimes}}(\mathcal{P}r^L) \underset{f^*}{\overset{(\mathcal{D}(k)\otimes_{Sp}-)}{\rightleftarrows}} Mod_{\mathcal{D}(k)^{\otimes}}(\mathcal{P}r^L) \tag{6.2.2}$$

with f^* the forgetful map given by the composition with f. Notice that the objects in the left side are stable $(\infty,1)$-categories because the adjunction is defined over the forgetful functors to $\mathcal{P}r^L$. By definition, a k-linear stable $(\infty,1)$-category is an object in $Mod_{\mathcal{D}(k)^{\otimes}}(\mathcal{P}r^L)$.

At the same time, there is a canonical way to assign an $(\infty,1)$-category to a dg-category. More precisely, given a small dg-category T we can apply the Dold-Kan construction to the positive truncations of the complexes of morphisms in T to get mapping spaces. Since the Dold-Kan functor is right-lax monoidal (via the Alexander-Whitney map), this construction provides a new simplicial category which happens to be enriched over Kan-complexes. By takings its simplicial nerve we obtain an $(\infty,1)$-category $N_{dg}(T)$. The details of this mechanism can be found in [100, Section 1.3.1]. Moreover, the assignement $T \mapsto N_{dg}(T)$ provides a right Quillen functor between the model category of dg-categories categories with the Dwyer-Kan model equivalences and the model category of simplicial sets with the Joyal's model structure [100, 1.3.1.20]. Following the discussion in 2.2.1 and since these model structures are combinatorial, this assignement provides a functor between the $(\infty,1)$-categories

$$N_{dg} : \mathcal{D}g(k) \to Cat_{\infty} \tag{6.2.3}$$

By the properties of the Dold-Kan correspondence, N_{dg} preserves the notion of "homotopy category"[8]. Moreover, using the arguments in (2.2.1), the combinatorial property implies that N_{dg} has a left adjoint and therefore preserves limits. In particular, for a bigger universe we also have a well-defined map

$$N^{big}_{dg} : \mathcal{D}g(k)^{big} \to Cat^{big}_{\infty} \tag{6.2.4}$$

Following [140] we have the notion of a locally presentable dg-category. By definition, these are big dg-categories that can be obtained as accessible reflexive localizations of big dg-categories of the form $\widehat{T_0}$ for some small dg-category T_0. Alternatively, we can describe them as the dg-categories of cofibrant-fibrant objects of a Bousfield localization of the left proper combinatorial model category $Ch(k)^{T_0}$ for some small dg-category T_0. Together with the colimit preserving maps, they form a (non-full) subcategory $\mathcal{D}g^{lp}(k)$ of $\mathcal{D}g(k)^{big}$. In particular, the $(\infty,1)$-category $\mathcal{D}g^{cc}(k)$ introduced in the previous section has a non-full embedding in $\mathcal{D}g^{lp}(k)$. As explained in the proof of [140, Lemma 2.3] a big dg-category having all colimits is locally presentable if and only if $N^{big}_{dg}(T)$ is in $\mathcal{P}r^L$. In particular, as the notions of colimit are compatible, the restriction

$$N^L_{dg} : \mathcal{D}g^{lp}(k) \to \mathcal{P}r^L \tag{6.2.5}$$

is well-defined.

For a dg-category of the form $\widehat{T_0}$, the $(\infty,1)$-category $N^L_{dg}(\widehat{T_0})$ can be identified with the underlying $(\infty,1)$-category of the combinatorial model category $Ch(k)^{T_0}$ which is compactly generated. In particular, since $Ch(k)^{T_0}$ is stable (in the sense of model categories), we find that $N^L_{dg}(\widehat{T_0})$ is a stable compactly generated $(\infty,1)$-category[9]. In fact, a dg-category $T \in \mathcal{D}g^{lp}(k)$ is in $\mathcal{D}g^{cc}(k)$ if and only if $N^L_{dg}(\widehat{T_0})$ is compactly generated. More generally, we can identify the functor N^L_{dg} with the map sending a Bousfield localization of $Ch(k)^{T_0}$ to its underlying $(\infty,1)$-category. In particular we find that N^L_{dg} factors through the full subcategory of $\mathcal{P}r^L$ spanned by the stable presentable $(\infty,1)$-categories $\mathcal{P}r^L_{Stb}$. In particular, N^L_{dg} restricts to $\mathcal{D}g^{cc}(k) \to \mathcal{P}r^L_{\omega,Stb} \subseteq \mathcal{P}r^L_{\omega}$.

[8] Recall that $\pi_n(DK(A)) \simeq H_n(A)$, where DK denotes the Dold-Kan map

[9] In the condition of having all limits and colimits, the property of being stable depends only on the fact the suspension functor is invertible at the level of the homotopy category

Remark 6.2.1. It follows from the fact that $Ch(k)^{T_0}$ is a stable model category and from the properties of the Dold-Kan construction that N_{dg}^{L} is conservative, for it preserves the notion of homotopy category and using stability, we see that it also reflects fully-faithfulness. More generally, the restriction of N_{dg} to dg-categories satisfying stability is conservative.

Remark 6.2.2. By the previous remark, since N_{dg}^{L} and more generally N_{dg} (restricted to big stable dg-categories) are conservative, and both commute with limits and the non-full inclusion $\mathcal{P}r^L \subseteq Cat_{\infty}^{big}$ (respectively the inclusion of big stable dg-categories inside all big dg-categories) preserves limits, we find that $\mathcal{D}g^{lp}(k)$ also has all small limits and that the inclusion $\mathcal{D}g^{lp}(k) \subseteq \mathcal{D}g(k)^{big}$ also preserves them.

We now come to the expected relation between the Morita theory of dg-categories and the theory of stable presentable $(\infty,1)$-categories: the map $N_{dg}^{L} : \mathcal{D}g^{lp}(k) \to \mathcal{P}r_{Stb}^{L}$ is expected to factor through the forgetful functor $f^* : Mod_{\mathcal{D}(k)^{\otimes}}(\mathcal{P}r^L) \to \mathcal{P}r_{Stb}^{L}$

$$\begin{tikzcd} \mathcal{D}g^{lp}(k) \arrow[r, dashed, "\theta"] & Mod_{\mathcal{D}(k)^{\otimes}}(\mathcal{P}r^L) \arrow[r, "f^*"] & \mathcal{P}r_{Stb}^{L} \end{tikzcd} \tag{6.2.6}$$

and this factorization θ is expected to be an equivalence of $(\infty,1)$-categories. In this case, the restriction

$$\begin{tikzcd} \mathcal{D}g^{cc}(k) \arrow[r, "\sim"] & Mod_{\mathcal{D}(k)^{\otimes}}(\mathcal{P}r_{\omega}^L) \end{tikzcd} \tag{6.2.7}$$

will provide a link between the Morita theory of dg-categories (as described in the previous section) and the theory of k-linear compactly generated $(\infty,1)$-categories. The following diagram is an attempt to schematize this landscape

$$\begin{tikzcd}
 & & & \infty(SpectralCats/Morita) \arrow[dl, "\beta"', "\sim"] \arrow[d, "\alpha"', "\sim"] & \\
 & Mod_{Sp^{\otimes}}(\mathcal{P}r^L) \arrow[dl, "\sim"'] & \mathcal{P}r_{\omega,Stb}^{L} \arrow[l, hook', "nonfull"'] \arrow[r, "\sim", "\gamma"'] & Cat_{\infty}^{\mathcal{E}x,idem} \arrow[r, hook] & Cat_{\infty}^{\mathcal{E}x} \\
\mathcal{P}r_{Stb}^{L} & Mod_{\mathcal{D}(k)^{\otimes}}(\mathcal{P}r^L) \arrow[l] \arrow[u, "f^*"] & Mod_{\mathcal{D}(k)^{\otimes}}(\mathcal{P}r_{\omega}^L) \arrow[l, hook', "nonfull"'] \arrow[u, "f^*"'] & & \\
 & \mathcal{D}g^{lp}(k) \arrow[ul, "N_{dg}^{L}"] \arrow[u, dashed, "\sim", "\theta"'] & \mathcal{D}g^{cc}(k) \arrow[l, hook', "nonfull"'] \arrow[u, dashed, "\sim", "\theta"'] \arrow[r, "\sim", "w"'] & \mathcal{D}g(k)^{idem} \arrow[r, hook] & \mathcal{D}g(k) \\
 & & & N(Cat_{Ch(k)})[W_{Mor}^{-1}] \arrow[ul, "u"', "\sim"] \arrow[u, "\sim", "v"'] &
\end{tikzcd} \tag{6.2.8}$$

Here $\infty(Spectral/Morita)$ (resp. $N(Cat_{Ch(k)})[W_{Mor}^{-1}]$) denotes the $(\infty,1)$-category associated to the Morita model structure on the small spectral categories (resp. small dg-categories). The map β is defined by sending a spectral category $\mathcal{C}$ to the stable $(\infty,1)$-category associated to the stable model category of $\mathcal{C}$-modules in spectra (ie, functors from $\mathcal{C}$ to the model category of spectra, together with the projective structure). The map γ is the equivalence discussed in 2.1.23 obtained by taking the full-subcategory of compact objects. The map α is the composition $\gamma \circ \beta$ and the fact that it is an equivalence is due to the Theorem 4.23 of [18]. The map θ is the expected equivalence and the maps u, v and w are the dg-analogues of α,β and γ, and the fact that they are equivalences results from the main results of [132, 139, 141] as indicated in the previous section. It is also important to remark that θ should respect the natural monoidal structures.

In the recent work [36] L. Cohn constructs a map θ satisfying these expected properties for any ring k.[10].

As the map γ is monoidal, it induces an equivalence between the $(\infty,1)$-categories of modules

$$Mod_{\mathcal{D}(k)^{\otimes}}(\mathcal{P}r^L_{\omega}) \xrightarrow{\sim} Mod_{\mathcal{D}(k)^{\omega,\otimes}}(Cat_{\infty}^{\mathcal{E}x,idem}) \tag{6.2.9}$$

so that the construction of θ can be approached at the level of small categories: the arguments in [36, Theorem 5.1 and Lemma 3.11] together with the enriched Dold-Kan correspondence of [?] provide an equivalence

$$\theta' : \mathcal{D}g(k)^{idem} \xrightarrow{\sim} Mod_{\mathcal{D}(k)^{\omega,\otimes}}(Cat_{\infty}^{\mathcal{E}x,idem}) \tag{6.2.10}$$

which we can easily check to make the diagram

$$\begin{tikzcd}
Mod_{\mathcal{D}(k)^{\otimes}}(\mathcal{P}r^L_{\omega}) \arrow[r, "\gamma", "\sim"'] & Mod_{\mathcal{D}(k)^{\omega,\otimes}}(Cat_{\infty}^{\mathcal{E}x,idem}) \\
\mathcal{D}g^{cc}(k) \arrow[u, dashed, "\sim", "\theta"'] \arrow[r, "\sim", "w"'] & \mathcal{D}g(k)^{idem} \arrow[u, "\sim", "\theta'"']
\end{tikzcd} \tag{6.2.11}$$

commute.

We hope this discussion clarifies the decision to work with dg-categories. For a quasi-compact and quasi-separated scheme X over k we have $\theta(L_{qcoh}(X)) \simeq \mathcal{D}(X)$ where $L_{qcoh}(X)$ is the derived dg-category of X (see the next section) and $\mathcal{D}(X)$ is the stable presentable symmetric monoidal derived $(\infty,1)$-category of X as in [100, Def. 1.3.5.8].

6.3 Dg-Categories and Noncommutative Geometry

6.3.1 From Schemes to dg-algebras (over a ring k)- Part I

Let k be a ring. Given a quasi-compact and separated k-scheme $(X, \mathcal{O}_X)$ we consider $Qcoh(X) \subseteq \mathcal{O}_X - Mod$ the subcategory of quasi-coherent sheaves on X. Under some general conditions, the natural tensor product in $\mathcal{O}_X - Mod$ is closed for quasi-coherent sheaves (see the Prop. 9.1.1 of [62]-Chap. 1). It results from a theorem of Deligne (see [65]-Appendix, Prop 2.2) that $Qcoh(X)$ is a *Grothendieck abelian category* (generally - see [38, Lemma 2.1.7] - for any scheme X there is an infinite cardinal κ such that $Qcoh(X)$ is κ-presentable) so that we can apply to $C(Qcoh(X))$ the Theorem 2.2 of [70] which tells us that the category of unbounded complexes on a Grothendieck abelian category can be equipped with a model structure, with cofibrations given by the monomorphisms and the weak-equivalences the quasi-isomorphisms of complexes. By the Proposition 2.12 of loc.cit, every fibrant-object is a complex of injectives and every bounded above complex of injectives is fibrant. Since X is defined over k, $\mathcal{O}_X$ is a sheaf of k-algebras and each $\mathcal{O}_X$-module is naturally a sheaf of k-modules. This induces a canonical action of $Ch(k)$ on $C(Qcoh(X))$ compatible with the model structure. By definition, the *dg-derived category of* X is the dg-category $L_{qcoh(X)} := Int(C(Qcoh(X)))$. Following the Remark 6.1.14, its associated homotopy category $[L_{qcoh(X)}]$ is canonically equivalent to the classical derived category of quasi-coherent sheaves on X.

Remark 6.3.1. In general, for any quasi-compact scheme X, the correct derived dg-category to consider is full subcategory of the derived category of $\mathcal{O}_X$-modules with quasi-coherent cohomology. When X is separated, this agrees with $L_{qcoh(X)}$.

It is compactly generated (in the sense of Neeman [108]) and thanks the results of [137] we know that its compact objects are perfect complexes of quasi-coherent sheaves. We write $L_{pe}(X)$ for the full subcategory of $L_{qcoh(X)}$ spanned by the perfect complexes and the general theory gives us a canonical

[10] The results in [36] are announced for a field k of characteristic zero but the proof works for any ring. I thank the author for several conversations on the subject

equivalence $\widehat{L_{pe}(X)} \simeq L_{qcoh(X)}$. By construction $L_{pe}(X)$ is an idempotent complete dg-category and we will understand it as the natural noncommutative incarnation of the scheme X. The philosophical importance of the following result is evident

Theorem 6.3.2. *(Bondal-Van den Bergh [23]-Thm 3.1.1) Let X be a quasi-compact and quasi-separated scheme over a ring k. Then $L_{pe}(X)$ has a compact generator.*

Together with the preceeding discussion, this result implies that for any quasi-compact and quasi-separated scheme X over k, the dg-category of perfect complexes $L_{pe}(X)$ is of the form $Perf(B)$ for some dg-algebra B.

6.3.2 Smooth and Proper Dg-categories

The geometric notions of smoothness and properness can be adapted to the world of dg-categories, in a way compatible with $L_{pe}(-)$. Recall that a dg-category T is said to be *locally perfect* it is enriched over perfect complexes of k-modules. T is said to be *proper* if it is locally perfect and it has a compact generator. We say that T is *smooth* if the object in $\widehat{T \otimes^{\mathbb{L}} T^{op}}$ defined the formula $(x, y) \mapsto T(x, y)$, is compact. Finally, we say that a dg-category is *saturated* if it is smooth, proper and idempotent complete. One can check ([141]-Lemma 2.6-(2)) that a dg-category T is proper (resp. smooth) if and only if its idempotent completion is proper (resp. smooth). Of course, these notions are also invariant under the operation $(-)^{op}$. This implies that a dg-category of the form $Perf(B)$ is proper (resp. smooth) if and only B is perfect as a complex of k-modules (meaning, the $B^{op} \otimes^{\mathbb{L}} B$-module defined by the formula $(\bullet, \bullet) \mapsto B$ is compact).

Example 6.3.3. This notion of smoothness is compatible with the classical geometrical notion: a morphism $Spec(A) \to Spec(k)$ is smooth (meaning, A is regular over k) if and only if the dg-category $Perf(A)$ is smooth. This is proved using a famous theorem of J.P.Serre [126, IV-37, Thm 9]: a commutative ring is regular if and only if it is of finite global homological dimension.

In the same spirit, and thanks to [141, Lemma 3.27] we have a machine to produce smooth and proper dg-categories: for any scheme X smooth and proper over a ring k, the dg-category $L_{pe}(X)$ is smooth and proper.

In 6.1.2 we explained how the notion of pseudo-perfectness can be used to describe the mapping spaces in $\mathcal{D}g(k)^{idem}$. Notice now that an object $E \in \widehat{T}$ is pseudo-perfect (over T relatively to 1_k) if it has values in compact complexes of k-modules. The distinction between being compact and pseudo-perfect is the key to understand the notions of smooth and proper as the following results from [141] suggest:

- [141]-Lemma 2.8-(1): A dg-category T is locally perfect if and only if for any dg-category T', we have an an inclusion of subcategories

$$(\widehat{T \otimes^{\mathbb{L}} T'})_c \subseteq (\widehat{T \otimes^{\mathbb{L}} T'})_{pspe} \tag{6.3.1}$$

- [141]-Lemma 2.8-(2): A dg-category T is smooth if and only if for any dg-category T', we have an an inclusion of subcategories

$$(\widehat{T \otimes^{\mathbb{L}} T'})_{pspe} \subseteq (\widehat{T \otimes^{\mathbb{L}} T'})_c \tag{6.3.2}$$

- [141]-Lemma 2.8-(3): From the two previous items, a dg-category T is smooth and proper iff it has a compact generator and for any dg-category T', the subcategories of $\widehat{T \otimes^{\mathbb{L}} T'}$ spanned by compact, respectively pseudo-perfect modules, coincide.

Remark 6.3.4. Recall from 6.1.2 that the mapping spaces $Map_{\mathcal{D}g(k)^{idem}}(T,T')$ are given by the maximal ∞-groupoids in $pspe(T,T')_\infty$ - the full subcategory of $(T,T')_\infty$ spanned by the pseudo-perfect modules. It follows that if T is smooth and proper, we can identify $pspe(T,T')_\infty$ with the full subcategory $(T,T')^\omega_\infty$ spanned by the compact modules.

The notions of smooth and proper are related to the notion of finite type:

- [141]-Corollary 2.13: Any smooth and proper dg-category is of finite type;
- [141]-Proposition 2.14: Any dg-category of finite type is smooth.

To conclude this section we recall another important characterization of smoothness and properness given by the following result due to B. Toën

Proposition 6.3.5. *(see [11]-Lectures on dg-categories and [140, Prop. 1.5]) An object $T \in \mathcal{D}g(k)^{idem}$ is smooth and proper if and only if it is dualizable with respect to the symmetric monoidal structure $\mathcal{D}g(k)^{idem,\otimes}$. In particular, the dual of a dg-category $\widehat{T}_c \in \mathcal{D}g(k)^{idem}$ is the opposite $(\widehat{T^{op}})_c$.*

6.3.3 From Schemes to Noncommutative Spaces (over a ring k) - Part II

Following [141], the notion of finite type should be understood as the correct notion of smoothness for noncommutative spaces, while the smooth dg-categories should only be understood as "formally smooth"' noncommutative spaces. Finally, we are ready to introduce our smooth noncommutative geometric objects.

Definition 6.3.6. *Let k be a ring. We define the $(\infty,1)$-category of* smooth noncommutative spaces *over k- $\mathcal{N}cS(k)$ - to be the* opposite *of $\mathcal{D}g(k)^{ft}$. It has a natural symmetric monoidal structure $\mathcal{N}cS(k)^{\otimes}$ induced from the one in $\mathcal{D}g(k)^{ft,\otimes}$, with unit object given by $L_{pe}(k)$.*

Notation 6.3.7. We will denote our smooth noncommutative spaces using caligraphic letters $\mathcal{X}$,$\mathcal{U}$, $\mathcal{V}$, $\mathcal{W}$, etc. For a smooth noncommutative space $\mathcal{X} \in \mathcal{N}cS$ we will denote by $T_{\mathcal{X}}$ its associated dg-category of finite type and by $A_{\mathcal{X}}$ a compact dg-algebra such that $T_{\mathcal{X}} \simeq Perf(A_{\mathcal{X}})$.

We will say that a smooth noncommutative space $\mathcal{X}$ is *smooth and proper* if its associated dg-category $T_{\mathcal{X}}$ is smooth and proper. We will let $\mathcal{N}cS(k)^{sp}$ denote the full subcategory of $\mathcal{N}cS(k)$ spanned by the smooth and proper noncommutative spaces. Since the smooth and proper dg-categories correspond to the dualizable objects in $\mathcal{D}g(k)^{ft}$, the subcategory $\mathcal{N}cS(k)^{sp}$ is closed under tensor products.

It follows immediately from the properties of $\mathcal{D}g(k)^{ft}$ that $\mathcal{N}cS(k)$ admits pullbacks, together with finite direct sums and a zero object. Moreover, the tensor product commutes with limits. In particular, if $\mathcal{X}$ and $\mathcal{Y}$ are two smooth noncommutative spaces, the mapping space $Map_{\mathcal{N}cS(k)}(\mathcal{X},\mathcal{Y})$ is given by the ∞-groupoid $pspe(A_{\mathcal{Y}},A_{\mathcal{X}})^{\simeq}_\infty$ of pseudo-perfect $A_{\mathcal{Y}} \otimes^{\mathbb{L}} A^{op}_{\mathcal{X}}$-dg-modules and equivalences between them.

We now explain how the formula $X \mapsto L_{pe}(X)$ can be properly arranged as an ∞-functor. We define it for the smooth affine schemes of finite type over k, whose 1-category we denote by $AffSm^{ft}(k)$. Recall that the full subcategory of 0-truncated objects in $CAlg(\mathcal{D}(k))^{cn}$ is equivalent to the nerve of the category of classical associative rings. In particular, we can identify the nerve of the category of commutative smooth k-algebras of finite type $N(SmCommAlg_k) \simeq N(AffSm^{ft}(k))^{op}$ with a full subcategory of $CAlg(\mathcal{D}(k))^{cn}$. Let L denote the composition

$$N(SmCommAlg_k) \hookrightarrow CAlg(\mathcal{D}(k))^{cn} \longrightarrow Alg_{\mathcal{A}ss}(\mathcal{D}(k))^{cn} \hookrightarrow Alg_{\mathcal{A}ss}(\mathcal{D}(k)) \xrightarrow{Perf} \mathcal{D}g(k)^{idem} \tag{6.3.3}$$

where $CAlg(\mathcal{D}(k))^{cn} \to Alg_{\mathcal{A}ss}(\mathcal{D}(k))^{cn}$ is the restriction of the forgetful functor to connective objects. The following is a key result:

Proposition 6.3.8. *Let A be a classical commutative smooth k-algebra of finite type. Then, $L(A)$ is a dg-category of finite type. In other words, L provides a well-defined functor $N(SmCommAlg_k) \to \mathcal{D}g(k)^{ft}$.*

Proof. If A is smooth as a classical commutative k-algebra it is smooth as a dg-category (Example 6.3.3) which by definition means it is compact as a $A \otimes_k A^{op}$-dg-module. Following the Remark 6.1.6 the category of $A \otimes_k A^{op}$-dg-modules can be naturally identified with the category of A-bimodules $BiMod(A, A)(Ch(k))$. Using the strictification results of 3.9.2 the underlying $(\infty, 1)$-category of $BiMod(A, A)(Ch(k))$ is equivalent to ${}_ABMod_A(\mathcal{D}(k)) \simeq Mod_A^{\mathcal{A}ss}(\mathcal{D}(k))$.

Of course, if A is compact in $Mod_A^{\mathcal{A}ss}(\mathcal{D}(k))$ and since $A \otimes_k A^{op}$ is also compact (it is a generator), the kernel of the multiplication map $I \to A \otimes_k A^{op} \to A$ will also be compact. Following the Example 3.11.2 we can now identify I with the relative cotangent complex $\mathbb{L}_{A/k} \in Mod_A^{\mathcal{A}ss}(\mathcal{D}(k))$. The Lemma 3.11.1 completes the proof. □

Using this, we define L_{pe} as the opposite of L

$$L_{pe} : N(\mathit{AffSm}^{ft}(k)) \to \mathcal{N}cS(k) \tag{6.3.4}$$

To conclude this section we observe that L_{pe} can be promoted to a monoidal functor

$$L_{pe}^{\otimes} : N(\mathit{AffSm}^{ft}(k))^{\times} \to \mathcal{N}cS(k)^{\otimes} \tag{6.3.5}$$

where $N(\mathit{AffSm}^{ft}(k))^{\times}$ is the cartesian structure in $N(\mathit{AffSm}^{ft}(k))$ which corresponds to the coproduct of classical commutative smooth k-algebras which is, well-known, given by the classical tensor product over k.

It follows from 3.2.6 and the fact that the tensor product in $\mathcal{D}(k)$ is compatible with the t-structure, that the composition $CAlg(\mathcal{D}(k))^{cn} \to Alg_{\mathcal{A}ss}(\mathcal{D}(k))^{cn} \subseteq Alg_{\mathcal{A}ss}(\mathcal{D}(k))$ is monoidal. Moreover, the functor $Perf$ is monoidal because it is the composition of monoidal functors - 6.1.11 and 6.1.20. We are left to check that the inclusion $N(SmCommAlg_k) \to CAlg(\mathcal{D}(k))^{cn}$ is monoidal. In other words, that for a commutative smooth k-algebra of finite type over k, the classical tensor product agrees with the derived tensor product. But this is true since smooth k-algebras are flat over k.

6.4 The Motivic $\mathbb{A}^1$-Homotopy Theory of Kontsevich's Noncommutative Spaces over a ring k

We will now use our main results to fabricate a motivic $\mathbb{A}^1$-homotopy theory for smooth noncommutative spaces over a ring k. In this section we proceed in analogy with the construction of the motivic stable homotopy for schemes as described in the previous chapter of this work. Recall from the Remark 5.1.1 that these constructions only depend on the category of *affine* smooth schemes of finite type over k.

Remark 6.4.1. There is a natural way to extend the functor L_{pe} to non-affine schemes. To do this, we observe that the classical category of schemes can be identified with a full subcategory of $\mathcal{P}^{big}(N(\mathit{AffSm}^{ft}(k)))$, by the identification of a scheme with its "functor of points". The universal property of (big) presheaves provides a colimit preserving map

$$\begin{tikzcd} N(\mathit{AffSm}^{ft}(k)) \arrow[r, "L_{pe}"] \arrow[d] & \mathcal{N}cS(k) \arrow[d] \\ \mathcal{P}^{big}(N(\mathit{AffSm}^{ft}(k))) \arrow[r, dashed] & \mathcal{P}^{big}(\mathcal{N}cS(k)) \end{tikzcd} \tag{6.4.1}$$

The Lemma 3.27 in [141] implies that the image through this map of any smooth and proper scheme X over k is representable in $\mathcal{P}^{big}(\mathcal{N}cS(k))$. This should remain true without the properness condition.

To start with, we need to introduce an appropriate analogue for the Nisnevich topology, for the interval $\mathbb{A}^1$ and for the projective space $\mathbb{P}^1$. For the last two we have natural choices - $L_{pe}(\mathbb{A}^1)$ and $L_{pe}(\mathbb{P}^1)$: the first is a dg-category of finite type because $\mathbb{A}^1$ is smooth affine over k; the second, $L_{pe}(\mathbb{P}^1)$, is of finite type because the canonical morphism $\mathbb{P}^1 \to Spec(k)$ is smooth and proper (see 6.3.3). The analogue of the Nisnevich topology requires a more careful discussion.

6.4.1 The noncommutative version of the Nisnevich Topology

To obtain our noncommutative analogue for the Nisnevich topology we isolate the formal properties of the commutative squares in $\mathcal{N}cS(k)$

$$\begin{tikzcd} L_{pe}(p^{-1}(U)) \arrow[r] \arrow[d] & L_{pe}(V) \arrow[d] \\ L_{pe}(U) \arrow[r] & L_{pe}(X) \end{tikzcd} \tag{6.4.2}$$

induced by the Nisnevich squares of schemes. Following the list of properties given in Section 5, we start with the notion of an open embedding. For that we need some preparations. Recall that an *exact sequence* in $\mathcal{D}g(k)^{idem}$ is the data of a commutative square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d, "g"] \\ * \arrow[r] & C \end{tikzcd} \tag{6.4.3}$$

where $*$ is the zero object in $\mathcal{D}g(k)^{idem}$, such that f fully-faithful and the diagram is a pushout. Since $\mathcal{D}g(k)^{idem}$ is a reflexive localization of $\mathcal{D}g(k)$, this pushout C is canonically equivalent to the idempotent completion of the pushout B/A computed in $\mathcal{D}g(k)$. Of course, using the equivalence (6.1.30), the previous diagram is an exact sequence if and only if the diagram

$$\begin{tikzcd} \widehat{A} \arrow[r, "\widehat{f}"] \arrow[d] & \widehat{B} \arrow[d, "\widehat{g}"] \\ * \arrow[r] & \widehat{C} \end{tikzcd} \tag{6.4.4}$$

is an exact sequence in $\mathcal{D}g^{cc}(k)$ in the same sense. Thanks to the works of B.Keller in [82], we know that this notion of exact sequence extends the notion given by Verdier [148].

Proposition 6.4.2. *(B. Keller [82]) The following conditions are equivalent:*

1. *a diagram as above is an exact sequence;*
2. *the functor $\widehat{f}$ induces an equivalence of $[\widehat{A}]$ with a triangulated subcategory of the triangulated category $[\widehat{B}]$ and $\widehat{g}$ exhibits the homotopy category $[\widehat{C}]$ as the Verdier quotient $[\widehat{B}]/[\widehat{A}]$;*
3. *the functor f induces an equivalence of $[A]$ with a triangulated subcategory of the triangulated category $[B]$ and the canonical map from the Verdier quotient $[B]/[A] \hookrightarrow [C]$ is cofinal (see our discussion in 2.1.24).*

Remark 6.4.3. Following the discussion in 6.2, we can use the functor $N_{dg}^L : \mathcal{D}g^{cc}(k) \to \mathcal{P}r^L_{\omega,Stb}$ to relate exact sequences of dg-categories in the above sense to exact sequences of stable presentable $(\infty,1)$-categories in the sense of 2.1.24. As explained in 6.2.1 N_{dg}^L is conservative, preserves fully-faithfulness and preserves the notion of "homotopy category" (see the Remark [100, 1.3.1.11]). This result, together with the Propositions 6.4.2 and 2.1.9 implies that a sequence of dg-categories $\widehat{A} \to \widehat{B} \to \widehat{C}$ in $\mathcal{D}g^{cc}(k)$ is exact in the sense discussed in this section if and only if its image $N_{dg}^L(\widehat{A}) \to N_{dg}^L(\widehat{B}) \to N_{dg}^L(\widehat{C})$ in $\mathcal{P}r^L_{\omega,Stb}$ is exact in the sense discussed in 2.1.24. It follows also that $\widehat{A}$ has a compact generator if and only if $h(N_{dg}^L(\widehat{A}))$ has a compact generator.

Remark 6.4.4. It is common to find in the literature the terminology of *strict exact sequence* to denote an exact sequence (6.4.3) in $\mathcal{D}g(k)^{idem}$ which, appart from being a pushout square, is also a pullback in $\mathcal{D}g(k)^{idem}$. It follows again from the results of [82] that in terms of the associated homotopy triangulated categories this corresponds to the additional condition that $[\widehat{A}]$ is thick in $[\widehat{B}]$. It follows however that when working in $\mathcal{D}g(k)^{idem}$ this terminology is unnecessary because every exact sequence is strict. This follows from the properties of the functor N_{dg}^L together with the Corollary 2.1.12.

Let us now come back to the definition of open immersion. Thanks to the results of Thomason in [137, Section 5] and to the work of B.Keller in [82], we know that for a quasi-compact and quasi-separated scheme X with a quasi-compact open embedding $j : U \hookrightarrow X$, the restriction map $j^* : L_{qcoh}(X) \to L_{qcoh}(U)$ fits in a strict exact sequence in $\mathcal{D}g^{cc}(k)$

$$\begin{tikzcd} L_{qcoh}(X)_{X-U} \arrow[r] \arrow[d] & L_{qcoh}(X) \arrow[d, "j^*"] \\ * \arrow[r] & L_{qcoh}(U) \end{tikzcd} \tag{6.4.5}$$

where $L_{qcoh}(X)_{X-U}$ is by definition the kernel of the restriction j^*. It is also well-known that this kernel has a compact generator (see the proof of [140, Prop. 3.9]). Of course, using the equivalence $\mathcal{D}g(k)^{idem} \simeq \mathcal{D}g^{cc}(k)$, we can reformulate this in terms of an exact sequence in $\mathcal{D}g(k)^{idem}$

$$\begin{tikzcd} (L_{qcoh}(X)_{X-U})_c \arrow[r] \arrow[d] & L_{pe}(X) \arrow[d, "j^*"] \\ * \arrow[r] & L_{pe}(U) \end{tikzcd} \tag{6.4.6}$$

where $(L_{qcoh}(X)_{X-U})_c$ has a compact generator.

Remark 6.4.5. More generally, and as explained in [51, Prop. 2.9], if T is a dg-category of finite type and k is an object in T, then quotient of T by the sub-dg-category generated by k is again a dg-category of finite type.

This motivates the following definition:

Definition 6.4.6. *Let $f : \mathcal{U} \to \mathcal{X}$ be a morphisms of smooth noncommutative spaces over k. We say that f is an* open immersion *if there exists a dg-category with a compact generator $K_{\mathcal{X}-\mathcal{U}} \in \mathcal{D}g(k)^{idem}$ (see 6.1.3) together with a fully-faithful map $K_{\mathcal{X}-\mathcal{U}} \hookrightarrow T_{\mathcal{X}}$ such that the opposite of f in $\mathcal{D}g(k)^{ft}$ fits in a exact sequence in $\mathcal{D}g(k)^{idem}$:*

$$\begin{tikzcd} K_{\mathcal{X}-\mathcal{U}} \arrow[r] \arrow[d] & T_{\mathcal{X}} \arrow[d] \\ * \arrow[r] & T_{\mathcal{U}} \end{tikzcd} \tag{6.4.7}$$

It follows from the Remark 6.4.4 that this diagram is also a pullback square.

Definition 6.4.7. *We will say that a commutative diagram in* $\mathcal{N}cS(k)$

$$\begin{tikzcd} \mathcal{W} \arrow[r] \arrow[d] & \mathcal{V} \arrow[d] \\ \mathcal{U} \arrow[r] & \mathcal{X} \end{tikzcd} \tag{6.4.8}$$

is a Nisnevich square of smooth noncommutative spaces *if the following conditions hold:*

1. *The maps* $\mathcal{U} \to \mathcal{X}$ *and* $\mathcal{W} \to \mathcal{V}$ *are open immersions;*
2. *The associated map* $T_{\mathcal{X}} \to T_{\mathcal{V}}$ *sends the compact generator of* $K_{\mathcal{X}-\mathcal{U}} \subseteq T_{\mathcal{X}}$ *to the compact generator of* $K_{\mathcal{V}-\mathcal{W}} \subseteq T_{\mathcal{V}}$ *and induces an equivalence* $K_{\mathcal{X}-\mathcal{U}} \simeq K_{\mathcal{V}-\mathcal{W}}$*;*
3. *The diagram is a pushout.*

Convention 6.4.8. *We will adopt the convention that if* $\mathcal{X}$ *is a smooth noncommutative space whose underlying dg-category* $T_{\mathcal{X}}$ *is a zero object of* $\mathcal{D}g(k)^{ft}$*, the empty set forms a Nisnevich square of* $\mathcal{X}$*.*

Using the duality between smooth noncommutative spaces and dg-categories, a Nisnevich square corresponds to the data of a commutative diagram in $\mathcal{D}g(k)^{idem}$

$$\begin{tikzcd} K_{\mathcal{X}-\mathcal{U}} \arrow[r] \arrow[d] & * \arrow[d] \\ T_{\mathcal{X}} \arrow[r] \arrow[d] & T_{\mathcal{U}} \arrow[d] \\ T_{\mathcal{V}} \arrow[r] & T_{\mathcal{W}} \\ K_{\mathcal{V}-\mathcal{W}} \arrow[r] \arrow[u] & * \arrow[u] \end{tikzcd} \tag{6.4.9}$$

where:

1) all $T_{\mathcal{X}}$, $T_{\mathcal{U}}$, $T_{\mathcal{V}}$ and $T_{\mathcal{W}}$ are of finite type;

2) Both $K_{\mathcal{X}-\mathcal{U}}$ and $K_{\mathcal{V}-\mathcal{W}}$ belong to $\mathcal{D}g(k)^{idem}$, have a compact generator and the maps $K_{\mathcal{X}-\mathcal{U}} \to T_{\mathcal{X}}$ and $K_{\mathcal{V}-\mathcal{W}} \to T_{\mathcal{V}}$ are fully-faithful;

3) The associated map $T_{\mathcal{X}} \to T_{\mathcal{V}}$ sends the compact generator of $K_{\mathcal{X}-\mathcal{U}} \subseteq T_{\mathcal{X}}$ to the compact generator of $K_{\mathcal{V}-\mathcal{W}} \subseteq T_{\mathcal{V}}$ and induces an equivalence $K_{\mathcal{X}-\mathcal{U}} \simeq K_{\mathcal{V}-\mathcal{W}}$;

4) the upper and lower squares are pushouts and pullbacks in $\mathcal{D}g(k)^{idem}$ (see 6.4.4) and the middle square is a pullback in $\mathcal{D}g(k)^{ft}$ and therefore in $\mathcal{D}g(k)^{idem}$ (see the Remark 6.1.28).

These conditions also imply that the middle square is a pushout in $\mathcal{D}g(k)^{idem}$. Indeed, because the exterior diagrams are pushouts we can write $T_{\mathcal{W}} \simeq T_{\mathcal{V}} \coprod_{K_{\mathcal{V}-\mathcal{W}}} *$ and $T_{\mathcal{U}} \simeq T_{\mathcal{X}} \coprod_{K_{\mathcal{X}-\mathcal{U}}} *$. Together with the fact that $K_{\mathcal{X}-\mathcal{U}}$ and $K_{\mathcal{V}-\mathcal{W}}$ are equivalent, we have

$$T_{\mathcal{V}} \coprod_{T_{\mathcal{X}}} T_{\mathcal{U}} \simeq T_{\mathcal{V}} \coprod_{T_{\mathcal{X}}} (T_{\mathcal{X}} \coprod_{K_{\mathcal{X}-\mathcal{U}}} *) \simeq T_{\mathcal{V}} \coprod_{K_{\mathcal{X}-\mathcal{U}}} * \simeq T_{\mathcal{V}} \coprod_{K_{\mathcal{V}-\mathcal{W}}} * \simeq T_{\mathcal{W}} \tag{6.4.10}$$

Corollary 6.4.9. *Every Nisnevich square in* $\mathcal{N}cS(k)$ *is a pullback.*

Remark 6.4.10. Let $\mathcal{U} \to \mathcal{X}$ be an open immersion of smooth noncommutative spaces. If the associated dg-category $K_{\mathcal{X}-\mathcal{U}} \in \mathcal{D}g(k)^{idem}$ is of finite type we can then see it as the dg-category $T_{\mathcal{Z}} = K_{\mathcal{X}-\mathcal{U}}$ dual to a smooth noncommutative space $\mathcal{Z}$. Of course, since the zero map $T_{\mathcal{Z}} \to *$ is a quotient of $T_{\mathcal{Z}}$ by itself, its dual $* \to \mathcal{Z}$ is an open immersion. Moreover, since the diagram $K_{\mathcal{X}-\mathcal{U}} \hookrightarrow T_{\mathcal{X}} \to T_{\mathcal{U}}$ is also a fiber sequence (see 6.4.4), the square of smooth noncommutative spaces

$$\begin{tikzcd} \mathcal{U} \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ * \arrow[r] & \mathcal{Z} \end{tikzcd} \tag{6.4.11}$$

is a pushout and therefore, Nisnevich.

Example 6.4.11. The notions of *semi-orthogonal decomposition* and *exceptional collection* for triangulated categories (see [21]) have an immediate translation to the setting of dg-categories in terms of *split short exact sequence* in $\mathcal{D}g(k)^{idem}$. Recall that an exact sequence in $\mathcal{D}g(k)^{idem}$

$$\begin{tikzcd} I \arrow[r, "f"] \arrow[d] & T \arrow[d, "g"] \\ * \arrow[r] & I' \end{tikzcd} \tag{6.4.12}$$

is said to *split* if the functor f (resp. g) admits a right adjoint j (resp. fully-faithful right adjoint i). Following the Remark 6.4.10 if $\mathcal{X}$ is a smooth noncommutative space, every semi-ortogonal decomposition of the associated dg-category $T_{\mathcal{X}}$ given by dg-categories I, I' of finite type provides the data dual to a Nisnevich square

$$\begin{tikzcd} I \arrow[r] \arrow[d] & * \arrow[d] \\ T_{\mathcal{X}} \arrow[r] & I' \end{tikzcd} \tag{6.4.13}$$

Example 6.4.12. The previous example will be particularly important to us in the case $\mathcal{X} = L_{pe}(\mathbb{P}^1)$. Thanks to the results of [12] we know that $\mathbb{P}^n$ admits an exceptional collection generated by the twisting sheaves $\langle \mathcal{O}, ..., \mathcal{O}(-n) \rangle$. By the previous example, the diagram in $\mathcal{D}g(k)^{idem}$ associated to the split exact sequence

$$\begin{tikzcd} Perf(k) \arrow[r] \arrow[d] & * \arrow[d] \\ L_{pe}(\mathbb{P}^1) \arrow[r] & Perf(k) \end{tikzcd} \tag{6.4.14}$$

provides the data of a Nisnevich square.

We now prove that our Nisnevich squares are compatible with the monoidal product of smooth noncommutative spaces. For that we will need the following preliminary result

Lemma 6.4.13. *Let*

$$\begin{tikzcd} \mathcal{W} \arrow[r] \arrow[d] & \mathcal{V} \arrow[d] \\ \mathcal{U} \arrow[r] & \mathcal{X} \end{tikzcd} \tag{6.4.15}$$

be a Nisnevich square of smooth noncommutative spaces and let

$$\begin{tikzcd} \widehat{T_{\mathcal{X}}} \arrow[r] \arrow[d] & \widehat{T_{\mathcal{U}}} \arrow[d] \\ \widehat{T_{\mathcal{V}}} \arrow[r] & \widehat{T_{\mathcal{W}}} \end{tikzcd} \tag{6.4.16}$$

be its associated pullback diagram in $\mathcal{D}g^{cc}(k)$. Then the image of (6.4.16) through the (non-full) inclusion $\mathcal{D}g^{cc}(k) \to \mathcal{D}g^{c}(k)$ remains a pullback diagram.

Proof. As discussed in 6.2.2, the $(\infty,1)$-category $\mathcal{D}g^{lp}(k)$ has all limits and the (non-full) inclusion $\mathcal{D}g^{lp}(k) \subseteq \mathcal{D}g(k)^{big}$ preserves them. By definition, $\mathcal{D}g^{c}(k)$ is the full subcategory of $\mathcal{D}g^{lp}(k)$ spanned by the locally presentable dg-categories of the form $\widehat{T}$ for some small dg-category T. Therefore, we are reduced to showing that the (non-full) inclusion $\mathcal{D}g^{cc}(k) \subseteq \mathcal{D}g^{lp}(k)$ preserves the pullback diagrams (6.4.16) associated to Nisnevich squares. This statement is the dg-analogue of the Proposition 2.1.10.

Following the discussion in 6.2, the functor N_{dg}^{L} provides a commutative square

$$\begin{tikzcd} \mathcal{D}g(k)^{idem} \simeq \mathcal{D}g(k)^{cc} \arrow[r, hook, "non-full"] \arrow[d, "N_{dg}^{L}"] & \mathcal{D}g(k)^{lp} \arrow[d, "N_{dg}^{L}"] \\ \mathcal{P}r^{L}_{\omega,Stb} \arrow[r, hook, "non-full"] & \mathcal{P}r^{L}_{Stb} \end{tikzcd} \tag{6.4.17}$$

and as explained in 6.2.1 N_{dg}^{L} is conservative, preserves fully-faithfulness and preserves the notion of "homotopy category" (see the Remark [100, 1.3.1.11]). It follows, as explained in the Remark 6.4.3 that N_{dg}^{L} preserves the notions of exact sequence. It follows also that $\widehat{A}$ has a compact generator if and only if $h(N_{dg}^{L}(\widehat{A}))$ has a compact generator.

Consider now the pullback diagram (6.4.16) associated to a Nisnevich covering and let $\widehat{K} \simeq \widehat{K_{\mathcal{X}-\mathcal{U}}} \simeq \widehat{K_{\mathcal{V}-\mathcal{W}}}$ be the dg-category (with a compact generator) in $\mathcal{D}g^{cc}(k)$ associated to the open immersions. We find a diagram in $\mathcal{P}r^{L}_{\omega,Stb}$

$$\begin{tikzcd} & N_{dg}^{L}(\widehat{T_{\mathcal{X}}}) \arrow[r] \arrow[d] & N_{dg}^{L}(\widehat{T_{\mathcal{U}}}) \arrow[d] \\ N_{dg}^{L}(\widehat{K}) \arrow[r, hook] & N_{dg}^{L}(\widehat{T_{\mathcal{V}}}) \arrow[r] & N_{dg}^{L}(\widehat{T_{\mathcal{W}}}) \end{tikzcd} \tag{6.4.18}$$

Since N_{dg}^{L} commutes with limits, this diagram remains a pullback in $\mathcal{P}r^{L}_{\omega,Stb}$ and we find ourselves facing the conditions of the Proposition 2.1.10 so that the diagram remains a pullback after the inclusion in $\mathcal{P}r^{L}_{Stb}$. Finally, since N_{dg}^{L} is conservative, the commutativity of (6.4.17) implies that (6.4.16) remains a pullback in $\mathcal{D}g(k)^{lp}$. This concludes the proof. $\square$

We can now state the main result:

Proposition 6.4.14. *1) Let $\mathcal{U} \to \mathcal{X}$ be an open immersion of smooth noncommutative spaces. Then, for any smooth noncommutative space $\mathcal{Y}$, the product map*

$$\mathcal{U} \otimes \mathcal{Y} \to \mathcal{X} \otimes \mathcal{Y} \tag{6.4.19}$$

is also an open immersion;

2) Let

$$\begin{tikzcd} \mathcal{W} \arrow[r] \arrow[d] & \mathcal{V} \arrow[d] \\ \mathcal{U} \arrow[r] & \mathcal{X} \end{tikzcd} \tag{6.4.20}$$

be a Nisnevich square of smooth noncommutative spaces. Then, for any smooth noncommutative space $\mathcal{Y}$, the square

$$\begin{tikzcd} \mathcal{W}\otimes\mathcal{Y} \arrow[r] \arrow[d] & \mathcal{V}\otimes\mathcal{Y} \arrow[d] \\ \mathcal{U}\otimes\mathcal{Y} \arrow[r] & \mathcal{X}\otimes\mathcal{Y} \end{tikzcd} \tag{6.4.21}$$

remains a Nisnevich square.

Proof. To prove 1), let

$$\begin{tikzcd} K_{\mathcal{X}-\mathcal{U}} \arrow[r] \arrow[d] & T_{\mathcal{X}} \arrow[d] \\ * \arrow[r] & T_{\mathcal{U}} \end{tikzcd} \tag{6.4.22}$$

be the data in $\mathcal{D}g(k)^{idem}$ corresponding to the open immersion. We are reduced to prove that by tensoring with $T_{\mathcal{Y}}$ (in $\mathcal{D}g(k)^{idem}$) the diagram

$$\begin{tikzcd} K_{\mathcal{X}-\mathcal{U}}\otimes T_{\mathcal{Y}} \arrow[r] \arrow[d] & T_{\mathcal{X}}\otimes T_{\mathcal{Y}} \arrow[d] \\ *\otimes T_{\mathcal{Y}} \arrow[r] & T_{\mathcal{U}}\otimes T_{\mathcal{Y}} \end{tikzcd} \tag{6.4.23}$$

remains the data of an open immersion. Observe first that since the monoidal structure in $\mathcal{D}g(k)^{idem}$ is compatible with colimits, $*\otimes T_{\mathcal{Y}}$ is again a zero object. To complete the proof it suffices to check that (i) $K_{\mathcal{X}-\mathcal{U}}\otimes T_{\mathcal{Y}}$ remains a dg-category having a compact generator; (ii) the map $K_{\mathcal{X}-\mathcal{U}}\otimes T_{\mathcal{Y}} \to T_{\mathcal{X}}\otimes T_{\mathcal{Y}}$ remains fully-faithful and (iii) the diagram (6.4.23) is a pushout. The first assertion follows from the Remark 6.1.26. The second is obvious by the definition of fully-faithful and the construction of tensor products. The third follows from the Proposition 1.6.3 in [44].

Let us now prove 2). It follows from 1) that both $\mathcal{W}\otimes\mathcal{Y} \to \mathcal{V}\otimes\mathcal{Y}$ and $\mathcal{U}\otimes\mathcal{Y} \to \mathcal{X}\otimes\mathcal{Y}$ remain open immersions, corresponding the quotients by the subcategories $K_{\mathcal{X}-\mathcal{U}}\otimes T_{\mathcal{Y}}$ and $K_{\mathcal{V}-\mathcal{W}}\otimes T_{\mathcal{Y}}$. Since the map $K_{\mathcal{X}-\mathcal{U}} \to K_{\mathcal{V}-\mathcal{W}}$ is an equivalence, the tensor product with the identity of $T_{\mathcal{Y}}$

$$K_{\mathcal{X}-\mathcal{U}}\otimes T_{\mathcal{Y}} \to K_{\mathcal{V}-\mathcal{W}}\otimes T_{\mathcal{Y}} \tag{6.4.24}$$

remains an equivalence. We are now left to prove that the diagram (6.4.21) remains a pushout. This is equivalent to prove that associated diagram of dg-categories

$$\begin{tikzcd} T_{\mathcal{X}}\otimes T_{\mathcal{Y}} \arrow[r] \arrow[d] & T_{\mathcal{U}}\otimes T_{\mathcal{Y}} \arrow[d] \\ T_{\mathcal{V}}\otimes T_{\mathcal{Y}} \arrow[r] & T_{\mathcal{W}}\otimes T_{\mathcal{Y}} \end{tikzcd} \tag{6.4.25}$$

remains a pullback in $\mathcal{D}g(k)^{ft}$. Since all the dg-categories in this diagram are of finite type we can find dg-algebras $T_{\mathcal{X}} = Perf(A_{\mathcal{X}})$, $T_{\mathcal{V}} = Perf(A_{\mathcal{V}})$, $T_{\mathcal{U}} = Perf(A_{\mathcal{U}})$, $T_{\mathcal{W}} = Perf(A_{\mathcal{W}})$ and $T_{\mathcal{Y}} = Perf(A_{\mathcal{Y}})$. It follows that the previous diagram is a pullback if and only if the diagram

$$\begin{tikzcd} \widehat{A_{\mathcal{X}} \otimes A_{\mathcal{Y}}} \arrow[r] \arrow[d] & \widehat{A_{\mathcal{U}} \otimes A_{\mathcal{Y}}} \arrow[d] \\ \widehat{A_{\mathcal{V}} \otimes A_{\mathcal{Y}}} \arrow[r] & \widehat{A_{\mathcal{W}} \otimes A_{\mathcal{Y}}} \end{tikzcd} \tag{6.4.26}$$

is a pullback in $\mathcal{D}g^{cc}(k)$. By the hypothesis, the diagram

$$\begin{tikzcd} \widehat{A_{\mathcal{X}}} \arrow[r] \arrow[d] & \widehat{A_{\mathcal{U}}} \arrow[d] \\ \widehat{A_{\mathcal{V}}} \arrow[r] & \widehat{A_{\mathcal{W}}} \end{tikzcd} \tag{6.4.27}$$

is a pullback and so, thanks to [139, Theorem 7.2-1)] and to the Lemma 6.4.13, we have equivalences

$$\widehat{A_{\mathcal{X}} \otimes A_{\mathcal{Y}}} \simeq \mathbb{R}\underline{Hom}_c(\widehat{A_{\mathcal{Y}}}, \widehat{A_{\mathcal{X}}}) \simeq \mathbb{R}\underline{Hom}_c(\widehat{A_{\mathcal{Y}}}, \widehat{A_{\mathcal{V}}} \times_{\widehat{A_{\mathcal{W}}}} \widehat{A_{\mathcal{U}}}) \simeq \tag{6.4.28}$$

$$\mathbb{R}\underline{Hom}_c(\widehat{A_{\mathcal{Y}}}, \widehat{A_{\mathcal{V}}}) \times_{\mathbb{R}\underline{Hom}_c(\widehat{A_{\mathcal{Y}}}, \widehat{A_{\mathcal{W}}})} \mathbb{R}\underline{Hom}_c(\widehat{A_{\mathcal{Y}}}, \widehat{A_{\mathcal{U}}}) \simeq \widehat{A_{\mathcal{V}} \otimes A_{\mathcal{Y}}} \times_{\widehat{A_{\mathcal{W}} \otimes A_{\mathcal{Y}}}} \widehat{A_{\mathcal{U}} \otimes A_{\mathcal{Y}}} \tag{6.4.29}$$

□

Remark 6.4.15. It follows from the proof that $T_{\mathcal{Y}}$ does not need to be of finite type. It is enough the existence of a compact generator.

To conclude this section we prove that our notion of Nisnevich squares of smooth noncommutative spaces is compatible with the classical notion for schemes.

Proposition 6.4.16. *If X is an affine smooth scheme of finite type over k and*

$$\begin{tikzcd} p^{-1}(U) \arrow[r] \arrow[d] & V \arrow[d, "p"] \\ U \arrow[r, "i"] & X \end{tikzcd} \tag{6.4.30}$$

is a Nisnevich square in $N(AffSm^{ft}(k))$, then the induced diagram in $\mathcal{N}cS(k)$

$$\begin{tikzcd} L_{pe}(p^{-1}(U)) \arrow[r] \arrow[d] & L_{pe}(V) \arrow[d] \\ L_{pe}(U) \arrow[r] & L_{pe}(X) \end{tikzcd} \tag{6.4.31}$$

is a Nisnevich square of smooth noncommutative spaces.

Proof. Indeed, it is immediate that both maps $L_{pe}(p^{-1}(U)) \to L_{pe}(V)$ and $L_{pe}(U) \to L_{pe}(X)$ are open immersions of smooth noncommutative spaces. This is exactly the example that motivated the definition. They correspond to the quotient maps in $\mathcal{D}g(k)^{idem}$

$$L_{pe}(X) \longrightarrow L_{pe}(X)/L_{pe}(X)_{X-U} \qquad \text{and} \qquad L_{pe}(V) \longrightarrow L_{pe}(V)/L_{pe}(V)_{V-p^{-1}(U)} \tag{6.4.32}$$

We are left to check that:

1) The square in $\mathcal{D}g(k)^{idem}$

$$\begin{tikzcd} L_{pe}(X) \arrow[r] \arrow[d] & L_{pe}(U) \arrow[d] \\ L_{pe}(V) \arrow[r] & L_{pe}(p^{-1}(U)) \end{tikzcd} \tag{6.4.33}$$

is a pullback;

2) the map $L_{pe}(X) \to L_{pe}(V)$ in $\mathcal{D}g(k)^{idem}$ induces an equivalence $L_{pe}(X)_{X-U} \simeq L_{pe}(V)_{V-p^{-1}(U)}$;

The fact that (6.4.33) is a pullback follows from the fact that perfect complexes satisfy descent for the étale topology (which is a refinement of the Nisnevich topology). This result was originally proven by Hirschowitz and Simpson in [128]. See also [144] for further details.

The assertion 2) follows from 1) together with the fact that both $L_{pe}(X)_{X-U}$ and $L_{pe}(V)_{V-p^{-1}(U)}$ are by definition, the kernels of the quotient maps (6.4.32). ☐

Remark 6.4.17. In fact, it can be proved that if a pullback diagram like (6.4.30) induces a Nisnevich square of smooth noncommutative spaces then it is a Nisnevich square in the classical sense. This can be deduced using the equivalence $L_{qcoh}(X)_{X-U} \simeq L_{qcoh}(V)_{p^{-1}(X-U)}$ together with the equivalences $L_{qcoh}(X)_{X-U} \simeq L_{qcoh}(\widehat{X}_{X-U})$ and $L_{qcoh}(V)_{p^{-1}(X-U)} \simeq L_{qcoh}(\widehat{V}_{p^{-1}(X-U)})$ where $\widehat{X}_{X-U}$, respectively, $\widehat{V}_{p^{-1}(X-U)}$, denotes the formal completion of X (resp. V) at the closed subset $X-U$ (resp. $p^{-1}(X-U)$) (see [57, Prop. 7.1.3 and Prop. 6.8.2]). In particular this shows that the new notion of Nisnevich square is not really a weaker form of the original notion.

Remark 6.4.18. This non-commutative incarnation of the Nisnevich topology is not an actual Grothendieck topology. The pushout of a Nisnevich covering of a dg-category of finite type T along a functor $T \to T'$ is not a Nisnevich covering of T' for it does not have remain a pullback.

6.4.2 The Motivic Stable Homotopy Theory of Noncommutative Spaces

Now that we have an analogue for the Nisnevich topology in the noncommutative setting, compatible with the classical notion for schemes, we can finally conclude our task. We apply the same formula that produces the theory of Morel-Voevodsky. We start with $\mathcal{N}cS(k)^{\otimes}$ and consider its free cocompletion $\mathcal{P}^{big}(\mathcal{N}cS(k))$ together with the natural unique monoidal product extending the monoidal operation in $\mathcal{N}cS(k)$, compatible with colimits on each variable and making the inclusion $j : \mathcal{N}cS(k) \to \mathcal{P}^{big}(\mathcal{N}cS(k))$ monoidal (see 3.2.7). In particular, $j(L_{pe}(k))$ is the unit object. Next step, consider the localization $\mathcal{P}^{big}_{Nis}(\mathcal{N}cS(k))$ of $\mathcal{P}^{big}(\mathcal{N}cS(k))$ along the set of all edges $j(\mathcal{U}) \coprod_{j(\mathcal{W})} j(\mathcal{V}) \to j(\mathcal{X})$ running over all the Nisnevich squares of smooth noncommutative space

$$\begin{tikzcd} \mathcal{W} \arrow[r] \arrow[d] & \mathcal{V} \arrow[d] \\ \mathcal{U} \arrow[r] & \mathcal{X} \end{tikzcd} \tag{6.4.34}$$

The theory of localization for presentable $(\infty, 1)$-categories [99, 5.5.4.15] implies that $\mathcal{P}^{big}_{Nis}(\mathcal{N}cS(k))$ is an accessible reflexive localization of $\mathcal{P}^{big}(\mathcal{N}cS(k))$. The same result, together with the fact that the Nisnevich squares are pushouts squares, implies that every representable $j(\mathcal{X})$ is in $\mathcal{P}^{big}_{Nis}(\mathcal{N}cS(k))$. Moreover, and thanks to the Proposition 6.4.14, we deduce that this localization is monoidal. Finally, and in analogy with the commutative case, we consider the localization

$$l^{nc}_{\mathbb{A}^1} : \mathcal{P}^{big}_{Nis}(\mathcal{N}cS(k)) \to \mathcal{H}_{nc}(k) \tag{6.4.35}$$

taken with respect to the set of all maps

$$j(Id_{\mathcal{X}}) \otimes j(L_{pe}(p)) : j(\mathcal{X}) \otimes j(L_{pe}(\mathbb{A}^1_k)) \longrightarrow j(\mathcal{X}) \otimes j(L_{pe}(Spec(k))) \tag{6.4.36}$$

with $\mathcal{X}$ running over $\mathcal{N}cS(k)$. Here, $p : \mathbb{A}^1 \to Spec(k)$ is the canonical projection and the tensor product is computed in $\mathcal{P}^{big}_{Nis}(\mathcal{N}cS(k))$. [11] Again, this is an accessible reflective localization of $\mathcal{P}^{big}_{Nis}(\mathcal{N}cS(k))$ and it follows immediately from the definition of the localizing set that it is monoidal. With this we have a sequence of monoidal localizations

$$\mathcal{N}cS(k)^{\otimes} \xrightarrow{j} \mathcal{P}^{big}(\mathcal{N}cS(k))^{\otimes} \longrightarrow \mathcal{P}^{big}_{Nis}(\mathcal{N}cS(k))^{\otimes} \longrightarrow \mathcal{H}_{nc}(k)^{\otimes} \tag{6.4.37}$$

and by construction, $\mathcal{H}_{nc}(k)$ is a presentable symmetric monoidal $(\infty,1)$-category (see the Remark 3.6.1) and has a final object which we can identify with the image of the zero object of $\mathcal{N}cS(k)$ through the yoneda's map. Again, in analogy with the classical situation, we consider the universal pointing map

$$()^{nc}_{+} : \mathcal{H}_{nc}(k)^{\otimes} \to \mathcal{H}_{nc}(k)^{\wedge(\otimes)}_{*} \tag{6.4.38}$$

which is an equivalence because of our Convention 6.4.8: when we localize with respect to the Nisnevich topology with 6.4.8 the $(\infty,1)$-category $\mathcal{H}_{nc}(k)$ becomes pointed.

Finally, the compatibility between the classical and the new Nisnevich squares [12] and the respective $\mathbb{A}^1$ and $L_{pe}(\mathbb{A}^1)$-localizations, we deduce the existence of uniquely determined monoidal colimit preserving functors that make the diagram homotopy commutative

$$\begin{tikzcd}
N(AffSm^{ft}(k))^{\times} \arrow[r, "L_{pe}"] \arrow[d, "j^{\otimes}"] & \mathcal{N}cS(k)^{\otimes} \arrow[d, "j^{\otimes}"] \\
\mathcal{P}^{big}(N(AffSm^{ft}(k)))^{\times} \arrow[r, dashed, "(L_{pe})_!"] \arrow[d] & \mathcal{P}^{big}(\mathcal{N}cS(k))^{\otimes} \arrow[d] \\
Sh^{big}_{Nis}(N(AffSm^{ft}(k)))^{\times} \arrow[r, dashed] \arrow[d, "l^{\times}_{\mathbb{A}^1}"] & \mathcal{P}^{big}_{Nis}(\mathcal{N}cS(k))^{\otimes} \arrow[d, "l^{nc,\otimes}_{\mathbb{A}^1}"] \\
\mathcal{H}(S)^{\times} \arrow[r, dashed] \arrow[d, "()_+"] & \mathcal{H}_{nc}(k)^{\otimes} \\
\mathcal{H}(k)^{\wedge}_{*} \arrow[ur, dashed, "\psi^{\otimes}"] &
\end{tikzcd} \tag{6.4.39}$$

If we proceed according to the classical construction, the next step would be to stabilize the theory, first with respect to S^1 (the ordinary stabilization) and then with respect to the Tate circle. It happens that the inner properties of the noncommutative world make both these steps unnacessary.

Proposition 6.4.19. *The presentable pointed symmetric monoidal $(\infty,1)$-category $\mathcal{H}_{nc}(k)^{\otimes}$ is stable. Moreover, the Tate circle $\psi(\mathbb{G}_m)$ is already an invertible object.*

Recall that in $\mathcal{H}(k)^{\wedge}_{*}$ we have an equivalence $(\mathbb{P}^1,\infty) \simeq S^1 \wedge \mathbb{G}_m$ with $\mathbb{G}_m$ pointed at 1. Since the functor $\psi^{\otimes}$ is monoidal and commutes with colimits, we also have $\psi((\mathbb{P}^1,\infty)) \simeq S^1 \wedge \psi(\mathbb{G}_m)$. In particular, the Proposition 6.4.19 will follow immediately from the following lemma (using the Remark 4.2.17).

[11] Of course, since j is monoidal and the representable objects are Nisnevich local, this is the same as localizing with respect to the class of all maps $j(\mathcal{X} \otimes L_{pe}(\mathbb{A}^1_k)) \longrightarrow j(\mathcal{X} \otimes L_{pe}(Spec(k)))$.

[12] Recall that the collection of classical Nisnevich squares forms a basis for the Nisnevich topology

Lemma 6.4.20. *The object $\psi((\mathbb{P}^1,\infty)) \in \mathcal{H}_{nc}(k)^{\otimes}$ is invertible and equivalent to a unit of the monoidal structure.*

Proof. By definition, we have

$$(\mathbb{P}^1,\infty) := cofiber_{\mathcal{H}(k)}[l_{\mathbb{A}^1}(\infty : Spec(k) \to \mathbb{P}^1)] \tag{6.4.40}$$

where $\infty : Spec(k) \to \mathbb{P}^1$ is the point at infinity. By diagram chasing, the fact that $L_{pe}(\mathbb{P}^1)$ is a dg-category of finite type, and the fact that all the relevant maps commute with colimits we find

$$\psi((\mathbb{P}^1,\infty)) \simeq l^{nc}_{\mathbb{A}^1}(cofiber_{\mathcal{P}_{Nis}(\mathcal{N}cS(k))}[j(L_{pe}(\infty))]) \tag{6.4.41}$$

We claim that the last cofiber is the unit for the monoidal structure in $\mathcal{P}^{big}_{Nis}(\mathcal{N}cS(k))$, which, because the Yoneda's functor is monoidal, corresponds to $j(Perf(k))$. To see this, we observe first that map $L_{pe}(\infty) : Perf(k) = L_{pe}(k) \to L_{pe}(\mathbb{P}^1)$ in $\mathcal{N}cS(k)$ corresponds in fact to the pullback map $L_{pe}(\mathbb{P}^1) \to Perf(k)$ along ∞ in $\mathcal{D}g(k)^{idem}$. Recall the existence of an exceptional collection in $L_{pe}(\mathbb{P}^1)$ generated by the sheaves $\mathcal{O}$ and $\mathcal{O}(-1)$. Since the pullback preserves structural sheaves, the map $Perf(k) \to L_{pe}(\mathbb{P}^1)$ in $\mathcal{N}cS(k)$ fits in the Nisnevich square of the Example 6.4.12

$$\begin{tikzcd} Perf(k) \arrow[r] \arrow[d] & L_{pe}(\mathbb{P}^1) \arrow[d] \\ * \arrow[r] & Perf(k) \end{tikzcd} \tag{6.4.42}$$

dual to the split exact sequence provided by the exceptional collection. Finally, since in $\mathcal{P}^{big}_{Nis}(\mathcal{N}cS(k))$ every Nisnevich square is forced to become a pushout, we have

$$cofiber_{\mathcal{P}_{Nis}(\mathcal{N}cS(k))}[j(Perf(k) \to L_{pe}(\mathbb{P}^1))] \simeq Perf(k) \tag{6.4.43}$$

which concludes the proof. □

It also follows that we have a canonical equivalence

$$\mathcal{H}_{nc}(k)^{\otimes}[(\psi(\mathbb{P}^1,\infty)^{-1})] \simeq \mathcal{H}_{nc}(k)^{\otimes} \tag{6.4.44}$$

and for this reason, we reset the notations to match the classical one

$$\mathcal{SH}_{nc}(k)^{\otimes} := \mathcal{H}_{nc}(k)^{\otimes} \tag{6.4.45}$$

Remark 6.4.21. Let $\mathcal{C}$ be an $(\infty,1)$-category with a zero object 0. Recall that a *split exact sequence* in $\mathcal{C}$ is the data of a pushout square

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d] & B \arrow[d, "p"] \\ 0 \arrow[r] & C \end{tikzcd} \tag{6.4.46}$$

together with maps $u : C \to B$ and $v : B \to A$ in $\mathcal{C}$ with $p \circ u \sim id_C$ and $v \circ i \sim id_A$. We can now see that if $\mathcal{C}$ is a stable $(\infty,1)$-category, the data of a split exact sequence provides an equivalence $B \simeq A \oplus C$. To see this, it is enough to check that $i \circ v + u \circ p \sim id_B$, or, equivalently, that $(id_B - u \circ p) \sim i \circ v$. To see the last, we use the fact that $\mathcal{C}$ is stable and therefore the previous square is also a pullback. In this case, since $p \circ (id_B - u \circ p) \sim (p - p \circ u \circ p) \sim (p - Id_C \circ p) \sim 0$, we can find a factorization δ(unique up to a contractible space)

$$\begin{tikzcd} B \arrow[dr, dashed, "\delta"] \arrow[drr, bend left, "id_B - u\circ p"] \arrow[ddr, bend right] & & \\ & A \arrow[r, "i"] \arrow[d] & B \arrow[d, "p"] \\ & 0 \arrow[r] & C \end{tikzcd} \tag{6.4.47}$$

with $(id_B - u \circ p) \sim i \circ \delta$. But, since $v \circ i \sim id_A$, we have $\delta \sim v \circ i \circ \delta \sim v \circ (id_B - u \circ p) \sim v$.

Remark 6.4.22. Let $\mathcal{X}$ be smooth noncommutative space whose associated dg-category $T_{\mathcal{X}}$ admits an exceptional collection generated by $n+1$ elements. By the Remarks 6.4.11 and 6.4.12, this provides to the data of n different Nisnevich coverings. These are sent to split exact sequences in $\mathcal{SH}_{nc}(k)$ which we now know is stable. Using the Remark 6.4.21 we find that the image of $\mathcal{X}$ in $\mathcal{SH}_{nc}(k)$ decomposes as a direct sum of $n+1$ copies of the unit $1 = l^{nc}_{\mathbb{A}^1}(Perf(k))$. In particular the smooth noncommutative space $L_{pe}(\mathbb{P}^n)$ becomes equivalent to the direct sum $\underbrace{1 \oplus ... \oplus 1}_{n+1}$ in $\mathcal{SH}_{nc}(k)^{\otimes}$.

Finally, our universal property for inverting an object in a presentable symmetric monoidal $(\infty,1)$-category ensures the existence of a unique monoidal colimit map $\mathcal{L}^{\otimes}$ extending the diagram (6.4.39) to

$$\begin{tikzcd} N(AffSm^{ft}(k))^{\times} \arrow[r, "L_{pe}^{\otimes}"] \arrow[d] & \mathcal{N}cS(k)^{\otimes} \arrow[d] \\ \mathcal{SH}(k)^{\otimes} \arrow[r, dashed, "\mathcal{L}^{\otimes}"] & \mathcal{SH}_{nc}(k)^{\otimes} \end{tikzcd} \tag{6.4.48}$$

relating the classical stable homotopy theory of schemes with our new theory. From now we assume k is Noetherian of finite Krull dimension.

Remark 6.4.23. Using the same arguments of 5.4 we can describe the symmetric monoidal $(\infty,1)$-category $\mathcal{SH}_{nc}(k)^{\otimes}$ using presheaves of spectra. More precisely, we can start from the $(\infty,1)$-category of smooth noncommutative spaces $\mathcal{N}cS(k)$ and consider the very big $(\infty,1)$-category $Fun(\mathcal{N}cS(k)^{op}, \widehat{Sp})$. Using the equivalence $Fun(\mathcal{N}cS(k)^{op}, \widehat{Sp}) \simeq Stab(\mathcal{P}^{big}(\mathcal{N}cS(k))_*)$ together with the Remark 4.2.17 we obtain a canonical monoidal structure $Fun(\mathcal{N}cS(k)^{op}, \widehat{Sp})^{\otimes}$ defined by the inversion $\mathcal{P}^{big}(\mathcal{N}cS(k))_*^{\wedge(\otimes)}[(S^1)^{-1}]^{\otimes}$. We proceed, and perform the localizations with respect to the noncommutative version of the Nisnevich topology and $L_{pe}(\mathbb{A}^1)$. More precisely, and using the same notations as in 5.4 we localize with respect to the class of all canonical maps

$$\delta_{\Sigma^{\infty}_{+}\circ j(\mathcal{U})}(K) \coprod_{\delta_{\Sigma^{\infty}_{+}\circ j(\mathcal{W})}(K)} \delta_{\Sigma^{\infty}_{+}\circ j(\mathcal{V})}(K) \to \delta_{\Sigma^{\infty}_{+}\circ j(\mathcal{X})}(K) \tag{6.4.49}$$

with K in $(\widehat{Sp})^{\omega}$ and $\mathcal{W}$,$\mathcal{V}$,$\mathcal{U}$ and $\mathcal{X}$ part of a Nisnevich square of noncommutative smooth spaces. For the $\mathbb{A}^1$ localization, we localize with respect to the class of all induced maps

$$\delta_{\Sigma^{\infty}_{+}\circ j(\mathcal{X}\otimes L_{pe}(\mathbb{A}^1))}(K) \to \delta_{\Sigma^{\infty}_{+}\circ j(\mathcal{X})}(K) \tag{6.4.50}$$

with $\mathcal{X}$ in $\mathcal{N}cS(k)$ and $K \in (\widehat{Sp})^{\omega}$. By the same argument, these are monoidal reflexive localizations. We denote the result as $Fun_{Nis, L_{pe}(\mathbb{A}^1)}(\mathcal{N}cS(k)^{op}, \widehat{Sp})^{\otimes}$. It is a stable presentable symmetric monoidal $(\infty,1)$-category and by the Prop. 6.4.19 and the universal properties involved, it is canonically monoidal equivalent to $\mathcal{SH}_{nc}(k)^{\otimes}$.

Using this equivalence and the definition of $\mathcal{N}cS(k)$, we can identify an object $F \in \mathcal{SH}_{nc}(k)^{\otimes}$ with a functor $\mathcal{D}g(k)^{ft} \to \widehat{Sp}$ satisfying $L_{pe}(\mathbb{A}^1)$-invariance, having a descent property with respect to the Nisnevich squares and because of the convention 6.4.8, satisfying $F(0) = *$.

Remark 6.4.24. (Strictification) It is also important to remark that an object F in $Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp})$ can always be identified up to equivalence with an actual strict functor F_s from the category of dg-categories endowed with the Morita model structure of [131] to some combinatorial model category whose underlying $(\infty, 1)$-category is $\widehat{Sp}$ (for instance, the big model category of symmetric spectra Sp^{Σ} of [72]), with F_s sending Morita equivalences to weak-equivalences and commuting with filtered homotopy colimits. Indeed, as explained in 6.1.27, $\mathcal{D}g(k)^{ft}$ generates $\mathcal{D}g(k)^{idem}$ under filtered colimits. Since Sp admits all small filtered colimits, using [99, Thm 5.3.5.10] we find an equivalence of $(\infty, 1)$-categories between $Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp})$ and $Fun_{\omega}(\mathcal{D}g(k)^{idem}, \widehat{Sp})$ - the full subcategory of $Fun(\mathcal{D}g(k)^{idem}, \widehat{Sp})$ spanned by the functors that preserve filtered colimits. Moreover, we have also seen that $\mathcal{D}g(k)^{idem}$ is the underlying $(\infty, 1)$-category of the Morita model structure for small dg-categories (see the discussion in 6.1.2). Finally, with the appropriate universe considerations, we can use the strictification result of [100, 1.3.4.25] and the characterization of homotopy limits and colimits in a model category as limits and colimits in its underlying $(\infty, 1)$-category [100, 1.3.4.24] to deduce the existence of a canonical equivalence between $Fun_{\omega}(\mathcal{D}g(k)^{idem}, \widehat{Sp})$ and the localization along the levelwise equivalences of the category of strict functors from the category of dg-categories to the strict model for spectra Sp^{Σ}, which commute with filtered homotopy colimits and send Morita weak-equivalences to weak-equivalences in Sp^{Σ}.

Remark 6.4.25. To conclude this chapter we remark the existence of a family of compact generators in $\mathcal{SH}_{nc}(k)$. In this case, as the construction ends after the $\mathbb{A}^1$-localization, the same arguments used in the proof of 5.3.3 are enough to show that the collection of noncommutative motives in the image of the canonical map $\mathcal{N}cS(k) \to \mathcal{SH}_{nc}(k)$ form a family of compact generators.

K-Theory and Non-commutative Motives

The results in the previous chapter establish an homotopy commutative diagram of colimit preserving monoidal functors extending the functor L_{pe}

$$\begin{tikzcd}
N(\mathit{AffSm}^{ft}(k))^{\times} \arrow[rr, "L_{pe}^{\otimes}"] \arrow[d, hook, "(\Sigma_+^{\infty}\circ j)^{\otimes}"] & & \mathcal{N}cS(k)^{\otimes} \arrow[d, hook, "(\Sigma_+^{\infty}\circ j_{nc})^{\otimes}"] \\
Fun(N(\mathit{AffSm}^{ft}(k))^{op}, \widehat{Sp})^{\otimes} \arrow[rr, dashed] \arrow[d, "l_{Nis}^{\otimes}"] & & Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp})^{\otimes} \arrow[d, "l_{Nis}^{nc,\otimes}"'] \\
Fun_{Nis}(N(\mathit{AffSm}^{ft}(k))^{op}, \widehat{Sp})^{\otimes} \arrow[rr, dashed] \arrow[d, "l_{\mathbb{A}^1}^{\otimes}"] & & Fun_{Nis}(\mathcal{D}g(k)^{ft}, \widehat{Sp})^{\otimes} \arrow[d, "l_{\mathbb{A}^1}^{nc,\otimes}"'] \\
Fun_{Nis,\mathbb{A}^1}(N(\mathit{AffSm}^{ft}(k))^{op}, \widehat{Sp})^{\otimes} \arrow[rr, dashed] \arrow[d, "\Sigma_{\mathbb{G}_m}^{\otimes}"] & & Fun_{Nis,L_{pe}(\mathbb{A}^1)}(\mathcal{D}g(k)^{ft}, \widehat{Sp})^{\otimes} \arrow[d, "\sim"] \\
\mathcal{SH}(k)^{\otimes} \arrow[rr, dashed, "\mathcal{L}^{\otimes}"] & & \mathcal{SH}_{nc}(k)^{\otimes}
\end{tikzcd} \tag{7.0.1}$$

thus providing a canonical mechanism to compare the theory of Morel-Voevodsky with our new approach.

Our goal in this last chapter is to explore how this bridge can be used to give a canonical interpretation to the various flavours of algebraic K-theory of schemes and dg-categories. In order to state our results, we observe first that, due to the Adjoint Functor Theorem ([99, Corollary 5.5.2.9]), each of the dotted monoidal functors in (7.0.1) has a right adjoint. This is because at each level, the source and target $(\infty, 1)$-categories are presentable and each dotted map is, by construction, colimit-preserving. Furthermore, since each dotted map is monoidal, these right adjoints are lax-monoidal (see [100, 7.3.2.7]). In this case, together with the lax-monoidal inclusions associated to the reflexive monoidal localizations, we have a new commutative diagram of lax-monoidal functors

$$\begin{tikzcd}
Fun(N(\mathit{AffSm}^{ft}(k))^{op}, \widehat{Sp})^{\otimes} & Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp})^{\otimes} \arrow[l, "\mathcal{M}_1^{\otimes}"'] \\
Fun_{Nis}(N(\mathit{AffSm}^{ft}(k))^{op}, \widehat{Sp})^{\otimes} \arrow[u, hook] & Fun_{Nis}(\mathcal{D}g(k)^{ft}, \widehat{Sp})^{\otimes} \arrow[l, "\mathcal{M}_2^{\otimes}"'] \arrow[u, hook] \\
Fun_{Nis,\mathbb{A}^1}(N(\mathit{AffSm}^{ft}(k))^{op}, \widehat{Sp})^{\otimes} \arrow[u, hook] & Fun_{Nis,L_{pe}(\mathbb{A}^1)}(\mathcal{D}g(k)^{ft}, \widehat{Sp})^{\otimes} \arrow[l, "\mathcal{M}_3^{\otimes}"'] \arrow[u, hook] \\
\mathcal{SH}(k)^{\otimes} \arrow[u, "\Omega^{\infty,\otimes}_{\mathbb{G}_m}"] & \mathcal{SH}_{nc}(k)^{\otimes} \arrow[l, "\mathcal{M}^{\otimes}"'] \arrow[u, "\sim"']
\end{tikzcd} \tag{7.0.2}$$

Let us present some remarks that will be useful along this chapter.

Remark 7.0.26. The first functor $\mathcal{M}_1$ commutes with small colimits. We can deduce this either from the fact that colimits in $Fun(N(\mathit{AffSm}^{ft}(k))^{op}, \widehat{Sp})$ and in $Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp})$ are computed objectwise (see [99, 5.1.2.3]) or from the spectral enriched version of Yoneda's lemma (5.4.1).

Remark 7.0.27. All the symmetric monoidal $(\infty, 1)$-categories appearing in the previous diagram are stable and presentable. Stability follows because pushouts of local objects remain local, thanks to the fact that all colimits are computed objectwise in spectra. Therefore all these are closed monoidal. In particular, recall that if $\mathcal{C}_0 \subseteq \mathcal{C}$ is a monoidal reflexive localization and if $\mathcal{C}$ admits internal-homs $\underline{Hom}_{\mathcal{C}}$ then $\mathcal{C}_0$ admits internal-homs: given X and Y local, we can easily see that $\underline{Hom}_{\mathcal{C}}(X, Y)$ is also local and works as an internal-hom in $\mathcal{C}_0$.

We observe that each functor $\mathcal{M}_*$ is compatible with the respective internal-homs, in the sense that at each level, for every object X on the left and F on the right, we have

$$\mathcal{M}_*(\underline{Hom}_*(\mathcal{L}_*(X), F)) \simeq \underline{Hom}_*(X, \mathcal{M}_*(F)) \tag{7.0.3}$$

where $\mathcal{L}_*$ denotes the respective monoidal left adjoint appearing in the diagram (7.0.1)

Remark 7.0.28. Thanks to the enriched version of Yoneda's lemma for spectral presheaves (see the Remark 5.4.1), given an object $F \in Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp})$, we have for each scheme X an equivalence of spectra

$$Map^{Sp}_{Fun(N(\mathit{AffSm}^{ft}(k))^{op},\widehat{Sp})}(\Sigma^{\infty}_{+}\circ j(X), \mathcal{M}_1(F)) \simeq Map^{Sp}_{Fun(\mathcal{D}g(k)^{ft},\widehat{Sp})}(\Sigma^{\infty}_{+}\circ j_{nc}(L_{pe}(X)), F) \simeq F(L_{pe}(X)) \tag{7.0.4}$$

so that $\mathcal{M}_1(F)$ can be thought of as a restriction of F to the commutative world. The same is valid for $\mathcal{M}_2$ and $\mathcal{M}_3$ because the upper vertical arrows are inclusions.

This mechanism allows us to restrict noncommutative invariants to the commutative world. In this chapter we will be interested in the restriction of the various algebraic K-theories of dg-categories. As we shall explain below, all of them live as objects in $Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp})$. There are two of primary relevance to us, namely, K^c encoding Waldhausen's connective K-theory (Section 7.1.2) and K^S encoding the non-connective K-theory of dg-categories defined by means of Schlichting's framework in [122] (Section 7.1.3). By construction, the latest comes naturally equipped with a canonical natural transformation $K^c \to K^S$ which is an equivalence in the connective part. For the first one, it follows immediately from the spectral version of Yoneda's lemma and from the definition in [137, Section 3] that $\mathcal{M}_1(K^c)$ recovers the connective algebraic K-theory of schemes. The second one, by the comparison result [122, Theorem 7.1], recovers the non-connective K-theory of schemes of Bass-Thomason-Trobaugh of [137]. The constructon of K^S in [34] using the methods of [122] is somehow ad-hoc. Our first main result explains how the non-connective version of K-theory K^S can be canonically obtained from the connective version K^c as a result of forcing our noncommutative-world version of Nisnevich descent.

Theorem 7.0.29. *The canonical morphism $K^c \to K^S$ presents non-connective K-theory of dg-categories as the (noncommutative) Nisnevich localization of connective K-theory.*

To prove this result we will first check that K^S is Nisnevich local. This follows from the well-known localization theorem for non-connective K-theory (see the Corollary 7.1.6 below). The rest of the proof will require a careful discussion concerning the behavior of the noncommutative Nisnevich localization. There are two main ingredients:

Step 1) *Every Nisnevich local $F : \mathcal{D}g(k)^{ft} \to \widehat{Sp}$ is determined by its connective part by means of the Bass exact sequences.* More precisely, we show that every Nisnevich local functor $F : \mathcal{D}g(k)^{ft} \to \widehat{Sp}$ satisfies the familiar Bass exact sequences for any integer n. We will see that the proof in [137] can be easily adapted to our setting. Namely, we start by showing that every Nisnevich local F satisfies the Projective Bundle theorem. This result is central and appears as a consequence of one of the most important features of the noncommutative world, namely, the fact that Nisnevich coverings of non-geometrical origin are allowed, in particular, those appearing from semi-orthogonal decompositions and exceptional collections. The projective bundle theorem is a direct consequence of the existence of an exceptional collection on $L_{pe}(\mathbb{P}^1)$ generated by the sheaves $\mathcal{O}_{\mathbb{P}^1}$ and $\mathcal{O}_{\mathbb{P}^1}(-1)$ (see [12]). Its existence forces the image of $L_{pe}(\mathbb{P}^1)$ in $Fun_{Nis}(\mathcal{D}g(k)^{ft}, \widehat{Sp})$ to become equivalent to the direct sum $L_{pe}(k) \oplus L_{pe}(k)$. To complete the proof we proceed as in [137, Theorem 6.1] and explain how this direct sum decomposition can be suitably adapted in order to extract the familiar Bass exact sequences out of the classical Nisnevich covering of $\mathbb{P}^1$ by two affine lines.

Step 2) *The connective truncation of the localization map $K^c \to l^{nc}_{Nis}(K^c)$ is an equivalence* [1]. In other words, the information stored in the connective part of $l_{Nis}(K^c)$ remains the information of connective K-theory. We will prove something a bit more general, namely, that this property holds not only for K^c but for the whole class of functors $F : \mathcal{D}g(k)^{ft} \to \widehat{Sp}$ satisfying the formal properties of K^c, namely, having values in connective spectra and sending Nisnevich squares of dg-categories to pullback squares of connective spectra (for K^c this follows from the fibration theorem of Waldhausen [153, 1.6.4] - see Prop. 7.1.4 below.). These will be called *connectively-Nisnevich local.* We prove that the connective truncation functor induces a canonical equivalence between the theory of connective-Nisnevich functors and that of Nisnevich functors (see 7.2.8). For this we will show that if F is connectively-Nisnevich local, its noncommutative Nisnevich localization $l^{nc}_{Nis}(F)$ is equivalent to F^B - the more familiar B-construction of Thomason of [137, Def. 6.4].

Remark 7.0.30. Since the functor $\mathcal{M}_2$ in the diagram (7.0.2) sends Nisnevich local objects to Nisnevich local objects, our Theorem 7.0.29 provides a new proof that the spectral presheaf giving the Bass-Thomason-Trobaugh K-theory of schemes satisfies Nisnevich descent.

We can now go one step further and consider the $\mathbb{A}^1$-localization of K^S. We will prove that

Theorem 7.0.31. $\mathcal{M}_3(l^{nc}_{\mathbb{A}^1}(K^S))$ *is the Nisnevich local $\mathbb{A}^1$-invariant spectral presheaf giving Weibel's homotopy invariant K-theory of schemes of [154]. In particular, $\mathcal{M}(l^{nc}_{\mathbb{A}^1}K^S)$ is canonically equivalent to the object $\mathcal{K}H$ in $\mathcal{SH}(k)$ studied in [150] and in [29] representing homotopy invariant algebraic K-theory of schemes.*

---

[1] Recall that $\widehat{Sp}$ has a natural t-structure $(\widehat{Sp}_{\geq 0}, \widehat{Sp}_{\leq -1})$ with $\widehat{Sp}_{\geq 0}$ the full subcategory spanned by connective spectra. As a consequence, the inclusion $\widehat{Sp}_{\geq 0} \subseteq \widehat{Sp}$ (resp. $\widehat{Sp}_{\leq -1} \subseteq \widehat{Sp}$) admits a right adjoint $\tau_{\geq 0}$ (resp. left adjoint $\tau_{\leq -1}$). In particular, we have an induced adjunction

$$\begin{tikzcd} Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp}_{\geq 0}) \arrow[r, hook, shift left] & Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp}) \arrow[l, bend left, "\tau_{\geq 0}"] \end{tikzcd}$$

with $\tau_{\geq 0}$ a right adjoint to the inclusion.

The proof of this result follows immediately from the results in [29] and from our Theorem 7.0.29 using a nice description of the $\mathbb{A}^1$-localization functors. This will be done in Section 7.3.

Our second main result in this section is a new representability theorem for K-theory.

Theorem 7.0.32. *The further localization $l_{\mathbb{A}^1}^{nc}(K^S)$ is a unit for the monoidal structure in $\mathcal{SH}_{nc}(k)^{\otimes}$.*

In [17], the author constructs an $\mathbb{A}^1$-equivalence between the split and the standard versions of Waldhausen's S-construction. In Section 7.4 we will explain how this $\mathbb{A}^1$-equivalence appears in our context and how the theorem follows as a consequence.

We deduce the following immediate corollaries

Corollary 7.0.33. *(Kontsevich) Let $\mathcal{X}$ and $\mathcal{Y}$ be two noncommutative spaces with $\mathcal{Y}$ smooth and proper. Then, there is a natural equivalence of spectra*

$$Map^{Sp}_{\mathcal{SH}_{nc}(k)}(\mathcal{X}, \mathcal{Y}) \simeq (l_{\mathbb{A}^1}^{nc} K^S)(T_{\mathcal{X}} \otimes \check{T_{\mathcal{Y}}}) \tag{7.0.5}$$

where we identify $\mathcal{X}$ and $\mathcal{Y}$ with their images in $\mathcal{SH}_{nc}(k)$ and where $T_{\mathcal{X}}$ (resp. $\check{T_{\mathcal{Y}}}$) denotes the dg-category of finite type associated to $\mathcal{X}$ (resp. the dual of the dg-category associated to $\mathcal{Y}$).

Proof. This follows directly from the spectral version of the Yoneda's lemma and from our Theorems 7.0.29 and 7.0.32, together with the fact that a smooth and proper noncommutative space is dualizable. □

Remark 7.0.34. We direct the reader to the Prop. 9.3.4 for an extension of the Theorem 7.0.32 and the Corollary 7.0.33 to non-commutative motives over a more general base scheme.

Corollary 7.0.35. *The object $\mathcal{K}H \in \mathcal{SH}(k)$ representing homotopy algebraic K-theory is equivalent to $\mathcal{M}(1_{nc})$. In particular, for each scheme X we have an equivalence of spectra*

$$KH(X) \simeq Map^{Sp}_{\mathcal{SH}(k)}(\Sigma^{\infty}_{+} \circ j(X), \mathcal{K}H) \simeq Map^{Sp}_{\mathcal{SH}_{nc}(k)}(\Sigma^{\infty}_{+} \circ j_{nc}(L_{pe}(X)), 1_{nc}) \tag{7.0.6}$$

At this point we should emphasize that a different representability result for connective K-theory is already known from the thesis of G. Tabuada [133] and for non-connective K-theory from his later works with D.C. Cisinski [34]. Our setting and proofs are independent of theirs. In the next chapter of this thesis we describe the relation between the two approaches. The main advantage of our theory is the existence of a canonical comparison with the original approach of Morel-Voevodsky and our new representabilty theorem brings some immediate consequences to the nature of this comparison. Namely, since $\mathcal{M}$ is lax-monoidal ([100, 7.3.2.7]), the object $\mathcal{K}H \simeq \mathcal{M}(1_{nc})$ acquires a canonical structure of commutative algebra-object in $\mathcal{SH}(k)$ induced by the trivial algebra structure on the unit object 1_{nc}. In this case, the comparison functor $\mathcal{L}^{\otimes} : \mathcal{SH}(k)^{\otimes} \to \mathcal{SH}_{nc}(k)^{\otimes}$ admits a canonical colimit preserving monoidal factorization (see our discussion in 3.3.9):

$$\begin{tikzcd}
\mathcal{SH}(k)^{\otimes} \arrow[r, "\mathcal{L}^{\otimes}"] \arrow[d, "-\otimes\mathcal{K}H"] & \mathcal{SH}_{nc}(k)^{\otimes} \arrow[d, "-\otimes\mathcal{L}(\mathcal{K}H)"'] \arrow[dr, "(-\otimes 1_{nc})\simeq Id"] & \\
Mod_{\mathcal{K}H}(\mathcal{SH}(k))^{\otimes} \arrow[r] & Mod_{\mathcal{L}(\mathcal{K}H)}(\mathcal{SH}_{nc}(k))^{\otimes} \arrow[r, "-\otimes_{\mathcal{L}(\mathcal{K}H)}1_{nc}"'] & Mod_{1_{nc}}(\mathcal{SH}_{nc}(k))^{\otimes}
\end{tikzcd} \tag{7.0.7}$$

where the first lower map is the monoidal functor induced by $\mathcal{L}$ at the level of modules and the last map is base-change with respect to the canonical morphisms of algebra objects given by the counit of the adjunction $\mathcal{L}(\mathcal{K}H) \simeq \mathcal{L} \circ \mathcal{M}(1_{nc}) \to 1_{nc}$ [2]. We will write $\mathcal{L}_{\mathcal{K}H}$ for this factorization.

[2] Notice that the adjunction $(\mathcal{L}, \mathcal{M})$ extends to an adjunction between the $(\infty, 1)$-categories of commutative algebra-objects, so that this counit map is a morphism of algebras. In particular, we can perform base-change with respect to it.

Warning 7.0.36. We will not prove here that the commutative algebra structure in $\mathcal{K}H$ obtained from our arguments is the same as the one already appearing in the literature and deduced from different methods (for instance, see [60, 106]). However, we believe that the arguments used in [106] also work in the ∞-categorical setting, so that our algebra structure should match the standard one.

Our representability result has the following corollary showing that under the existence of resolutions of singularities the passage to the noncommutative world produces no loss of information from the K-theoretic viewpoint.

Corollary 7.0.37. *Let k be a field admitting resolutions of singularities. Then the canonical map*

$$\mathcal{L}_{\mathcal{K}H} : Mod_{\mathcal{K}H}(\mathcal{SH}(k)) \to \mathcal{SH}_{nc}(k) \tag{7.0.8}$$

is fully faithful.

Proof. Thanks to the main results of [116] the family of dualizable objects in $\mathcal{SH}(k)$ is a family of ω-compact generators for the stable $(\infty, 1)$-category $\mathcal{SH}(k)$ in the sense of the Proposition 2.1.2. See the Prop. 3.8.3. One can now easily check that the collection of all objects in the stable $(\infty, 1)$-category $Mod_{\mathcal{K}H}(\mathcal{SH}(k))$ of the form $X \otimes \mathcal{K}H$ with X dualizable in $\mathcal{SH}(k)$ is again a family of ω-compact generators in the sense of Prop.2.1.2. Since the functor $(- \otimes \mathcal{K}H)$ is monoidal, the objects $X \otimes \mathcal{K}H$ are dualizable in $Mod_{\mathcal{K}H}(\mathcal{SH}(k))$ and as $\mathcal{L}_{\mathcal{K}H}$ is monoidal, their image in $\mathcal{SH}_{nc}(k)$ is dualizable and therefore compact (using the fact the monoidal structure is compatible with colimits in each variable). By the Proposition 2.1.7 we are now reduced to showing that $\mathcal{L}_{\mathcal{K}H}$ is fully faithful when restricted to the full subcategory spanned by all the objects of the form $X \otimes \mathcal{K}H$ with X dualizable in $\mathcal{SH}(k)$. This follows from the canonical chain of equivalences

$$Map_{Mod_{\mathcal{K}H}(\mathcal{SH}(k))}(X \otimes \mathcal{K}H, Y \otimes \mathcal{K}H) \simeq Map_{\mathcal{SH}(k)}(X, Y \otimes \mathcal{K}H) \tag{7.0.9}$$

$$\simeq Map_{\mathcal{SH}(k)}(X \otimes \check{Y}, \mathcal{K}H) \simeq Map_{\mathcal{SH}_{nc}(k)}(\mathcal{L}(X \otimes \check{Y}), 1_{nc}) \tag{7.0.10}$$

$$\simeq Map_{\mathcal{SH}_{nc}(k)}(\mathcal{L}(X) \otimes \mathcal{L}(\check{Y}), 1_{nc}) \simeq Map_{\mathcal{SH}(k)}(\mathcal{L}(X) \otimes \mathcal{L}(\check{Y}), 1_{nc}) \tag{7.0.11}$$

$$\simeq Map_{\mathcal{SH}_{nc}(k)}(\mathcal{L}(X)\otimes, \mathcal{L}(Y)) \tag{7.0.12}$$

where we use the adjunction properties, the fact that $\mathcal{K}H \simeq \mathcal{M}(1_{nc})$ and the fact that $\mathcal{L}$ is monoidal and therefore preserves dualizable objects. This concludes the proof. □

Although this result is new in the literature, its content has been known for a while. I think particularly of B.Toen, M. Vaquie and G. Vezzosi and also of D-C. Cisinski and G. Tabuada. Later on in Chapter 9 we will explain our attempts to extend it to to the theories of motives over a general base scheme not necessarily affine.

7.1 K-theory Preliminaries

7.1.1 Connective K-theory - an historical overview

K-theory was discovered by A. Grothendieck during his attempts to generalize the classical Riemman-Roch Theorem (see [27, 53]). Given an abelian category E he was led to consider an abelian group $K_0(E)$ together with a map $\theta : Obj(E) \to K_0(E)$ universal with respect to the following property: for any exact sequence $a \to b \to c$ in E we have $\theta(b) = \theta(a) + \theta(c)$.

The essential insight leading to the introduction of higher K-theory groups is the observation by Quillen [112] that the group law on $K_0(E)$ can be understood as the π_0-reminiscent part of a grouplike homotopy commutative law on a certain space $K(E)$. Following his ideas, for any "exact category" E we are able to define such a K-theory space $K(E)$ whose homotopy groups $\pi_n(K(E))$ we interpret as level n K-theoretic information. In particular, this methodology allows us to attach a K-theory space to every scheme X using the canonical structure of exact category on $E = Vect(X)$.

An important step in this historical account is a theorem by Segal [125, 3.4] (and its later formulation in terms of model categories in [28]) establishing an equivalence between the homotopy theory of grouplike homotopy commutative algebras in spaces and the homotopy theory of connective spectra. This is the reason why connective spectra is commonly used as the natural target for K-theory and the origin of the term "connective". In the modern days this equivalence can be stated by means of an equivalence of $(\infty, 1)$-categories, namely, between the $(\infty, 1)$-category $CAlg^{grplike}(S)$ and the $(\infty, 1)$-category $Sp_{\geq 0}$ (see [100, Theorem 5.2.6.10 and Remark 5.2.6.26]).

Technical reasons and possible further applications led Waldhausen [153] to extend the domain of K-theory from exact categories to what we nowadays call "Waldhausen categories". Grosso modo, these are triples $(\mathcal{C}, W, Cof(\mathcal{C}))$ where $\mathcal{C}$ is a classical category having a zero object and both W and $Cof(\mathcal{C})$ are classes of morphisms in $\mathcal{C}$, respectively called "weak-equivalences" and "cofibrations". These triples are subject to certain conditions which we will not specify here. The core of Waldhausen's method to construct a K-theory space out of this data is the algorithm known as the "S-Construction" which we review here very briefly:

Construction 7.1.1. (S-Construction) Let $Ar[n]$ be the category of arrows in the linear category $[n]$. In more explicit terms it can be described as the category where objects are pairs (i, j) with $i \leq j$ and there is one morphism $(i, j) \to (l, k)$ everytime $i \leq l$ and $j \leq k$. Let now $(\mathcal{C}, W, Cof(\mathcal{C}))$ be a Waldhausen category. We let $S_n(\mathcal{C})$ denote the full subcategory of all functors $Fun(Ar[n], \mathcal{C})$ spanned by those functors A verifying:

1. $A(i, i)$ is a zero object of $\mathcal{C}$ for all $0 \leq i \leq n$;
2. for any i the maps $A(i, j) \to A(i, k)$ with $j \leq k$ are cofibrations in $\mathcal{C}$;
3. for any $i \leq j \leq k$ the induced diagram

$$\begin{tikzcd} A(i,j) \arrow[r] \arrow[d] & A(i,k) \arrow[d] \\ 0 = A(j,j) \arrow[r] & A(j,k) \end{tikzcd} \tag{7.1.1}$$

is a pushout $\mathcal{C}$.

In other words, the objects in $S_n(\mathcal{C})$ can be identified with sequences of cofibrations of length $n-1$ plus the datum of the sucessive quotients. In particular, $S_0(\mathcal{C})$ is the category with a single object and $S_1(\mathcal{C})$ is equivalent to $\mathcal{C}$. Moreover, the collection of categories $\{S_n(\mathcal{C})\}_{n \in \mathbb{N}}$ assembles together to form a simplicial category $S_\bullet(\mathcal{C})$ carrying at each level a canonical structure of Waldhausen category whose weak-equivalences W_n are the levelwise weak-equivalences in $\mathcal{C}$. By definition, the K-theory space of $\mathcal{C}$ is the simplicial set $K^c(\mathcal{C}) := \Omega\, colim_{\Delta^{op}} N(S_n(\mathcal{C})^{W_n})$ where $S_n(\mathcal{C})^{W_n}$ denotes the subcategory of $S_n(\mathcal{C})$ contaning all the objects and only those morphisms which are weak-equivalences and N is the standard nerve functor. By iterating this procedure we can produce a connective spectrum. For the complete details see [153].

There is a natural notion of exact functor between Waldhausen categories providing a category $Wald_{Classic}$ and the K-theory assignement can be understood as a functor

$$K^c_{Wald} : N(Wald_{Classic}) \longrightarrow Sp^{\Sigma} \tag{7.1.2}$$

where Sp^{Σ} is a model category for the $(\infty, 1)$-category Sp.

Many Waldhausen categories used in practice appear as subcategories of a Quillen model category [113] with the cofibrations and weak-equivalences therein. We will denote by $Wald^{Model}_{Classic}$ the full subcategory of $Wald_{Classic}$ spanned by those Waldhausen categories falling into this list of examples. These Waldhausen categories have a special advantage - the factorization axioms for the model category allow us to change the Construction 7.1.1 to consider all morphisms in $\mathcal{C}$, not only the cofibrations.

This first era of connective K-theory finishes with the works of Thomason-Trobaugh in [137] where the machinery of Waldhausen is applied to schemes and it is proven that the connective K-theory of a scheme X introduced by Quillen can be recovered from the K-theory attached to the Waldhausen structure on the category of perfect complexes on the scheme.

The current era begins with the observation that the K-theory of a Waldhausen datum $(\mathcal{C}, W, Cof(\mathcal{C}))$ is not an invariant of the classical 1-categorical localization $\mathcal{C}[W^{-1}]$: there are examples of pairs of Waldhausen categories with the same homotopy categories but with different K-theory spaces (see [121]). The crucial results of Toën-Vezzosi in [142] allow us to identify the world of $(\infty,1)$-categories as the natural ultimate domain for K-theory. They prove that if the underlying $(\infty,1)$-categories associated to a pair of Waldhausen categories (via the ∞-localization) are equivalent then the associated K-theory spaces are equivalent. Moreover, in the same paper, the authors remark that the classical S-construction of Waldhausen can be lifted to the setting of $(\infty,1)$-categories. Following this insight, in [9] the author introduces the notion of a *Waldhausen* $(\infty,1)$*-category* (which, grosso modo are pairs of $(\infty,1)$-categories $(\mathcal{A}_0, \mathcal{A})$ with $\mathcal{A}_0$ a full subcategory of $\mathcal{A}$ containing its maximal ∞-groupoid, together with extra conditions on this pair) and develops this ∞-version of the S-construction. The collection of Waldhausen $(\infty,1)$-categories forms itself an $(\infty,1)$-category $\mathcal{W}ald_\infty$ and the result of this new ∞-version of the S-construction can be encoded as an ∞-functor $K^c_{Barwick} : \mathcal{W}ald_\infty \to Sp_{\geq 0}$. Moreover, there is a canonical ∞-functor linking the classical theory to this new approach

$$N(Wald^{Model}_{Classic}) \longrightarrow \mathcal{W}ald_\infty \tag{7.1.3}$$

sending a classical Waldhausen datum $(\mathcal{C}, W, Cof(\mathcal{C}))$ to the ∞-localization $N(\mathcal{C})[W^{-1}]$ together with its smallest subcategory containing the equivalences and the images of the cofibrations under the localization functor (see [9, Example 2.12]). The author then proves that the two S-constructions, respectively, the classical and the new ∞-version agree by means of this assignement and therefore produce the same K-theory ([9, 10.6.2]). Up to our days this framework seems to be the most natural and general domain for connective K-theory. However, we should remark that a different ∞-categorical domain has been established in the paper [18] where the authors study K-theory spaces associated to pointed $(\infty,1)$-categories having all finite colimits, whose collection forms an $(\infty,1)$-category $Cat_\infty(\omega)_*$. They generalize the classical S-construction to this new domain obtaining a new ∞-functor $K^c_{BGT} : Cat_\infty(\omega)_* \to Sp_{\geq 0}$,and prove that for any Waldhausen category $\mathcal{C}$ with equivalences W (appearing as a subcategory of a model category), the K-theory space which their method assigns to the ∞-localization $N(\mathcal{C})[W^{-1}]$ is equivalent to the classical K-theory space attached to $\mathcal{C}$ through the classical methods of Waldhausen. This framework is of course related to the wider framework of [9]: following the Example [9, 2.9], every pointed $(\infty,1)$-category with finite colimits has a naturally associated Waldhausen $(\infty,1)$-category. Again, this assignement can be properly understood as an ∞-functor

$$\Psi : Cat_\infty(\omega)_* \longrightarrow \mathcal{W}ald_\infty \tag{7.1.4}$$

We summarize this fast historical briefing with the existence of a diagram of $(\infty,1)$-categories

$$\begin{tikzcd}
 & Cat_\infty(\omega)_* \arrow[dr, "\Psi"] \arrow[dd, dotted, "K^c_{B.G.T.}"] & \\
N(Wald^{Model}_{Classic}) \arrow[ur] \arrow[rr] \arrow[dr, "K^c_{Wald}"'] & & \mathcal{W}ald_\infty \arrow[dl, "K^c_{Barwick}"] \\
 & Sp_{\geq 0} &
\end{tikzcd} \tag{7.1.5}$$

whose commutativity follows from the results in [9] and in [18] and from the agreement of the two ∞-categorical versions of the S-construction via Ψ. This agreement follows from the very definition of the two procedures. Consult [9, Section 5] and [18, Section 7.1] for the complete details.

7.1.2 Connective K-theory of dg-categories

Our goal in this section is to explain how to define the connective K-theory of a dg-category and how to present this assignement as an ∞-functor $K^c : \mathcal{D}g(k)^{idem} \to \widehat{Sp}_{\geq 0}$ commuting with filtered colimits. One possible way is to use the classical theory of Waldhausen categories. As discussed in the Remark 6.4.24, the data of an object $F \in Fun_\omega(\mathcal{D}g(k)^{idem}, \widehat{Sp})$ corresponds in a essentially unique way to the data of an actual strict functor F_s from the category of small dg-categories $Cat_{Ch(k)}$ with the Morita model structure to some combinatorial model category whose underlying $(\infty,1)$-category is $\widehat{Sp}$ such that 1) F_s sends Morita equivalences to weak-equivalences and 2) F_s preserves filtered homotopy colimits. In the case of connective K-theory such a functor can be obtained by composing the strict functor $K^c_{Wald} : Wald_{Classic} \to Sp^\Sigma$ of the previous section with the functor $Cat_{Ch(k)} \to Wald^{Model}_{Classic}$ defined by sending a small dg-category T to the strict category of perfect cofibrant dg-modules (obtained by forgetting the dg-enrichement), with its natural structure of Waldhausen category given by the weak-equivalences of T-dg-modules and the cofibrations of the module structure therein. This is well-defined because perfect modules are stable under homotopy pushouts and satisfy the "cube lemma" [69, 5.2.6]. The conditions 1) and 2) are also well-known to be satisfied (for instance see [17, Section 2.2]). For the most part of this work it will be enough to work with the ∞-functor $K^c : \mathcal{D}g(k)^{ft} \to \widehat{Sp}_{\geq 0}$ associated to this composition via the Remark 6.4.24 or its canonical ω-continuous extension $\mathcal{D}g(k)^{idem} \to \widehat{Sp}$. However, some of our purposes (namely the Theorem 7.0.32) will require an alternative approach. More precisely, and in the same spirit of [18, Section 7.1] for stable ∞-categories, we will need to have a description of the Waldhausen's S-construction within the setting of dg-categories.

Construction 7.1.2. Let $Ar[n]_k$ be the dg-category obtained as the k-linearization of the category $Ar[n]$ described in the Construction 7.1.1. More precisely, its objects are the objects in $Ar[n]$ and its complexes of morphisms are all given by k seen as a complex concentrated in degree zero. For each n the dg-category $Ar[n]_k$ is locally cofibrant (meaning, enriched over cofibrant complexes - see 6.1.1) so that for any locally cofibrant dg-category T we have $Ar[n]_k \otimes^{\mathbb{L}} T \simeq Ar[n]_k \otimes T$ (recall our discussion in 6.1.1 about the derived monoidal structure in the $(\infty,1)$-category $\mathcal{D}g(k)$).

Recall also from [139] that the symmetric monoidal $(\infty,1)$-category $\mathcal{D}g(k)^\otimes$ admits an internal-hom $\mathbb{R}\underline{Hom}(A,B)$ given by the full sub-dg-category of right-quasi-representable cofibrant $A \otimes^{\mathbb{L}} B^{op}$-dg-modules. If T is a locally cofibrant dg-category and $\widehat{T}_c$ is its idempotent-completion (which, as explained in the Remark 6.1.18, we can always assume to be locally cofibrant), we find a canonical equivalence in $\mathcal{D}g(k)$ between $\mathbb{R}\underline{Hom}(A, \widehat{T}_c)$ and $\widehat{A^{op} \otimes^{\mathbb{L}} T}_{pspe}$ - the full sub-dg-category of cofibrant pseudo-perfect $A \otimes^{\mathbb{L}} T^{op}$-dg-modules (by definition, these are cofibrant dg-modules E such that for any object $a \in A$, the T^{op}- module $E(A,-)$ is perfect). With this, we have $\mathbb{R}\underline{Hom}(Ar[n]_k, \widehat{T}_c) \simeq \widehat{Ar[n]^{op}_k \otimes^{\mathbb{L}} T}_{pspe} \simeq \widehat{Ar[n]^{op}_k \otimes T}_{pspe}$ so that the objects in this internal-hom can be identified with $Ar[n]$-indexed diagrams in the underlying strict category of perfect cofibrant T^{op}-modules (obtained by forgetting the dg-enrichement). We now set $S^{dg}_n(T)$ to be the full sub-dg-category of $\mathbb{R}\underline{Hom}(Ar[n]_k, \widehat{T}_c)$ spanned by those diagrams satisfying the conditions in the construction 7.1.1. These conditions make sense for the same reasons the functor $Cat_{Ch(k)} \to Wald^{Model}_{Classic}$ of the previous section also makes sense (see [17, Section 2.2]). Again, the collection of dg-categories $S^{dg}_n(T)$ for $n \geq 0$ forms a simplicial object in dg-categories and by considering each level as a category (omitting its dg-enrichement) we can recover the K-theory of T as $\Omega\, colim_{\Delta^{op}} N(S^{dg}_n(T)^{W_n})$ where W_n is the class of maps in $S^{dg}_n(T)$ given by the levelwise weak-equivalences of dg-modules and $S^{dg}_n(T)^{W_n}$ is the full subcategory of $S^{dg}_n(T)$ spanned by all the objects and only those morphisms which are in W_n.

Let now $[n]_k$ be the dg-category obtained as the k-linearization of the ordered category $[n] = \{0 \leq 1 \leq .... \leq n\}$. This dg-category is again locally cofibrant and for the same reasons as above the underlying category obtained from $\mathbb{R}\underline{Hom}([n]_k, \widehat{T}_c)$ by forgetting the dg-enrichement is the category of sequences of perfect cofibrant T^{op}-dg-modules of length $n+1$. As cofibers of maps are essentially uniquely determined up to isomorphism, we have a canonical equivalence of categories between $S_n^{dg}(T)$ and $\mathbb{R}\underline{Hom}([n-1]_k, \widehat{T}_c)$. Since the model structure on T^{op}-dg-modules satisfies the "cube lemma" [69, 5.2.6] (because $Ch(k)$ satisfies it for the projective model structure) this equivalence becomes an equivalence of pairs $(S_n^{dg}(T), W_n)$ and $(\mathbb{R}\underline{Hom}([n-1]_k, \widehat{T}_c), W_n')$, where we consider both dg-categories as categories by forgetting the dg-enrichements and where W_n' denotes the class of maps of sequences which are levelwise given by weak-equivalences of dg-modules. Thanks to this equivalence we find a homotopy equivalence of simplicial sets between $N(S_n^{dg}(T)^{W_n})$ and $N(\mathbb{R}\underline{Hom}([n-1]_k, \widehat{T}_c)^{W_n'})$. This is a dg-version of [100, 1.2.2.4]. Finally, and thanks to the main theorem of [139] the latter is exactly the mapping space $Map_{\mathcal{D}g(k)}([n-1]_k, \widehat{T}_c)$ which by adjunction is equivalent to $Map_{\mathcal{D}g(k)^{idem}}(\widehat{([n-1]_k)}_c, \widehat{T}_c)$. Under this chain of equivalences this family of mapping spaces for $n \geq 0$ inherits the structure of a simplicial object in the $(\infty,1)$-category of spaces and the K-theory space of T can finally be rewritten as

$$\Omega\, colim_{[n]\in\Delta^{op}} Map_{\mathcal{D}g(k)^{idem}}(\widehat{([n-1]_k)}_c, \widehat{T}_c) \tag{7.1.6}$$

This concludes the construction.

To conclude this section we remark two important properties of K^c. The first should be well known to the reader:

Proposition 7.1.3. *([153]) The ∞-functor $K^c : \mathcal{D}g(k)^{idem} \to \widehat{Sp}$ sends exact sequences of dg-categories to fiber sequences in $\widehat{Sp}_{\geq 0}$.*

Proof. This follows from the so-called Waldhausen's Fibration Theorem [153, 1.6.4] and [137, 1.8.2], together with the dictionary between homotopy limits and homotopy colimits in the model category of spectra and limits and colimits in the $(\infty,1)$-category $\widehat{Sp}$ (see [100, 1.3.4.23 and 1.3.4.24]). □

The second is a consequence of this first and will be very important to us:

Proposition 7.1.4. *K^c sends Nisnevich squares of noncommutative smooth spaces to pullback squares of connective spectra.*

Proof. Let

$$\begin{tikzcd} T_{\mathcal{X}} \arrow[r] \arrow[d] & T_{\mathcal{U}} \arrow[d] \\ T_{\mathcal{V}} \arrow[r] & T_{\mathcal{W}} \end{tikzcd} \tag{7.1.7}$$

be a Nisnevich square of dg-categories. By definition, there are dg-categories $K_{\mathcal{X}-\mathcal{U}}$ and $K_{\mathcal{V}-\mathcal{W}}$ in $\mathcal{D}g(k)^{idem}$, having compact generators, and such that the maps $T_{\mathcal{X}} \to T_{\mathcal{U}}$ and $T_{\mathcal{V}} \to T_{\mathcal{W}}$ fit into strict short exact sequences in $\mathcal{D}g(k)^{idem}$ (see 6.4.6 and the Remark 6.4.4)

$$\begin{tikzcd} K_{\mathcal{X}-\mathcal{U}} \arrow[r] \arrow[d] & T_{\mathcal{X}} \arrow[d] \\ 0 \arrow[r] & T_{\mathcal{U}} \end{tikzcd} \qquad \begin{tikzcd} K_{\mathcal{V}-\mathcal{W}} \arrow[r] \arrow[d] & T_{\mathcal{V}} \arrow[d] \\ 0 \arrow[r] & T_{\mathcal{W}} \end{tikzcd} \tag{7.1.8}$$

Again by the definition of an open immersion and because of 7.1.3 we have pullback squares of connective spectra

$$\begin{tikzcd} K^c(K_{\mathcal{X}-\mathcal{U}}) \arrow[r] \arrow[d] & K^c(T_{\mathcal{X}}) \arrow[d] \\ 0 \arrow[r] & K^c(T_{\mathcal{U}}) \end{tikzcd} \qquad \begin{tikzcd} K^c(K_{\mathcal{V}-\mathcal{W}}) \arrow[r] \arrow[d] & K^c(T_{\mathcal{V}}) \arrow[d] \\ 0 \arrow[r] & K^c(T_{\mathcal{W}}) \end{tikzcd} \tag{7.1.9}$$

With these properties in mind, we aim to show that the diagram

$$\begin{tikzcd} K^c(T_{\mathcal{X}}) \arrow[r] \arrow[d] & K^c(T_{\mathcal{U}}) \arrow[d] \\ K^c(T_{\mathcal{V}}) \arrow[r] & K^c(T_{\mathcal{W}}) \end{tikzcd} \tag{7.1.10}$$

is a pullback of connective spectra. For that purpose we consider the pullback squares

$$\begin{tikzcd} K^c(K_{\mathcal{X}-\mathcal{U}}) \arrow[r] \arrow[d] & K^c(K_{\mathcal{V}-\mathcal{W}}) \arrow[r] \arrow[d] & 0 \arrow[d] \\ K^c(T_{\mathcal{X}}) \arrow[r, dashed] & K^c(T_{\mathcal{V}}) \times_{K^c(T_{\mathcal{W}})} K^c(T_{\mathcal{U}}) \arrow[r] \arrow[d] & K^c(T_{\mathcal{U}}) \arrow[d] \\ & K^c(T_{\mathcal{V}}) \arrow[r] & K^c(T_{\mathcal{W}}) \end{tikzcd} \tag{7.1.11}$$

from which we extract a morphism of fiber sequences

$$\begin{tikzcd} K^c(K_{\mathcal{X}-\mathcal{U}}) \arrow[r] \arrow[d] & K^c(K_{\mathcal{V}-\mathcal{W}}) \arrow[d] \\ K^c(T_{\mathcal{X}}) \arrow[r, dashed] \arrow[d] & K^c(T_{\mathcal{V}}) \times_{K^c(T_{\mathcal{W}})} K^c(T_{\mathcal{U}}) \arrow[d] \\ K^c(T_{\mathcal{U}}) \arrow[r, equal] & K^c(T_{\mathcal{U}}) \end{tikzcd} \tag{7.1.12}$$

To conclude, since the square (7.1.7) is Nisnevich, by definition, the canonical morphism $K_{\mathcal{X}-\mathcal{U}} \to K_{\mathcal{V}-\mathcal{W}}$ is an equivalence in $\mathcal{D}g(k)^{idem}$ so that the top map is an equivalence $K^c(K_{\mathcal{X}-\mathcal{U}}) \simeq K^c(K_{\mathcal{V}-\mathcal{W}})$. Using the associated long exact sequences we conclude that the canonical morphism

$$K^c(T_{\mathcal{X}}) \dashrightarrow K^c(T_{\mathcal{V}}) \times_{K^c(T_{\mathcal{W}})} K^c(T_{\mathcal{U}}) \tag{7.1.13}$$

is also an equivalence, thus concluding the proof.

□

7.1.3 Non-connective K-Theory

The first attempts to define negative K-theory groups date back to the works of Bass in [10] and Karoubi in [80]. The motivation to look for these groups is very simple: the higher K-theory groups of an exact sequence of Waldhausen categories do not fit in a long exact sequence. The full solution to this problem appeared in the legendary paper of Thomason-Trobaugh [137] where the author provides a mechanism to extend the connective spectrum K^c of Waldhausen to a new non-connective spectrum K^B whose connective part recovers the classical data. His attention focuses on the K-theory of schemes and recovers the negative groups of Bass (by passing to the homotopy groups). Moreover, it satisfies the property people were waiting for [137, Thm 7.4]: for any reasonable scheme X with an open subscheme $U \subseteq X$ with complement Z, there is a pullback-pushout sequence of spectra $K(X \text{ on } Z) \to K(X) \to K(U)$ where $K(X \text{ on } Z)$ is the K-theory spectrum associated to the category of perfect complexes on X supported on Z. Moreover, he proves that his non-connected version of K-theory satisfies descent with respect to the classical Nisnevich topology for schemes (see [137, Thm 10.8]).

More recently, Schlichting [122] introduced a mechanism that allows us to define non-connective versions of K-theory in a wide range of situations and in [35, Section 6 and 7] the authors applied this algorithm to the context of dg-categories. The result is a procedure that sends Morita equivalences of dg-categories to weak-equivalences of spectra and commutes with filtered homotopy colimits (for instance, see [17, 2.12]) and comes canonically equipped with a natural transformation from connective K-theory inducing an equivalence in the connective part. By applying the arguments of the Remark 6.4.24 their construction can be encoded in a unique way in the form of an ω-continuous ∞-functor $K^S : \mathcal{D}g(k)^{idem} \to \widehat{Sp}$ together with a natural transformation $K^c \to K^S$ with $\tau_{\geq 0}K^c \simeq \tau_{\geq 0}K^S$. The *motto* of non-connective K-theory can now be stated as

Proposition 7.1.5. *K^S sends exact sequences in $\mathcal{D}g(k)^{idem}$ to cofiber/fiber sequences in $\widehat{Sp}$.*

Proof. This follows from [122, 12.1 Thm 9] and from the adaptation of the Schlichting's setup to dg-categories in [35, Section 6], together with the fact that our notion of exact sequences in $\mathcal{D}g(k)^{idem}$ agrees with the notion of exact sequences in [35] (see 6.4.2-(3)). To conclude use again the dictionary between homotopy limits and homotopy colimits in a model category and limits and colimits on the underlying $(\infty, 1)$-category. $\square$

Using the same arguments as in Prop. 7.1.4, we find

Corollary 7.1.6. *K^S is Nisnevich local.*

The method of Thomason (the so called B-construction) and the methods of Schlichting to create non-connective extensions of K-thery are somehow *ad hoc*. In this thesis we will show how these two constructions can both be understood as explicit models for the same process, namely, the Nisnevich "sheafification" [3] in the noncommutative world.

7.2 Non-connective K-theory is the Nisnevich localization of connective K-theory

In this section we give the proof of Theorem 7.0.29. As explained in the introduction, it goes in two steps. First, in 7.2.0.1, we prove that every Nisnevich local functor $F : \mathcal{D}g(k)^{ft} \to \widehat{Sp}$ satisfies the familiar Bass exact sequences for any integer n. The second step requires a more careful discussion. In 7.2.0.2 we introduce the notion of *connective-Nisnevich descent* for functors $F : \mathcal{D}g(k)^{ft} \to \widehat{Sp}$ with values in $\widehat{Sp}_{\geq 0}$. We will see (Prop.7.2.6 below) that the full subcategory $Fun_{Nis_{\geq 0}}(\mathcal{D}g(k)^{ft}, \widehat{Sp}_{\geq 0})$ spanned by those functors satisfying this descent conditon, is an accessible reflexive localization of $Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp}_{\geq 0})$

$$\begin{tikzcd} Fun_{Nis_{\geq 0}}(\mathcal{D}g(k)^{ft}, \widehat{Sp}_{\geq 0}) \arrow[r, hook, "\alpha"] & Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp}_{\geq 0}) \arrow[l, bend right, "l_{nis_{\geq 0}}"'] \end{tikzcd} \tag{7.2.1}$$

and that as a consequence of the definition the connective truncation of a Nisnevich local is connectively-Nisnevich local, and we have a natural factorization $\overline{\tau_{\geq 0}}$

$$\begin{tikzcd} Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp}_{\geq 0}) & Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp}) \arrow[l, "\tau_{\geq 0}"] \\ Fun_{Nis_{\geq 0}}(\mathcal{D}g(k)^{ft}, \widehat{Sp}_{\geq 0}) \arrow[u, hook, "\alpha"] & Fun_{Nis}(\mathcal{D}g(k)^{ft}, \widehat{Sp}) \arrow[u, hook, "\beta"'] \arrow[l, dashed, "\overline{\tau_{\geq 0}}"'] \end{tikzcd} \tag{7.2.2}$$

where α and β denote the inclusions. By abstract nonsense, the composition $i_! := l^{nc}_{nis} \circ i \circ \alpha$ provides a left adjoint to $\overline{\tau_{\geq 0}}$ and, because the diagram of right adjoints commutes, the diagram of left adjoints

[3] The noncommutative Nisnevich topology is not a Grothendieck topology.

$$\begin{tikzcd} Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp}_{\geq 0}) \arrow[r, hook, "i"] \arrow[d, "l_{nis_{\geq 0}}"] & Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp}) \arrow[d, "l^{nc}_{nis}"] \\ Fun_{Nis_{\geq 0}}(\mathcal{D}g(k)^{ft}, \widehat{Sp}_{\geq 0}) \arrow[r, dashed, "i_!"] & Fun_{Nis}(\mathcal{D}g(k)^{ft}, \widehat{Sp}) \end{tikzcd} \tag{7.2.3}$$

also commutes.

The second step in our strategy amounts to checking that the adjunction $(i_!, \overline{\tau_{\geq 0}})$ is an equivalence of $(\infty, 1)$-categories. At this point our task is greatly simplified by the first step: the fact that Nisnevich local objects satisfy the Bass exact sequences for any integer n implies that $\overline{\tau_{\geq 0}}$ is conservative. Therefore, we are reduced to prove that the counit of the adjunction $\overline{\tau_{\geq 0}} \circ i_! \to Id$ is an equivalence of functors. In other words, if F is already connectively-Nisnevich local, its Nisnevich localization preserves the connective part. In order to achieve this we will need a more explicit description of the noncommutative Nisnevich localization of a connectively-Nisnevich local F. Our main result is that the more familiar $(-)^B$ construction of Thomason-Trobaugh (which we reformulate in our setting) provides such an explicit model, namely, we prove that if F is connectively-Nisnevich local, $\tau_{\geq 0}(F^B)$ is naturally equivalent to F and F^B is Nisnevich local and naturally equivalent to $l^{nc}_{nis}(F)$.

7.2.0.1 Nisnevich descent forces all the Bass Exact Sequences

In this section we prove that every Nisnevich local $F : \mathcal{D}g(k)^{ft} \to \widehat{Sp}$ satisfies the familiar Bass exact sequences for any integer n. Our proof follows the arguments of [137, 6.1]. The first step is to show that every Nisnevich local F satisfies the Projective Bundle theorem. As explained in the introduction, this follows from the existence of an exceptional collection in $L_{pe}(\mathbb{P}^1)$ generated by the twisting sheaves $\mathcal{O}_{\mathbb{P}^1}$ and $\mathcal{O}_{\mathbb{P}^1}(-1)$, which, following 6.4.12, provides a split short exact sequence of dg-categories

$$\begin{tikzcd} L_{pe}(k) \arrow[r, shift right, "i_{\mathcal{O}_{\mathbb{P}^1}}"'] \arrow[d] & L_{pe}(\mathbb{P}^1) \arrow[l, shift right] \arrow[d, shift right] \\ 0 \arrow[r] & L_{pe}(k) \arrow[u, shift right, "i_{\mathcal{O}_{\mathbb{P}^1}(-1)}"'] \end{tikzcd} \tag{7.2.4}$$

where the map $i_{\mathcal{O}_{\mathbb{P}^1}}$, resp. $i_{\mathcal{O}_{\mathbb{P}^1}(-1)}$, is the inclusion of the full triangulated subcategory generated by $\mathcal{O}_{\mathbb{P}^1}$, respectively $\mathcal{O}_{\mathbb{P}^1}(-1)$. In particular, since $\mathcal{D}g(k)^{ft}$ has direct sums, we extract canonical maps of dg-categories

$$L_{pe}(k) \oplus L_{pe}(k) \xrightarrow{\psi} L_{pe}(\mathbb{P}^1) \qquad L_{pe}(\mathbb{P}^1) \xrightarrow{\phi} L_{pe}(k) \oplus L_{pe}(k) \tag{7.2.5}$$

We observe now that these maps become mutually inverse once we consider them in $Fun_{Nis}(\mathcal{D}g(k)^{ft}, \widehat{Sp})$ via Yoneda's embedding. Indeed, the split exact sequence in (7.2.4), or more precisely, its opposite in $\mathcal{N}cS(k)$, induces a split exact sequence in $Fun_{Nis}(\mathcal{D}g(k)^{ft}, \widehat{Sp})$

$$\begin{tikzcd} l^{nc}_{Nis} \circ \Sigma^{\infty}_{+} \circ j_{nc}(L_{pe}(k)) \arrow[r, shift right] \arrow[d] & l^{nc}_{Nis} \circ \Sigma^{\infty}_{+} \circ j_{nc}(L_{pe}(\mathbb{P}^1)) \arrow[l, shift right] \arrow[d, shift right] \\ 0 \arrow[r] & l^{nc}_{Nis} \circ \Sigma^{\infty}_{+} \circ j_{nc}(L_{pe}(k)) \arrow[u, shift right] \end{tikzcd} \tag{7.2.6}$$

This is because $Fun_{Nis}(\mathcal{D}g(k)^{ft}, \widehat{Sp})$ is stable, together with the effects of the Nisnevich localization. Also because of stability we know that l^{nc}_{Nis} preserves direct sums. In particular, we have canonical maps

$$l^{nc}_{Nis} \circ \Sigma^{\infty}_{+} \circ j(L_{pe}(\mathbb{P}^1)) \to l^{nc}_{Nis} \circ \Sigma^{\infty}_{+} \circ j_{nc}(L_{pe}(k)) \oplus l^{nc}_{Nis} \circ \Sigma^{\infty}_{+} \circ j_{nc}(L_{pe}(k)) \tag{7.2.7}$$

$$l^{nc}_{Nis} \circ \Sigma^\infty_+ \circ j_{nc}(L_{pe}(k)) \oplus l^{nc}_{Nis} \circ \Sigma^\infty_+ \circ j_{nc}(L_{pe}(k)) \to l^{nc}_{Nis} \circ \Sigma^\infty_+ \circ j(L_{pe}(\mathbb{P}^1)) \tag{7.2.8}$$

which can be identified with the image under $\Sigma^\infty_+ \circ j_{nc}$ of the opposites of the canonical maps of dg-categories in (7.2.5), respectively. This is because in $\mathcal{N}cS(k)$ finite sums are the same as finite products (see the end of our discussion in 6.1.2), because Yoneda's embedding commutes with finite products and because the pointing map $\mathcal{S} \to \mathcal{S}_*$ and the suspension Σ^∞ commute with all colimits.

This time, and as explained in 6.4.21 and 6.4.22, because $Fun_{Nis}(\mathcal{D}g(k)^{ft}, \widehat{Sp})$ is stable, these canonical maps are inverses to each other. In other words, we have a direct sum decomposition

$$l^{nc}_{Nis}\circ\Sigma^\infty_+\circ j(L_{pe}(\mathbb{P}^1)) \simeq l^{nc}_{Nis}\circ\Sigma^\infty_+\circ j_{nc}(L_{pe}(k))\oplus l^{nc}_{Nis}\circ\Sigma^\infty_+\circ j_{nc}(L_{pe}(k)) \simeq l^{nc}_{Nis}\circ\Sigma^\infty_+\circ j_{nc}(L_{pe}(k)\oplus L_{pe}(k)) \tag{7.2.9}$$

where the first (resp. second) component can be identified with the part of $L_{pe}(\mathbb{P}^1)$ generated by $\mathcal{O}_{\mathbb{P}^1}$ (resp. $\mathcal{O}_{\mathbb{P}^1}(-1)$).

In particular, if we denote by $\underline{\mathrm{Hom}}$ the internal-hom in $Fun_{Nis}(\mathcal{D}g(k)^{ft}, \widehat{Sp})$ we find

Corollary 7.2.1. *Let F be a Nisnevich local functor $\mathcal{D}g(k)^{ft} \to \widehat{Sp}$. Then F satisfies the projective bundle theorem. In other words, we have*

$$\underline{Hom}(l^{nc}_{Nis} \circ \Sigma^\infty_+ \circ j(L_{pe}(\mathbb{P}^1)), F) \simeq \underline{Hom}(l^{nc}_{Nis} \circ \Sigma^\infty_+ \circ j(L_{pe}(k)), F) \oplus \underline{Hom}(l^{nc}_{Nis} \circ \Sigma^\infty_+ \circ j(L_{pe}(k)), F)$$

$$\simeq F \oplus F$$

As in [137, 6.1] we can now re-adapt this direct sum decomposition to a new one, suitably choosen to extract the Bass exact sequences out of the classical Zariski (therefore Nisnevich) covering of $\mathbb{P}^1$ given by

$$\begin{tikzcd} \mathbb{G}_m \arrow[r, hook, "i"] \arrow[d, hook, "j"] & \mathbb{A}^1 \arrow[d, hook, "\alpha"] \\ \mathbb{A}^1 \arrow[r, hook, "\beta"] & \mathbb{P}^1 \end{tikzcd} \tag{7.2.10}$$

The basic ingredient is the induced pullback diagram of dg-categories

$$\begin{tikzcd} L_{pe}(\mathbb{P}^1) \arrow[r, "\alpha^*"] \arrow[d, "\beta^*"] & L_{pe}(\mathbb{A}^1) \arrow[d, "j^*"] \\ L_{pe}(\mathbb{A}^1) \arrow[r, "i^*"] & L_{pe}(\mathbb{G}_m) \end{tikzcd} \tag{7.2.11}$$

together with the composition

$$\begin{tikzcd} L_{pe}(k) \oplus L_{pe}(k) \arrow[dr, "\psi"] \arrow[drr, bend left, "\alpha^*\circ\psi"] \arrow[ddr, bend right, "\beta^*\circ\psi"'] & & \\ & L_{pe}(\mathbb{P}^1) \arrow[r, "\alpha^*"] \arrow[d, "\beta^*"] & L_{pe}(\mathbb{A}^1) \arrow[d, "j^*"] \\ & L_{pe}(\mathbb{A}^1) \arrow[r, "i^*"] & L_{pe}(\mathbb{G}_m) \end{tikzcd} \tag{7.2.12}$$

More precisely, we will focus on the diagram in $Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp})$ induced by the opposite of the above diagram, namely,

$$\begin{tikzcd}
\Sigma^{\infty}_{+} \circ j_{nc}(L_{pe}(\mathbb{G}_m)) \arrow[r, "L_{pe}(i)"] \arrow[d, "L_{pe}(j)"] & \Sigma^{\infty}_{+} \circ j_{nc}(L_{pe}(\mathbb{A}^1)) \arrow[d, "L_{pe}(\alpha)"] \arrow[ddr, bend left] & \\
\Sigma^{\infty}_{+} \circ j_{nc}(L_{pe}(\mathbb{A}^1)) \arrow[r, "L_{pe}(\beta)"] \arrow[drr, bend right] & \Sigma^{\infty}_{+} \circ j_{nc}(L_{pe}(\mathbb{P}^1)) \arrow[dr, "\Sigma^{\infty}_{+} \circ j_{nc}(\psi^{op})"] & \\
& & \Sigma^{\infty}_{+} \circ j_{nc}(L_{pe}(k) \oplus L_{pe}(k))
\end{tikzcd} \tag{7.2.13}$$

Remark 7.2.2. It follows from 6.4.16, from the effects of the Nisnevich localization and from the above discussion that the exterior commutative square in (7.2.13) becomes a pushout-pullback square in $Fun_{Nis}(\mathcal{D}g(k)^{ft}, \widehat{Sp})$.

In order to extract the Bass exact sequences, we consider a different direct sum decomposition of $l^{nc}_{Nis} \circ \Sigma^{\infty}_{+} \circ j_{nc}(L_{pe}(\mathbb{P}^1))$. For that purpose, let us start by introducing a bit of notation. We let i_1, i_2 denote the canonical inclusions $L_{pe}(k) \to L_{pe}(k) \oplus L_{pe}(k)$ in $\mathcal{D}g(k)^{ft}$, and let π_1, π_2 denote the projections $L_{pe}(k) \oplus L_{pe}(k) \to L_{pe}(k)$. At the same time, let i_1^{op} and i_2^{op} denote the associated projections in $\mathcal{N}cS(k)$ and π_1^{op} and π_2^{op} the canonical inclusions. Since Yoneda's map $\Sigma^{\infty}_{+} \circ j_{nc}$ commutes with direct sums, the maps $\Sigma^{\infty}_{+} \circ j_{nc}(i_1^{op})$ and $\Sigma^{\infty}_{+} \circ j_{nc}(i_1^{op})$ can be identified with the canonical projections

$$\Sigma^{\infty}_{+} \circ j_{nc}(L_{pe}(k)) \oplus \Sigma^{\infty}_{+} \circ j_{nc}(L_{pe}(k)) \to \Sigma^{\infty}_{+} \circ j_{nc}(L_{pe}(k)) \tag{7.2.14}$$

and $\Sigma^{\infty}_{+} \circ j_{nc}(\pi_1^{op})$ and $\Sigma^{\infty}_{+} \circ j_{nc}(\pi_2^{op})$ with the canonical inclusions

$$\Sigma^{\infty}_{+} \circ j_{nc}(L_{pe}(k)) \to \Sigma^{\infty}_{+} \circ j_{nc}(L_{pe}(k)) \oplus \Sigma^{\infty}_{+} \circ j_{nc}(L_{pe}(k)) \tag{7.2.15}$$

in $Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp})$.

Let us proceed. To achieve the new decomposition, we compose the decomposition we had before with an equivalence Θ in $Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp})$

$$\Sigma^{\infty}_{+} \circ j_{nc}(L_{pe}(k)) \oplus \Sigma^{\infty}_{+} \circ j_{nc}(L_{pe}(k)) \xrightarrow{\Theta} \Sigma^{\infty}_{+} \circ j_{nc}(L_{pe}(k)) \oplus \Sigma^{\infty}_{+} \circ j_{nc}(L_{pe}(k)) \tag{7.2.16}$$

defined to be the map

$$\begin{tikzcd}
& \Sigma^{\infty}_{+} \circ j_{nc}(L_{pe}(k)) \\
\Sigma^{\infty}_{+} \circ j_{nc}(L_{pe}(k)) \oplus \Sigma^{\infty}_{+} \circ j_{nc}(L_{pe}(k)) \arrow[ur, "\delta_1"] \arrow[dr, "\delta_2"'] \arrow[r, dashed, "\Theta"] & \Sigma^{\infty}_{+} \circ j_{nc}(L_{pe}(k)) \oplus \Sigma^{\infty}_{+} \circ j_{nc}(L_{pe}(k)) \arrow[u, "\Sigma^{\infty}_{+} \circ j_{nc}(i_1^{op})"'] \arrow[d, "\Sigma^{\infty}_{+} \circ j_{nc}(i_2^{op})"] \\
& \Sigma^{\infty}_{+} \circ j_{nc}(L_{pe}(k))
\end{tikzcd} \tag{7.2.17}$$

obtained from the universal property of the direct sum, where:

- δ_1 it is the canonical dotted arrow in the diagram

$$\begin{tikzcd}
\Sigma^\infty_+ \circ j_{nc}(L_{pe}(k)) \arrow[d, "\Sigma^\infty_+\circ j_{nc}(\pi_1^{op})"'] \arrow[dr, "id"] & \\
\Sigma^\infty_+ \circ j_{nc}(L_{pe}(k)) \oplus \Sigma^\infty_+ \circ j_{nc}(L_{pe}(k)) \arrow[r, dashed] & \Sigma^\infty_+ \circ j_{nc}(L_{pe}(k)) \\
\Sigma^\infty_+ \circ j_{nc}(L_{pe}(k)) \arrow[u, "\Sigma^\infty_+\circ j_{nc}(\pi_2^{op})"] \arrow[ur, "0"'] &
\end{tikzcd} \tag{7.2.18}$$

- δ_2 is the canonical map obtained from

$$\begin{tikzcd}
\Sigma^\infty_+ \circ j_{nc}(L_{pe}(k)) \arrow[d, "\Sigma^\infty_+\circ j_{nc}(\pi_1^{op})"'] \arrow[dr, "id"] & \\
\Sigma^\infty_+ \circ j_{nc}(L_{pe}(k)) \oplus \Sigma^\infty_+ \circ j_{nc}(L_{pe}(k)) \arrow[r, dashed] & \Sigma^\infty_+ \circ j_{nc}(L_{pe}(k)) \\
\Sigma^\infty_+ \circ j_{nc}(L_{pe}(k)) \arrow[u, "\Sigma^\infty_+\circ j_{nc}(\pi_2^{op})"] \arrow[ur, "-id"'] &
\end{tikzcd} \tag{7.2.19}$$

Of course, it follows from this definition that Θ is an equivalence with inverse equal to itself. Finally, we consider the composition

$$\begin{tikzcd}
\Sigma^\infty_+ \circ j_{nc}(L_{pe}(\mathbb{G}_m)) \arrow[r, "L_{pe}(i)"] \arrow[d, "L_{pe}(j)"] & \Sigma^\infty_+ \circ j_{nc}(L_{pe}(\mathbb{A}^1)) \arrow[d, "L_{pe}(\alpha)"] \arrow[ddr, bend left] & \\
\Sigma^\infty_+ \circ j_{nc}(L_{pe}(\mathbb{A}^1)) \arrow[r, "L_{pe}(\beta)"] \arrow[drr, bend right] & \Sigma^\infty_+ \circ j_{nc}(L_{pe}(\mathbb{P}^1)) \arrow[dr, "\Theta\circ\Sigma^\infty_+\circ j_{nc}(\psi^{op})"] & \\
& & \Sigma^\infty_+ \circ j_{nc}(L_{pe}(k)) \oplus \Sigma^\infty_+ \circ j_{nc}(L_{pe}(k))
\end{tikzcd} \tag{7.2.20}$$

which again, as in the Remark 7.2.2, provides a pushout-pullback square in $Fun_{Nis}(\mathcal{D}g(k)^{ft}, \widehat{Sp})$. The important point of this new decomposition is the fact that both maps $\Theta \circ (\Sigma^\infty_+ \circ j_{nc}(\psi^{op} \circ L_{pe}(\alpha)))$ and $\Theta \circ (\Sigma^\infty_+ \circ j_{nc}(\psi^{op} \circ L_{pe}(\beta)))$ become simpler. In fact, since $\alpha^*(\mathcal{O}_{\mathbb{P}^1}) = \alpha^*(\mathcal{O}_{\mathbb{P}^1}(-1)) = \beta^*(\mathcal{O}_{\mathbb{P}^1}) = \alpha^*(\mathcal{O}_{\mathbb{P}^1}(-1)) = \mathcal{O}_{\mathbb{A}^1}$, we find that

- The composition $\Sigma^\infty_+ \circ j_{nc}(i_1^{op}) \circ \Theta \circ (\Sigma^\infty_+ \circ j_{nc}(\psi^{op} \circ L_{pe}(\alpha)))$ can be identified with the map $\Sigma^\infty_+ \circ j_{nc}(L_{pe}(p))$ induced by pullback along the canonical projection $p : \mathbb{A}^1 \to Spec(k)$. Indeed, we have

$$\Sigma^\infty_+ \circ j_{nc}(i_1^{op}) \circ \Theta \circ (\Sigma^\infty_+ \circ j_{nc}(\psi^{op} \circ L_{pe}(\alpha))) \simeq \tag{7.2.21}$$
$$\simeq \delta_1 \circ (\Sigma^\infty_+ \circ j_{nc}(\psi^{op} \circ L_{pe}(\alpha))) \simeq \tag{7.2.22}$$
$$\simeq \delta_1 \circ (\Sigma^\infty_+ \circ j_{nc}(\pi_1^{op} \circ i_1^{op} + \pi_2^{op} \circ i_2^{op})) \circ (\Sigma^\infty_+ \circ j_{nc}(\psi^{op} \circ L_{pe}(\alpha))) \simeq \tag{7.2.23}$$
$$\simeq \Sigma^\infty_+ \circ j_{nc}(i_1^{op} \circ \psi^{op} \circ L_{pe}(\alpha)) + 0 \simeq \tag{7.2.24}$$
$$\simeq \Sigma^\infty_+ \circ j_{nc}((\alpha^* \circ \psi \circ i_1)^{op}) \simeq \tag{7.2.25}$$
$$\simeq \Sigma^\infty_+ \circ j_{nc}((p^*)^{op}) \simeq \tag{7.2.26}$$
$$\simeq \Sigma^\infty_+ \circ j_{nc}(L_{pe}(p)) \tag{7.2.27}$$

The same holds for the composition $\Sigma^\infty_+ \circ j_{nc}(i_1^{op}) \circ \Theta \circ (\Sigma^\infty_+ \circ j_{nc}(\psi^{op} \circ L_{pe}(\beta)))$;

- The maps $\Sigma^{\infty}_{+}\circ j_{nc}(i_2^{op})\circ\Theta\circ(\Sigma^{\infty}_{+}\circ j_{nc}(\psi^{op}\circ L_{pe}(\alpha)))$ and $\Sigma^{\infty}_{+}\circ j_{nc}(i_2^{op})\circ\Theta\circ(\Sigma^{\infty}_{+}\circ j_{nc}(\psi^{op}\circ L_{pe}(\beta)))$ are zero. Indeed, we have

$$\Sigma^{\infty}_{+}\circ j_{nc}(i_2^{op})\circ\Theta\circ(\Sigma^{\infty}_{+}\circ j_{nc}(\psi^{op}\circ L_{pe}(\alpha)))\simeq \tag{7.2.28}$$
$$\simeq \delta_2\circ(\Sigma^{\infty}_{+}\circ j_{nc}(\psi^{op}\circ L_{pe}(\alpha)))\simeq \tag{7.2.29}$$
$$\simeq \delta_2\circ(\Sigma^{\infty}_{+}\circ j_{nc}(\pi_1^{op}\circ i_1^{op}+\pi_2^{op}\circ i_2^{op}))\circ(\Sigma^{\infty}_{+}\circ j_{nc}(\psi^{op}\circ L_{pe}(\alpha)))\simeq \tag{7.2.30}$$
$$\simeq Id\circ(\Sigma^{\infty}_{+}\circ j_{nc}(i_1^{op}\circ\psi^{op}\circ L_{pe}(\alpha)))+(-Id)\circ(\Sigma^{\infty}_{+}\circ j_{nc}(i_1^{op}\circ\psi^{op}\circ L_{pe}(\alpha)))\simeq \tag{7.2.31}$$
$$\simeq \Sigma^{\infty}_{+}\circ j_{nc}((\alpha^*\circ\psi\circ i_1)^{op})-\Sigma^{\infty}_{+}\circ j_{nc}((\alpha^*\circ\psi\circ i_2)^{op}) \tag{7.2.32}$$

  But since $\alpha^*(\mathcal{O}_{\mathbb{P}^1})=\alpha^*(\mathcal{O}_{\mathbb{P}^1}(-1))=\mathcal{O}_{\mathbb{A}^1}$, we have $\alpha^*\circ\psi\circ i_1\simeq\alpha^*\circ\psi\circ i_2$ so that the last difference is zero. The same argument holds for β^*.

From these two facts combined we conclude that $\Theta\circ(\Sigma^{\infty}_{+}\circ j_{nc}(\psi^{op}\circ L_{pe}(\alpha)))$ is equivalent to the sum $\Sigma^{\infty}_{+}\circ j_{nc}(L_{pe}(p))\oplus 0$ so that the outer commutative square of the diagram (7.2.20) can now be written as

$$\begin{tikzcd}
\Sigma^{\infty}_{+}\circ j_{nc}(L_{pe}(\mathbb{G}_m)) \arrow[r, "L_{pe}(i)"] \arrow[d, "L_{pe}(j)"] & \Sigma^{\infty}_{+}\circ j_{nc}(L_{pe}(\mathbb{A}^1)) \arrow[d, "\Sigma^{\infty}_{+}\circ j_{nc}(L_{pe}(p))\oplus 0"] \\
\Sigma^{\infty}_{+}\circ j_{nc}(L_{pe}(\mathbb{A}^1)) \arrow[r, "\Sigma^{\infty}_{+}\circ j_{nc}(L_{pe}(p))\oplus 0"] & \Sigma^{\infty}_{+}\circ j_{nc}(L_{pe}(k))\oplus\Sigma^{\infty}_{+}\circ j_{nc}(L_{pe}(k))
\end{tikzcd} \tag{7.2.33}$$

We are almost done. To proceed, we rewrite the diagram 7.2.20 as

$$\begin{tikzcd}
\Sigma^{\infty}_{+}\circ j_{nc}(L_{pe}(\mathbb{G}_m)) \arrow[r] \arrow[d] & \Sigma^{\infty}_{+}\circ j_{nc}(L_{pe}(\mathbb{A}^1))\oplus\Sigma^{\infty}_{+}\circ j_{nc}(L_{pe}(\mathbb{A}^1)) \arrow[d, "(\Sigma^{\infty}_{+}\circ j_{nc}(L_{pe}(\alpha)),-\Sigma^{\infty}_{+}\circ j_{nc}(L_{pe}(\beta)))"'] \arrow[ddr, bend left, "(\Sigma^{\infty}_{+}\circ j_{nc}(L_{pe}(p)),-\Sigma^{\infty}_{+}\circ j_{nc}(L_{pe}(p)))\oplus 0"] & \\
0 \arrow[r] \arrow[ddrr, bend right] & \Sigma^{\infty}_{+}\circ j_{nc}(L_{pe}(\mathbb{P}^1)) \arrow[dr, "\Theta\circ\Sigma^{\infty}_{+}\circ j_{nc}(\psi^{op})"] & \\
 & & \Sigma^{\infty}_{+}\circ j_{nc}(L_{pe}(k)\oplus L_{pe}(k))
\end{tikzcd} \tag{7.2.34}$$

where of course, since Yoneda's map $\Sigma^{\infty}_{+}\circ j_{nc}$ commutes with direct sums, we have

$$\Sigma^{\infty}_{+}\circ j_{nc}(L_{pe}(\mathbb{A}^1))\oplus\Sigma^{\infty}_{+}\circ j_{nc}(L_{pe}(\mathbb{A}^1))\simeq\Sigma^{\infty}_{+}\circ j_{nc}(L_{pe}(\mathbb{A}^1)\oplus L_{pe}(\mathbb{A}^1)) \tag{7.2.35}$$

We observe that both the inner and the outer squares become pullback-pushouts once we pass to the Nisnevich localization. Moreover, the map $\Theta\circ(\Sigma^{\infty}_{+}\circ j_{nc}(\psi^{op}))$ becomes an equivalence.

In a different direction, we also observe that the pullback map of dg-categories $p^*:L_{pe}(k)\to L_{pe}(\mathbb{A}^1)$ admits a left inverse $s^*:L_{pe}(\mathbb{A}^1)\to L_{pe}(k)$ given by the pullback along the zero section $s: Spec(k)\to\mathbb{A}^1$, [4]. In terms of noncommutative spaces, this can be rephrased by saying that $L_{pe}(p)$ has a right inverse $L_{pe}(s)$. We can use this right-inverse to construct a right inverse to the first projection of $(\Sigma^{\infty}_{+}\circ j_{nc}(L_{pe}(p)),-\Sigma^{\infty}_{+}\circ j_{nc}(L_{pe}(p)))\oplus 0$, namely, we consider the map $(\Sigma^{\infty}_{+}\circ j_{nc}(L_{pe}(s)),0)$ induced by the universal property of the direct sum in $Fun(\mathcal{D}g(k)^{ft},\widehat{Sp})$

[4] which in terms of rings is given by the evaluation at zero $ev_0: k[T]\to k$

$$\begin{tikzcd}
 & \Sigma^\infty_+ \circ j_{nc}(L_{pe}(\mathbb{A}^1)) \\
\Sigma^\infty_+ \circ j_{nc}(L_{pe}(k)) \arrow[ur, bend left, "\Sigma^\infty_+\circ j_{nc}(L_{pe}(s))"] \arrow[dr, bend right, "0"'] \arrow[r, dashed, "(\Sigma^\infty_+\circ j_{nc}(L_{pe}(s)){,}0)"] & \Sigma^\infty_+ \circ j_{nc}(L_{pe}(\mathbb{A}^1)) \oplus \Sigma^\infty_+ \circ j_{nc}(L_{pe}(\mathbb{A}^1)) \arrow[u] \arrow[d] \\
 & \Sigma^\infty_+ \circ j_{nc}(L_{pe}(\mathbb{A}^1))
\end{tikzcd} \tag{7.2.36}$$

It is immediate to check that the composition $\Sigma^\infty_+\circ j_{nc}(i_1^{op})\circ((\Sigma^\infty_+\circ j_{nc}(L_{pe}(p)), -\Sigma^\infty_+\circ j_{nc}(L_{pe}(p)))\oplus 0)\circ(\Sigma^\infty_+\circ j_{nc}((L_{pe}(s)), 0)$ is the identity, so that $\Sigma^\infty_+\circ j_{nc}(i_1^{op})\circ((\Sigma^\infty_+\circ j_{nc}(L_{pe}(p)), -\Sigma^\infty_+\circ j_{nc}(L_{pe}(p)))\oplus 0)$ has a right inverse that we can picture as a dotted arrow

$$\begin{tikzcd}
\Sigma^\infty_+ \circ j_{nc}(L_{pe}(\mathbb{G}_m)) \arrow[r] \arrow[d] & \Sigma^\infty_+ \circ j_{nc}(L_{pe}(\mathbb{A}^1)) \oplus \Sigma^\infty_+ \circ j_{nc}(L_{pe}(\mathbb{A}^1)) \arrow[d, "(L_{pe}(\alpha){,}-L_{pe}(\beta))"] \arrow[dd, bend left=60, "(L_{pe}(p){,}-L_{pe}(p))\oplus 0"] \\
0 \arrow[r] & \Sigma^\infty_+ \circ j_{nc}(L_{pe}(\mathbb{P}^1)) \arrow[d, "\Theta\circ(\Sigma^\infty_+\circ j_{nc}(\psi^{op}))"] \\
 & \Sigma^\infty_+ \circ j_{nc}(L_{pe}(k) \oplus L_{pe}(k)) \arrow[d, "\Sigma^\infty_p\circ j_{nc}(i_1^{op})"] \\
 & \Sigma^\infty_+ \circ j_{nc}(L_{pe}(k)) \arrow[uuu, dashed, bend right=90]
\end{tikzcd} \tag{7.2.37}$$

At the same time, the preceding discussion implies that the second projection

$$\begin{tikzcd}
\Sigma^\infty_+ \circ j_{nc}(L_{pe}(\mathbb{G}_m)) \arrow[r] \arrow[d] & \Sigma^\infty_+ \circ j_{nc}(L_{pe}(\mathbb{A}^1)) \oplus \Sigma^\infty_+ \circ j_{nc}(L_{pe}(\mathbb{A}^1)) \arrow[d, "(L_{pe}(\alpha){,}-L_{pe}(\beta))"] \arrow[ddd, bend left=60, "0"] \\
0 \arrow[r] & \Sigma^\infty_+ \circ j_{nc}(L_{pe}(\mathbb{P}^1)) \arrow[d, "\Theta\circ(\Sigma^\infty_+\circ j_{nc}(\psi^{op}))"] \\
 & \Sigma^\infty_+ \circ j_{nc}(L_{pe}(k) \oplus L_{pe}(k)) \arrow[d, "\Sigma^\infty_+\circ j_{nc}(i_2^{op})"] \\
 & \Sigma^\infty_+ \circ j_{nc}(L_{pe}(k))
\end{tikzcd} \tag{7.2.38}$$

is just the zero map.

We now explain how to extract the familiar Bass exact sequence from these two diagrams. Given any object $F \in Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp})$ and a noncommutative space $\mathcal{X}$, we set the notation $F_{\mathcal{X}} := \underline{Hom}(\Sigma^\infty_+ \circ j_{nc}(\mathcal{X}), F)$. By enriched Yoneda, this is the functor given by $F(\mathcal{X} \otimes -)$. To proceed, we consider the image of the diagram (7.2.37) under the functor $\underline{Hom}(-, F)$, to find a diagram

$$\begin{tikzcd}
F_{L_{pe}(k)} \simeq F \arrow[r, "i_1^F"] & F \oplus F \simeq F_{L_{pe}(k) \oplus L_{pe}(k)} \arrow[r] & F_{L_{pe}(\mathbb{P}^1)} \arrow[r] \arrow[d] & F_{L_{pe}(\mathbb{A}^1) \oplus L_{pe}(\mathbb{A}^1)} \simeq F_{L_{pe}(\mathbb{A}^1)} \oplus F_{L_{pe}(\mathbb{A}^1)} \arrow[d] \arrow[lll, bend right] \\
 & & 0 \arrow[r] & F_{L_{pe}(\mathbb{G}_m)}
\end{tikzcd} \tag{7.2.39}$$

where the first map $i_1^F : F \to F \oplus F$ can be identified with the canonical inclusion in the first coordinate and the composition $F \longrightarrow F_{L_{pe}(\mathbb{A}^1)} \oplus F_{L_{pe}(\mathbb{A}^1)} \dashrightarrow F$ is the identity.

From this we can produce a new commutative diagram by taking successive pushouts

$$\begin{tikzcd}
F \arrow[r, "i_1^F"] \arrow[d] & F \oplus F \arrow[r] \arrow[d] & F_{L_{pe}(\mathbb{P}^1)} \arrow[r] \arrow[d] & F_{L_{pe}(\mathbb{A}^1)} \oplus F_{L_{pe}(\mathbb{A}^1)} \arrow[d] \arrow[ddr, bend left] & \\
0 \arrow[r] & F \arrow[r] & F \coprod_{F \oplus F} F_{L_{pe}(\mathbb{P}^1)} \arrow[r] \arrow[d] & F_{L_{pe}(\mathbb{A}^1)} \coprod_F F_{L_{pe}(\mathbb{A}^1)} \arrow[dr, dashed] & \\
 & & 0 \arrow[rr] & & F_{L_{pe}(\mathbb{G}_m)}
\end{tikzcd} \tag{7.2.40}$$

and we notice that the vertical map $F \oplus F \to F$ can be identified with the projection in the second coordinate.

In particular, if we denote as $U(F)$ the pullback

$$\begin{tikzcd}
U(F) \arrow[r] \arrow[d] & F_{L_{pe}(\mathbb{A}^1)} \coprod_F F_{L_{pe}(\mathbb{A}^1)} \arrow[d] \\
0 \arrow[r] & F_{L_{pe}(\mathbb{G}_m)}
\end{tikzcd} \tag{7.2.41}$$

we find a canonical map

$$F \coprod\nolimits_{F \oplus F} F_{L_{pe}(\mathbb{P}^1)} \dashrightarrow U(F) \tag{7.2.42}$$

induced from the diagram (7.2.40) using the universal property of the pullback .

At the same time, if we apply $\underline{Hom}(-, F)$ to the diagram (7.2.38) we find a new commutative diagram

$$
\begin{tikzcd}
F \arrow[d, "i_2^F"] \arrow[drrr, bend left, "0"] \arrow[dd, bend right, "id"'] & & & \\
F \oplus F \arrow[r] \arrow[d] & F_{L_{pe}(\mathbb{P}^1)} \arrow[rr] \arrow[d] & & F_{L_{pe}(\mathbb{A}^1)} \oplus F_{L_{pe}(\mathbb{A}^1)} \arrow[d] \\
F \arrow[r] & F \coprod_{F\oplus F} F_{L_{pe}(\mathbb{P}^1)} \arrow[rr] \arrow[dr, dashed] \arrow[dd] & & F_{L_{pe}(\mathbb{A}^1)} \coprod_F F_{L_{pe}(\mathbb{A}^1)} \arrow[dd] \\
& & U(F) \arrow[ur] \arrow[dl] & \\
& 0 \arrow[rr] & & F_{L_{pe}(\mathbb{G}_m)}
\end{tikzcd}
\tag{7.2.43}
$$

and discover that the map $F \to F \coprod_{F\oplus F} F_{L_{pe}(\mathbb{P}^1)} \to U(F)$ admits a natural factorization

$$
\begin{tikzcd}
& \Omega F_{L_{pe}(\mathbb{G}_m)} \arrow[r] \arrow[d] & 0 \arrow[d] \\
F \arrow[r] \arrow[ur, dashed, "\sigma_F"] & U(F) \arrow[r] \arrow[d] & F_{L_{pe}(\mathbb{A}^1)} \coprod_F F_{L_{pe}(\mathbb{A}^1)} \arrow[d] \\
& 0 \arrow[r] & F_{L_{pe}(\mathbb{G}_m)}
\end{tikzcd}
\tag{7.2.44}
$$

because $\Omega F_{L_{pe}(\mathbb{G}_m)}$ is the fiber of $U(F) \to F_{L_{pe}(\mathbb{A}^1)} \coprod_F F_{L_{pe}(\mathbb{A}^1)}$. This concludes the preliminary steps.

From now on, we suppose that F is Nisnevich local. In this case, by the Corollary 7.2.1, the map $F \oplus F \to F_{L_{pe}(\mathbb{P}^1)}$ is an equivalence and the commutative square

$$
\begin{tikzcd}
F_{L_{pe}(\mathbb{P}^1)} \arrow[r] \arrow[d] & F_{L_{pe}(\mathbb{A}^1)} \oplus F_{L_{pe}(\mathbb{A}^1)} \arrow[d] \\
0 \arrow[r] & F_{L_{pe}(\mathbb{G}_m)}
\end{tikzcd}
\tag{7.2.45}
$$

is a pushout-pullback because the image of the square (7.2.10) under L_{pe} is a Nisnevich square of noncommutative spaces. Using these two facts we conclude that the canonical maps constructed above, $F \to F \coprod_{F\oplus F} F_{L_{pe}(\mathbb{P}^1)}$ and $F \coprod_{F\oplus F} F_{L_{pe}(\mathbb{P}^1)} \to U(F)$ are equivalences so that the diagram

$$
\begin{tikzcd}
F \arrow[r] \arrow[d] & F_{L_{pe}(\mathbb{A}^1)} \coprod_F F_{L_{pe}(\mathbb{A}^1)} \arrow[d] \\
0 \arrow[r] & F_{L_{pe}(\mathbb{G}_m)}
\end{tikzcd}
\tag{7.2.46}
$$

is a pullback-pushout. In particular, as in the diagram (7.2.44), we find the existence of a section

$$\begin{tikzcd} & \Omega F_{L_{pe}(\mathbb{G}_m)} \arrow[r] \arrow[d] & 0 \arrow[d] \\ F \arrow[ur, dashed, "\sigma_F"] \arrow[r, "Id"] & F \arrow[r] \arrow[d] & F_{L_{pe}(\mathbb{A}^1)} \coprod_F F_{L_{pe}(\mathbb{A}^1)} \arrow[d] \\ & 0 \arrow[r] & F_{L_{pe}(\mathbb{G}_m)} \end{tikzcd} \tag{7.2.47}$$

We are almost done. To conclude, we consider the induced pullback-pushout square

$$\begin{tikzcd} F_{L_{pe}(\mathbb{A}^1)} \coprod_F F_{L_{pe}(\mathbb{A}^1)} \arrow[r] \arrow[d] & 0 \arrow[d] \\ F_{L_{pe}(\mathbb{G}_m)} \arrow[r] & \Sigma F \end{tikzcd} \tag{7.2.48}$$

where now, the suspension $\Sigma(\sigma_F)$ makes $\Sigma(F)$ a retract of $F_{L_{pe}(\mathbb{G}_m)}$. We are done now. Since the evaluation maps commute with colimits and, by definition of $F_{(-)}$, we have for each $T_{\mathcal{X}} \in \mathcal{D}g(k)^{ft}$ a pullback-pushout diagram in $\widehat{Sp}$

$$\begin{tikzcd} F(L_{pe}(\mathbb{A}^1) \otimes T_{\mathcal{X}}) \coprod_{F(T_{\mathcal{X}})} F(L_{pe}(\mathbb{A}^1) \otimes T_{\mathcal{X}}) \arrow[r] \arrow[d] & 0 \arrow[d] \\ F(L_{pe}(\mathbb{G}_m) \otimes T_{\mathcal{X}}) \arrow[r] & \Sigma F(T_{\mathcal{X}}) \end{tikzcd} \tag{7.2.49}$$

and therefore a long exact sequence of abelian groups

$$... \to \pi_n(F(L_{pe}(\mathbb{A}^1) \otimes T_{\mathcal{X}}) \coprod_{F(T_{\mathcal{X}})} F(L_{pe}(\mathbb{A}^1) \otimes T_{\mathcal{X}})) \to \pi_n(F(L_{pe}(\mathbb{G}_m) \otimes T_{\mathcal{X}})) \to \pi_n(\Sigma F(T_{\mathcal{X}})) = \pi_{n-1}(F(T_{\mathcal{X}})) \to ... \tag{7.2.50}$$

and because of the existence of $\Sigma(\sigma_F)$, the maps $\pi_n(F(L_{pe}(\mathbb{G}_m) \otimes T_{\mathcal{X}})) \to \pi_n(\Sigma F(T_{\mathcal{X}})) = \pi_{n-1}(F(T_{\mathcal{X}}))$ are necessarily surjective, so that the long exact sequence breaks up into short exact sequences

$$0 \to \pi_n(F(L_{pe}(\mathbb{A}^1) \otimes T_{\mathcal{X}}) \coprod_{F(T_{\mathcal{X}})} F(L_{pe}(\mathbb{A}^1) \otimes T_{\mathcal{X}})) \to \pi_n(F(L_{pe}(\mathbb{G}_m) \otimes T_{\mathcal{X}})) \to \pi_n(\Sigma F(T_{\mathcal{X}})) = \pi_{n-1}(F(T_{\mathcal{X}})) \to 0 \tag{7.2.51}$$

$\forall n \in \mathbb{Z}$.

At the same time, since the square

$$\begin{tikzcd} F \arrow[r, "i_1^F"] \arrow[d] & F \oplus F \arrow[r] & F_{L_{pe}(\mathbb{A}^1)} \oplus F_{L_{pe}(\mathbb{A}^1)} \arrow[d] \\ 0 \arrow[rr] & & F_{L_{pe}(\mathbb{A}^1)} \coprod_F F_{L_{pe}(\mathbb{A}^1)} \end{tikzcd} \tag{7.2.52}$$

is also a pullback-pushout and the top map $F \to F_{L_{pe}(\mathbb{A}^1)} \oplus F_{L_{pe}(\mathbb{A}^1)}$ admits a left inverse, the associated long exact sequence

$$... \to \pi_n(F(L_{pe}(\mathbb{A}^1) \otimes T_{\mathcal{X}}) \coprod_{F(T_{\mathcal{X}})} F(L_{pe}(\mathbb{A}^1) \otimes T_{\mathcal{X}})) \to \pi_n(F(T_{\mathcal{X}})) \to \pi_n(F(L_{pe}(\mathbb{A}^1) \otimes T_{\mathcal{X}}) \oplus F(L_{pe}(\mathbb{A}^1) \otimes T_{\mathcal{X}})) \to ... \tag{7.2.53}$$

breaks up into short exact sequences

$$0 \to \pi_n(F(T_{\mathcal{X}})) \to \pi_n(F(L_{pe}(\mathbb{A}^1)\otimes T_{\mathcal{X}}) \oplus F(L_{pe}(\mathbb{A}^1)\otimes T_{\mathcal{X}})) \to \pi_n(F(L_{pe}(\mathbb{A}^1)\otimes T_{\mathcal{X}}) \coprod_{F(\mathcal{X})} F(L_{pe}(\mathbb{A}^1)\otimes T_{\mathcal{X}})) \to 0 \tag{7.2.54}$$

Combining the two short exact sequences (7.2.51) and (7.2.54) we find the familar exact sequences of Bass-Thomason-Trobaugh

$$0 \to \pi_n(F(T_{\mathcal{X}})) \to \pi_n(F(L_{pe}(\mathbb{A}^1)\otimes T_{\mathcal{X}}) \oplus F(L_{pe}(\mathbb{A}^1)\otimes T_{\mathcal{X}})) \to \tag{7.2.55}$$

$$\to \pi_n(F(L_{pe}(\mathbb{G}_m)\otimes T_{\mathcal{X}})) \to \pi_{n-1}(F(T_{\mathcal{X}})) \to 0 \tag{7.2.56}$$

This concludes this section.

7.2.0.2 Nisnevich vs Connective-Nisnevich descent and the Thomason-Trobaugh $(-)^B$-Construction

In this section we study the class of functors sharing the same formal properties of K^c, namely, the one of sending Nisnevich squares to pullback squares of connective spectra. This will take us through a small digression aiming to understand how the truncation functor $\tau_{\geq 0}$ interacts with the Nisnevich localization.

Definition 7.2.3. *Let $F \in Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp}_{\geq 0})$. We say that F is* connectively-Nisnevich local *if for any Nisnevich square of dg-categories*

$$\begin{tikzcd} T_{\mathcal{X}} \arrow[r] \arrow[d] & T_{\mathcal{U}} \arrow[d] \\ T_{\mathcal{V}} \arrow[r] & T_{\mathcal{W}} \end{tikzcd} \tag{7.2.57}$$

the induced square

$$\begin{tikzcd} F(T_{\mathcal{X}}) \arrow[r] \arrow[d] & F(T_{\mathcal{U}}) \arrow[d] \\ F(T_{\mathcal{V}}) \arrow[r] & F(T_{\mathcal{W}}) \end{tikzcd} \tag{7.2.58}$$

is a pullback of connective spectra.

Remark 7.2.4. It follows that if F belongs to $Fun_{Nis}(Dg^{ft}, \widehat{Sp})$, its connective truncation $\tau_{\geq 0}(F)$ is connectively-Nisnevich local. This is because $\tau_{\geq 0}$ acts objectwise and is a right adjoint to the inclusion of connective spectra into all spectra, thus preserving pullbacks.

It is also convenient to isolate the following small technical remark:

Remark 7.2.5. Let $\mathcal{C}$ be a stable $(\infty,1)$-category and let $\mathcal{C}_0 \subseteq \mathcal{C}$ be a subcategory such that the inclusion preserves direct sums. Then, if

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d] & B \arrow[d, "p"] \\ 0 \arrow[r] & C \end{tikzcd} \tag{7.2.59}$$

is a pullback square in $\mathcal{C}_0$ such that

- the map i admits a left inverse v;
- the map p admits a right inverse u;

- the sum $i \circ v + u \circ p$ is homotopic to the identity,

we conclude, by the same arguments given in the Remark 6.4.21, that $B \simeq A \oplus C$. Moreover, under the hypothesis that the inclusion preserves direct sums, the square remains a pullback after the inclusion $\mathcal{C}_0 \subseteq \mathcal{C}$ and therefore a pushout. In particular, it becomes a split exact sequence in $\mathcal{C}$. This holds for any universe.

In particular, for any pullback square of dg-categories associated to a Nisnevich square of non-commutative spaces (7.2.57) such that $T_{\mathcal{U}}$ is zero and the sequence splits, the induced diagram of connective spectra (7.2.58) makes $F(T_{\mathcal{V}})$ canonically equivalent to the direct sum $F(T_{\mathcal{X}}) \oplus F(T_{\mathcal{W}})$ in $\widehat{Sp}$.

We let $Fun_{Nis_{\geq 0}}(\mathcal{D}g(k)^{ft}, \widehat{Sp}_{\geq 0})$ denote the full subcategory of $Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp}_{\geq 0})$ spanned by the connectively-Nisnevich local functors. For technical reasons it is convenient to observe that the inclusion $Fun_{Nis_{\geq 0}}(\mathcal{D}g(k)^{ft}, \widehat{Sp}_{\geq 0}) \subseteq Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp}_{\geq 0})$ admits a left adjoint $l_{nis_{\geq 0}}$. More precisely

Proposition 7.2.6. *$Fun_{Nis_{\geq 0}}(\mathcal{D}g(k)^{ft}, \widehat{Sp}_{\geq 0})$ is an accessible reflexive localization of $Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp}_{\geq 0})$.*

Proof. We evoke the Proposition 5.5.4.15 of [99] so that we are reduced to showing the existence of a small class of maps S in $Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp}_{\geq 0})$ such that an object F is connectively-Nisnevich local if and only if it is local with respect to the maps in S.

To define S, we ask the reader to bring back to recollect our discussion and notations in 5.4 and in 6.4.23. Using the same notations, we define S to be the collection of all maps

$$\delta_{\Sigma^\infty_+ \circ j_{nc}(\mathcal{U})}(K) \coprod_{\delta_{\Sigma^\infty_+ \circ j_{nc}(\mathcal{W})}(K)} \delta_{\Sigma^\infty_+ \circ j_{nc}(\mathcal{V})}(K) \to \delta_{\Sigma^\infty_+ \circ j_{nc}(\mathcal{X})}(K) \tag{7.2.60}$$

given by the universal property of the pushout, this time with K in $\widehat{Sp}_{\geq 0} \cap (\widehat{Sp})^\omega$ [5] and $\mathcal{W}$,$\mathcal{V}$,$\mathcal{U}$ and $\mathcal{X}$ part of a Nisnevich square of noncommutative smooth spaces. As before, the fact that S satisfies the required property follows directly from the definition of the functors $\delta_{\Sigma^\infty_+ \circ j_{nc}(-)}$ as left adjoints to Map^{Sp} and from the enriched version of Yoneda's lemma.

□

It follows directly from the definition of the class S in the previous proof and from the description of the class of maps that generate the Nisnevich localization in $Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp})$ (see 6.4.23) that the inclusion

$$i : Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp}_{\geq 0}) \hookrightarrow Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp}) \tag{7.2.61}$$

sends connective-Nisnevich local equivalences to Nisnevich local equivalences. In particular, the universal property of the localization provides us with a canonical colimit preserving map

$$\begin{tikzcd}
Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp}_{\geq 0}) \arrow[r, hook, "i"] \arrow[d, "l_{nis_{\geq 0}}"] & Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp}) \arrow[d, "l^{nc}_{nis}"] \\
Fun_{Nis_{\geq 0}}(\mathcal{D}g(k)^{ft}, \widehat{Sp}_{\geq 0}) \arrow[r, dashed] & Fun_{Nis}(\mathcal{D}g(k)^{ft}, \widehat{Sp})
\end{tikzcd} \tag{7.2.62}$$

rendering the diagram commutative. Moreover, since the localizations are presentable, the Adjoint Functor Theorem implies the existence of a right adjoint which makes the associated diagram of right adjoints

[5] Here $(\widehat{Sp})^\omega$ denotes the full subcategory of $\widehat{Sp}$ spanned by the compact objects. Recall that $\widehat{Sp} \simeq Ind((\widehat{Sp})^\omega)$.

$$\begin{tikzcd}
Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp}_{\geq 0}) & Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp}) \arrow[l, "\tau_{\geq 0}"'] \\
Fun_{Nis_{\geq 0}}(\mathcal{D}g(k)^{ft}, \widehat{Sp}_{\geq 0}) \arrow[u, hook, "\alpha"] & Fun_{Nis}(\mathcal{D}g(k)^{ft}, \widehat{Sp}) \arrow[l, dashed] \arrow[u, hook, "\beta"']
\end{tikzcd} \tag{7.2.63}$$

commute. At the same time, the Remark 7.2.4 implies the existence of the two commutative diagrams (7.2.2) and (7.2.3). By comparison with the new diagrams, we find that the canonical colimit preserving map $Fun_{Nis_{\geq 0}}(\mathcal{D}g(k)^{ft}, \widehat{Sp}_{\geq 0}) \dashrightarrow Fun_{Nis}(\mathcal{D}g(k)^{ft}, \widehat{Sp})$ can be identified with the composition $i_! := l^{nc}_{Nis} \circ i \circ \alpha$ and that its right adjoint can be identified with $\overline{\tau_{\geq 0}}$, the restriction of the truncation functor $\tau_{\geq 0}$ to the Nisnevich local functors.

Our goal is to prove that this adjunction

$$\begin{tikzcd}
Fun_{Nis_{\geq 0}}(\mathcal{D}g(k)^{ft}, \widehat{Sp}_{\geq 0}) \arrow[r, "i_!"] & Fun_{Nis}(\mathcal{D}g(k)^{ft}, \widehat{Sp}) \arrow[l, bend left, "\overline{\tau_{\geq 0}}"]
\end{tikzcd} \tag{7.2.64}$$

is an equivalence. Our results from 7.2.0.1 already provide one step towards this:

Proposition 7.2.7. *The functor $\overline{\tau_{\geq 0}}$ is conservative.*

Proof. Recall from 7.2.0.1 that for any Nisnevich local F we can construct a pullback-pushout square

$$\begin{tikzcd}
F_{L_{pe}(\mathbb{A}^1)} \coprod_F F_{L_{pe}(\mathbb{A}^1)} \arrow[r] \arrow[d] & 0 \arrow[d] \\
F_{L_{pe}(\mathbb{G}_m)} \arrow[r] & \Sigma F
\end{tikzcd} \tag{7.2.65}$$

such that for any $T_{\mathcal{X}} \in \mathcal{D}g(k)^{ft}$, the associated long exact sequence of homotopy groups breaks up into short exact sequences for any $n \in \mathbb{N}$

$$0 \to \pi_n(F(L_{pe}(\mathbb{A}^1) \otimes T_{\mathcal{X}}) \coprod_{F(T_{\mathcal{X}})} F(L_{pe}(\mathbb{A}^1) \otimes T_{\mathcal{X}})) \to \pi_n(F(L_{pe}(\mathbb{G}_m) \otimes T_{\mathcal{X}})) \to \pi_{n-1}(F(T_{\mathcal{X}})) \to 0 \tag{7.2.66}$$

Therefore, given a morphism $f : F \to G$ in $Fun_{Nis}(\mathcal{D}g(k)^{ft}, \widehat{Sp})$, we have an induced diagram

$$\begin{tikzcd}
& G_{L_{pe}(\mathbb{A}^1)} \coprod_G G_{L_{pe}(\mathbb{A}^1)} \arrow[rr] \arrow[dd] & & 0 \arrow[dd] \\
F_{L_{pe}(\mathbb{A}^1)} \coprod_F F_{L_{pe}(\mathbb{A}^1)} \arrow[ur] \arrow[rr] \arrow[dd] & & 0 \arrow[ur] \arrow[dd] & \\
& G_{L_{pe}(\mathbb{G}_m)} \arrow[rr] & & \Sigma G \\
F_{L_{pe}(\mathbb{G}_m)} \arrow[ur] \arrow[rr] & & \Sigma F \arrow[ur] &
\end{tikzcd} \tag{7.2.67}$$

which induces natural maps of short exact sequences

$$\begin{tikzcd}
\pi_n(F(L_{pe}(\mathbb{A}^1)\otimes T_{\mathfrak{X}})\coprod_{F(T_{\mathfrak{X}})} F(L_{pe}(\mathbb{A}^1)\otimes T_{\mathfrak{X}})) \arrow[r] \arrow[d] & \pi_n(F(L_{pe}(\mathbb{G}_m)\otimes T_{\mathfrak{X}})) \arrow[r] \arrow[d] & \pi_{n-1}(F(T_{\mathfrak{X}})) \arrow[d] \\
\pi_n(G(L_{pe}(\mathbb{A}^1)\otimes T_{\mathfrak{X}})\coprod_{G(T_{\mathfrak{X}})} G(L_{pe}(\mathbb{A}^1)\otimes T_{\mathfrak{X}})) \arrow[r] & \pi_n(G(L_{pe}(\mathbb{G}_m)\otimes T_{\mathfrak{X}})) \arrow[r] & \pi_{n-1}(G(T_{\mathfrak{X}}))
\end{tikzcd} \tag{7.2.68}$$

In particular, if f is an equivalence in the connective part, by induction on $n = 0, -1, -2, ...$, we conclude that f is an equivalence. □

With this result, in order to prove that $i_!$ is an equivalence we are reduced to showing that the counit of the adjunction $\overline{\tau_{\geq 0}} \circ i_! \to Id$ is a natural equivalence of functors. Notice that since α and i are fully-faithful, this amounts to show that for any F connectively-Nisnevich local, the canonical map $i \circ \tau_{\geq 0} \circ l^{nc}_{Nis} \circ i \circ \alpha(F) \to i \circ \alpha(F)$ is an equivalence. Of course, to achieve this we will need a more explicit description of the noncommutative Nisnevich localization functor l^{nc}_{Nis} restricted to connectively-Nisnevich local objects. There is a naive candidate, namely, the familiar $(-)^B$ construction of Thomason-Trobaugh [137, 6.4]. Our goal to the end of this section is to prove the following proposition confirming that this guess is correct:

Proposition 7.2.8. *There is an accessible localization functor $(-)^B : Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp}) \to Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp})$ encoding the B-construction of [137, 6.4] such that for any $F \in Fun_{Nis_{\geq 0}}(\mathcal{D}g(k)^{ft}, \widehat{Sp})$ we have:*

- $\overline{\tau_{\geq 0}}(i \circ \alpha(F)^B) \simeq F$.
- *the object $(i \circ \alpha(F))^B$ is Nisnevich local;*
- *there is a canonical equivalence $(i \circ \alpha(F))^B \simeq l^{nc}_{Nis}((i \circ \alpha(F))$;*

In particular, the natural transformation $\overline{\tau_{\geq 0}} \circ i_! \to Id$ is an equivalence. Together with the Proposition 7.2.7 we have an equivalence of $(\infty, 1)$-categories between the theory of connectively-Nisnevich local functors and the theory of Nisnevich local functors.

With these results available we can already uncover the proof of our first main theorem:

Proof of the Theorem 7.0.29:
Thanks to the Corollary 7.1.6 we already know that K^S is Nisnevich local. In this case, and by the universal property of the localization, the canonical map $K^c \to K^S$ admits a canonical uniquely determined factorization

$$\begin{tikzcd}
K^c \arrow[d] \arrow[dr] & \\
l^{nc}_{Nis}(K^c) \arrow[r, dashed] & K^S
\end{tikzcd} \tag{7.2.69}$$

so that we are reduced to showing that this canonical morphism $l^{nc}_{Nis}(K^c) \to K^S$ is an equivalence. But since these are Nisnevich local objects and since we now know by the Prop. 7.2.8 that the truncation functor $\tau_{\geq 0}$ is an equivalence when restricted to Nisnevich locals, it suffices to check that the induced map $\tau_{\geq 0} l^{nc}_{Nis}(K^c) \to \tau_{\geq 0} K^S$ is an equivalence. But this follows because all the morphisms in the image of the commutative diagram (7.2.69) become equivalences after applying $\tau_{\geq 0}$. This follows from the construction of K^S and again by the results in the Proposition 7.2.8.

□

We now start our small journey towards the proof of the Proposition 7.2.8. To start with we need to specify how the B-Construction of [137, 6.4] can be formulated in our setting:

Construction 7.2.9. (Thomason-Trobaugh $(-)^B$-Construction) We begin by asking the reader to recall the diagrams constructed in 7.2.0.1, or more precisely, that for any $F \in Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp})$, we found a commutative diagram

$$\begin{tikzcd} & \Omega F_{L_{pe}(\mathbb{G}_m)} \arrow[r] \arrow[d] & 0 \arrow[d] \\ F \arrow[r, "\alpha_F"] \arrow[ur, dashed, "\sigma_F"] & U(F) \arrow[r] \arrow[d] & F_{L_{pe}(\mathbb{A}^1)} \coprod_F F_{L_{pe}(\mathbb{A}^1)} \arrow[d] \\ & 0 \arrow[r] & F_{L_{pe}(\mathbb{G}_m)} \end{tikzcd} \tag{7.2.70}$$

where both squares are pushout-pullbacks. Iterating this construction, we find a sequence of canonical maps

$$F \xrightarrow{\alpha_F} U(F) \xrightarrow{\alpha_{U(F)}} U(U(F)) \xrightarrow{\alpha_{U^2(F)}} .... \tag{7.2.71}$$

and we define F^B to be the colimit for sequence (which is of course unique up to canonical equivalence). The assignement $F \mapsto F^B$ provides an endofunctor $(-)^B$ of the $(\infty, 1)$-category $Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp})$. To see this we can use the fact the monoidal structure in $Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp})$ admits internal-homs $\underline{Hom}$. More precisely, we consider the diagram of natural transformations induced by the image of the diagram (7.2.37) under the first entry of $Hom(-,-)$. With this, and keeping the notations we have been using, we define f_1 to be the functor cofiber of $Id = (-)_{L_{pe}(k)} \to (-)_{L_{pe}(\mathbb{A}^1)} \oplus (-)_{L_{pe}(\mathbb{A}^1)}$. The universal property of the cofiber gives us a canonical natural transformation $f_1 \to (-)_{L_{pe}(\mathbb{G}_m)}$ and define a new functor U as the fiber of this map (recall that colimits and limits in the category of functors are determined objectwise). Finally, we consider $(-)^B$ as the colimit of the natural transformations

$$\begin{tikzcd} Id \arrow[r, "\alpha"] \arrow[dr] & U = Id \circ U \arrow[r] \arrow[d] & U^2 = Id \circ U^2 \arrow[r] \arrow[dl] & .... \\ & (-)^B & & \end{tikzcd} \tag{7.2.72}$$

We prove that for any F the object F^B satisfies the exact sequences of Bass-Thomason-Trobaugh for any $n \in \mathbb{Z}$. The proof requires some technical steps:

Lemma 7.2.10. *The functor U commutes with small colimits.*

Proof. Let $\{F_i\}_{i\in I}$ be a diagram in $\mathcal{C}$. Then, by definition we have a pullback diagram

$$\begin{tikzcd} U(colim_I\, F_i) \arrow[r] \arrow[d] & (colim_I\, F_i)_{L_{pe}(\mathbb{A}^1)} \coprod_{(colim_I\, F_i)} (colim_I\, F_i)_{L_{pe}(\mathbb{A}^1)} \arrow[d] \\ 0 \arrow[r] & (colim_I\, F_i)_{L_{pe}(\mathbb{G}_m)} \end{tikzcd} \tag{7.2.73}$$

but since $(-)_{L_{pe}(\mathbb{G}_m)}$ and $(-)_{L_{pe}(\mathbb{A}^1)}$ commute with all colimits (thanks to Yoneda's lemma and the fact the evaluation map commutes with small colimits), this diagram is equivalent to

$$\begin{tikzcd} U(colim_I\, F_i) \arrow[r] \arrow[d] & colim_I\, ((F_i)_{L_{pe}(\mathbb{A}^1)} \coprod_{F_i} (F_i)_{L_{pe}(\mathbb{A}^1)}) \arrow[d] \\ 0 \arrow[r] & colim_I\, ((F_i)_{L_{pe}(\mathbb{G}_m)}) \end{tikzcd} \tag{7.2.74}$$

and since in the stable context colimits commute with pullbacks we find a canonical equivalence

$$U(colim_I\, F_i) \simeq colim_I\, U(F_i) \tag{7.2.75}$$

□

Lemma 7.2.11. *The two maps $U = Id \circ U \to U^2$ and $U = U \circ Id \to U^2$ induced by the natural transformation $Id \to U$, are homotopic.*

Proof. We are reduced to showing that for any F the natural maps $U(\alpha_F), \alpha_{U(F)} : U(F) \to U^2(F)$ are homotopic. Recall that by definition $\alpha_{U(F)}$ is determined by the universal property of pullbacks, as being the essentially unique map that makes the diagram

$$\begin{tikzcd}
U(F) \arrow[dr, dashed, "\alpha_{U(F)}"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& U^2(F) \arrow[r] \arrow[d] & U(F)_{L_{pe}(\mathbb{A}^1)} \coprod_{U(F)} U(F)_{L_{pe}(\mathbb{A}^1)} \arrow[d] \\
& 0 \arrow[r] & U(F)_{L_{pe}(\mathbb{G}_m)}
\end{tikzcd} \tag{7.2.76}$$

commute. In this case, as U commutes with colimits by the Lemma 7.2.10 and as $Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp})$ is stable, U also preserves pullbacks. In this case, and as we have equivalences $U(F_{\mathbb{G}_m}) \simeq U(F)_{\mathbb{G}_m}$ and $U(F_{L_{pe}(\mathbb{A}^1)}) \simeq U(F)_{L_{pe}(\mathbb{A}^1)}$, the diagram in (7.2.76) is in fact equivalent to the image of the diagram

$$\begin{tikzcd}
F \arrow[dr, dashed, "\alpha_F"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& U(F) \arrow[r] \arrow[d] & F_{L_{pe}(\mathbb{A}^1)} \coprod_F F_{L_{pe}(\mathbb{A}^1)} \arrow[d] \\
& 0 \arrow[r] & F_{L_{pe}(\mathbb{G}_m)}
\end{tikzcd} \tag{7.2.77}$$

under U, where the inner commutative square is a pullback so that $U(\alpha_F)$ is necessarily homotopic to $\alpha_{U(F)}$. □

These lemmas have the following consequences:

Proposition 7.2.12. *The natural transformation $(-)^B \circ Id \to (-)^B \circ U$ is an equivalence.*

Proof. This amounts to check that for any F the natural map $F^B \to U(F)^B$ is an equivalence. By construction, this is the map induced at the colimit level by the morphism of diagrams

$$\begin{tikzcd}
\ldots & \ldots \\
U^2(F) \arrow[u, "\alpha_{U^2(F)}"] \arrow[r, "U^2(\alpha_F)"] & U(F) \arrow[u, "\alpha_{U^3(F)}"'] \\
U(F) \arrow[u, "\alpha_{U(F)}"] \arrow[r, "U(\alpha_F)"] & U(F) \arrow[u, "\alpha_{U^2(F)}"'] \\
F \arrow[u, "\alpha_F"] \arrow[r, "\alpha_F"] & U(F) \arrow[u, "\alpha_{U(F)}"']
\end{tikzcd} \tag{7.2.78}$$

By iterating the Lemma 7.2.11 we find that for any $k \geq 0$ the maps $U^k(\alpha_F)$ and $\alpha_{U^k(F)}$ are homotopic so that, by cofinality, the map $F^B \to U(F)^B$ induced between the colimit of each column is an equivalence. □

Proposition 7.2.13. *The natural map $(-)^B \circ U \to U \circ (-)^B$ is an equivalence.*

Proof. It is enough to show that for any F the natural map $U(F)^B \to U(F^B)$ is an equivalence. As F^B can be obtained as a colimit for the sequence (7.2.71) and as U commutes with colimits, $U(F^B)$ is the colimit of

$$U(F) \xrightarrow{U(\alpha_F)} U^2(F) \xrightarrow{U(\alpha_{U(F)})} U^3(F) \xrightarrow{U(\alpha_{U^2(F)})} .... \tag{7.2.79}$$

and again, by using the Lemma 7.2.11 together with cofinality, we deduce that this colimit is equivalent to $U(F)^B$.

□

We can now put these results together and show that

Corollary 7.2.14. *For any object $F \in Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp})$ the object F^B satisfies the Bass-Thomason-Trobaugh exact sequences for any $n \in \mathbb{Z}$.*

Proof. By combining the Propositions 7.2.12 and 7.2.13 we deduce that the canonical map $F^B \to U(F^B)$ is an equivalence. Therefore, we have a pullback-pushout square

$$\begin{tikzcd} (F^B)_{L_{pe}(\mathbb{A}^1)} \coprod_{F^B} (F^B)_{L_{pe}(\mathbb{A}^1)} \arrow[r] \arrow[d] & 0 \arrow[d] \\ (F^B)_{L_{pe}(\mathbb{G}_m)} \arrow[r] & \Sigma F^B \end{tikzcd} \tag{7.2.80}$$

and using exactly the same arguments as in 7.2.0.1 we find that for any $T_{\mathcal{X}} \in \mathcal{D}g(k)^{ft}$, the associated long exact sequence breaks up into short exact sequences

$$0 \to \pi_n(F^B(L_{pe}(\mathbb{A}^1) \otimes T_{\mathcal{X}}) \coprod_{F^B(T_{\mathcal{X}})} F^B(L_{pe}(\mathbb{A}^1) \otimes T_{\mathcal{X}})) \to \pi_n(F^B(L_{pe}(\mathbb{G}_m) \otimes T_{\mathcal{X}})) \to \pi_{n-1}(F^B(T_{\mathcal{X}})) \to 0 \tag{7.2.81}$$

and again by the same arguments we are able to extract the familiar exact sequences of Bass-Thomason-Trobaugh, for all $n \in \mathbb{Z}$. □

Remark 7.2.15. As the canonical map $F^B \to U(F^B)$ is an equivalence it follows from the Construction 7.2.9 that when we construct the diagram (7.2.70) with F^B

$$\begin{tikzcd} & \Omega(F^B_{L_{pe}(\mathbb{G}_m)}) \arrow[rr] \arrow[d] & & 0 \arrow[d] \\ F^B \arrow[r] \arrow[ur, dashed, "\sigma_{F^B}"] & U(F^B) \arrow[r] \arrow[d] & (F^B)_{L_{pe}(\mathbb{A}^1)} \coprod_{F^B} (F^B)_{L_{pe}(\mathbb{A}^1)} \arrow[d] \\ & 0 \arrow[r] & (F^B)_{L_{pe}(\mathbb{G}_m)} \end{tikzcd} \tag{7.2.82}$$

the section σ_{F^B} makes F^B a retract of $\Omega(F^B_{L_{pe}(\mathbb{G}_m)})$. In particular, by iteratively applying the construction $\Omega(-)_{L_{pe}(\mathbb{G}_m)}$ we find (because $\Sigma^\infty_+ \circ j_{nc}$ is monoidal) that for any $n \geq 1$, the composition

$$F^B \to \Omega(F^B_{L_{pe}(\mathbb{G}_m)}) \to ... \to \Omega^n(F^B_{L_{pe}(\mathbb{G}_m)^{\otimes n}}) \to ... \to \Omega(F^B_{L_{pe}(\mathbb{G}_m)}) \to F^B \tag{7.2.83}$$

is the identity map so that, for any $n \geq 1$, F^B is a rectract of $\Omega^n(F^B_{L_{pe}(\mathbb{G}_m)^{\otimes n}})$. Equivalently, for any $n \geq 1$, the suspension $\Sigma^n F^B$ is a retract of $(F^B)_{L_{pe}(\mathbb{G}_m)^{\otimes n}}$.

We will now show that the construction $(-)^B$ defines a localization:

Proposition 7.2.16. *The functor $(-)^B : Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp}) \to Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp})$ of the Construction 7.2.9 is an accessible localization functor.*

This result follows from the Lemmas 7.2.10 and 7.2.11 together with the following general result:

Lemma 7.2.17. *Let $\mathcal{C}$ be a presentable $(\infty,1)$-category and let $U : \mathcal{C} \to \mathcal{C}$ be an colimit preserving endofunctor of $\mathcal{C}$, together with a natural transformation $f : Id_{\mathcal{C}} \to U$ such that the two obvious maps $U \circ Id_{\mathcal{C}} \to U^2$ and $Id_{\mathcal{C}} \circ U \to U^2$ are equivalent. Let*

$$\begin{tikzcd} Id \arrow[r, "f"] \arrow[dr, "i_0"'] & U = Id_{\mathcal{C}} \circ U \arrow[r] \arrow[d, "i_1"] & U^2 = Id_{\mathcal{C}} \circ U^2 \arrow[r] \arrow[dl, "i_2"] & .... \\ & T & & \end{tikzcd} \tag{7.2.84}$$

be a colimit cone for the horizontal sequence (indexed by $\mathbb{N}$), necessarily in $Fun^L(\mathcal{C}, \mathcal{C})$. Then, the functor $T : \mathcal{C} \to \mathcal{C}$ provides a reflexive localization of $\mathcal{C}$. Moreover, since T commutes with small colimits the localization is accessible.

Proof. The proof requires some preliminaries. To start with we observe that the arguments in the proof of the Propositions 7.2.12 and 7.2.13 apply mutatis-mutandis to this general situation so that we have natural equivalences $T \circ Id_{\mathcal{C}} \simeq T \circ U$ and $T \circ U \simeq U \circ T$. These two facts combined force the canonical maps

$$T \longrightarrow U \circ T \longrightarrow U^2 \circ T \longrightarrow ... \tag{7.2.85}$$

to be equivalences.

Let us now explain the main proof. For this purpose we will use the description of a reflexive localization functor given in [99, 5.2.7.4-(3)]. Namely, for a functor $T : \mathcal{C} \to \mathcal{C}$ from an $(\infty,1)$-category $\mathcal{C}$ to itself to provide a reflexive localization of $\mathcal{C}$ (which we recall, means that T factors as $\mathcal{C} \to \mathcal{C}_0 \subseteq \mathcal{C}$ with $\mathcal{C}_0$ a full subcategory of $\mathcal{C}$, $\mathcal{C}_0 \subseteq \mathcal{C}$ the inclusion and $\mathcal{C} \to \mathcal{C}_0$ a left adjoint to the inclusion $\mathcal{C}_0 \subseteq \mathcal{C}$) it is enough to have T equipped with a natural transformation $\alpha : Id_{\mathcal{C}} \to T$ such that for every object $X \in \mathcal{C}$, the morphisms $\alpha_{T(X)}$ and $T(\alpha_X)$ are equivalences .

In our case, we let α be the canonical natural transformation $i_0 : Id_{\mathcal{C}} \to T$ appearing in the colimit cone (7.2.84). We show that for any $X \in \mathcal{C}$, the maps $(i_0)_{T(X)}$ and $T((i_0)_X)$ are equivalences. The first follows immediately from our preliminaries: since all the maps in the sequence

$$T(X) \to U(T(X)) \to U^2(T(X)) \to ... \tag{7.2.86}$$

are equivalences and $(i_0)_{T(X)}$ is by definition the first structural map in the colimit cone of this sequence, it is also an equivalence.

Let us now discuss $T((i_0)_X)$. By construction of the functor T, this is the map $colim_{n\in\mathbb{N}} U^n((i_0)_X) : T(X) \to T(T(X))$ induced by the universal property of colimits by means of the morphism of sequences

$$\begin{tikzcd} X \arrow[r] \arrow[d, "(i_0)_X"] & U(X) \arrow[r] \arrow[d, dashed, "U((i_0)_X)"] & U^2(X) \arrow[r] \arrow[d, dashed, "U^2((i_0)_X)"] & ... \\ T(X) \arrow[r] & U(T(X)) \arrow[r] & U^2(T(X)) \arrow[r] & ... \end{tikzcd} \tag{7.2.87}$$

We will prove that

(i) For any X there is a canonical homotopy between the maps $U((i_0)_X)$ and $(i_0)_{U(X)}$. By induction we get canonical homotopies between $U^n((i_0)_X)$ and $(i_0)_{U^n(X)}$;

(ii) For any diagram $I \to \mathcal{C}$ in $\mathcal{C}$ (denoted as $\{X_k\}_{k\in I}$), there is a canonical homotopy between the maps $colim_{k\in I}\,((i_0)_{X_k})$ and $(i_0)_{colim_{k\in I}\,(X_k)}$.

so that by combining these two results we get

$$T((i_0)_X) \simeq colim_{n\in\mathbb{N}}\, U^n((i_0)_X) \simeq colim_{n\in\mathbb{N}}\,((i_0)_{U^n(X)}) \simeq (i_0)_{colim_{n\in\mathbb{N}}\,(U^n(X))} \simeq (i_0)_{T(X)} \quad (7.2.88)$$

and since we already know that $(i_0)_{T(X)}$ is an equivalence, we deduce the same for $T((i_0)_X)$.

To prove (i) we observe the existence of a canonical commutative triangle

$$\begin{tikzcd} T(U(X)) \arrow[r, dashed] & U(T(X)) \\ U(X) \arrow[u, "(i_0)_{U(X)}"] \arrow[ur, "U((i_0)_X)"'] & \end{tikzcd} \quad (7.2.89)$$

provided by the universal property of the colimit defining T. As explained in the preliminaries this dotted map is an equivalence so that the commutativity of this diagram holds the desired homotopy.

Let us now prove (ii). Let $I^{\triangleright} \to \mathcal{C}$ be a colimit diagram in $\mathcal{C}$ (which, by abusing the notation we denote as $\{X_k, \phi_k : X_k \to colim_{k\in I}\, X_k\}_{k\in I}$). Since i_0 is a natural transformation we find for any $k \in$I a commutative diagram

$$\begin{tikzcd} X_k \arrow[r, "\phi_k"] \arrow[d, "(i_0)_{(X_k)}"'] & colim_{k\in I}\, X_k \arrow[d, "(i_0)_{(colim_{k\in I}\, X_k)}"] \\ T(X_k) \arrow[r, "T(\phi_k)"] & T(colim_{k\in I}\, X_k) \end{tikzcd} \quad (7.2.90)$$

and the universal property of colimits allows us to factor the lower horizontal arrows as

$$\begin{tikzcd} T(X_k) \arrow[r] & colim_{k\in I}\, T(X_k) \arrow[r, dashed, "\theta"] & T(colim_{k\in I}\, X_k) \end{tikzcd} \quad (7.2.91)$$

where the dotted map θ is essentially unique. More importantly, since by construction T commutes with colimits, θ is an equivalence.

At the same time, the map $colim_{k\in I}\,(i_0)_{(X_k)}$ is by definition the essentially unique map $colim_{k\in I}\, X_k \to colim_{k\in I}\, T(X_k)$, induced by the universal property colimits, that makes the diagrams

$$\begin{tikzcd} X_k \arrow[r, "\phi_k"] \arrow[d, "(i_0)_{(X_k)}"'] & colim_{k\in I}\, X_k \arrow[d, dashed, "colim_{k\in I}\,(i_0)_{(X_k)}"] \\ T(X_k) \arrow[r] & colim_{k\in I} T(X_k) \end{tikzcd} \quad (7.2.92)$$

commute. Finally, since θ is an equivalence, the commutativity of (7.2.90) implies the commutativity of

$$\begin{tikzcd} X_k \arrow[r, "\phi_k"] \arrow[d, "(i_0)_{(X_k)}"'] & colim_{k\in I}\, X_k \arrow[d, "\theta^{-1}\circ(i_0)_{(colim_{k\in I}\, X_k)}"] \\ T(X_k) \arrow[r] & colim_{k\in I}\, T(X_k) \end{tikzcd} \quad (7.2.93)$$

so that by the uniqueness property that defines $colim_{k\in I}(i_0)(X_k)$, the diagram

$$\begin{tikzcd} colim_{k\in I}\,(X_k) \arrow[d, "colim_{k\in I}\,(i_0)_{(X_k)}"'] \arrow[dr, "(i_0)_{(colim_{k\in I}\,X_k)}"] & \\ colim_{k\in I}\,T(X_k) \arrow[r, "\theta"] & T(colim_{k\in I}\,X_k) \end{tikzcd} \tag{7.2.94}$$

must commute. This provides the desired homotopy and concludes (ii) and the proof.

□

Remark 7.2.18. It follows from the Proposition 7.2.16 and from the Construction 7.2.9 that an object $F \in Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp})$ is local with respect to the localization $(-)^B$ if and only if the diagram

$$\begin{tikzcd} F \arrow[r] \arrow[d] & F_{L_{pe}(\mathbb{A}^1)} \coprod_F F_{L_{pe}(\mathbb{A}^1)} \arrow[d] \\ 0 \arrow[r] & F_{L_{pe}(\mathbb{G}_m)} \end{tikzcd} \tag{7.2.95}$$

is a pullback-pushout square. In particular, the discussion in 7.2.0.1 implies that any Nisnevich local object F is $(-)^B$-local.

We now come to a series of technical steps in order prove each of the items in 7.2.8. First thing, we give a precise sense to what it means for a functor F with connective values to satisfy all the Bass exact sequences for $n \geq 1$.

Definition 7.2.19. *Let $F \in Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp}_{\geq 0})$ and consider its associated diagram (7.2.70) (constructed in $Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp})$, where we identify F with its inclusion). We say that F satisfies all Bass exact sequences for $n \geq 1$ if the canonical induced map of connective functors $F \to \tau_{\geq 0}U(F)$ is an equivalence, or, in other words, since $\tau_{\geq 0}$ commutes with limits and because of the definition of $U(F)$, if the diagram (7.2.95) is a pullback in $Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp}_{\geq 0})$.*

Remark 7.2.20. Let $F \in Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp}_{\geq 0})$ and consider the pullback-pushout diagram in $Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp})$

$$\begin{tikzcd} \Omega(F_{L_{pe}(\mathbb{A}^1)} \coprod_F F_{L_{pe}(\mathbb{A}^1)}) \arrow[r] \arrow[d] & \Omega(F_{L_{pe}(\mathbb{G}_m)}) \arrow[d] \\ 0 \arrow[r] & U(F) \end{tikzcd} \tag{7.2.96}$$

Since, $\tau_{\geq 0}$ preserves pullbacks, we obtain a pullback diagram in $Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp}_{\geq 0})$

$$\begin{tikzcd} \tau_{\geq 0}\Omega(F_{L_{pe}(\mathbb{A}^1)} \coprod_F F_{L_{pe}(\mathbb{A}^1)}) \arrow[r] \arrow[d] & \tau_{\geq 0}\Omega(F_{L_{pe}(\mathbb{G}_m)}) \arrow[d] \\ 0 \arrow[r] & \tau_{\geq 0}U(F) \end{tikzcd} \tag{7.2.97}$$

If F satisfies the condition in the previous definition, then the zero truncation of the composition

$$\begin{tikzcd} F \arrow[r, dashed, "\sigma_F"] & \Omega(F_{L_{pe}(\mathbb{G}_m)}) \arrow[r] & U(F) \end{tikzcd} \tag{7.2.98}$$

makes F a retract of $\tau_{\geq 0}\Omega(F_{L_{pe}(\mathbb{G}_m)})$. With this, and as before, once evaluated at $T_{\mathcal{X}} \in \mathcal{D}g(k)^{ft}$, the long exact sequence associated to the pullback (7.2.97) splits up into short exact sequences

$$0 \to \pi_n(F(L_{pe}(\mathbb{A}^1) \otimes T_{\mathcal{X}}) \coprod_{F(T_{\mathcal{X}})} F(L_{pe}(\mathbb{A}^1) \otimes T_{\mathcal{X}})) \to \pi_n(F(L_{pe}(\mathbb{G}_m) \otimes T_{\mathcal{X}})) \to \pi_{n-1}(F(T_{\mathcal{X}})) \to 0 \tag{7.2.99}$$

$\forall n \geq 1$, and again by the same arguments, we can extract the exact sequences of Bass-Thomason.

Lemma 7.2.21. *If $F \in Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp})$ has connective values and satisties all the Bass exact sequences for $n \geq 1$ (in the sense of the Definition 7.2.19), then the canonical map $F \simeq \tau_{\geq 0}F \to \tau_{\geq 0}F^B$ is an equivalence.*

Proof. Assuming that F satisfies the condition in the Definition 7.2.19, meaning the canonical map $F \to \tau_{\geq 0}U(F)$ is an equivalence, we will show that for any $k \geq 2$, the canonical map $F \to \tau_{\geq 0}U^k(F)$ is an equivalence. Once we have this, the conclusion of the lemma will follow from the fact $\tau_{\geq 0}$ commutes with filtered colimits (because the t-structure in $\widehat{Sp}$ is determined by the stable homotopy groups and these commute with filtered colimits), so that

$$\tau_{\geq 0}(F^B) \simeq \tau_{\geq 0}(colim_{i\in\mathbb{N}}U^i(F)) \simeq colim_{i\in\mathbb{N}}\tau_{\geq 0}(U^i(F)) \simeq colim_{i\in\mathbb{N}}F \simeq F \tag{7.2.100}$$

So, let us prove the assertion for $k = 2$. By definition, we have a pullback-pushout square in $Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp})$

$$\begin{tikzcd} U^2(F) \arrow[r] \arrow[d] & U(F)_{L_{pe}(\mathbb{A}^1)} \coprod_{U(F)} U(F)_{L_{pe}(\mathbb{A}^1)} \arrow[d] \\ 0 \arrow[r] & U(F)_{L_{pe}(\mathbb{G}_m)} \end{tikzcd} \tag{7.2.101}$$

and since $\tau_{\geq 0}$ preserves pullbacks, we find

$$\tau_{\geq 0}U^2(F) \simeq \tau_{\geq 0}(U(F)_{L_{pe}(\mathbb{A}^1)} \coprod_{U(F)} U(F)_{L_{pe}(\mathbb{A}^1)}) \times_{\tau_{\geq 0}(U(F)_{L_{pe}(\mathbb{G}_m)})} 0 \tag{7.2.102}$$

We observe that

(i) $\tau_{\geq 0}(U(F)_{L_{pe}(\mathbb{G}_m)}) \simeq F_{L_{pe}(\mathbb{G}_m)}$.

(ii) $\tau_{\geq 0}(U(F)_{L_{pe}(\mathbb{A}^1)} \coprod_{U(F)} U(F)_{L_{pe}(\mathbb{A}^1)}) \simeq F_{L_{pe}(\mathbb{A}^1)} \coprod_F F_{L_{pe}(\mathbb{A}^1)}$

To deduce the first equivalence, we use the equivalence $\tau_{\geq 0}U(F) \simeq F$ together with the fact that $(-)_{L_{pe}(\mathbb{G}_m)}$ commutes with $\tau_{\geq 0}$. The second equivalence requires a more sophisticated discussion. Recall from the section 7.2.0.1 that for any $G \in Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp})$ we are able to construct a pushout square in $Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp})$

$$\begin{tikzcd} G \arrow[r, "i_1^G"] \arrow[d] & G \oplus G \arrow[r] & G_{L_{pe}(\mathbb{P}^1)} \arrow[r] & G_{L_{pe}(\mathbb{A}^1)} \oplus G_{L_{pe}(\mathbb{A}^1)} \arrow[d] \\ 0 \arrow[rrr] & & & G_{L_{pe}(\mathbb{A}^1)} \coprod_G G_{L_{pe}(\mathbb{A}^1)} \end{tikzcd} \tag{7.2.103}$$

such that the top horizontal composition admits a left inverse. Applying this construction to $G = F$ and to $G = U(F)$, we construct a map between the associated pullback-pushout squares

$$\begin{tikzcd}
 & U(F) \arrow[rr] \arrow[dd] & & U(F)_{L_{pe}(\mathbb{A}^1)} \oplus U(F)_{L_{pe}(\mathbb{A}^1)} \arrow[dd] \\
F \arrow[ur] \arrow[rr] \arrow[dd] & & F_{L_{pe}(\mathbb{A}^1)} \oplus F_{L_{pe}(\mathbb{A}^1)} \arrow[ur] \arrow[dd] & \\
 & 0 \arrow[rr] & & U(F)_{L_{pe}(\mathbb{A}^1)} \coprod_{U(F)} U(F)_{L_{pe}(\mathbb{A}^1)} \\
0 \arrow[ur] \arrow[rr] & & F_{L_{pe}(\mathbb{A}^1)} \coprod_F F_{L_{pe}(\mathbb{A}^1)} \arrow[ur] &
\end{tikzcd} \tag{7.2.104}$$

(obtained using the natural transformation $\underline{Hom}(-,F) \to \underline{Hom}(-,U(F))$ induced by canonical morphism $F \to U(F)$).

Both the front and back faces are pullback-pushouts and both the top horizontal maps admite left-inverses.

Finally, since $\tau_{\geq 0}U(F) \simeq F$ and because the top horizontal maps admit left-inverses, the long exact sequences associated to each square breaks up into short exact sequences, and for each $n \geq 0$ and each $T_{\mathcal{X}} \in \mathcal{D}g(k)^{ft}$ we find natural maps of short exact sequences

$$\begin{tikzcd}
\pi_n(U(F)(T_{\mathcal{X}})) \arrow[r] & \pi_n((U(F)_{L_{pe}(\mathbb{A}^1)} \oplus U(F)_{L_{pe}(\mathbb{A}^1)})(T_{\mathcal{X}})) \arrow[r] & \pi_n((U(F)_{L_{pe}(\mathbb{A}^1)} \coprod_{U(F)} U(F)_{L_{pe}(\mathbb{A}^1)})(T_{\mathcal{X}})) \\
\pi_n(F(T_{\mathcal{X}})) \arrow[r] \arrow[u, "\sim"] & \pi_n((F_{L_{pe}(\mathbb{A}^1)} \oplus F_{L_{pe}(\mathbb{A}^1)})(T_{\mathcal{X}})) \arrow[r] \arrow[u, "\sim"] & \pi_n((F_{L_{pe}(\mathbb{A}^1)} \coprod_F F_{L_{pe}(\mathbb{A}^1)})(T_{\mathcal{X}})) \arrow[u]
\end{tikzcd} \tag{7.2.105}$$

implying the equivalence in (ii).

Finally, we deal with the case $k > 2$. Applying the same strategy for $G = F$ and $G = U^k(F)$, we consider the analogue of the diagram (7.2.104) induced by the canonical morphism $F \to U^k(F)$. By induction, we deduce that $\tau_{\geq 0}U^{k+1}(F) \simeq F$. This concludes the proof.

□

Proposition 7.2.22. *Let F be a connectively-Nisnevich local object. Then, it satisfies the Projective Bundle Theorem and all the Bass exact sequences for $n \geq 1$. In particular, by the Lemma 7.2.21 we have $F \simeq \tau_{\geq 0}F \simeq \tau_{\geq 0}F^B$.*

Proof. To start with, we prove that if is F connectively-Nisnevich local then it satisfies the Projective bundle theorem. Indeed, we can use the arguments used in 7.2.0.1 together with the definition of being connectively-Nisnevich local to construct a pullback diagram in $Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp}_{\geq 0})$

$$\begin{tikzcd}
F \simeq F_{L_{pe}(k)} \arrow[r] \arrow[d] & F_{L_{pe}(\mathbb{P}^1)} \arrow[d] \\
0 \arrow[r] & F \simeq F_{L_{pe}(k)}
\end{tikzcd} \tag{7.2.106}$$

with splittings, which, as explained in the Remark 7.2.5, provide a canonical equivalence $F_{L_{pe}(\mathbb{P}^1)} \simeq F \oplus F$ in $Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp})$. Secondly, and again by the definition of connectively-Nisnevich local, we can easily deduce that the canonical diagram

$$\begin{tikzcd} F_{L_{pe}(\mathbb{P}^1)} \arrow[r] \arrow[d] & F_{L_{pe}(\mathbb{A}^1)} \oplus F_{L_{pe}(\mathbb{A}^1)} \arrow[d] \\ 0 \arrow[r] & F_{L_{pe}(\mathbb{G}_m)} \end{tikzcd} \tag{7.2.107}$$

associated to the covering of $\mathbb{P}^1$ by two affine lines (7.2.10) is a pullback in $Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp}_{\geq 0})$.

With these two ingredients we prove that if is F connectively-Nisnevich local then it satisfies all the Bass exact sequences for $n \geq 1$ in the sense of the Definition 7.2.19, namely, we show that the canonical map $F \simeq \tau_{\geq 0}F \to \tau_{\geq 0}U(F)$ is an equivalence, or, in other words, that the diagram (7.2.95) is a pullback within connective functors.

Consider the pushout squares in $Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp})$ described in (7.2.40). More precisely, since F satisfies the Projective bundle theorem, we are interested in the pullback-pushout square

$$\begin{tikzcd} F \oplus F \simeq F_{L_{pe}(\mathbb{P}^1)} \arrow[r] \arrow[d] & F_{L_{pe}(\mathbb{A}^1)} \oplus F_{L_{pe}(\mathbb{A}^1)} \arrow[d] \\ F \simeq F \coprod_{F \oplus F} F_{L_{pe}(\mathbb{P}^1)} \arrow[r] & F_{L_{pe}(\mathbb{A}^1)} \coprod_F F_{L_{pe}(\mathbb{A}^1)} \end{tikzcd} \tag{7.2.108}$$

which, in particular, is a pullback square in $Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp}_{\geq 0})$ once truncated at level zero. Combining with the pullback square (7.2.107) we find a series of pullback squares in $Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp}_{\geq 0})$.

$$\begin{tikzcd} \Omega F \arrow[r] \arrow[d] & 0 \arrow[d] & \\ \Omega(F_{L_{pe}(\mathbb{A}^1)} \coprod_F F_{L_{pe}(\mathbb{A}^1)}) \arrow[r] \arrow[dd] & \Omega(F_{L_{pe}(\mathbb{G}_m)}) \arrow[r] \arrow[d] & 0 \arrow[d] \\ & F_{L_{pe}(\mathbb{P}^1)} \arrow[r] \arrow[d] & F_{L_{pe}(\mathbb{A}^1)} \oplus F_{L_{pe}(\mathbb{A}^1)} \arrow[d] \\ 0 \arrow[r] & F \arrow[r] & F_{L_{pe}(\mathbb{A}^1)} \coprod_F F_{L_{pe}(\mathbb{A}^1)} \end{tikzcd} \tag{7.2.109}$$

Now comes the important ingredient: since the diagram (7.2.107) is a pullback, we can still deduce (as before) the existence of a canonical map σ_F such that the composition

$$\begin{tikzcd} F \arrow[r, dashed, "\sigma_F"] & \Omega F_{L_{pe}(\mathbb{G}_m)} \arrow[r] & F_{L_{pe}(\mathbb{P}^1)} \arrow[r] & F \end{tikzcd} \tag{7.2.110}$$

is the identity. We now explain how the existence of this section allows us to prove that the diagram (7.2.95) is a pullback. More precisely, by using σ_F at each copy of F in (7.2.95) and applying the construction $\Omega(-)_{L_{pe}(\mathbb{G}_m)}$ we find the square (7.2.95) as a retract of the square

$$\begin{tikzcd} \Omega F_{L_{pe}(\mathbb{G}_m)} \arrow[r] \arrow[d] & (\Omega F_{L_{pe}(\mathbb{G}_m)})_{L_{pe}(\mathbb{A}^1)} \coprod_{\Omega(F)_{L_{pe}(\mathbb{G}_m)}} (\Omega F_{L_{pe}(\mathbb{G}_m)})_{L_{pe}(\mathbb{A}^1)} \arrow[d] \\ 0 \arrow[r] & \Omega(F_{L_{pe}(\mathbb{G}_m)})_{L_{pe}(\mathbb{G}_m)} \end{tikzcd} \tag{7.2.111}$$

but since both Ω and $\underline{Hom}(\Sigma^\infty_+ \circ j_{nc}(L_{pe}(\mathbb{G}_m)), -)$ commute with colimits, we can easily indentify this last square with the image of the top left pullback square in (7.2.109) under $\underline{Hom}(\Sigma^\infty_+ \circ j_{nc}(L_{pe}(\mathbb{G}_m)), -)$ and conclude that this is also a pullback square. We conclude the proof using the fact that the rectract of a pullback square is a pullback.

$\square$

We now address the second item of the Proposition 7.2.8, namely,

Proposition 7.2.23. *Let $F \in Fun_{Nis_{\geq 0}}(\mathcal{D}g(k)^{ft}, \widehat{Sp}_{\geq 0})$. Then, the object $(i \circ \alpha(F))^B$ is Nisnevich local.*

The proof of this proposition is based on a very helpful criterium to decide if a given F is Nisnevich local by studying its truncations $\tau_{\geq 0}\Sigma^n F$, namely:

Lemma 7.2.24. *Let F be any object in $Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp})$. Then, if for any $n \geq 0$ the truncations $\tau_{\geq 0}\Sigma^n F$ are connectively-Nisnevich local, the object F itself is Nisnevich local.*

This lemma follows from a somewhat more general situation, which we isolate in the following remark:

Remark 7.2.25. Let $\mathcal{C}$ be a stable $(\infty, 1)$-category with a right-complete t-structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ and let $\tau_{\geq n}$ and $\tau_{\leq n}$ denote the associated truncation functors (see [100, Section 1.2.1] for the complete details or our discussion in section 2.1.25 for a fast review of the subject). We observe that a commutative square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd} \tag{7.2.112}$$

in $\mathcal{C}$ is a pullback (therefore pushout) if and only if for any $n \leq 0$ the truncated squares

$$\begin{tikzcd} \tau_{\geq n}A \arrow[r] \arrow[d] & \tau_{\geq n}B \arrow[d] \\ \tau_{\geq n}C \arrow[r] & \tau_{\geq n}D \end{tikzcd} \tag{7.2.113}$$

are pullbacks in $\mathcal{C}_{\geq n}$. Indeed, if we let H denote the pullback of the square in $\mathcal{C}$, we want to show that the canonical map $A \to H$ in $\mathcal{C}$ induced by the universal property of the pullback, is an equivalence. But, since the truncation functors $\tau_{\geq n}$ are right adjoints to the inclusions $\mathcal{C}_{\geq n} \subseteq \mathcal{C}$, $\tau_{\geq n}H$ is a pullback for the square in $\mathcal{C}_{\geq n}$ and therefore the induced maps $\tau_{\geq n}A \to \tau_{\geq n}H$ are equivalences for all $n \leq 0$. To conclude, we are reduce to show that if a map $f : X \to Y$ in $\mathcal{C}$ induces equivalences $\tau_{\geq n}X \simeq \tau_{\geq n}Y$ for all $n \leq 0$ then the map f itself is an equivalence. To see this, and because $\mathcal{C}$ is stable if suffices to check that the fiber $fib(f)$ is equivalent to zero. This fiber fits in pullback-pushout square

$$\begin{tikzcd} fib(f) \arrow[r] \arrow[d] & X \arrow[d] \\ 0 \arrow[r] & Y \end{tikzcd} \tag{7.2.114}$$

and since $\tau_{\geq n}$ commutes with pullbacks and the maps $\tau_{\geq n}X \to \tau_{\geq n}Y$ are equivalences, we find that for any $n \leq 0$ we have $\tau_{\geq n}fib(f) \simeq 0$. Finally, we use the canonical pullback-pushout squares in $\mathcal{C}$

$$\begin{tikzcd} \tau_{\geq n}fib(f) \arrow[r] \arrow[d] & fib(f) \arrow[d] \\ 0 \arrow[r] & \tau_{\leq n-1}fib(f) \end{tikzcd} \tag{7.2.115}$$

to deduce that for all $n \leq -1$ the map $fib(f) \to \tau_{\leq n}fib(f)$ is an equivalence. In particular $fib(f)$ belongs to the intersection $\cap_n \mathcal{C}_{\leq n}$ so that, since the t-structure is assumed to be right-complete, we have $fib(f) \simeq 0$.

In particular, since the truncations $\tau_{\geq n}$ can be obtained as the compositions $\Omega^n \circ \tau_{\geq 0} \circ \Sigma^n$ and since Ω commutes with limits, the previous discussion implies that for the square (7.2.112) to be a pullback in $\mathcal{C}$ it suffices to have for each $n \geq 0$, the induced square

$$\begin{tikzcd} \tau_{\geq 0}\Sigma^n A \arrow[r] \arrow[d] & \tau_{\geq 0}\Sigma^n B \arrow[d] \\ \tau_{\geq 0}\Sigma^n C \arrow[r] & \tau_{\geq 0}\Sigma^n D \end{tikzcd} \tag{7.2.116}$$

a pullback in $\mathcal{C}_{\geq 0}$.

Proof of the Lemma 7.2.24: Just apply the Remark 7.2.25 to the commutative squares of spectra

$$\begin{tikzcd} F(T_{\mathcal{X}}) \arrow[r] \arrow[d] & F(T_{\mathcal{U}}) \arrow[d] \\ F(T_{\mathcal{V}}) \arrow[r] & F(T_{\mathcal{W}}) \end{tikzcd} \tag{7.2.117}$$

induced by the Nisnevich squares of noncommutative spaces. The discussion therein works because the t-structure in $\widehat{Sp}$ is known to be right-complete (see [100, 1.4.3.6]).

□

Proof of the Proposition 7.2.23: As explained in the Remark 7.2.15, for any $n \geq 1$, the suspension $\Sigma^n F^B$ is a retract of $(F^B)_{L_{pe}(\mathbb{G}_m)^{\otimes n}}$. In particular, $\tau_{\geq 0}\Sigma^n F^B$ is a retract of $\tau_{\geq 0}((F^B)_{L_{pe}(\mathbb{G}_m)^{\otimes n}})$ which is a mere notation for $\tau_{\geq 0}\underline{Hom}(\Sigma^\infty_+ \circ j_{nc}(L_{pe}(\mathbb{G}_m)^{\otimes n}), F^B)$ so that

$$\tau_{\geq 0}((F^B)_{L_{pe}(\mathbb{G}_m)^{\otimes n}}) \simeq \underline{Hom}(\Sigma^\infty_+ \circ j_{nc}(L_{pe}(\mathbb{G}_m)^{\otimes n}), \tau_{\geq 0}F^B) \simeq \underline{Hom}(\Sigma^\infty_+ \circ j_{nc}(L_{pe}(\mathbb{G}_m)^{\otimes n}), F) \tag{7.2.118}$$

where the first equivalence follows because the t-structure in $Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp})$ is determined object-wise by the t-structure in Sp and the second follows from the Proposition 7.2.22. In particular, since F is connectively-Nisnevich local, $\underline{Hom}(\Sigma^\infty_+ \circ j_{nc}(L_{pe}(\mathbb{G}_m)^{\otimes n}), F)$ is also connectively-Nisnevich local so that $\tau_{\geq 0}\Sigma^n F^B$ is the retract of a connectively-Nisnevich local and therefore, it is itself local [6]. We conclude using the Lemma 7.2.24, observing that for $n = 0$ the condition follows by the hypothesis that F is connectively-Nisnevich local.

□

Finally,

Corollary 7.2.26. *Let F be any object in $Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp})$. Then, there is a canonical equivalence*

$$(i \circ \alpha(F))^B \simeq l^{nc}_{Nis}((i \circ \alpha(F)) \tag{7.2.119}$$

Proof. This follows from the Proposition 7.2.16, the Remark 7.2.18 and the Proposition 7.2.23, using the universal properties of the two localizations. □

Proof of the Proposition 7.2.8: The three items correspond, respectively to the Propositions 7.2.22, 7.2.23 and to the Corollary 7.2.26. The conclusion now follows from the universal property of the two localizations.

□

[6] In general, the retract of a local object in a reflexive localization is local. This is, ultimately, because the retract of an equivalence is an equivalence.

7.3 Comparing the commutative and the noncommutative $\mathbb{A}^1$-localizations

In this section we prove the Theorem 7.0.31. We start by asking the reader to recall the diagrams (7.0.1) and (7.0.2) and to recall that after the Theorem 7.0.29, together with Yoneda's lemma, $\mathcal{M}_2(l^{nc}_{Nis}(K^c))$ is the Bass-Thomason-Trobaugh K-theory of schemes. Recall also that, by definition[7], Weibel's homotopy invariant K-theory of [154] is the "commutative" localization $l_{\mathbb{A}^1}(\mathcal{M}_2(l^{nc}_{Nis}(K^c))$. With these ingredientes the conclusion of 7.0.31 will follow if we prove that the commutative and noncommutative versions of the $\mathbb{A}^1$-localizations make the diagram

$$\begin{tikzcd} Fun_{Nis}(N(AffSm^{ft}(k))^{op}, \widehat{Sp}) \arrow[d, "l_{\mathbb{A}^1}"] & Fun_{Nis}(\mathcal{D}g(k)^{ft}, \widehat{Sp}) \arrow[l, "\mathcal{M}_2"'] \arrow[d, "l^{nc}_{\mathbb{A}^1}"] \\ Fun_{Nis,\mathbb{A}^1}(N(AffSm^{ft}(k))^{op}, \widehat{Sp}) & Fun_{Nis,L_{pe}(\mathbb{A}^1)}(\mathcal{D}g(k)^{ft}, \widehat{Sp}) \arrow[l, "\mathcal{M}_3"'] \end{tikzcd} \tag{7.3.1}$$

commute. In fact, we will be able to prove something slightly more general. We begin by recalling a well-known explicit formula for the $\mathbb{A}^1$-localization of presheaves of spectra. Let $\Delta_{\mathbb{A}^1}$ be the cosimplicial affine scheme given by

$$\Delta^n_{\mathbb{A}^1} := Spec(k[t_0, ..., t_n]/(t_0 + ... + t_n - 1)) \tag{7.3.2}$$

Notice that at each level we have (non-canonical) isomorphisms $\Delta^n_{\mathbb{A}^1} \simeq (\mathbb{A}^1_k)^n$. After [29], the endofunctor of $\mathcal{C} = Fun(N(AffSm^{ft}(k))^{op}, \widehat{Sp})$ defined by the formula

$$F \mapsto colim_{n\in\Delta^{op}} \underline{Hom}(\Delta^n_{\mathbb{A}^1}, F) \tag{7.3.3}$$

with $\underline{Hom}$ the internal-hom for presheaves of spectra, is an explicit model for the $\mathbb{A}^1$-localization in the commutative world. To see that this indeed gives something $\mathbb{A}^1$-local we use the $\mathbb{A}^1$-homotopy $m : \mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$ between the identity of $\mathbb{A}^1$ and the constant map at zero. The map m is given by the usual multiplication. It follows from this explicit description that the $\mathbb{A}^1$-localization preserves Nisnevich local objects (this is because in a stable context, sifted colimits commute with pullbacks and the Nisnevich local condition is determined by certain squares being pullbacks).

The important point now is that this mechanism applies mutadis-mutandis in the noncommutative world. Indeed, by taking the composition

$$\Delta^{nc}_{\mathbb{A}^1} : \Delta \xrightarrow{\Delta_{\mathbb{A}^1}} N(AffSm^{ft}(k)) \xrightarrow{L_{pe}} \mathcal{N}cS(k) \tag{7.3.4}$$

we obtain a cosimplicial noncommutative space and as L_{pe} is monoidal we get $(\Delta^{nc,n}_{\mathbb{A}^1}) \simeq L_{pe}(\mathbb{A}^1)^{\otimes_n}$. Moreover, we can use exactly the same arguments to prove that the endofunctor of $\mathcal{C} = Fun(\mathcal{N}cS(k)^{op}, \widehat{Sp})$ defined by the formula

$$F \mapsto colim_{n\in\Delta^{op}} \underline{Hom}(\Delta^{nc,n}_{\mathbb{A}^1}, F) \tag{7.3.5}$$

is an explicit model for the noncommutative $\mathbb{A}^1$-localization functor on spectral presheaves and also by the same arguments, we conclude that Nisnevich local objects are preserved under this localization.

With this we can now reduce the proof that the diagram 7.3.1 commutes to the proof that the following diagram commutes

[7] Either we take it as a definition or as a consequence of the explicit formula given in this section.

$$\begin{tikzcd} Fun(N(AffSm^{ft}(k))^{op}, \widehat{Sp}) \arrow[d, "l_{\mathbb{A}^1}"] & Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp}) \arrow[l, "\mathcal{M}_1"'] \arrow[d, "l_{\mathbb{A}^1}^{nc}"] \\ Fun_{\mathbb{A}^1}(N(AffSm^{ft}(k))^{op}, \widehat{Sp}) & Fun_{L_{pe}(\mathbb{A}^1)}(\mathcal{D}g(k)^{ft}, \widehat{Sp}) \arrow[l, "\mathcal{M}'"'] \end{tikzcd} \tag{7.3.6}$$

where the lower part corresponds to the reflexive $\mathbb{A}^1$-localizations and $\mathcal{M}'$ is the right adjoint of this context obtained by the same formal arguments as $\mathcal{M}_2$ and $\mathcal{M}_3$. The commutativity of this diagram is measured by the existence of a canonical natural transformation of functors $l_{\mathbb{A}^1} \circ \mathcal{M}_1 \to \mathcal{M}' \circ l_{\mathbb{A}^1}^{nc}$ induced by the fact that $\mathcal{M}'$ sends $L_{pe}(\mathbb{A}^1)$-local objects to $\mathbb{A}^1$-local objects, together with the universal property of $l_{\mathbb{A}^1}$. The diagram commutes if and only if this natural transformation is an equivalence of functors. In particular, since the diagram of right adjoints commutes

$$\begin{tikzcd} Fun(N(AffSm^{ft}(k))^{op}, \widehat{Sp}) & Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp}) \arrow[l, "\mathcal{M}_1"'] \\ Fun_{\mathbb{A}^1}(N(AffSm^{ft}(k))^{op}, \widehat{Sp}) \arrow[u, hook, "\alpha"] & Fun_{L_{pe}(\mathbb{A}^1)}(\mathcal{D}g(k)^{ft}, \widehat{Sp}) \arrow[l, "\mathcal{M}'"'] \arrow[u, hook, "\beta"] \end{tikzcd} \tag{7.3.7}$$

and the vertical maps are fully-faithful, it will be enough to show that the induced natural transformation $\alpha \circ l_{\mathbb{A}^1} \circ \mathcal{M}_1 \to \alpha \circ \mathcal{M}' \circ l_{\mathbb{A}^1}^{nc}$ is an equivalence. But now, using our explicit descriptions for the $\mathbb{A}^1$-localization functors we know that for each $F \in Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp})$ we have

$$\alpha \circ l_{\mathbb{A}^1}(\mathcal{M}_1(F)) \simeq colim_{n\in\Delta^{op}}\, \underline{Hom}(\Sigma^{\infty}_{+} \circ j(\mathbb{A}^1)^{\otimes^n}, \mathcal{M}_1(F)) \simeq \tag{7.3.8}$$

$$\simeq colim_{n\in\Delta^{op}}\, \mathcal{M}_1(\underline{Hom}(\Sigma^{\infty}_{+} \circ j_{nc}(L_{pe}(\mathbb{A}^1))^{\otimes^n}, F) \simeq \tag{7.3.9}$$

$$\simeq \mathcal{M}_1(colim_{n\in\Delta^{op}}\, \underline{Hom}(\Sigma^{\infty}_{+} \circ j_{nc}(L_{pe}(\mathbb{A}^1))^{\otimes^n}, F) \simeq \tag{7.3.10}$$

$$\mathcal{M}_1(\beta \circ l_{\mathbb{A}^1}^{nc}(F)) \simeq \alpha \circ \mathcal{M}' \circ l_{\mathbb{A}^1}^{nc}(F) \tag{7.3.11}$$

where the first and penultimate equivalences follow from the explicit formulas for the $\mathbb{A}^1$-localizations, the middle equivalences follow, respectively, from the Remarks 7.0.27 and 7.0.26 and the last equivalence follows from the commutativity of the diagram (7.3.7).

In particular, when applied to $F = l_{Nis}(K^c)$ we conclude the proof of the Theorem 7.0.31.

7.4 The $\mathbb{A}^1$-localization of non-connective K-theory is the unit non-commutative motive

In this section we prove Theorem 7.0.32. We start by gathering some necessary preliminary remarks. To start with, and as explained in the Remark 6.4.23 we have two different equivalent ways to construct $\mathcal{SH}_{nc}(k)$: one by using presheaves of spaces, forcing Nisnevich descent and $\mathbb{A}^1$-invariance and a second one by using presheaves of spectra and forcing again the Nisnevich and $\mathbb{A}^1$-localizations. These two approaches are related by means of a commutative diagram of monoidal functors

$$\begin{tikzcd}
 & \mathcal{N}cS(k) \arrow[dl] \arrow[dr] & \\
Fun(\mathcal{D}g(k)^{ft},\widehat{\mathcal{S}}) \arrow[rr, "\Sigma^\infty_+"] \arrow[d, "l^{nc}_{0,Nis}"] & & Fun(\mathcal{D}g(k)^{ft},\widehat{Sp}) \arrow[d, "l^{nc}_{Nis}"] \\
Fun_{Nis}(\mathcal{D}g(k)^{ft},\widehat{\mathcal{S}}) \arrow[rr, "\Sigma^\infty_{+,Nis}"] \arrow[d, "l^{nc}_{0,\mathbb{A}^1}"] & & Fun_{Nis}(\mathcal{D}g(k)^{ft},\widehat{Sp}) \arrow[d, "l^{nc}_{\mathbb{A}^1}"] \\
\mathcal{SH}_{nc}(k) := Fun_{Nis,L_{pe}(\mathbb{A}^1)}(\mathcal{D}g(k)^{ft},\widehat{\mathcal{S}}) \arrow[rr, "\Sigma^\infty_{+,Nis,\mathbb{A}^1}", "\sim"'] & & Fun_{Nis,L_{pe}(\mathbb{A}^1)}(\mathcal{D}g(k)^{ft},\widehat{Sp})
\end{tikzcd} \tag{7.4.1}$$

induced by the universal properties involved and the last induced $\Sigma^\infty_{+,Nis,\mathbb{A}^1}$ is an equivalence because of the results in the Proposition 6.4.19. To be completely precise we have to check that the class of maps with respect to which we localize the theory of presheaves of spaces is sent to the class of maps with respect to which we localize spectral presheaves. Following the description of the last given in the Remark 6.4.23 it is enough to see that for any representable object $j(\mathcal{X})$ we have $\Sigma^\infty_+ j(\mathcal{X}) \simeq \delta_{j(\mathcal{X})}(S)$ where the S is the sphere spectrum. This is because $Map^{Sp}(-)$ is an internal-hom in $\widehat{Sp}$ and the sphere spectrum is a unit for the monoidal structure.

In this section we will be considering the associated commutative diagram of right adjoints

$$\begin{tikzcd}
Fun(\mathcal{D}g(k)^{ft},\widehat{\mathcal{S}}) & Fun(\mathcal{D}g(k)^{ft},\widehat{Sp}) \arrow[l, "\Omega^\infty"'] \\
Fun_{Nis}(\mathcal{D}g(k)^{ft},\widehat{\mathcal{S}}) \arrow[u, hook] & Fun_{Nis}(\mathcal{D}g(k)^{ft},\widehat{Sp}) \arrow[l, "\Omega^\infty_{Nis}"'] \arrow[u, hook] \\
Fun_{Nis,L_{pe}(\mathbb{A}^1)}(\mathcal{D}g(k)^{ft},\widehat{\mathcal{S}}) \arrow[u, hook] & Fun_{Nis,L_{pe}(\mathbb{A}^1)}(\mathcal{D}g(k)^{ft},\widehat{Sp}) \arrow[l, "\sim", "\Omega^\infty_{Nis,\mathbb{A}^1}"'] \arrow[u, hook]
\end{tikzcd} \tag{7.4.2}$$

where again the last map is an equivalence. We will now explain how to use this diagram to reduce the proof that $l^{nc}_{\mathbb{A}^1}(K^S)$ is unit for the monoidal structure in $Fun_{Nis,L_{pe}(\mathbb{A}^1)}(\mathcal{D}g(k)^{ft},\widehat{Sp})$ to the proof that $l^{nc}_{0,\mathbb{A}^1}(\Omega^\infty(K^c))$ is a unit for the monoidal structure in $Fun_{Nis,L_{pe}(\mathbb{A}^1)}(\mathcal{D}g(k)^{ft},\widehat{\mathcal{S}})$. This will require some preliminaries. First we recall that thanks to the Prop. 7.2.8 we have an equivalence

$$Fun_{Nis\geq 0}(\mathcal{D}g(k)^{ft},\widehat{Sp}_{\geq 0}) \underset{\tau_{\geq 0}}{\overset{\sim}{\longleftarrow}} Fun_{Nis}(\mathcal{D}g(k)^{ft},\widehat{Sp}) \tag{7.4.3}$$

This equivalence provides a compatibility for the $\mathbb{A}^1$-localizations, in the sense that the diagram

$$\begin{tikzcd}
Fun_{Nis\geq 0}(\mathcal{D}g(k)^{ft},\widehat{Sp}_{\geq 0}) \arrow[d, "l^{nc}_{\geq 0,\mathbb{A}^1}"] & Fun_{Nis}(\mathcal{D}g(k)^{ft},\widehat{Sp}) \arrow[l, "\sim", "\tau_{\geq 0}"'] \arrow[d, "l^{nc}_{\mathbb{A}^1}"] \\
Fun_{Nis\geq 0,L_{pe}(\mathbb{A}^1)}(\mathcal{D}g(k)^{ft},\widehat{Sp}_{\geq 0}) & Fun_{Nis,L_{pe}(\mathbb{A}^1)}(\mathcal{D}g(k)^{ft},\widehat{Sp}) \arrow[l, "\sim", "\tau_{\geq 0}"']
\end{tikzcd} \tag{7.4.4}$$

commutes. Here $l^{nc}_{\geq 0,\mathbb{A}^1}$ is the (noncommutative) $\mathbb{A}^1$-localization functor for connectively-Nisnevich local presheaves.

The second preliminary result is a consequence of the equivalence between $\widehat{Sp}_{\geq 0}$ and the $(\infty,1)$-category of grouplike commutative algebra objects $CAlg^{grplike}(\widehat{\mathcal{S}})$ (see [100, 5.1.3.17]) and the equivalence of this last one with $Fun^{Segal-grplike}(N(Fin_*),\widehat{\mathcal{S}})$ - the full subcategory of the $(\infty,1)$-category $Fun(N(Fin_*),\widehat{\mathcal{S}})$ spanned by those functors satisfying the standard Segal condition and which are grouplike (see [100, 2.4.2.5]). See the final discussion in this section where this notion is discussed.

We can easily check that this equivalence induces equivalences

$$Fun^{Segal-grplike}(N(Fin_*),Fun_{Nis}(\mathcal{D}g(k)^{ft},\widehat{\mathcal{S}}))\simeq Fun_{Nis\geq 0}(\mathcal{D}g(k)^{ft},\widehat{Sp}_{\geq 0}) \tag{7.4.5}$$

and

$$Fun^{Segal-grplike}(N(Fin_*),Fun_{Nis,L_{pe}(\mathbb{A}^1)}(\mathcal{D}g(k)^{ft},\widehat{\mathcal{S}}))\simeq Fun_{Nis\geq 0,L_{pe}(\mathbb{A}^1)}(\mathcal{D}g(k)^{ft},\widehat{Sp}_{\geq 0}) \tag{7.4.6}$$

and we claim that the $\mathbb{A}^1$-localization functor $l^{nc}_{\geq 0,\mathbb{A}^1}$ can be identified along this equivalence with the functor induced by the levelwise application of the $\mathbb{A}^1$-localization functor for spaces $l^{nc}_{0,\mathbb{A}^1}$. To confirm that this is indeed the case we observe first that the composition with $l^{nc}_{0,\mathbb{A}^1}$ produces a left-adjoint to the inclusion

$$Fun(N(Fin_*),Fun_{Nis,L_{pe}(\mathbb{A}^1)}(\mathcal{D}g(k)^{ft},\widehat{\mathcal{S}}))\subseteq Fun(N(Fin_*),Fun_{Nis}(\mathcal{D}g(k)^{ft},\widehat{\mathcal{S}})) \tag{7.4.7}$$

so that it suffices to check that this left-adjoint preserves Segal-grouplike objects. To prove this we will need an explicit descriptin of the $\mathbb{A}^1$-localization functor of Nisnevich local objects $\mathcal{D}g(k)^{ft}\to\widehat{\mathcal{S}}$. Unfortunately, the explicit formula (7.3.3) will not work directly in the unstable case because when we apply the formula to a Nisnevich object the result might not be Nisnevich local. In any case the formula defines a reflexive localization of $Fun(\mathcal{D}g(k)^{ft},\widehat{\mathcal{S}})$ so that that we have the following situation: a presentable $(\infty,1)$-category $\mathcal{C}:=Fun(\mathcal{D}g(k)^{ft},\widehat{\mathcal{S}})$ together with two reflexive accessible localizations:

$$\begin{tikzcd} \mathcal{C}_1 \arrow[r, hook, "i_1"'] & \mathcal{C} \arrow[l, bend right, "L_1"'] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C}_2 \arrow[r, hook, "i_2"'] & \mathcal{C} \arrow[l, bend right, "L_2"'] \end{tikzcd}$$

with (L_1,i_1) corresponding to the localization produced by the formula (7.3.3) and (L_2,i_2) corresponding to the Nisnevich localization. Let $f_1:=i_1\circ L_1$ and $f_2:=i_2\circ L_2$. Our goal is to describe $\mathcal{C}_1\cap\mathcal{C}_2$ as an accessible reflexive localization of $\mathcal{C}_2$ and to understand how the left adjoint L_1 has to be modified in order to produce a left adjoint to the inclusion $\mathcal{C}_1\cap\mathcal{C}_2\subseteq\mathcal{C}_2$. The idea is that the intersection localization functor can be obtained by an infinite iteration of the composition $f_2\circ f_1$. We observe that:

a) f_1 commutes with colimits (his is because $L_{pe}(\mathbb{A})$ is completely compact as an object in $\mathcal{C}$ and because sifted colimits commute with colimits);

b) f_2 commutes with filtered colimits (this is because Nisnevich coverings are defined via a pullback condition and filtered colimits preserve pullbacks);

c) Let us denote by S_{12} the class of maps $F\to E$ in $\mathcal{C}_2$ such for any object $X\in\mathcal{C}_1\cap\mathcal{C}_2$ the composition $Map(E,X)\to Map(F,X)$ is an equivalence. As the generating $\mathbb{A}^1$-equivalences, by definition, live in the Nisnevich local category, S_{12} corresponds to the strongly saturated closured of this class (see [99, 55.4.15]). In particular, L_2 sends f_1-equivalences to maps in S_{12}

We will now follow [105, Lemma 1-3.20, Lemma 2.2.6] and produce a new localization of $\mathcal{C}$ that will give the right answer. We start by considering the endofunctor $\mathcal{C}$ defined by the formula

$$F \mapsto V(F) := colim_{n\in\mathbb{N}}(\underbrace{(f_2 \circ f_1) \circ ... \circ (f_2 \circ f_1)}_{n}(F)) \tag{7.4.8}$$

and we consider its restriction to $\mathcal{C}_2$ given by the composition

$$\tilde{V} : \mathcal{C}_2 \xhookrightarrow{i_1} \mathcal{C} \xrightarrow{V} \mathcal{C} \xrightarrow{L_2} \mathcal{C}_2$$

We have the following lemma:

Lemma 7.4.1. *The endofunctor $\tilde{V} : \mathcal{C}_2 \to \mathcal{C}_2$ is a localization functor of $\mathcal{C}_2$ with local objects corresponding to the intersection $\mathcal{C}_1 \cap \mathcal{C}_2$.*

Proof. This lemma was proved in [105, Lemma 1-3.20, Lemma 2.2.6] in the case where

$$\mathcal{C} = Fun(N(AffSm^{ft}(k))^{op}, \widehat{\mathcal{S}})$$

with $f_1(F) := colim_{n\in\Delta^{op}}\underline{Hom}(\Delta^n_{\mathbb{A}^1}, F)$[8] and f_2 is the endofunctor corresponding to the Nisnevich localization. As we shall now explain the same proof works also in our context.

The key ingredients to prove the lemma are the properties a)-c) above, together with the explicit description of f_1. We will only sketch the main steps. We leave it to the reader to check that the formula (7.4.8) indeed defines a localization functor. This follows essentially by using cofinality arguments. We now need to prove that this localization indeed provides a left adjoint to the inclusion $\mathcal{C}_1 \cap \mathcal{C}_2 \subseteq \mathcal{C}_2$. For this purpose we observe that the canonical map $F \to \tilde{V}(F)$ is in S_{12} and that $\tilde{V}(F)$ is $\mathbb{A}^1$-Nisnevich-local. The first follows by the definition of $\mathbb{A}^1$-equivalence in $\mathcal{C}_2$: take X an object in $\mathcal{C}_1 \cap \mathcal{C}_2$ and it is immedite from the definitions to see that the composition map $Map_2(\tilde{V}(F), X) \to Map_2(F, X)$ is an equivalence in $\mathcal{C}_2$. The second requires us to use the explicit description of f_1: $L_{pe}(\mathbb{A}^1)$ is an interval-object and each of the inclusions $i_0, i_1 : L_{pe}(k) \to L_{pe}(\mathbb{A}^1)$ admits a left inverse given by the projection $p : L_{pe}(\mathbb{A}^1) \to L_{pe}(k)$. In particular, for every non-commutative space $\mathcal{X}$ we have $Map_{Nis}(\mathcal{X}, \tilde{V}(F))$ as a retract of $Map_{Nis}(\mathcal{X} \otimes L_{pe}(\mathbb{A}^1), \tilde{V}(F))$. It suffices then to show that the composition

$$Map_{Nis}(\mathcal{X} \otimes L_{pe}(\mathbb{A}^1), \tilde{V}(F)) \to Map_{Nis}(\mathcal{X}, \tilde{V}(F)) \to Map_{Nis}(\mathcal{X} \otimes L_{pe}(\mathbb{A}^1), \tilde{V}(F))$$

is homotopic to the identity. As both $\mathcal{X}$ and $L_{pe}(\mathbb{A}^1)$ are compact, this composition can be obtained as

$$colim_n\,(\,Map_{\mathcal{C}}(\mathcal{X}\otimes L_{pe}(\mathbb{A}^1), (f_2\circ f_1)^n(i_2F)) \to Map_{\mathcal{C}}(\mathcal{X}, (f_2\circ f_1)^n(i_2F)) \to Map_{\mathcal{C}}(\mathcal{X}\otimes L_{pe}(\mathbb{A}^1), (f_2\circ f_1)^n(i_2F))\,)$$

To conclude we use the fact that the composition $i \circ p$ is strongly $\mathbb{A}^1$-homotopic to the identity so that $f_1(i \circ p) \simeq f_1(id_{\mathcal{X}\otimes L_{pe}(\mathbb{A}^1)})$ and in particular, $f_2 \circ f_1(i \circ p) \simeq f_2 \circ f_1(id_{\mathcal{X}\otimes L_{pe}(\mathbb{A}^1)})$. Using this we see that the composition

$$Map_{\mathcal{C}}(\mathcal{X}\otimes L_{pe}(\mathbb{A}^1), (f_2\circ f_1)^n(i_2F)) \to Map_{\mathcal{C}}(\mathcal{X}, (f_2\circ f_1)^n(i_2F)) \to Map_{\mathcal{C}}(\mathcal{X}\otimes L_{pe}(\mathbb{A}^1), (f_2\circ f_1)^n(i_2F)) \to$$

$$\to Map_{\mathcal{C}}(\mathcal{X} \otimes L_{pe}(\mathbb{A}^1), (f_2 \circ f_1)^{n+1}(i_2F))$$

becomes the identity map when we take the colimit. $\square$

[8] In the original formulation of this result the authors use a different description of $f_1(F)$ that follows from the fact that the geometric realization of a simplicial space is homotopy equivalent to the diagonal of the underlying bisimplicial set.

This description can now be used to prove that the composition with $l^{nc}_{0,\mathbb{A}^1}$ preserves the Segal-grouplike condition. Indeed, this follows immediately from 1)this explicit description together with 2) the fact that products in $Fun_{Nis}(\mathcal{D}g(k)^{ft}, \mathcal{S})$ are computed objectwise in spaces; 3) the fact that in spaces both sifted and filtered colimits commute with finite products (see [99, 5.5.8.11,5.5.8.12] for the sifted case) and finally 4),the fact that the Nisnevich localization commutes with finite products - this is a consequence of [99, 5.5.4.15] together with the fact that in $\mathcal{D}g(k)^{idem}$ finite products are the same as finite coproducts so that the product of a Nisnevich square of dg-categories of finite type with a dg-category of finite type remains of finite type.

Finally, the grouplike condition follows also from this, together with the functoriality of $l^{nc}_{0,\mathbb{A}^1}$. As a summary of this discussion, we concluded the existence of a commutative diagram

$$\begin{tikzcd}
Fun^{Segal-grplike}(N(Fin_*), Fun_{Nis}(\mathcal{D}g(k)^{ft}, \widehat{\mathcal{S}})) \arrow[d, "(l^{nc}_{0,\mathbb{A}^1}\circ -)"] & Fun_{Nis\geq 0}(\mathcal{D}g(k)^{ft}, \widehat{Sp}_{\geq 0}) \arrow[l, "\sim"'] \arrow[d, "l^{nc}_{\geq 0,\mathbb{A}^1}"] \\
Fun^{Segal-grplike}(N(Fin_*), Fun_{Nis,L_{pe}(\mathbb{A}^1)}(\mathcal{D}g(k)^{ft}, \widehat{\mathcal{S}})) & Fun_{Nis\geq 0,L_{pe}(\mathbb{A}^1)}(\mathcal{D}g(k)^{ft}, \widehat{Sp}_{\geq 0}) \arrow[l, "\sim"']
\end{tikzcd} \tag{7.4.9}$$

Finally, combining the commutativity of this diagram with the diagram (7.4.4) we obtain the commutativity of the diagram

$$\begin{tikzcd}
Fun_{Nis}(\mathcal{D}g(k)^{ft}, \widehat{\mathcal{S}}) \arrow[d, "l^{nc}_{0,\mathbb{A}^1}"] & Fun_{Nis}(\mathcal{D}g(k)^{ft}, \widehat{Sp}) \arrow[l, "\Omega^{\infty}_{Nis}"] \arrow[d, "l^{nc}_{\mathbb{A}^1}"] \\
Fun_{Nis,L_{pe}(\mathbb{A}^1)}(\mathcal{D}g(k)^{ft}, \widehat{\mathcal{S}}) & Fun_{Nis,L_{pe}(\mathbb{A}^1)}(\mathcal{D}g(k)^{ft}, \widehat{Sp}) \arrow[l, "\sim"', "\Omega^{\infty}_{Nis,\mathbb{A}^1}"]
\end{tikzcd} \tag{7.4.10}$$

This follows because Ω^{∞}_{Nis} can now be identified with the evaluation at $\langle 1\rangle \in N(Fin_*)$ by means of the commutativity and form of the diagrams (7.4.2), (7.4.4) and (7.4.9).

The following lemma is the last step in our preliminaries:

Lemma 7.4.2. *Let F be a connectively-Nisnevich local object in $Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp})$. Then, $\Omega^{\infty}(F)$ is Nisnevich local and the canonical map $\Omega^{\infty}(F) \simeq l^{nc}_{0,Nis}(\Omega^{\infty}(F)) \to \Omega_{Nis}(l^{nc}_{Nis}(F))$ is an equivalence in $Fun_{Nis}(\mathcal{D}g(k)^{ft}, \widehat{\mathcal{S}})$.*

Proof. The proof depends on two observations. The first is that if F is connectively-Nisnevich local, the looping $\Omega^{\infty}(F)$ is Nisnevich local as a functor $\mathcal{D}g(k)^{ft} \to \widehat{\mathcal{S}}$. This is because the composition $\widehat{Sp}_{\geq 0} \hookrightarrow \widehat{Sp} \xrightarrow{\Omega^{\infty}} \widehat{\mathcal{S}}$ preserves limits (one possible way to see this is to use the equivalence between connective spectra and grouplike commutative algebras in $\widehat{\mathcal{S}}$ for the cartesian product [100, Theorem 5.2.6.10 and Remark 5.2.6.26] and the fact that this equivalence identifies the looping functor Ω^{∞} with the forgetful functor which we know as a left adjoint and therefore commutates with limits. The conclusion now follows from the definition of connectively-Nisnevich local. The second observation is that the looping functor Ω^{∞} only captures the connective part of a spectrum. This follows from the very definition of the canonical t-structure in $\widehat{Sp}$ (see [100, 1.4.3.4]) In particular, since F is connectively-Nisnevich local, our Proposition 7.2.8 implies that the canonical morphism $F \to l^{nc}_{Nis}(F)$ is an equivalence in the connective part so that its image under Ω^{∞} is an equivalence. Putting together these two observations we have equivalences fitting in a commutative diagram

$$\begin{tikzcd}
\Omega^{\infty}(F) \arrow[d, "\sim"] \arrow[dr, "\sim"] & \\
l^{nc}_{0,Nis}(\Omega^{\infty}(F)) \arrow[r, "\delta"] & \Omega^{\infty}(l^{nc}_{Nis}(F))
\end{tikzcd} \tag{7.4.11}$$

so that the canonical map δ induced by the universal property of the localization is also an equivalence. □

Finally, we uncover the formulas

$$\Omega^{\infty}_{Nis,\mathbb{A}^1}(l^{nc}_{\mathbb{A}^1}(K^S)) \simeq l^{nc}_{0,\mathbb{A}^1}(\Omega^{\infty}_{Nis}(K^S)) \simeq l_{0,\mathbb{A}^1}(\Omega^{\infty}(K^c)) \tag{7.4.12}$$

where the first equivalence follows from the preceeding discussion and the last one follows from the previous lemma.

The first task is done. Now we explain the equivalence between $l_{0,\mathbb{A}^1}(\Omega^{\infty}(K^c))$ and the unit for the monoidal structure in $Fun_{Nis,L_{pe}(\mathbb{A}^1)}(\mathcal{D}g(k)^{ft},\widehat{\mathcal{S}})$.

Our starting point is the formula (7.1.6) describing the K-theory space of an idempotent complete dg-category T by means of a colimit of mapping spaces. Since colimits and limits of functors are determined objetwise, the functor $\Omega^{\infty}K^c$ can itself be written as $\Omega\, colim_{[n]\in\Delta^{op}}\, Seq$ where Seq is the object in the $(\infty,1)$-category $Fun(\Delta^{op}, Fun(\mathcal{D}g(k)^{idem},\widehat{\mathcal{S}}))$ resulting from the last stage of the Construction 7.1.2.

Remark 7.4.3. More precisely, at the end the Construction 7.1.2 we obtained a functor

$$N(Cat_{Ch(k)}) \to Fun(N(\Delta^{op}), N(\widehat{\Delta}_{big})) \to Fun(N(\Delta^{op}),\widehat{\mathcal{S}}) \tag{7.4.13}$$

where the second map is induced by the localization functor $N(\widehat{\Delta}_{big}) \to \widehat{\mathcal{S}}$ with $\widehat{\Delta}_{big}$ the very big category big of simplicial sets equipped with the standard model structure. By the description of each space at level n as a mapping space we conclude that this composition sends Morita equivalences of dg-categories to equivalences and therefore by the universal property the localization extends to a unique functor $\mathcal{D}g(k)^{idem} \to Fun(N(\Delta)^{op},\mathcal{S})$ which, using the equivalence between $Fun(\mathcal{D}g(k)^{idem}, Fun(N(\Delta^{op}),\widehat{\mathcal{S}})$ and $Fun(N(\Delta^{op}), Fun(\mathcal{D}g(k)^{idem},\widehat{\mathcal{S}})$ gives what we call Seq.

The value of Seq at zero is the constant functor with value $*$ and its value at $n \geq 1$ is $Map_{\mathcal{D}g(k)^{idem}}((\widehat{[n-1]_k})_c,-)$. The boundary and degeneracy maps are obtained from the S-construction as explained in the Construction 7.1.2. We observe now that the dg-categories $(\widehat{[n-1]_k})_c$, for any $n \geq 0$, are of finite type so that each level of the simplicial object Seq is in the full subcategory of ω-continuous functors. Moreover, we can think of the dg-categories $\widehat{([n]_k)_c}$ as non-commutative spaces I_n so that by means of the Yoneda's map $j_{nc} : \mathcal{N}cS(k) \hookrightarrow Fun(\mathcal{D}g(k)^{ft},\mathcal{S})$ we can identify Seq_n with the representable $Map_{\mathcal{N}cS(k)}(-,I_{n-1})$. In particular, since the Yoneda's map is fully-faithfull, the simplicial object Seq is the image through j_{nc} of a uniquely determined simplicial object $Seq_{nc} \in Fun(\Delta^{op},\mathcal{N}cS(k))$ whose value at level n is the noncommutative space I_{n-1}. Finally, with these notations we can write $\Omega^{\infty}K^c$ as $\Omega\, colim_{[n]\in\Delta^{op}}\, j_{nc}\circ Seq_{nc}$ so that our main goal is to understand the localization $l^{nc}_{0,\mathbb{A}^1}(\Omega\, colim_{[n]\in\Delta^{op}}\, j_{nc}\circ Seq_{nc})$. As the zero level of the simplicial object $j_{nc}\circ Seq_{nc}$ is contractible, the realization $colim_{[n]\in\Delta^{op}}\, j_{nc}\circ Seq_{nc}$ is 1-connective [9].

We have the following lemma:

Lemma 7.4.4. *The canonical map*

$$l^{nc}_{0,\mathbb{A}^1}(\Omega\, colim_{[n]\in\Delta^{op}}\, j_{nc}\circ Seq_{nc}) \to \Omega\, l^{nc}_{0,\mathbb{A}^1}(colim_{[n]\in\Delta^{op}}\, j_{nc}\circ Seq_{nc}) \tag{7.4.14}$$

is an equivalence.

Proof. The key observation is that the presheaf of spaces $colim_{[n]\in\Delta^{op}}\, j_{nc}\circ Seq_{nc}$ is the zero level of a presheaf of connective spectra (for instance, as constructed by Waldhausen in [153]). The important point is that this spectral presheaf satisfies Nisnevich descent as a result of the Waldhausen's localization theorem [153, 1.6.4]. In particular, it is Nisnevich local. The result now follows from the commutativity of the diagram (7.4.10). □

[9]Recall that a space is said to be n-connective if it is non-empty and all its homotopy groups for $i < n$ are zero.

Our main goal now is to understand the simplicial object Seq. Following Waldhausen [153] we recall the existence of a weaker version of the S-construction that considers only those sequence of cofibrations that split. More precisely, and using the same terminology as in the Construction 7.1.2 we denote by $\mathbb{R}\underline{Hom}^{split}(Ar[n]_k, \widehat{T}_c)$ the full sub dg-category of $\mathbb{R}\underline{Hom}(Ar[n]_k, \widehat{T}_c)$ spanned by those $Ar[n]$-indexed diagrams satisfying the conditions given in the Construction 7.1.1 and where the top sequence is given by the canonical inclusions $E_1 \to E_1 \oplus E_2 \to E_1 \oplus E_2 \oplus E_3 \to ... \to E_1 \oplus ... \oplus E_n$ for some list of perfect modules $(E_1, ..., E_n)$. These are called *split cofibrations*. As in the standard S-construction, the categories subjacent to $\mathbb{R}\underline{Hom}^{split}(Ar[n]_k, \widehat{T}_c)$ carries a notion of weak-equivalences W_n^{Split} and assemble to form a simplical space $[n] \to N(\mathbb{R}\underline{Hom}^{split}(Ar[n]_k, \widehat{T}_c)^{W_n^{Split}})$.

As in the Construction 7.1.2 we can now describe these spaces in a somewhat more simple form. As the dg-categories 1_k are cofibrant (see [132]) they are also locally-cofibrant and for any $n \geq 0$ the coproduct $\coprod_{i=1}^n 1_k$ is an homotopy coproduct. Moreover, for any locally-cofibrant dg-category T we have equivalences $\mathbb{R}\underline{Hom}(\coprod_{i=1}^n 1_k, \widehat{T}_c) \simeq \prod_{i=1}^n (\widehat{1_k \otimes^{\mathbb{L}} T})_{pspe} \simeq \prod_{i=1}^n (\widehat{1_k \otimes T})_{pspe} \simeq \prod_{i=1}^n \widehat{T}_c$ In this case, for every $n \geq 0$ and for every dg-category T there is an equivalence between the category subjacent to $\mathbb{R}\underline{Hom}^{split}(Ar[n]_k, \widehat{T}_c)$ and the category subjacent to $\mathbb{R}\underline{Hom}(\coprod_{i=1}^n 1_k, \widehat{T}_c)$, defined by sending a sequence $E_1 \to E_1 \oplus E_2 \to E_1 \oplus E_2 \oplus E_3 \to ... \to E_1 \oplus ... \oplus E_n$ to the sucessive quotients $(E_1, ..., E_n)$. This correspondence is functorial and defines an equivalence because of the universal property of direct sums. Moreover, and again thanks to the cube lemma, this equivalence preserves the natural notions of weak-equivalences. Finally, and again due to the main theorem of [139] we found the spaces $N(\mathbb{R}\underline{Hom}^{split}(Ar[n]_k, \widehat{T}_c)^{W_n^{Split}})$ and $Map_{\mathcal{D}g(k)^{idem}}(\oplus_{i=1}^n \widehat{(1_k)}_c, \widehat{T}_c)$ to be equivalent so that by the same arguments as in the Remark 7.4.3 we obtain a simplicial object $Split \in Fun(N(\Delta)^{op}, Fun(\mathcal{D}g(k)^{idem}, \widehat{\mathcal{S}})$, which, because the dg-categories $\oplus_{i=1}^n \widehat{(1_k)}_c$ are of finite type, lives in the full subcategory of ω-continuous functors, therefore being an object in $Fun(N(\Delta^{op}), \mathcal{P}(\mathcal{N}cS(k)))$. Moreover, for each $n \geq 0$ $Split_n$ is representable by the noncommutative space associated to the dg-category $\oplus_{i=1}^n \widehat{(1_k)}_c$ so that by Yoneda the whole simplicial object $Split$ is of the form $j_{nc} \circ \Theta$ for a simplicial object $\Theta \in Fun(N\Delta^{op}), \mathcal{N}cS(k))$ with level n given by $\oplus_{i=1}^n \widehat{(1_k)}_c$.

Finally, the inclusion of split cofibrations into all sequences of morphisms provides a strict map of simplicial objects in the model category $\widehat{\Delta}$ between $[n] \to N(\mathbb{R}\underline{Hom}^{split}(Ar[n]_k, \widehat{T}_c)^{W_n})$ and $[n] \to N(S_n^{dg}(T)^{W_n})$ and we define λ

$$\lambda : j_{nc}(\Theta) \simeq Split \to Seq \tag{7.4.15}$$

to be the image of this map under the composition in (7.4.13). This is where the result of [17] becomes crucial:

Proposition 7.4.5. *[17, Prop. 4.6] The map λ is a levelwise noncommutative $\mathbb{A}^1$-equivalence in $Fun(\mathcal{D}g(k)^{ft}, \mathcal{S})$.*

Proof. In [17, Prop. 4.6] the author uses an inductive argument to prove that for any $n \geq 0$ the map λ_n is an $\mathbb{A}^1$-equivalence.

For $n = 1$, λ_1 is an equivalence. For $n = 2$ we need some further adaptation to our case. Namely, we are required to construct a noncommutative $\mathbb{A}^1$-homotopy between the identity of the noncommutative space I_{2-1} and the zero map. Such an homotopy corresponds to a co-homotopy in $\mathcal{D}g(k)^{idem}$, namely, a map $H : \widehat{([1]_k)}_c \to \widehat{([1]_k)}_c \otimes^{\mathbb{L}} L_{pe}(\mathbb{A}^1)$ in $\mathcal{D}g(k)^{idem}$ fitting in a commutative diagram

$$\begin{tikzcd}
 & & \widehat{([1]_k)}_c \\
\widehat{([1]_k)}_c \arrow[r, "H"] \arrow[urr, bend left, "Id"] \arrow[drr, bend right, "0"'] & \widehat{([1]_k)}_c \otimes^{\mathbb{L}} L_{pe}(\mathbb{A}^1) \arrow[ur, "ev_1"] \arrow[dr, "ev_0"'] & \\
 & & \widehat{([1]_k)}_c
\end{tikzcd} \tag{7.4.16}$$

Recall that $L_{pe}(\mathbb{A}^1)$ is canonically equivalent to $\widehat{k[X]}_c$ - the idempotent completion of the dg-category with one object and $k[X]$ concentrated in degree zero as endomorphisms. In this case the term in the middle is equivalent to $\widehat{(([1]_k) \otimes k[X])}_c$. We define H to be the map induced by the universal property of the idempotent completion $\widehat{(-)}_c : \mathcal{D}g(k) \to \mathcal{D}g(k)^{idem}$ by means of the composition

$$([1]_k) \to ([1]_k) \otimes k[X] \subseteq \widehat{(([1]_k) \otimes k[X])}_c \tag{7.4.17}$$

where the first map is obtained from the strict dg-functor defined by the identity on the objects, by the inclusion $k \subseteq k[X]$ on the endomorphisms of 0 and by the composition $k \subseteq k[X] \to k[X]$ on the complex of maps between 0 and 1 and on the endomorphisms of 1, where the last map is the multiplication by the variable X. This makes the diagram above commute and provides the required homotopy. We conclude as in [17, Prop. 4.6] to find that λ_2 is an $\mathbb{A}^1$-equivalence given by a strong $\mathbb{A}^1$-homotopy.

We now conclude with the induction step: it follows from the observation that the canonical map $Seq_n \to Seq_{n-1} \times_{Seq_n} Seq_2$ is an equivalence of presheaves. The conclusion now follows because the fiber product of strong $\mathbb{A}^1$-homotopy equivalences remains a strong $\mathbb{A}^1$-homotopy equivalence (it is easy to write down the homotopies for the fiber product) □

Finally, the fact that any colimit of $\mathbb{A}^1$-equivalences is an $\mathbb{A}^1$-equivalence gives us the following corollary:

Corollary 7.4.6. *The map induced by λ between the colimits $colim_{\Delta^{op}}\, j_{nc} \circ \Theta \to colim_{\Delta^{op}}\, j_{nc} \circ Seq_{nc}$ is an $\mathbb{A}^1$-equivalence. Moreover, and since $l^{nc}_{0,\mathbb{A}^1}$ commutes with colimits and representable objects are Nisnevich local, we have equivalences*

$$colim_{\Delta^{op}}\, l^{nc}_{0,\mathbb{A}^1} \circ j_{nc} \circ \Theta \simeq l^{nc}_{0,\mathbb{A}^1}(colim_{\Delta^{op}}\, j_{nc} \circ \Theta) \simeq l^{nc}_{0,\mathbb{A}^1}(colim_{\Delta^{op}}\, j_{nc} \circ Seq_{nc}) \simeq colim_{\Delta^{op}}\, l^{nc}_{0,\mathbb{A}^1} \circ j_{nc} \circ Seq_{nc} \tag{7.4.18}$$

in $\mathcal{SH}_{nc}(k)$.

Our next move requires a small preliminary digression. To start with, recall that any $(\infty, 1)$-category endowed with finite sums and an initial object or finite products and a final object, can be considered as a symmetric monoidal $(\infty, 1)$-category with respect to these two operations, respectively denoted as $\mathcal{C}^{\amalg}$ and $\mathcal{C}^{\times}$ (see [100, Sections 2.4.1 and 2.4.3]). Monoidal structures appearing from this mechanism are called, respectively, *cartesian* and *cocartesian*. In particular, if $\mathcal{C}$ has direct sums and a zero object, these monoidal structures coincide $\mathcal{C}^{\oplus}$ (this follows from the Proposition [100, 2.4.3.19]). In this particular situation the theory of algebras over a given ∞-operad $\mathcal{O}^{\otimes}$ gets simplified: the $(\infty, 1)$-category of $\mathcal{O}$-algebras on $\mathcal{C}^{\oplus}$ is equivalent to a full subcategory of $Fun(\mathcal{O}^{\otimes}, \mathcal{C})$, spanned by a class of functors satisfying the standard Segal conditions (see [100, 2.4.2.1, 2.4.2.5]). In the particular case of associative algebras, and since the category Δ^{op} is a "model" for the associative operad (see [100, 4.1.2.6, 4.1.2.10, 4.1.2.14] for the precise statement) an associative algebra in $\mathcal{C}^{\oplus}$ is just a simplicial object in $\mathcal{C}$ satisfying the Segal condition.

$$Alg_{\mathcal{A}ss}(\mathcal{C}) \simeq Fun^{Segal}(N(\Delta^{op}), \mathcal{C}) \tag{7.4.19}$$

We shall now come back to our situation and observe that

Lemma 7.4.7. *The simplicial object Θ satisfies the Segal conditions.*

Proof. As the Yoneda's embedding preserves limits and is fully-faithfull it suffices to check that *Split* satisfies the Segal conditions. But this is obvious from the definition of the simplicial structure given by the S-construction. At each level the map appearing in the Segal condition is the map sending a sequence of dg-modules $E_0 \to E_0 \oplus E_1 \to \ldots. E_0 \oplus \ldots \oplus E_{n-1}$ to the quotients $(E_0, \ldots, E_{n-1})$. □

We now characterize the simplicial object Θ in a somewhat more canonical fashion. An important aspect of a cocartesian symmetric monoidal structure $\mathcal{C}^{\coprod}$ is that any object X in $\mathcal{C}$ admits a unique algebra structure, determined by the codiagonal map $X \coprod X \to X$. More precisely (see [100, 2.4.3.16] for the general result), the forgetful map $Alg_{\mathcal{A}ss}(\mathcal{C}) \to \mathcal{C}$ is an equivalence of $(\infty,1)$-categories [10]. By choosing an inverse to this equivalence and composing with the equivalence (7.4.19) we obtain an ∞-functor

$$\mathcal{C} \to Alg_{\mathcal{A}ss}(\mathcal{C}) \simeq Fun^{Segal}(N(\Delta)^{op}, \mathcal{C}) \tag{7.4.20}$$

providing for any object in $\mathcal{C}$ a uniquely determined simplicial object, encoding the algebra structure induced by the codigonal[11]. Because of the Segal condition this simplicial object is a zero object of $\mathcal{C}$ in degree zero, X in degree one and more generally is $X^{\oplus n}$ in degree n. We now apply this discussion to $\mathcal{C} = \mathcal{N}cS(k)$ (it has direct sums and a zero object because $\mathcal{D}g(k)^{idem}$ has and the inclusion $\mathcal{D}g(k)^{ft} \subseteq \mathcal{D}g(k)^{idem}$ preserves them) and to $X = \widehat{(1_k)}_c$. Since the simplicial object Θ satisfies the Segal condition and its first level is equivalent to X, the equivalence (7.4.20) tells us that it is necessarily the simplicial object codifying the unique associative algebra structure on X given by the codiagonal.

With the Corollary 7.4.6 we are now reduced to study the colimit of the simplicial object $l^{nc}_{\mathbb{A}^1} \circ j_{nc} \circ \Theta$ in $\mathcal{SH}_{nc}(k)$. As the last is a stable $(\infty,1)$-category it has direct sums and therefore can be understood as the underlying $(\infty,1)$-category of a symmetric monoidal structure $\mathcal{SH}_{nc}(k)^{\oplus}$ which is simultaneously cartesian and cocartesian. As the canonical composition $\mathcal{N}cS(k) \to \mathcal{SH}_{nc}(k)$ preserves direct sums (this follows from 1) the fact the Yoneda functor preserves limits; 2) the fact representables are Nisnevich local; 3) the fact the $\mathbb{A}^1$-localization preserves finite products (as explained when confirming that it preserves the Segal conditions) and finally 4) the fact that $\mathcal{SH}_{nc}(k)$ is stable.) it can be lifted in a essentially unique way to a monoidal functor $\mathcal{N}cS(k)^{\oplus} \to \mathcal{SH}_{nc}(k)^{\oplus}$ ([100, Cor. 2.4.1.8]). This monoidal map allows us to transport algebras and provides a commutative diagram

$$\begin{tikzcd}
Fun^{Segal}(N(\Delta)^{op}, \mathcal{N}cS(k)) \arrow[r] \arrow[d, "\sim"] \arrow[dd, bend right=60, "ev_{[1]}"'] & Fun^{Segal}(N(\Delta)^{op}, \mathcal{SH}_{nc}(k)) \arrow[d, "\sim"] \arrow[dd, bend left=60, "ev_{[1]}"] \\
Alg_{\mathcal{A}ss}(\mathcal{N}cS(k)) \arrow[r] \arrow[d, "\sim"] & Alg_{\mathcal{A}ss}(\mathcal{SH}_{nc}(k)) \arrow[d, "\sim"] \\
\mathcal{N}cS(k) \arrow[r] & \mathcal{SH}_{nc}(k)
\end{tikzcd} \tag{7.4.21}$$

where the upper map is the composition with $\mathcal{N}cS(k) \to \mathcal{SH}_{nc}(k)$. It follows from the description of Θ above and from the commutativity of this diagram that the simplicial object $l^{nc}_{\mathbb{A}^1} \circ j_{nc} \circ \Theta$ in $\mathcal{SH}_{nc}(k)$ corresponds to the unique commutative algebra structure on $1_{nc} := l^{nc}_{\mathbb{A}^1} \circ j_{nc}(L_{pe}(k))$ created by the codiagonal.

Our next task is to study the theory of associative algebras on a stable $(\infty,1)$-category equipped with its natural simultaneously cartesian and cocartesian monoidal structure induced by the existence of direct sums. We recall some terminology. If $\mathcal{C}^{\otimes}$ is a cartesian symmetric monoidal structure, an associative algebra on $\mathcal{C}$ is said to be *grouplike* if the simplicial object which codifies it $A \in Fun^{Segal}(N(\Delta^{op}), \mathcal{C})$ is a groupoid object in $\mathcal{C}$ in the sense of the definition [99, 6.1.2.7]. We let $Alg^{grplike}_{\mathcal{A}ss}(\mathcal{C})$ denote the full subcategory of $Alg_{\mathcal{A}ss}(\mathcal{C})$ spanned by the grouplike associative algebras.

Let now Δ^{op}_{+} be the standard augmentation of the category Δ^{op}. Following [99, 6.1.2.11], an object $U_{+} \in Fun(\Delta^{op}_{+}, \mathcal{C})$ is said to be a *Cech nerve of the morphism* $U_0 \to U_{-1}$ if the restriction $U_{+}|_{N(\Delta^{op})}$ is a groupoid object and the commutative diagram

[10] Recall that the associative operad is unital.

[11] The fact that the multiplication can be identified with the codiagonal map follows from the simplicial identities and from the universal property defining the codiagonal.

$$\begin{tikzcd} U_1 \arrow[r] \arrow[d] & U_0 \arrow[d] \\ U_0 \arrow[r] & U_{-1} \end{tikzcd} \tag{7.4.22}$$

is a pullback diagram in $\mathcal{C}$. Again by [99, 6.1.2.11], a Cech nerve U_+ is determined by the map $U_0 \to U_{-1}$ in a essentially unique way as the right-Kan extension along the inclusion $N(\Delta^{op}_{+,\leq 0}) \subseteq N(\Delta^{op}_{+})$.

We have the following lemma:

Lemma 7.4.8. *Let $\mathcal{C}^{\otimes}$ be a cartesian symmetric monoidal $(\infty,1)$-category whose underlying $(\infty,1)$-category is stable. Then*

1. *The inclusion $Alg^{grplike}(\mathcal{C}) \subseteq Alg(\mathcal{C})$ is an equivalence;*
2. *For any object X in $\mathcal{C}$ the simplicial object associated to X by means of the composition (7.4.20) is a Cech nerve of the canonical morphism $0 \to \Sigma X$.*

Proof. The first assertion is true because in any stable $(\infty,1)$-category every morphism $f : X \to Y$ has an inverse $-f$ with respect to the additive structure[12]. In particular, for any object $X \in \mathcal{C}$ there is map $-Id_X$ providing an inverse for the algebra structure given by the codiagonal map $X \oplus X \to X$. More precisely, let X be an object in $\mathcal{C}$ and let U_X be the simplicial object associated to X by means of the mechanism (7.4.20). By construction this simplicial object satisfies the Segal condition and in particular we have $(U_X)_0 \simeq 0$ and $(U_X)_1 \simeq X$. We aim to prove that this simplicial object is a groupoid object. For that we observe that for a simplicial object A to be a groupoid object it is equivalent to ask for A to satisfy the Segal conditions and to ask for the induced map

$$A([2]) \xrightarrow{A(\partial_1)\times A(\partial_0)} A([1]) \times A([1]) \tag{7.4.23}$$

to be an equivalence. Indeed, if A is a groupoid object, by the description in [99, 6.1.2.6 - (4")] it satisfies these two requirements automatically. The converse follows by applying the same arguments as in the proof of [99, 6.1.2.6 - 4)' implies 3)], together with the observation that for the induction step to work we don't need the full condition in $4')$ but only the Segal condition. The induction basis is equivalent to the Segal conditions for $n = 2$ together with the condition that (7.4.23) is an equivalence.

In our case (7.4.23) is the map $\nabla \times id_X : X \oplus X \to X \oplus X$ where ∇ is the codigonal map $X \oplus X \to X$. Of course, since the identity of X admits an inverse $(-Id_X)$ the map $(\nabla \circ (Id_X \times (-Id_X))) \times Id_X$ is an explicit inverse for $\nabla \times id_X$.

Let us now prove 2). Again by construction, we know that the colimit of the truncation $(U_X)_{|_{N(\Delta_{\leq 1})^{op}}}$ is canonically equivalent to the suspension ΣX. Therefore U_X admits a canonical augmentation $(U_X)^+ : N(\Delta^{op}_+) \to \mathcal{C}$ with $(U_X)^+_{-1} \simeq \Sigma X$. It follows from 1) that U is a groupoid object and since $\mathcal{C}$ is stable, the diagram

$$\begin{tikzcd} (U_X)_1 \simeq X \arrow[r] \arrow[d] & (U_X)_0 \simeq 0 \arrow[d] \\ (U_X)_0 \simeq 0 \arrow[r] & (U_X)_{-1} \simeq \Sigma X \end{tikzcd} \tag{7.4.24}$$

is a pullback so that $(U_X)^+$ is the Cech nerve of the canonical map $0 \to \Sigma X$. $\square$

[12] More precisely $\pi_0 Map(X,Y)$ has a canonical structure of abelian group.

In particular, we find that the simplicial object $l^{nc}_{0,\mathbb{A}^1} \circ j_{nc} \circ \Theta$ is a Cech nerve of the canonical map $0 \to \Sigma 1_{nc}$. Finally, recall that a morphism $A \to B$ is said to be an effective epimorphism if the colimit of its Cech nerve is B. The following lemma holds the final step

Lemma 7.4.9. *Let $\mathcal{C}$ be a stable $(\infty, 1)$-category. Then, for any object X in $\mathcal{C}$, the canonical morphism $0 \to X$ is an effective epimorphism.*

Proof. Let $U : N(\Delta^{op}) \to \mathcal{C}$ be a simplicial object in $\mathcal{C}$. Then the colimit of U can be computed as the sequential colimit of the sucessive colimits of its truncations $U_{|_{N(\Delta^{op}_{\leq n})}}$. Using the descriptions of Cech nerves as right-Kan extensions (see above) we know that if U^+ is the Cech nerve of the map $0 \to X$, its level n is given by the n-fold fiber product of 0 over X. As $\mathcal{C}$ is stable this n-dimensional limit cube will also be a colimit n-cube so that the colimit of the truncation at level n will necessarily be X (See the Proposition [100, 1.2.4.13]). Since this holds for every $n \geq 0$ the colimit of the Cech nerve is necessarily canonically equivalent to X. $\square$

We are done. Since $\mathcal{SH}_{nc}(k)$ is stable we have $colim_{\Delta^{op}}\, l^{nc}_{0,\mathbb{A}^1} \circ j_{nc} \circ \Theta \simeq \Sigma 1_{nc}$ so that, by the lemma 7.4.4 we have $l^{nc}_{0,\mathbb{A}^1}(\Omega^\infty(K^c))$ is equivalent to $\Omega\Sigma 1_{nc} \simeq 1_{nc}$.

CHAPTER **8**

Localizing Noncommutative Motives and the comparison with the approach of Cisinski-Tabuada

In this chapter we explain the relation between our approach to noncommutative motives and the approach already studied by G. Tabuada in [133, 135] and Cisinski-Tabuada in [35, 34]. Both theories have the $(\infty,1)$-category $Fun_{\omega}(\mathcal{D}g(k)^{idem}, \widehat{Sp})$ as a common ground. To start with we observe that our version $\mathcal{SH}_{nc}(k)$ can be identified with the full subcategory spanned by those functors F sending Nisnevich squares of dg-categories to pullback-pushout squares in spectra and satisfying $\mathbb{A}^1$-invariance. Indeed, our original definition of $\mathcal{SH}_{nc}(k)$ as a localization of $Fun(\mathcal{D}g(k)^{ft}.\widehat{Sp})$ can be transported along the equivalence

$$Fun_{\omega}(\mathcal{D}g(k)^{idem}, \widehat{Sp}) \simeq Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp}) \tag{8.0.1}$$

Remark 8.0.10. We give a more precise description of this localization. Given a noncommutative smooth space $\mathcal{X}$ associated with a dg-category of finite type $T_{\mathcal{X}}$, the image of $\mathcal{X}$ in $Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp})$ under the spectral Yoneda's embedding is just the corepresentable $\Sigma^{\infty}_{+}Map_{\mathcal{D}g(k)^{ft}}(T_{\mathcal{X}}, -)$. Moreover, since $\mathcal{D}g(k)^{ft}$ is the full subcategory of compact objects in $\mathcal{D}g(k)^{idem}$ (6.1.27) the image of this corepresentable under the equivalence (8.0.1) is the corepresentable $\Sigma^{\infty}_{+}Map_{\mathcal{D}g(k)^{idem}}(T_{\mathcal{X}}, -)$. We consider $Fun_{\omega,Nis}(\mathcal{D}g(k)^{idem}, \widehat{Sp})$ the reflexive accessible localization of $Fun_{\omega}(\mathcal{D}g(k)^{idem}, \widehat{Sp})$ obtained by inverting the small set of all maps of the form

$$\delta_{\Sigma^{\infty}_{+}Map_{\mathcal{D}g(k)^{idem}}(T_{\mathcal{V}},-)}(K) \coprod_{\delta_{\Sigma^{\infty}_{+}Map_{\mathcal{D}g(k)^{idem}}(T_{\mathcal{W}},-)}(K)} \delta_{\Sigma^{\infty}_{+}Map_{\mathcal{D}g(k)^{idem}}(T_{\mathcal{U}},-)}(K) \to \tag{8.0.2}$$

$$\delta_{\Sigma^{\infty}_{+}Map_{\mathcal{D}g(k)^{idem}}(T_{\mathcal{X}},-)}(K) \tag{8.0.3}$$

for K a compact spectrum and induced by the Nisnevich squares

$$\begin{tikzcd} T_{\mathcal{X}} \arrow[r] \arrow[d] & T_{\mathcal{U}} \arrow[d] \\ T_{\mathcal{V}} \arrow[r] & T_{\mathcal{W}} \end{tikzcd} \tag{8.0.4}$$

of dg-categories as described in the discussion following the definition in 6.4.7. See the notations at section 5.4. By the theorem [99, 5.5.4.15] this is an accessible reflexive localization of $Fun_{\omega}(\mathcal{D}g(k)^{idem}, \widehat{Sp})$ and now by construction the local objects are those functors $F : \mathcal{D}g(k)^{idem} \to \widehat{Sp}$ commuting with filtered colimits and sending the classical Nisnevich squares of dg-categories of finite type to pullback-pushout squares. To conclude this discussion we remark that the existing left adjoint to the inclusion $Fun_{\omega,Nis}(\mathcal{D}g(k)^{idem}, \widehat{Sp}) \subseteq Fun_{\omega}(\mathcal{D}g(k)^{idem}, \widehat{Sp})$ fits in a commutative diagram

$$\begin{tikzcd}
Fun_{\omega}(\mathcal{D}g(k)^{idem}, \widehat{Sp}) \arrow[r, "\sim"] \arrow[d] & Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp}) \arrow[d, "l_{Nis}"] \\
Fun_{\omega,Nis}(\mathcal{D}g(k)^{idem}, \widehat{Sp}) \arrow[r, "\sim"] & Fun_{Nis}(\mathcal{D}g(k)^{ft}, \widehat{Sp})
\end{tikzcd} \tag{8.0.5}$$

We can now proceed in analogous terms and localize with respect to $\mathbb{A}^1$ to obtain our new description of $\mathcal{SH}_{nc}(k)$.

Tabuada's approach focuses on the full subcategory $Fun_{\omega,Loc}(\mathcal{D}g(k)^{idem}, \widehat{Sp})$ spanned by those functors sending exact sequences of dg-categories to fiber/cofiber sequences in spectra. His main theorem is the existence of a stable presentable $(\infty,1)$-category which we denote here as $\mathcal{M}^{Tab}_{Loc}$, together with a functor $\mathcal{D}g(k)^{idem} \to \mathcal{M}^{Tab}_{Loc}$ preserving filtered colimits, sending exact sequences to fiber/cofiber sequences and universal in this sense. We can also easily see that $\mathcal{M}^{Tab}_{Loc}$ is a stable presentable symmetric monoidal $(\infty,1)$-category with the monoidal structure extending the monoidal structure in $\mathcal{D}g(k)^{idem}$. This result was originally formulated using the language of derivators (see [101] for an introduction) but we can easily extend it to the setting of $(\infty,1)$-categories by applying the same construction and the general machinery developed by J. Lurie in [100, 99]. In particular we have an equivalence of $(\infty,1)$-categories

$$Fun_{\omega,Loc}(\mathcal{D}g(k)^{idem}, \widehat{Sp}) \simeq Fun^{L}(\mathcal{M}^{Tab}_{Loc}, \widehat{Sp}) \tag{8.0.6}$$

As we can see this is a theorem about a specific class of objects inside $Fun_{\omega}(\mathcal{D}g(k)^{idem}, \widehat{Sp})$, namely, those that satisfy localization. The comparison with our approach starts with the observation that any object F satisfying localization satisfies also our condition of Nisnevich descent so that we have an inclusion of full subcategories $Fun_{\omega,Loc}(\mathcal{D}g(k)^{idem}, \widehat{Sp}) \subseteq Fun_{\omega,Nis}(\mathcal{D}g(k)^{idem}, \widehat{Sp})$. In particular, we can identify

$$\mathcal{SH}^{Loc}_{nc}(k) := Fun_{\omega,Loc,\mathbb{A}^1}(\mathcal{D}g(k)^{idem}, \widehat{Sp}) \tag{8.0.7}$$

with a full subcategory of $\mathcal{SH}_{nc}(S)$. We summarize this in the following diagram

$$\begin{tikzcd}
 & Fun_{\omega}(\mathcal{D}g(k)^{idem}, \widehat{Sp}) & \\
Fun_{\omega,Nis}(\mathcal{D}g(k)^{idem}, \widehat{Sp}) \arrow[ur, hook] & & Fun_{\omega,Loc}(\mathcal{D}g(k)^{idem}, \widehat{Sp}) \arrow[ul, hook'] \arrow[ll, hook'] \\
Fun_{\omega,Nis,\mathbb{A}^1}(\mathcal{D}g(k)^{idem}, \widehat{Sp}) =: \mathcal{SH}_{nc}(k) \arrow[u, hook] & & \mathcal{SH}^{Loc}_{nc}(k) \arrow[u, hook] \arrow[ll, hook']
\end{tikzcd} \tag{8.0.8}$$

The second observation is that the construction $\mathcal{M}^{Tab}_{Loc}$ of Tabuada and the formula (8.0.6) admits analogues adapted to each of the full subcategories in this diagram. More precisely one can easily show the existence of new stable presentable symmetric monoidal $(\infty,1)$-categories $\mathcal{M}^{Tab}_{Nis}$, $\mathcal{M}^{Tab}_{Nis,\mathbb{A}^1}$, $\mathcal{M}^{Tab}_{Loc,\mathbb{A}^1}$ all equipped with ω-continuous monoidal functors from $\mathcal{D}g(k)^{idem}$, universal with respect to each of the obvious respective properties. In particular we find an equivalence

$$Fun^{L}(\mathcal{M}^{Tab}_{Nis,\mathbb{A}^1}, \widehat{Sp}) \simeq \mathcal{SH}_{nc}(k) \tag{8.0.9}$$

exhibiting the duality between our approach and the corresponding Nisnevich-$\mathbb{A}^1$-version of Tabuada's construction (recall that the very big $(\infty,1)$-category of big stable presentable $(\infty,1)$-categories has a natural symmetric monoidal structure [100, 4.8.2.10, 4.8.2.18 and 4.8.1.17] where the big $(\infty,1)$-category of spectra $\widehat{Sp}$ is a unit and $Fun^{L}(-,-)$ is the internal-hom).

To conclude this discussion, one can show that all the vertical inclusions in the diagram admit monoidal left adjoints and that, in particular, $\mathcal{SH}_{nc}^{Loc}(k)$ is endowed with an obvious universal property concerning Localizing descent. In fact it it enough to describe a monoidal left adjoint to the inclusion $\mathcal{SH}_{nc}^{Loc}(k) \subseteq Fun_{\omega}(\mathcal{D}g(k)^{idem}, \widehat{Sp})$ because, as Localizing descent forces Nisnevich descent (see the arguments in the proof of the Prop. 7.1.4), the universal properties involved will then provide a monoidal left adjoint to the inclusion $\mathcal{SH}_{nc}^{Loc}(k) \subseteq \mathcal{SH}_{nc}(k)$, relating our theory to the dual of localizing theory of Tabuada

$$l_{Loc} : \mathcal{SH}_{nc}(k) \to \mathcal{SH}_{nc}^{Loc}(k) \tag{8.0.10}$$

Construction 8.0.11. The existence of a monoidal left adjoint to the inclusion

$$\mathcal{SH}_{nc}^{Loc}(k) \subseteq Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp})$$

follows essentially by the arguments of Chapter 6. We first consider the canonical functor

$$\chi^{(-)} : (\mathcal{D}g(k)^{idem})^{op} \to \mathcal{P}^{big}(\mathcal{N}cS(k)) \tag{8.0.11}$$

defined by the formula

$$\chi^T := Map_{\mathcal{D}g(k)^{idem}}(T, -) \tag{8.0.12}$$

such that when T is of finite type this is just the Yoneda's inclusion $\mathcal{N}cS(k) \to \mathcal{P}(\mathcal{N}cS(k))$. We denote again by the same symbol $\chi^{(-)}$ the composition with the stabilization

$$\mathcal{P}^{big}(\mathcal{N}cS(k)) \to Stab(\mathcal{P}^{big}(\mathcal{N}cS(k))) \simeq Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp}) \simeq Fun_{\omega}(\mathcal{D}g(k)^{idem}, \widehat{Sp}) \tag{8.0.13}$$

and consider the localization of the last with respect to the following class of maps: if

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d, "g"] \\ * \arrow[r] & C \end{tikzcd} \tag{8.0.14}$$

is an exact sequence of dg-categories in sense discussed in section 6.4.1, and K is a compact object in $\widehat{Sp}$, we consider the canonical map

$$\delta_{\chi^C}(K) \to fiber(\delta_{\chi^B}(K) \to \delta_{\chi^A}(K)) \tag{8.0.15}$$

One can check that the collection of maps of this form is of small generation (see [133, Section 10]) and we consider the localization of $Fun_{\omega}(\mathcal{D}g(k)^{idem}, \widehat{Sp})$ with respect to it. The result is obvisouly equivalent to $Fun_{\omega,Loc}(\mathcal{D}g(k)^{idem}, \widehat{Sp})$. Then, by definition, $\mathcal{SH}_{nc}^{Loc}(k)$ is obtain by forcing $\mathbb{A}^1$-invariance via the standard procedure. The whole procedure is monoidal (this follows from the same arguments as in the proof of the Prop. 6.4.14-2)) so that $\mathcal{SH}_{nc}^{Loc}(k)$ acquires a symmetric monoidal structure $\mathcal{SH}_{nc}^{Loc,\otimes}(k)$ and the monoidal functor $\mathcal{N}cS(k) \to \mathcal{SH}_{nc}^{Loc}(k)$ is seen to have the obvious monoidal universal property with respect to exact sequences of dg-categories.

Remark 8.0.12. The previous construction can also be made using presheaves of spaces.

Remark 8.0.13. Notice also that $\mathcal{SH}_{nc}^{Loc}(k)$ is stable by the same reasons $\mathcal{SH}_{nc}(k)$ is: combine the arguments in the Prop 6.4.19 with the fact the universal map $l_{Loc} : \mathcal{SH}_{nc}(k) \to \mathcal{SH}_{nc}^{Loc}(k)$ is monoidal and commutes with colimits. Moreover, as in the Prop. 5.3.3 and in the Remark 6.4.25 we can check that collection of noncommutative motives of dg-categories of finite type forms a family of compact generators in $\mathcal{SH}_{nc}^{Loc}(k)$. Indeed, to see this it is enough to check that the inclusion of $\mathcal{SH}_{nc}^{Loc}(k)$ inside $Fun_{\omega}(\mathcal{D}g(k)^{idem}, \widehat{Sp})$ commutes with filtered colimits: if $\{F_i\}_{i\in I}$ is a filtered family of functors

$\mathcal{D}g(k)^{idem} \to \widehat{Sp}$, each one sending an exact sequence of dg-categories to a pullback square in spectra and satisfying $\mathbb{A}^1$-invariance, then the colimit functor $colim_i F_i$ will again satisfy $\mathbb{A}^1$-invariance and send exact sequences of dg-categories to pullbacks in spectra. This is because colimits of functors are computed objectwise and also because filtered colimits in spectra commute with pullbacks.

Remark 8.0.14. It follows from the universal property of $\mathcal{SH}_{nc}^{Loc}(k)$ and from the discussion in Section 6.4.1 that if X is a quasi-compact and quasi-separated scheme over k and $i : Z \to X$ is a closed subscheme (over k) with open complementary $j : U \to X$, the image of the exact sequence of dg-categories in the diagram (6.4.6) becomes a cofiber/fiber sequence in $\mathcal{SH}_{nc}^{Loc}(k)$. In particular, the image of $\chi^{L_{pe,Z}(X)}$ becomes the homotopy cofiber of the image of the inclusion $\chi^{L_{pe}(U)} \to \chi^{L_{pe}(X)}$.

Remark 8.0.15. Let $F \in Fun(\mathcal{D}g(k)^{ft}, \widehat{Sp})$. Suppose that the truncation $\tau_{\geq 0}F$ satisfies Localization (meaning, sends exact sequences of dg-categories to homotopy fiber sequences in $\widehat{Sp}_{\geq 0}$). Then, using exactly the same arguments used to prove the Proposition 7.2.23, we find that the $\bar{B}$-construction of Thomason is an explicit model for the localization in the Construction 8.0.11.

By the theorem 7.0.32, the unit in $\mathcal{SH}_{nc}(k)$ is equivalent to the homotopy invariant K-theory functor. In particular, it satisfies localization and therefore lives in the subcategory $\mathcal{SH}_{nc}^{Loc}(k)$ (where it is also a unit because l_{Loc} is monoidal). Combining this fact with the Corollary 7.0.33 and the fact that l_{Loc} is left adjoint to the inclusion, we find that l_{Loc} is an equivalence when restrict to the subcategory of noncommutative motives generated by smooth and proper dg-categories. Moreover, following the same arguments as in the proof of the Corollary 7.0.37 we have

Corollary 8.0.16. *Let k be a field admitting resolutions of singularities. Then the composition*

$$l_{Loc} \circ \mathcal{L}_{\mathcal{K}H} : Mod_{\mathcal{K}H}(\mathcal{SH}(k)) \to \mathcal{SH}_{nc}(k) \to \mathcal{SH}_{nc}^{Loc}(k) \tag{8.0.16}$$

is fully faithful.

As emphasized before, the main advantage (in fact, la *raison-d'être*) of our approach to noncommutative motives is the easy comparison with the motivic stable homotopy theory of schemes. The duality presented in this chapter explains why the original approach of Cisinski-Tabuada is not directly comparable. The second main advantage is the canonical way in which we extract non-connective K-theory out of connective K-theory.

Part III

- Six Operations and Noncommutative Motives

Chapter **9**

Preliminaries III - Functoriality and Base Change

In this section we explore some technical results that will be necessary to the discussion in the next chapter. In sections 9.1 and 9.2 we analyze the relative behavior of the theory of motives and noncommutative motives as the base ring varies. These results are then applied in section 9.3 to extend the theories of motives and noncommutative motives to general base schemes: for any base scheme S we introduce stable presentable symmetric monoidal $(\infty,1)$-categories $\mathcal{SH}^{\otimes}(S)$ and $\mathcal{SH}_{nc}^{\otimes}(S)$ encoding, respectively, the theories of commutative and noncommutative motives over S, together with a bridge $\mathcal{L}_S^{\otimes}:\mathcal{SH}^{\otimes}(S)\to\mathcal{SH}_{nc}^{\otimes}(S)$ relating the two. Moreover, we study the behavior under a change of S. In paralell we will also introduce symmetric monoidal $(\infty,1)$-categories $\mathcal{SH}_{nc}^{Loc,\otimes}(S)$ and promote the monoidal localization functor of the previous chapter to a natural transformation $\mathcal{SH}_{nc}^{\otimes}\to\mathcal{SH}_{nc}^{Loc,\otimes}$.

As a last preliminary step, in section 9.4, we combine the results of J. Ayoub in [6, 7] with the recents results in [93, 94] to describe the existence of a formalism of six operations for the system of symmetric monoidal $(\infty,1)$-categories $\mathcal{SH}^{\otimes}$.

9.1 Functorial Behavior of $k\mapsto\mathcal{SH}(k)^{\otimes}$ and $k\mapsto\mathcal{SH}_{nc}(k)^{\otimes}$

We choose universes $\mathbb{U}\in\mathbb{V}\in\mathbb{W}$ and $\mathbb{W}$ will be assumed to be an element of an even bigger universe which we allow ourselves to omit from the notations. We will write Cat_∞ (resp. $\mathcal{S}^{small}$), Cat_∞^{big} (resp. $\mathcal{S}$) and Cat_∞^{vbig} (resp. $\mathcal{S}^{vbig}$) to denote, respectively, the big $(\infty,1)$-category of small $(\infty,1)$-categories (resp. small spaces), the very big $(\infty,1)$-category of big $(\infty,1)$-categories (resp. big spaces) and the (very very big) $(\infty,1)$-category of very big $(\infty,1)$-categories (resp. very big spaces). We will write $\mathcal{P}r^L$ to denote the non-full subcategory of Cat_∞^{vbig} spanned by those very big $(\infty,1)$-categories which are $\mathbb{V}$-presentable.

Let $N(Aff)$ be the nerve of the category of (small) affine schemes and $N(BAff)$ be its full subcategory spanned by those affine schemes that are Noetherian and of finite Krull dimension. In section 9.1 we show that both the assignments $k\mapsto\mathcal{SH}(k)^{\otimes}$ and $k\mapsto\mathcal{SH}_{nc}(k)^{\otimes}$ can be encoded as ∞-functors ∞-functors $\mathcal{SH}^{\otimes},\mathcal{SH}_{nc}^{\otimes}:N(BAff)^{op}\to CAlg(\mathcal{P}r_{Stb}^L)$ and that the collection of monoidal functors $\mathcal{L}_k^{\otimes}:\mathcal{SH}(S)^{\otimes}\to\mathcal{SH}_{nc}(k)^{\otimes}$ provides a natural transformation $\mathcal{SH}^{\otimes}\to\mathcal{SH}_{nc}^{\otimes}$. This is the same as saying that the $\mathcal{L}_k^{\otimes}$ are compatible under base change. In section 9.2 we prove that both the ∞-functors $\mathcal{SH}^{\otimes}$ and $\mathcal{SH}_{nc}^{\otimes}$ are stacks of symmetric monoidal ∞-categories with respect to the Zariski topology in $N(BAff)$. Finally in 9.3 we use these descent properties to explain how the extend our definitions to any Noetherian scheme of finite Krull dimension.

We organize this section in two steps:

Step 1) The construction of the ∞-functors $\mathcal{SH}^{\otimes},\mathcal{SH}_{nc}^{\otimes}:N(Aff)^{op}\to CAlg(\mathcal{P}r_{Stb}^L)$, encoding respectively the assignments $k\mapsto\mathcal{SH}(k)^{\otimes}$ and $k\mapsto\mathcal{SH}_{nc}(k)^{\otimes}$. This of course uses the fact that for any morphism of rings $u : k \to k'$ the base-change along u is compatible with both the commutative and noncommutative definitions of Nisnevich topology and affine line.

Step 2) We check that the collection of monoidal functors $\mathcal{L}_k^{\otimes} : \mathcal{SH}(S)^{\otimes} \to \mathcal{SH}_{nc}(k)^{\otimes}$ provides a natural transformation $\mathcal{SH}^{\otimes} \to \mathcal{SH}_{nc}^{\otimes}$

Step 1)

We start by dealing with the assignment $k \to \mathcal{SH}(k)^{\otimes}$. Again let $N(\mathit{Aff})$ be the nerve of the category of small affine schemes. It is a big $(\infty, 1)$-category. Let

$$Fun(\Delta[1], N(\mathit{Aff})^{op}) \to N(\mathit{Aff})^{op} \tag{9.1.1}$$

be the evaluation at 1. We let Sm denote the full subcategory of $Fun(\Delta[1], N(\mathit{Aff})^{op})$ spanned by those morphisms $Spec(A) \to Spec(k)$ in $N(\mathit{Aff})$ such that k is Noetherian and of finite Krull dimension and A is smooth of finite type over k. For each $Spec(k) \in N(B\mathit{Aff})^{op}$, its fiber over the composition $Sm \subseteq Fun(\Delta[1], N(\mathit{Aff})^{op}) \to N(\mathit{Aff})^{op}$ is equivalent to the nerve of the category of smooth affine schemes of finite type over k, $N(\mathit{AffSm}^{ft}(k))$. One can easily check that this composition is a cocartesian fibration because smooth algebras are stable under base change. Therefore, using the ∞-categorical version of the Grothendieck construction (see [99, Chapter 3]) this can be arranged as an ∞-functor

$$Sm : N(B\mathit{Aff})^{op} \to Cat_{\infty}^{big} \tag{9.1.2}$$

sending $Spec(k)$ to the nerve $N(\mathit{AffSm}^{ft}(k))$. As for any ring k the later category has finite products and as for any morphism of affine schemes $Spec(k') \to Spec(k)$, the induced pullback functor $N(\mathit{AffSm}^{ft}(k)) \to N(\mathit{AffSm}^{ft}(k'))$ preserves finite products, using the equivalence between $(\infty, 1)$-categories with finite products and cartesian symmetric monoidal structures together with monoidal functors (see [100, Corollary 2.4.1.9], we find that our functor admits a lifting to an ∞-functor

$$Sm^{\times} : N(B\mathit{Aff})^{op} \to CAlg(Cat_{\infty}^{big}) \tag{9.1.3}$$

this time sending k to the nerve of the category $N(\mathit{AffSm}^{ft}(k))$ endowed with the tensor structure given by the cartesian product.

We now consider the composition

$$N(B\mathit{Aff})^{op} \longrightarrow CAlg(Cat_{\infty}^{big}) \xrightarrow{\mathcal{P}(-)} CAlg(Cat_{\infty}^{vbig}(\mathcal{K}))$$

where $\mathcal{K}$ is the collection of all small simplicial sets and $Cat_{\infty}^{vbig}(\mathcal{K})$ is the non-full subcategory of Cat_{∞}^{vbig} spanned by the $(\infty, 1)$-categories having all (big) colimits together with colimit preserving functors. By [100, 4.8.1.3, 4.8.1.4 and 4.8.1.9] this has a natural symmetric monoidal structure and the functor $S \mapsto \mathcal{P}(S)$ is monoidal with respect to this tensor product. As the full inclusion $\mathcal{P}r^L \subseteq Cat_{\infty}^{vbig}(\mathcal{K})$ is monoidal (see [100, 4.8.1.14]) and the $(\infty, 1)$-categories $\mathcal{P}(N(\mathit{AffSm}^{ft}(k)))$ are presentable, the previous composition factors as

$$\mathcal{P}Sm^{\times} : N(B\mathit{Aff})^{op} \to CAlg(\mathcal{P}r^L)$$

We now explain how to pass from the $(\infty, 1)$-categories $\mathcal{P}(N(\mathit{AffSm}^{ft}(k)))$ to the Nisnevich-$\mathbb{A}^1$-localizations $\mathcal{H}(k)$ in such a way that the functoriality with respect to the base is respected. For this purpose we will need to perform a small digression. Following the construction [100, 4.1.3.1], it is possible to construct an $(\infty, 1)$-category $\mathcal{W}Cat_{\infty}^{vbig}$ whose objects are pairs $(\mathcal{C}, W)$ with $\mathcal{C}$ a very big $(\infty, 1)$-category and W a class of arrows in $\mathcal{C}$ that is stable under homotopies, composition and contains all equivalences. By construction this $(\infty, 1)$-category comes equipped with a forgetful functor $\mathcal{W}Cat_{\infty}^{vbig} \to Cat_{\infty}^{vbig}$ defined by forgetting the class of arrows. Moreover, it is endowed with a cartesian symmetric monoidal structure given by the product of pairs and the forgeful functor is monoidal. Furthermore, the forgetful map admits a canonical fully-faithful left adjoint sending an $(\infty,1)$-category $\mathcal{C}$ to the pair $(\mathcal{C}, W)$ with W the collection of equivalences in $\mathcal{C}$. By [100, 4.1.3.2], this section admits a left adjoint. In other words, there is an ∞-functor $Loc : \mathcal{W}Cat_{\infty}^{vbig} \to Cat_{\infty}^{vbig}$ sending a pair $(\mathcal{C}, W)$ to an $(\infty,1)$-category $\mathcal{C}[W^{-1}]$ with the universal property of the localization.

We set $\mathcal{WP}r^L$ as a notation for the (non-full) subcategory of $\mathcal{W}Cat_{\infty}^{vbig}$ spanned by those pairs $(\mathcal{C}, W)$ with $\mathcal{C}$ a very big presentable $(\infty,1)$-category and W a strongly saturated class of arrows in $\mathcal{C}$ which is of small generation (see [99, 5.5.4.5, 5.5.4.7]). As a result of [99, 5.5.4.15 and 5.5.4.20], the composition of Loc with the inclusion $\mathcal{WP}r^L \subseteq \mathcal{W}Cat_{\infty}^{vbig}$ admits a factorization

$$\begin{tikzcd} \mathcal{WP}r^L \arrow[r, hook] \arrow[d, dashed] & \mathcal{W}Cat_{\infty}^{vbig} \arrow[d] \\ \mathcal{P}r^L \arrow[r, hook] & Cat_{\infty}^{vbig} \end{tikzcd} \tag{9.1.4}$$

Let us denote this new factorization as Loc^{pr}. Our main observation is that Loc^{pr} can be given the structure of monoidal functor. As already discussed in this section $\mathcal{P}r^L$ is closed under the tensor structure in $Cat_{\infty}^{vbig}(\mathcal{K})$ and the same arguments that allow the monoidal structure in $\mathcal{P}r^L$ can now be used to give a monoidal structure to $\mathcal{WP}r^L$: let $\mathcal{W}Cat_{\infty}^{\times} \to N(Fin_*)$ be the cocartesian fibration encoding the cartesian symmetric monoidal structures on pairs and let $\mathcal{W}Cat_{\infty}^{vbig}(\mathcal{K})^{\otimes}$ denote the non-full subcategory of $\mathcal{W}Cat_{\infty}^{\times}$ spanned by those sequences $((\mathcal{C}_1, W_1), ..., (\mathcal{C}_n, W_n))$ such that for every i, $\mathcal{C}_i$ has all colimits, together with those morphisms of pairs $u : ((\mathcal{C}_1, W_1), ..., (\mathcal{C}_n, W_n)) \to ((\mathcal{D}_1, S_1), ..., (\mathcal{D}_m, S_m))$ such that the corresponding maps

$$\Pi_{j \in f^{-1}(\{i\})}(\mathcal{C}_j, W_j) \to (\mathcal{D}_i, S_i) \text{ with } i \in \{1, ..., m\} \tag{9.1.5}$$

are given by functors that preserve colimits in each variable separately. In this notation, $f : \langle n \rangle \to \langle m \rangle$ is the projection of u in $N(Fin_*)$.

Proposition 9.1.1. *The composition $p : \mathcal{W}Cat_{\infty}^{vbig}(\mathcal{K})^{\otimes} \subseteq \mathcal{W}Cat_{\infty}^{\times} \to N(Fin_*)$ is a cocartesian fibration. Moreover, the non full inclusion $\mathcal{W}Cat_{\infty}^{vbig}(\mathcal{K})^{\otimes} \to \mathcal{W}Cat_{\infty}^{vbig,\times}$ is lax monoidal.*

Proof. Recall that the tensor product in $Cat_{\infty}^{vbig}(\mathcal{K})$ is defined by means of a universal property: if $\mathcal{C}$ and $\mathcal{C}'$ both have all colimits, we can fabricate a new $(\infty,1)$-category $\mathcal{C} \otimes \mathcal{C}'$ having all colimits and equipped with a functor $\mathcal{C} \times \mathcal{C}' \to \mathcal{C} \otimes \mathcal{C}'$ universal with respect to the property of preserving colimits separately in each variable (see [99, 5.3.6.2]). It follows that as soon as we construct an analogous universal property in the context of pairs, the proof of this proposition will follow using arguments similar to those in [100, 4.8.1.3]. Let us address the universal property. Let $(\mathcal{C}_1, W_1)$ and $(\mathcal{C}_2, W_2)$ be two pairs with $\mathcal{C}_1$ and $\mathcal{C}_2$ having all colimits. We define their tensor product as follows: the underlying $(\infty,1)$-category of the pair is the tensor product $\mathcal{C}_1 \otimes \mathcal{C}_2$ in $Cat_{\infty}^{vbig}(\mathcal{K})$. For the collection of morphisms we consider the image of the product collection $W_1 \times W_2$ in $\mathcal{C}_1 \times \mathcal{C}_2$ along the universal map $\mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{C}_1 \otimes \mathcal{C}_2$. We denote this collection as $W_1 \otimes W_2$ so that the product pair can be written as $(\mathcal{C}_1 \otimes \mathcal{C}_2, W_1 \otimes W_2)$. Of course, by definition the universal map provides a morphism of pairs $(\mathcal{C}_1, W_1) \times (\mathcal{C}_2, W_2) \to (\mathcal{C}_1 \otimes \mathcal{C}_2, W_1 \otimes W_2)$. We are left to check that for any pair $(\mathcal{D}, S)$ with $\mathcal{D}$ having all colimits, the composition along this morphism

$$Map_{\mathcal{W}Cat_{\infty}^{vbig}}((\mathcal{C}_1 \otimes \mathcal{C}_2, W_1 \otimes W_2), (\mathcal{D}; S)) \to Map_{\mathcal{W}Cat_{\infty}^{vbig}}((\mathcal{C}_1, W_1) \times (\mathcal{C}_2, W_2), (\mathcal{D}, S)) \tag{9.1.6}$$

induces an homotopy equivalence between the subspace of maps of pairs that preserve colimits (on the left side) and the subspace of maps that of pairs that preserve colimits separately in each variable (on the right side). But of course, this follows from the full universal property of the construction $\mathcal{C} \otimes \mathcal{C}'$ because the mapping spaces of pairs are summands of the mapping spaces in Cat_{∞}^{vbig} obtained by forgeting the collections of arrows, together with the fact that the definition of $W_1 \otimes W_2$ is compatible with this universal property.

□

Using the same arguments of [100, 4.8.1.14] we can also prove that the full subcategory $\mathcal{W}\mathcal{P}r^{L,\otimes}$ of $\mathcal{W}Cat_{\infty}^{vbig}(\mathcal{K})^{\otimes}$ spanned by those sequences $((\mathcal{C}_1, W_1), ..., (\mathcal{C}_n, W_n))$ where each $\mathcal{C}_i$ is presentable, is closed under this monoidal structure. Moreover, as we can identify $\mathcal{P}r^L$ with the full subcategory of $\mathcal{W}\mathcal{P}r^L$ spanned by those pairs $(\mathcal{C}, W)$ where W consists of the collection of all equivalences in $\mathcal{C}$ and as Loc^{pr} provides a left adjoint to this inclusion, we can use the arguments in the proof of [100, 4.8.1.14] to check that the formula $Loc^{pr}((\mathcal{C}, W)) \otimes Loc((\mathcal{D}, S)) \simeq Loc^{pr}((\mathcal{C}, W) \otimes (\mathcal{D}, S))$ holds. This implies the necessary conditions to apply [100, Prop. 2.2.1.9] and deduce that Loc^{pr} is a monoidal reflexive localization and that the induced monoidal structure on $\mathcal{P}r^L$ is the same we had before.

Following this discussion, Loc^{pr} extends to algebra-objects so that $CAlg(\mathcal{P}r^L)$ becomes a reflexive localization of $CAlg(\mathcal{W}Prl)$. At the same time, as the forgetful functor $\mathcal{W}Cat_{\infty}^{vbig} \to Cat_{\infty}^{vbig}$ commutes with products, it extends to a monoidal functor $\mathcal{W}Cat_{\infty}^{vbig,\times} \to Cat_{\infty}^{vbig,\times}$ so that the composition

$$\mathcal{W}\mathcal{P}r^{L,\otimes} \subseteq \mathcal{W}Cat_{\infty}(\mathcal{K})^{vbig,\otimes} \to \mathcal{W}Cat_{\infty}^{vbig,\times} \to Cat_{\infty}^{vbig,\times} \tag{9.1.7}$$

is lax monoidal. As explained above, this composition factors through $\mathcal{P}r^{L,\otimes}$ so the forgetful functor restricts to a lax monoidal functor $\mathcal{W}\mathcal{P}r^{L,\otimes} \to \mathcal{P}r^{L,\otimes}$. In particular this makes it possible to extend the forgetful functor to algebra-objects $for : CAlg(\mathcal{W}\mathcal{P}r^L) \to CAlg(\mathcal{P}r^L)$.

Let us now apply this discussion to our situation. Consider $\mathcal{P}Sm^{\times}$ the functor obtained at the end of the last step. Now, as the forgeful functor is lax monoidal it induces a map $CAlg(\mathcal{W}\mathcal{P}r^L) \to CAlg(\mathcal{P}r^L)$, so that we can identify a coherent choice[1] of a class of arrows on each $(\infty, 1)$-category $\mathcal{P}(N(AffSm^{ft}(k)))$ with the data of a lifting

$$\begin{tikzcd} & CAlg(\mathcal{W}\mathcal{P}r^L) \arrow[d, "for"] \\ N(BAff)^{op} \arrow[r, "\mathcal{P}Sm^{\times}"'] \arrow[ur, dashed] & CAlg(\mathcal{P}r^L) \end{tikzcd} \tag{9.1.8}$$

As it can easily be checked, the definition of a Nisnevich square of schemes is well-behaved under base-change. In this case, the data provided by the generating Nisnevich-equivalences [2]satisfies the necessary coherences so that it defines a lifting

$$\begin{tikzcd} & CAlg(\mathcal{W}\mathcal{P}r^L) \arrow[d, "for"] \\ N(BAff)^{op} \arrow[r, "\mathcal{P}Sm^{\times}"'] \arrow[ur, dashed, "\mathcal{P}Sm^{\times}_{Nis}"] & CAlg(\mathcal{P}r^L) \end{tikzcd} \tag{9.1.9}$$

and we define a new ∞-functor $\mathcal{H}^{\times}_{Nis}$ as the composition

$$N(BAff)^{op} \xrightarrow{\mathcal{P}Sm^{\times}_{Nis}} CAlg(\mathcal{W}\mathcal{P}r^L) \xrightarrow{Loc^{pr}} CAlg(\mathcal{P}r^L) \tag{9.1.10}$$

It follows from the universal property of the localization that for every ring k, the underlying $(\infty, 1)$-category of $\mathcal{H}^{\times}_{Nis}(k)$ recovers (up to equivalence) the full subcategory of $\mathcal{P}(N(AffSm^{ft}(k)))$ spanned by those presheaves that satisfy Nisnevich descent. We now repeat this procedure to obtain

[1] meaning, stable under base change

[2] for a fixed ring k, this is the collection of all maps $j(U) \coprod_{J(W)} j(V) \to j(X)$ for any Nisnevich square of smooth schemes over k, with j the Yoneda's embedding

the $\mathbb{A}^1$-localized version of the theory. Again, as the definition of affine line is compatible with base-change, the data of the generating $\mathbb{A}^1$-equivalences [3] satisfies the necessary coherences and therefore can be identified with a lifting

$$\begin{tikzcd} & CAlg(\mathcal{W}\mathcal{P}r^L) \arrow[d, "for"] \\ N(BAff)^{op} \arrow[ur, dashed, "\mathcal{H}^{\times}_{Nis,\mathbb{A}^1}"] \arrow[r, "\mathcal{H}^{\times}_{Nis}"] & CAlg(\mathcal{P}r^L) \end{tikzcd} \tag{9.1.11}$$

so that again by composition with Loc^{pr} we obtain a new ∞-functor $\mathcal{H}^{\times}$ sending a ring k to the cartesian presentable symmetric monoidal $(\infty,1)$-category $\mathcal{H}(k)^{\times}$.

Following the discussion in 5.2.1, we consider the composition with the pointing map

$$\mathcal{H}^{\wedge}_{*} : N(BAff)^{op} \xrightarrow{\mathcal{H}^{\times}} CAlg(\mathcal{P}r^L) \xrightarrow{(-)_*} CAlg(\mathcal{P}r^L_{Pt}) \hookrightarrow CAlg(\mathcal{P}r^L)$$

We come now to the last step of the construction, namely, the inversion of an object with respect to the monoidal structure. We recall our discussion in the section 4.1 where this procedure was developed. For any presentable symmetric monoidal $(\infty,1)$-category $\mathcal{C}^{\otimes}$ together with the choice of an object $X \in \mathcal{C}$, we consructed a new presentable symmetric monoidal $(\infty,1)$-category $\mathcal{C}^{\otimes}[X^{-1}]$ together with a monoidal map $\mathcal{C} \to \mathcal{C}^{\otimes}[X^{-1}]$ sending X to a $\otimes$-invertible object and universal in this sense within presentable symmetric monoidal $(\infty,1)$-categories. Using the same notations as there, the data of a pair $(\mathcal{C}^{\otimes}, X \in \mathcal{C})$ can be identified with a lifting

$$\begin{tikzcd} & CAlg(\mathcal{P}r^L)_{\mathcal{P}(free^{\otimes}(\Delta[0])^{\otimes}/.} \arrow[d] \\ \Delta[0] \arrow[ur, dashed, "(\mathcal{C}^{\otimes},X)"] \arrow[r] & CAlg(\mathcal{P}r^L) \end{tikzcd}$$

where $\Delta[0] \to CAlg(\mathcal{P}r^L)$ is the 0-simplex corresponding to $\mathcal{C}^{\otimes}$, $\mathcal{P}(free^{\otimes}(\Delta[0])^{\otimes}$ is the free cocomplete symmetric monoidal $(\infty,1)$-category generated by one object, and the map $CAlg(\mathcal{P}r^L)_{\mathcal{P}(free^{\otimes}(\Delta[0])^{\otimes}/.} \to CAlg(\mathcal{P}r^L)$ is the evaluation at the target. Under this identification, $\mathcal{C}^{\otimes}[X^{-1}]$ is, by definition, the target of the 0-cell in $CAlg(\mathcal{P}r^L)_{\mathcal{P}(free^{\otimes}(\Delta[0])^{\otimes}/.}$ given by the coproduct of $(\mathcal{C}^{\otimes}, X)$ with the functor

$$[*^{-1}] : \mathcal{P}(free^{\otimes}(\Delta[0])^{\otimes} \to \mathcal{P}(\mathcal{L}^{\otimes}_{(free^{\otimes}(\Delta[0]),*)}(free^{\otimes}(\Delta[0]))^{\otimes} \tag{9.1.12}$$

providing the universal monoidal colimit-preserving $\otimes$-inversion of the solo generating object in $\mathcal{P}(free^{\otimes}(\Delta[0])^{\otimes}$ (see our discussion in Section 4.1 for the details).

We are now interested in a relative formulation of this situation. More precisely, given a simplicial set K together with a diagram $\Phi : K \to CAlg(\mathcal{P}r^L)$ and the data of a family of objects $\{X_\alpha \in \Phi(\alpha)\}_{\alpha \in K_0}$ which is compatible with Φ (in the sense that for any arrow in $t : \alpha \to \alpha'$ in K we have equivalences $X_{\alpha'} \simeq \Phi(t)(X_\alpha)$), we want to define a new K-indexed diagram $\Phi[\{X_\alpha\}^{-1}_{\alpha\in K_0}] : K \to CAlg(\mathcal{P}r^L)$ together with a natural transformation $u : \Phi \to \Phi[\{X_\alpha\}^{-1}_{\alpha\in K_0}]$ such that for each $\alpha \in K_0$, the monoidal functor u_α exhibits $\Phi[\{X_\alpha\}^{-1}_{\alpha\in K_0}](\alpha)$ as the universal $\otimes$-inversion of the object X_α in $\Phi(\alpha)$ and for any arrow $t : \alpha \to \alpha'$ in K the induced functor $\Phi[\{X_\alpha\}^{-1}_{\alpha\in K_0}](t)$ is the one induced by the universal property of the universal $\otimes$-inversion.

Of course, the preceeding discussion for a single object generalizes to this case: First we observe that the choice of compatible collection of objects $\{X_\alpha\}_{\alpha\in K_0}$ is defined by the choice of a lifting

[3] For a fixed ring k this the collection of all projection maps $j(X \times \mathbb{A}^1_k) \to j(X)$, with X a smooth k-scheme.

$$\begin{tikzcd} & CAlg(\mathcal{P}r^L)_{\mathcal{P}(free^{\otimes}(\Delta[0])^{\otimes}/.} \arrow[d] \\ K \arrow[ur, dashed, "{(\Phi,\{X_\alpha\}_{\alpha\in K_0})}"] \arrow[r] & CAlg(\mathcal{P}r^L) \end{tikzcd}$$

At the same time, we have a constant functor $K \to CAlg(\mathcal{P}r^L)_{\mathcal{P}(free^{\otimes}(\Delta[0])^{\otimes}/.}$ sending every vertice of K to the universal $\otimes$-inversion $[*^{-1}]$ considered above. We will now use the same notation to denote this constant K-indexed diagram. Finally, we define a K-indexed diagram $\Phi[\{X_\alpha\}^{-1}_{\alpha\in K_0}]$ as the target of the coproduct

$$\Phi[\{X_\alpha\}^{-1}_{\alpha\in K_0}] := (\Phi, \{X_\alpha\}_{\alpha\in K_0}) \coprod [*^{-1}] \tag{9.1.13}$$

in $Fun(K, CAlg(\mathcal{P}r^L)_{\mathcal{P}(free^{\otimes}(\Delta[0])^{\otimes}/.})$. By definition, it comes equipped with a natural transformation $u : \Phi \to \Phi[\{X_\alpha\}^{-1}_{\alpha\in K_0}]$ that fits in a pushout diagram

$$\begin{tikzcd} \mathcal{P}(free^{\otimes}(\Delta[0])^{\otimes} \arrow[r, "{[*^{-1}]}"] \arrow[d, "\{X_\alpha\}_{\alpha\in K_0}"] & \mathcal{P}(\mathcal{L}^{\otimes}_{(free^{\otimes}(\Delta[0]),*)}(free^{\otimes}(\Delta[0]))^{\otimes} \arrow[d] \\ \Phi \arrow[r, "u"] & \Phi[\{X_\alpha\}^{-1}_{\alpha\in K_0}] \end{tikzcd} \tag{9.1.14}$$

in $Fun(K, CAlg(\mathcal{P}r^L))$ (with both the top entries understood as constant K-indexed diagrams). To conclude, as pushouts of diagrams are computed objectwise [99, 5.1.2.3], we get the expected universal property for u.

Let us now apply this discussion to our main goal. We consider the diagram $\mathcal{H}^{\wedge}_*$ together with the collection of objects $\{(\mathbb{P}^1_k, \infty)\}_{Spec(k)\in N(Aff)}$. This collection of objects is compatible in the sense specified above because the definition of projective line is stable under base change. Using the preceding discussion we set

$$\mathcal{SH}^{\otimes} := \mathcal{H}^{\wedge}_*[\{S^1 \wedge \mathbb{G}_m(k)\}^{-1}_{Spec(k)\in N(Aff)}] \tag{9.1.15}$$

The universal properties of our relative construction imply that for any ring k the presentable symmetric monoidal $(\infty,1)$-category $\mathcal{SH}^{\otimes}(Spec(k))$ is canonicaly equivalent to $\mathcal{SH}(k)^{\otimes}$. In particular, $\mathcal{SH}^{\otimes}$ factors through $CAlg(\mathcal{P}r^L_{Stb})$.

***

We now deal with the case $k \mapsto \mathcal{SH}_{nc}(k)^{\otimes}$. As in the previous case, the first step is to organize the assignment $k \mapsto \mathcal{N}cS(k)^{\otimes}$ as an ∞-functor

$$\mathcal{N}cS^{\otimes} : N(Aff)^{op} \to CAlg(Cat^{big}_{\infty}) \tag{9.1.16}$$

To achieve this we can proceed as follows. For any ring k we consider $Cat_{Ch(k)}$ the (big) category of small categories enriched in chain complexes of k-modules. If $k \to k'$ is a morphism of rings we have an induced base change functor $Ch(k) \to Ch(k')$ which extends to $Cat_{Ch(k)} \to Cat_{Ch(k')}$. This, we can check, makes the assignment $k \mapsto Cat_{Ch(k)}$ a pseudo-functor. Using the usual strictification procedure (back and forth using the Grothendieck construction) it can be strictified and encoded as a functor

$$Aff^{op} \to Cat^{big} \tag{9.1.17}$$

Moreover, for any ring k, $Cat_{Ch(k)}$ carries a natural symmetric monoidal structure induced by the tensor product of complexes and for any morphism $k \to k'$ the induced base-change is monoidal. Therefore, the previous functor can actually be seen as

$$Aff^{op} \to CAlg(Cat^{big}) \tag{9.1.18}$$

where $CAlg(Cat^{big})$ denotes the category of classical symmetric monoidal categories. As a third important ingredient we consider the model structure on $Cat_{Ch(k)}$ with weak-equivalences given by the Dwyer-Kan equivalences of dg-categories. As we know this model structure is not compatible with tensor products. In this case, we consider $Cat^{loc}_{Ch(k)}$ the full subcategory of $Cat_{Ch(k)}$ spanned by those dg-categories having cofibrant enriching complexes. As cofibrant complexes are levelwise projective (therefore flat) and stable under tensor products ($Ch(k)$ is a monoidal model category), $Cat^{loc-cof}_{Ch(k)}$ is closed under the monoidal structure in $Cat_{Ch(k)}$ and the tensor product of weak-equivalences is again a weak-equivalence. Moreover, as for any morphism of rings $k \to k'$ the induced base change $Ch(k) \to Ch(k')$ is a left Quillen functor, it preserves cofibrant objects and weak-equivallences between them ([69, 1.1.12]) so that the assignment $k \mapsto Cat^{loc-cof}_{Ch(k)}$ provides a functor

$$Aff^{op} \to CAlg(\mathcal{W}Cat^{big}) \tag{9.1.19}$$

where $CAlg(\mathcal{W}Cat^{big})$ is our notation for the category of classical symmetric monoidal categories equipped with a collection of morphisms stable under tensor products, together with monoidal functors compatible with the data of those collections.

By applying the nerve functor we get an ∞-functor

$$N(Aff)^{op} \to N(CAlg(\mathcal{W}Cat^{big})) \tag{9.1.20}$$

where the last $(\infty, 1)$-category is equivalent to the $CAlg(N(\mathcal{W}Cat^{big}))$ (see [100, 2.1.3.3]). Of course, there is a fully-faithful inclusion $N(\mathcal{W}Cat^{big}) \subseteq \mathcal{W}Cat^{big}_{\infty}$ which preserves cartesian products and therefore extends to an inclusion $CAlg(N(\mathcal{W}Cat^{big})) \subseteq CAlg(\mathcal{W}Cat^{big}_{\infty})$. We set $\mathcal{D}g^{\otimes^{\mathbb{L}}}$ as the composition

$$N(Aff)^{op} \to N(CAlg(\mathcal{W}Cat^{big})) \simeq CAlg(N(\mathcal{W}Cat^{big})) \subseteq CAlg(\mathcal{W}Cat^{big}_{\infty}) \to CAlg(Cat^{big}_{\infty}) \tag{9.1.21}$$

where the last arrow is the localization functor *Loc*.

We now consider the Morita model structure on $Cat_{Ch(k)}$ [131]. It is a Bousfield localization of the standard model structure considered in the discussion above. In particular, it has the same cofibrant objects. Again, the tensor product of dg-categories it not compatible with this model structure but everything works if we restrict to locally-cofibrant dg-categories. By applying the same arguments as above we end up with a functor

$$Aff^{op} \to CAlg(\mathcal{W}Cat^{big}) \tag{9.1.22}$$

(up to equivalence of categories) sending a ring k to the pair $(Cat^{loc-cof}_{Ch(k)}, W_{Morita})$ with $Cat^{loc-cof}_{Ch(k)}$ endowed with the tensor product of dg-categories and W_{Morita} the collection of Morita equivalences. We let $\mathcal{D}g^{idem,\otimes}$ denote the result of applying the formula in (9.1.21) to this functor.

As Dwyer-Kan equivalences of dg-categories are Morita equivalences, we have a canonical natural transformation from (9.1.19) to (9.1.22) which, as the nerve functor preserves internal-homs, induces by transport along (9.1.21) a natural transformation of ∞-functors $\mathcal{D}g^{\otimes^{\mathbb{L}}} \to \mathcal{D}g^{idem,\otimes}$.

Remark 9.1.2. For any ring k this natural transformation is nothing but the process of idempotent completion $\widehat{(-)}_{pe}$. More precisely we have commutative diagrams

$$\begin{tikzcd} N(Cat_{Ch(k)})[W_{DK}^{-1}] \arrow[r, "-\otimes_k k'"] \arrow[d, "\mathbb{L}Id"] & N(Cat_{Ch(k)})[W_{DK}^{-1}] \arrow[d, "\mathbb{L}Id"] \\ N(Cat_{Ch(k)})[W_{Morita}^{-1}] \arrow[r, "-\otimes_k k'"] & N(Cat_{Ch(k')})[W_{Morita}^{-1}] \end{tikzcd} \tag{9.1.23}$$

with $\mathbb{L}Id$ a left derived functor of the identity functor. By the definition of a Bousfield localization, this is given by a fibrant replacement in the Morita theory which we know corresponds to the process of idempotent completion

$$\begin{tikzcd} \mathcal{D}g(k) \simeq N(Cat_{Ch(k)})[W_{DK}^{-1}] \arrow[d, "\widehat{(-)}_{pe}"] \arrow[dr, "\mathbb{L}Id"] & \\ \mathcal{D}g^{idem}(k) \arrow[r, "\sim"] & N(Cat_{Ch(k)})[W_{Morita}^{-1}] \end{tikzcd} \tag{9.1.24}$$

so that the commutativity of these diagrams means that base-change commutes with idempotent completion. For more details we address reader to our survey in Section 6.1.2.

We can now apply the ∞-categorical version of the Grothendieck construction [99, Chapter 3] together with [100, 2.4.2.5] to encode $\mathcal{D}g^{idem,\otimes}$ as a cocartesian fibration (which we will denote using the same notation)

$$\begin{tikzcd} \mathcal{D}g^{idem,\otimes} \arrow[d] \\ N(Fin_*) \times N(\mathit{Aff})^{op} \end{tikzcd} \tag{9.1.25}$$

Finally, we let $\mathcal{D}g^{ft,\otimes}$ denote the full subcategory of $\mathcal{D}g^{idem,\otimes}$ spanned by those vertices $(T_1, ..., T_n, \langle n\rangle)$ for which each T_i is a dg-category of finite type (see [141] and our digression in Section 6.1.4). We prove now that the composition $\mathcal{D}g^{ft,\otimes} \subseteq \mathcal{D}g^{idem,\otimes} \to N(Fin_*) \times N(\mathit{Aff})^{op}$ is a cocartesian fibration. First we observe that the composition map $\mathcal{D}g^{ft,\otimes} \subseteq \mathcal{D}g^{idem,\otimes} \to N(Fin_*) \times N(\mathit{Aff})^{op} \to N(\mathit{Aff})^{op}$ is a cartesian fibration. This follows of course from the Grothendieck construction since for any morphism of rings $f : k \to k'$ the base change $\mathcal{D}g(k)^{idem} \to \mathcal{D}g(k')^{idem}$ has a right adjoint F_f^{idem} which we can understand as a forgetful functor

$$\begin{tikzcd} \mathcal{D}g^{idem}(k) \arrow[r, shift left, "(-\otimes_k^{\mathbb{L}} k')"] & \mathcal{D}g^{idem}(k') \arrow[l, shift left, "F_f^{idem}"] \end{tikzcd} \tag{9.1.26}$$

Remark 9.1.3. There exists also a forgetful functor $F_f : \mathcal{D}g(k') \to \mathcal{D}g(k)$ right adjoint to the base-change. Its existence can be deduced either from the adjoint functor theorem or from the fact that the standard forgetful functor for the strict theory of dg-categories is a right Quillen functor. As the base-change commutes with idempotent completion, we have a commutative diagram of right adjoints

$$\begin{tikzcd} \mathcal{D}g(k') \arrow[r, "F"', "f"] & \mathcal{D}g(k) \\ \mathcal{D}g^{idem}(k') \arrow[u, hook] \arrow[r, "F_f^{idem}"] & \mathcal{D}g^{idem}(k) \arrow[u, hook] \end{tikzcd} \tag{9.1.27}$$

where the vertical maps are the inclusion functors right adjoints to the idempotent completion. As both $\mathcal{D}g^{idem}(k)$ and $\mathcal{D}g^{idem}(k')$ are the underlying $(\infty, 1)$-categories of the Morita model structure on dg-categories, F_f^{idem} admits an explicit description as the composition of a fibrant replacement functor in k'-dg-categories P (with respect to the Morita model structure), the standard forgetful functor F_f, and finally, a cofibrant replacement functor Q k-dg-categories (again with respect to Morita). The fibrant replacement can be described by means of the assignement $T \mapsto \widehat{T}_{pe,k'}$ - meaning, the idempotent completion inside k'-dg-categories. For more details see the proof of [100, 1.3.4.21].

With this we can check that all the conditions in [100, 4.5.3.4] hold and conclude by observing that:

- for a fixed ring k dg-categories of finite type are stable under tensor product in $\mathcal{D}g(k)^{idem,\otimes}$ (see for instance [35, Theorem 4.3]).
- for a morphism of rings $f : k \to k'$ the induced base change functor $\mathcal{D}g(k)^{idem} \to \mathcal{D}g(k')^{idem}$ preserves dg-categories of finite type. This follows because the forgetful functor F_f^{idem} preserves filtered colimits. This will be proved in the Lemma 9.2.7 below.

To finish this discussion we apply again the Grothendieck construction and [100, 2.4.2.5] to present this cocartesian fibration as an ∞-functor

$$\mathcal{D}g^{ft,\otimes} : N(\mathit{Aff})^{op} \to CAlg(Cat_{\infty}^{big}) \tag{9.1.28}$$

and use the fact that the opposite $(\infty,1)$-category provides an ∞-functor $(-)^{op} : Cat_{\infty}^{big} \to Cat_{\infty}^{big}$ that commutes with cartesian products and therefore induces a functor $(-)^{op} : CAlg(Cat_{\infty}^{big}) \to CAlg(Cat_{\infty}^{big})$. With this we set

$$\mathcal{N}cS^{\otimes} := (-)^{op} \circ \mathcal{D}g^{ft,\otimes} \tag{9.1.29}$$

thus concluding our initial task.

To conclude this section we apply to $\mathcal{N}cS^{\otimes}$ exactly the same procedure described for the commutative case and with this obtain an ∞-functor

$$\mathcal{SH}_{nc}^{\otimes} : N(\mathit{Aff})^{op} \to CAlg(\mathcal{P}r_{Stb}^{L}) \tag{9.1.30}$$

satisfying the expected requirements. This time the key ingredients are that 1) both the definitions of noncommutative Nisnevich squares and noncommutative affine line, are stable under base-change (see our results in 6.4.14 and 6.4.15) and 2) by construction, we only invert the collection of unit objects $\{1_k\}_{k\in N(\mathit{Aff})}$ which, as base-change is given by monoidal functors, satisfies the necessary compatibility condition. To conclude we consider the restriction of $\mathcal{SH}_{nc}^{\otimes}$ to $N(B\mathit{Aff})$.

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Step 2)

In this section we want to present the collection of colimit preserving monoidal functors $\mathcal{L}_k^{\otimes}$: $\mathcal{SH}(k)^{\otimes} \to \mathcal{SH}_{nc}(k)^{\otimes}$ as part of a natural transformation $\mathcal{L}^{\otimes} : \mathcal{SH}^{\otimes} \to \mathcal{SH}_{nc}^{\otimes}$. The first step is to encode the family of monoidal functors $L_{pe,k}^{\otimes} : N(\mathit{AffSm}^{ft}(k)) \to \mathcal{N}cS(k)$ as a natural transformation $L_{pe}^{\otimes} : Sm^{\times} \to \mathcal{N}cS^{\otimes}$.

First, we consider the nerve $N(SmCommAlg_k)^{\otimes}$ of the category of smooth k-algebra equipped with the tensor product of algebras. Of course, base change allows us to encode the assignemt $k \mapsto N(SmCommAlg_k)^{\otimes}$ as ∞-functor

$$N(SmCommAlg_{(-)})^{\otimes} : N(\mathit{Aff})^{op} \to CAlg(Cat_{\infty}^{big}) \tag{9.1.31}$$

so that by composing it with $(-)^{op}$ we obtain $Sm^{\times}$.

Now we construct a natural transformation

$$L : N(SmCommAlg_{(-)})^{\otimes} \to \mathcal{D}g^{ft,\otimes} \tag{9.1.32}$$

whose opposite will be our $L_{pe}^{\otimes}$. Following our steps in 6.3.3 we construct L as a composition of three natural transformations

$$N(SmCommAlg_{(-)})^{\otimes} \subseteq Alg_{Ass}(\mathcal{D}(-))^{\otimes} \to \mathcal{D}g^{\otimes^{\mathbb{L}}} \to \mathcal{D}g^{idem,\otimes} \tag{9.1.33}$$

The last map is the one constructed above in the preliminaries to the construction of $\mathcal{N}cS^{\otimes}$. Let us explan the middle map. The reader can consult our discussion in Section 6.1.1 for a detailed background. As for a ring k the $(\infty,1)$-category $Alg_{Ass}(\mathcal{D}(k))$ of associative algebra objects in the derived $(\infty,1)$-category of k is equivalent to the underlying $(\infty,1)$-category of the model structure on k-dg-algebras - see [100, 4.1.4.4] - our strategy is to construct a natural transformation between the strict theories of dg-algebras and dg-categories and then apply the localization functor. For a ring k let $Alg(Ch(k))$ be (non-∞) category $Alg(Ch(k))$ of strict dg-algebras over k together with its monoidal structure induced by the tensor product of complexes. As cofibrant algebras have underlying cofibrant complexes (see [124]) and the tensor product of cofibrant complexes is cofibrant, the full subcategory $Alg(Ch(k))^{loc-cof}$ of dg-algebras having an underlying cofibrant complex, is closed under tensor products. Moreover, as cofibrant complexes are flat, the product of weak-equivalences is again a weak-equivalence. In this case, and using the fact that for any morphism of rings $f : k \to k'$ there is an induced monoidal base change $Alg(Ch(k)) \to Alg(Ch(k'))$ along which the notion of locally-cofibrant is preserved (because the base change of complexes is left Quillen), the assignment $k \mapsto Alg(Ch(k))^{loc-cof}$ can be presented as a functor

$$Aff^{op} \to CAlg(\mathcal{W}Cat^{big}) \tag{9.1.34}$$

which, after applying the nerve functor and composed with Loc (as in (9.1.21)) provides what we denoted as $Alg_{Ass}(\mathcal{D}(-))^{\otimes}$.

At the same time, for any ring k we have a natural functor $Alg(Ch(k)) \to Cat_{Ch(k)}$ sending a k-dg-algebra A to the dg-category one object and A as algebra of endomorphisms. We can easily check that this construction:

- is monoidal and compatible with base change;
- sends weak-equivalences of dg-algebras to Dwyer-Kan equivalences of dg-categories;
- by definition, sends locally-cofibrant dg-algebras to locally-cofibrant dg-categories

so that it defines a natural transformation (9.1.34)$\to$ (9.1.19). By definition the middle map is the transport of this arrow along Loc.

We now discuss the first map. This is the simplest one: for any ring k, the full subcategory of objects in $Alg_{Ass}(\mathcal{D}(k))$ concentrated in degre zero is equivalent to the nerve of the classical category of associative k-algebras. In particular, we have a natural inclusion $N(SmCommAlg_{(k)}) \subseteq Alg_{Ass}(\mathcal{D}(k))$ which is monoidal because we are restricting to smooth k-algebras and because these are flat, so that the classical tensor product and the derived tensor product are the same. We can again obtain the natural transformation using strictification arguments similar to the ones above and applying Loc. This concludes the construction of L.

Finally, we define $L_{pe}^{\otimes}$ as the image of L along the composition with $(-)^{op}$, restricted to $N(BAff)$. As a next step, we consider the image of $L_{pe}^{\otimes}$ along the composition with the functor $\mathcal{P}(-)^{\otimes}$

$$Fun(N(BAff)^{op}, CAlg(Cat_{\infty}^{big})) \to Fun(N(BAff)^{op}, CAlg(\mathcal{P}r^{L})) \tag{9.1.35}$$

to obtain a natural transformation $\mathcal{P}L_{pe}^{\otimes} : \mathcal{P}Sm^{\times} \to \mathcal{P}\mathcal{N}cS^{\otimes}$. Again, we use the fact that the choices of the generating equivalences respectively, for the commutative and noncommutative versions of the

Nisnevich topology, provide liftings $\mathcal{P}Sm^{\times}_{Nis}$ and $\mathcal{PN}cS^{\otimes}_{Nis}$, respectively, of the objects $\mathcal{P}Sm^{\times}$ and $\mathcal{PN}cS^{\otimes}$ along the composition with the forgeful functor

$$\begin{tikzcd} Fun(N(BAff)^{op}, CAlg(\mathcal{WP}r^L)) \arrow[d] \\ Fun(N(BAff)^{op}, CAlg(\mathcal{P}r^L)) \end{tikzcd} \tag{9.1.36}$$

The new ingredient is that the compatibility between the commutative and noncommutative versions of the Nisnevich topology along L_{pe} (see 6.4.16) defines an arrow $\mathcal{P}L^{\otimes}_{pe,Nis} : \mathcal{P}Sm^{\times}_{Nis} \to \mathcal{PN}cS^{\otimes}_{Nis}$ lifting $\mathcal{P}L^{\otimes}_{pe}$. Finally, we consider the image of this arrow along the composition with the localization functor Loc^{pr} to obtain a natural transformation $\mathcal{H}^{\times}_{Nis} \to \mathcal{H}^{\otimes}_{nc,Nis}$.

As again the notions of commutative and noncommutative affine line are, by definition, compatible with L_{pe}, we can apply the same procedure to the last arrow to and obtain a natural transformation $\mathcal{H}^{\times} \to \mathcal{H}^{\otimes}_{nc}$ between the $Nisnevich + \mathbb{A}^1$-versions of the theory. We can then consider the image of this arrow along the composition with the pointing functor $(-)_*$ and obtain a natural transformation $\mathcal{H}^{\wedge}_* \to \mathcal{H}^{\otimes}_{nc,*}$. To conclude our task, and thanks to our result in 6.4.20, this last natural transformation lifts to a morphism of pairs $(\mathcal{H}^{\wedge}_*, \{(\mathbb{P}^1_k, \infty)\}_{Spec(k)\in N(Aff)^{op}}) \to (\mathcal{H}^{\otimes}_{nc,*}, \{1_k\}_{Spec(k)\in N(Aff)^{op}})$ so that by the universal property of the coproduct we find the desired natural transformation $\mathcal{L}^{\otimes} : \mathcal{SH}^{\otimes} \to \mathcal{SH}^{\otimes}_{nc}$

This concludes our work in this section.

9.2 Descent Properties of $\mathcal{SH}^{\otimes}$ and $\mathcal{SH}^{\otimes}_{nc}$: Smooth base change and Zariski descent

Our main goal in this section is to prove the following result:

Proposition 9.2.1. *The ∞-functors $\mathcal{SH}^{\otimes}$ and $\mathcal{SH}^{\otimes}_{nc}$ satisfy descent with respect to the Zariski topology in $N(BAff)$*

This is well known to be equivalent to say that for any Zariski covering of the form

$$\begin{tikzcd} W = Spec(A \otimes_k R) \arrow[r, "j'"] \arrow[d, "p'"] & V = Spec(R) \arrow[d, hook, "p"] \\ U = Spec(A) \arrow[r, hook, "j"] & X = Spec(k) \end{tikzcd} \tag{9.2.1}$$

the induced diagrams

$$\begin{tikzcd} \mathcal{SH}^{\otimes}(k) \arrow[r] \arrow[d] & \mathcal{SH}^{\otimes}(A) \arrow[d] \\ \mathcal{SH}^{\otimes}(V) \arrow[r] & \mathcal{SH}^{\otimes}(A \otimes_k R) \end{tikzcd} \qquad \begin{tikzcd} \mathcal{SH}^{\otimes}_{nc}(k) \arrow[r] \arrow[d] & \mathcal{SH}^{\otimes}_{nc}(A) \arrow[d] \\ \mathcal{SH}^{\otimes}_{nc}(R) \arrow[r] & \mathcal{SH}^{\otimes}_{nc}(A \otimes_k R) \end{tikzcd} \tag{9.2.2}$$

are pullbacks in $CAlg(\mathcal{P}r^L_{\mathcal{S}tb})$.

Remark 9.2.2. Notice that as limits in the category of algebras are computed as limits in the underlying category ([100, 3.2.2.1, 3.2.2.5]), this is the same as saying that the underlying squares in $\mathcal{P}r^L_{\mathcal{S}tb}$ are pullbacks.

By the remark, we have to show that the induced maps

$$\mathcal{SH}(k) \to \mathcal{SH}(A) \times_{\mathcal{SH}(A\otimes_k R)} \mathcal{SH}(R) \tag{9.2.3}$$

and

$$\mathcal{SH}_{nc}(k) \to \mathcal{SH}_{nc}(A) \times_{\mathcal{SH}_{nc}(A\otimes_k R)} \mathcal{SH}_{nc}(R) \tag{9.2.4}$$

are equivalences of $(\infty,1)$-categories. For this purpose we will first show that the underlying functors of the monoidal maps in (9.2.2) have left adjoints so that the induced maps (9.2.3) and (9.2.4) will also admit left adjoints which we then prove to be equivalences. The construction of the left adjoints will occupy most of the remaining part of this section.

The first observation is that both the maps j and p are smooth maps of finite type (j and p are open immersions). Suppose $f : Spec(k') \to Spec(k)$ is a smooth map of finite type. In the commutative case, as the composition of smooth (finite type) morphisms of schemes is smooth (of finite type), the composition with f establishes a functor

$$(f \circ -) : \mathit{Affsm}(k') \to \mathit{Affsm}(k) \tag{9.2.5}$$

sending a smooth affine scheme of finite type over k', $Spec(A) \to Spec(k')$ to the composition $Spec(A) \to Spec(k') \to Spec(k)$. We can easily check from the universal property of pullbacks that this functor is a left adjoint to the pullback f^*. This functor verifies an important property, namely, for any $X \in \mathit{Affsm}(k')$ and $Y \in \mathit{Affsm}(k)$, the universal property of pullbacks gives us a canonical isomorphism

$$(f \circ -)(X \times_{Spec(k')} f^*(Y)) \simeq (f \circ -)(X) \times_{Spec(k)} Y \tag{9.2.6}$$

Another way to understand this formula is to say that $(f \circ -)$ is a map of $\mathit{Affsm}(k)^{\times}$-module objects in categories, with $\mathit{Affsm}(k')$ endowed with the action of $\mathit{Affsm}(k)^{\times}$ induced by the pullback functor f^*.

In the noncommutative case, for any map $f : Spec(k') \to Spec(k)$ we have an adjunction (9.1.26). Our next goal is to prove that when f is smooth then the forgetful functor F_f^{idem} in this adjunction will also preserve dg-categories of finite type. To prove this we will need to understand a bit better the properties of F_f^{idem}.

To start with let us contemplate some aspects of the strict theory of dg-categories. Let $f : k \to k'$ is a morphism of commutative rings. As we know, the base change functor $Cat_{Ch(k)} \to Cat_{Ch(k')}$ is a left adjoint to the standard forgetful functor obtained by composition with f. Let us denote it simply by f. What is also true is that f admits a right adjoint: for any k-dg-category T, the internal-hom in k-dg-categories $\underline{Hom}_k(1_{k'}, T)$ admits a natural structure of k'-dg-category and $T \to \phi(T)$ provides a right adjoint to f. An important example is the following: let $Ch_{dg,k'}(k')$ (resp. $Ch_{dg,k'}(k')$) denote the category of k-complexes (resp. k') seen as a dg-category over k (resp. k') via its internal hom. Then, we have a canonical equivalence

$$\phi(Ch_{dg,k}(k)) \simeq Ch_{dg,k'}(k') \tag{9.2.7}$$

We now collect (without proof) some standard relations involving the adjunctions f, $(- \otimes_k k')$ and ϕ holding at the strict level. The first is the projection formula: for any $T \in Cat_{Ch(k)}$ and any $S \in Cat_{Ch(k')}$ we have natural isomorphisms in $Cat_{Ch(k)}$

$$f((T \otimes_k k') \otimes S) \simeq f(S) \otimes T \tag{9.2.8}$$

This comes from the same formula for the tensor product of complexes. From this relation and using the adjunctions $((- \otimes_k k'), f)$ and (f, ϕ) we can easily deduce the existence of natural isomorphisms

$$f\underline{Hom}_{k'}(S, \phi(T)) \simeq \underline{Hom}_k(f(S), T) \tag{9.2.9}$$

and

$$f\underline{Hom}_{k'}(T\otimes_k k', S) \simeq \underline{Hom}_k(T, f(S)) \tag{9.2.10}$$

with $\underline{Hom}_k$ and $\underline{Hom}_{k'}$ the internal homs, respectively, in $Cat_{Ch(k')}$ and $Cat_{Ch(k)}$.

We now use these adjunctions to provide a series of useful remarks concerning the behavior of the forgetful functor F_f:

Remark 9.2.3. Let $f : k \to k'$ be a morphism of commutative rings. Then, the (big version of the) ∞-functor F_f preserves the construction of dg-modules. More precisely, if T is a small dg-category (assume locally-cofibrant) in $\mathcal{D}g(k')$, then we have a canonical equivalence of big k-dg-categories $F_f(\widehat{(T)}^{k'}) \simeq \widehat{F_f(T)}^{k}$ where on the left we have k'-dg-modules and on the right we mean k-dg-modules. Indeed, by definition of dg-modules $\widehat{(T)}^{k'}$ is the full sub k'-dg-category of $\underline{Hom}_{k'}(T^{op}, Ch_{dg,k'}(k'))$ spanned by those dg-functors that are cofibrant with respect to the projective model structure induced from the model structure on k'-complexes. By applying the standard forgetful functor together with the adjunction (f, ϕ) and the formulas (9.2.7) and (9.2.9) we have

$$f(\widehat{(T)}^{k'}) \subseteq f\underline{Hom}_{k'}(T^{op}, Ch_{dg,k'}(k')) \simeq \underline{Hom}_k(f(T)^{op}, Ch_{dg,k}(k)) \tag{9.2.11}$$

We now observe that the two possible induced projective model structures in these isomorphic categories have to be the same. This is because both have the same weak-equivalences and fibrations (the standard forgetful functor preserves and reflects them, by definition). Therefore, the subcategories of cofibrant objects are the same so that we have an isomorphism $f(\widehat{(T)}^{k'}) \simeq \widehat{f(T)}^{k}$. As F_f is obtained by composing f with a cofibrant replacement functor (see the Remark (9.1.3)) the left side side is equivalent to $F_f(\widehat{(T)}^{k'})$. To conclude, we use [139, Prop.3.2] deduce that right side is equivalent to $\widehat{F_f(T)}^{k}$

Remark 9.2.4. It follows from the previous remark that F_f preserves and reflects Morita equivalences in $\mathcal{D}g(k)^{idem}$. In particular, F_f^{idem} is conservative. Moreover, as F_f preserves the notion of homotopy category and quasi-fully faithfulnes, the previous remark implies that the forgetful functors F_f and F_f^{idem} of the Remark 9.1.3 are compatible with idempotent completions. More precisely, there is a natural tranformation rendering the diagrams

$$\begin{tikzcd} \mathcal{D}g(k') \arrow[r, "F_f"] \arrow[d, "\widehat{(-)}_{pe,k'}"] & \mathcal{D}g(k) \arrow[d, "\widehat{(-)}_{pe,k}"] \\ \mathcal{D}g^{idem}(k') \arrow[r, "F_f^{idem}"] & \mathcal{D}g^{idem}(k) \end{tikzcd} \tag{9.2.12}$$

commutative. This follows because perfect complexes are the compact objects in the homotopy categories of dg modules.

Remark 9.2.5. Both F_f and F_f^{idem} satisfy projection formulas as in (9.2.8). Let T (resp. S) be a locally cofibrant k-dg-category (resp. over k'). Then, as F_f can be identified with the standard forgetful functor followed by a cofibrant replacement Q we deduce DK-weak equivalences

$$F_f(S\otimes_{k'}(T\otimes_k k')) \simeq Q(f(S\otimes_{k'}(T\otimes_k k'))) \simeq Q(f(S)\otimes_k T) \simeq f(S)\otimes_k T \simeq Qf(S)\otimes T \tag{9.2.13}$$

where the middle map is the isomorphism in (9.2.8) and the last two maps are quasi-equivalences because T is assume locally cofibrant. The result for F_f^{idem} follows from the result for F_f together with the Remarks 9.1.3 and 9.2.4. Notice also that these formulas are natural in T and S.

Finally, we put these remarks together to prove that

Lemma 9.2.6. *For any morphism of commutative rings $f : k \to k'$, the ∞-functor F_f has a right adjoint $\widetilde{\phi}$. In particular, it preserves small colimits.*

Proof. Following what happens in the strict theory, there is a natural candidate for $\widetilde{\phi}$, namely, the internal-hom $\mathbb{R}\underline{Hom}_k(1_{k'}, -)$. Let T be an object in $\mathcal{D}g(k)$ (assume locally cofibrant). Using the description of internal-homs given by [139, Thm 6.1], we set $\widetilde{\phi}(T)$ as the full sub-dg-category of $\widehat{(T \otimes_k k')}^{k'}$ spanned by the quasi-representable objects. For any dg-category $A \in \mathcal{D}g(k')$ (again, assume A locally cofibrant) and thanks to [139, Thm 4.2] the mapping space $Map_{\mathcal{D}g(k')}(A, \widetilde{\phi}(T))$ can be described as follows: we regard the k'-dg-category $\widehat{(A \otimes (T \otimes_k k'))}^{k'}$ as a classical 1-category by forgetting its dg-enrichment and we consider the nerve of its subcategory spanned by all the objects together with those morphisms that are weak equivalences. Denoting this simplicial set as $N^W(\widehat{(A^{op} \otimes (T \otimes_k k'))}^{k'})$ the mapping space is obtained by considering its full simplicial set spanned by those vertices that correspond to right-quasi-representable dg-modules. it is clear that this construction does not depend on the dg-enrichement so that

$$N^W(\widehat{(A^{op} \otimes (T \otimes_k k'))}^{k'}) \simeq N^W(f(\widehat{(A^{op} \otimes (T \otimes_k k'))}^{k'})) \tag{9.2.14}$$

so that, using the Remarks 9.2.3 and 9.2.5 and the Prop. [139, Prop.3.2] together with the fact T is assumed to be locally cofibrant, the last is equivalent to

$$N^W(\widehat{(F_f(A)^{op} \otimes T)}^{k}) \tag{9.2.15}$$

Moreover, as the notion of right-quasi-representable is stable under this chain of equivalences, we can again use the description of mapping spaces [139, Thm 4.2] to conclude that the last simplicial set is a model for the mapping space $Map_{\mathcal{D}g(k)}(F_f(A), T)$.

To finish the prove one can use the Grothendieck construction of [99, Chapter 3] to exhibit F as a cocartesian fibration over $\Delta[1]$ and use the equivalence of the mapping spaces above to show that this fibration is also cartesian. To conclude we use again the methods of [99, Chapter 3] to extract a map of simplicial sets $\widetilde{\phi}$ right adjoint to F_f.

□

As a corollary

Lemma 9.2.7. *For any morphism of commutative rings $f : k \to k'$, the ∞-functor F_f^{idem} preserves small colimits.*

Proof. The result follows immediately from the previous lemma together with the commutativity of both diagrams (9.1.27) and (9.2.12) and the definition of colimits in the Morita theory (being a reflexive localization).

□

Remark 9.2.8. Another interesting way to prove the previous proposition is to use the equivalence between $\mathcal{D}g(k)^{idem}$ and the $(\infty, 1)$-category of modules in $\mathcal{P}r^L_\omega$ over the derived ∞-category of k. This was recently established in [36]. See also our survey in section 6.2. The result follows because colimits of modules are computed by means of the forgetful functor. See [100, 3.4.4.6, 4.8.7.9 and 4.8.7.11].

Remark 9.2.9. An interesting consequence of the previous lemma, together with the fact F_f^{idem} is conservative (as explained above), is that using the ∞-version of the Barr-Beck Theorem [100, 4.7.4.5], for any morphism of commutative rings $k \to k'$ the forgetful functor F^{idem} makes $\mathcal{D}g^{idem}(k')$ monadic over over $\mathcal{D}g^{idem}(k)$. In particular, for any commutative ring k, $\mathcal{D}g^{idem}(k)$ is monadic over $\mathcal{D}g^{idem}(\mathbb{Z})$.

We can finally accomplish our goal:

Proposition 9.2.10. *Let $f : Spec(k') \to Spec(k)$ be a smooth map of finite type. Then the forgetful functor $F_f^{idem} : \mathcal{D}g^{idem}(k') \to \mathcal{D}g^{idem}(k)$ preserves dg-categories of finite type.*

Proof. In order to prove this result we will use the description of dg-categories of finite type over the ring k' as compact objects in the $(\infty,1)$-category $\mathcal{D}g^{idem}(k')$. The last is the underlying $(\infty,1)$- category of the Morita model structure on strict k'-dg-categories and as this model structure is compactly generated (see [131, Thm 5.1] and [141, Def. 2.1 and Prop. 2.2]), we conclude that dg-categories of finite type over k' are the same (up to equivalence) as retracts of finite strict I-cell objects in $Cat_{Ch(k')}$ with respect to the Morita model structure. As retacts are functorial, we are reduced to prove the proposition for dg-categories obtained as finite I-cells. Let then T be a k'-dg-category of finite type obtained as a cell object

$$\emptyset = T_0 \to T_1 \to ... \to T_n = T \tag{9.2.16}$$

where for each $0 \leq i < n$, the dg-functor $T_i \to T_{i+1}$ is obtained from a pushout diagram

$$\begin{tikzcd} T_i \arrow[r] & T_{i+1} \\ A \arrow[u] \arrow[r] & B \arrow[u] \end{tikzcd} \tag{9.2.17}$$

where $A \to B$ is an element in the set of generating cofibrations of the Morita Model structure over k'. According to [131, Thm 5.1] and [132], these can only be of two types:

- the unique functor $\emptyset \to 1_{k'}$ where $1_{k'}$ is the k'-dg-category with one object and k' in degree zero as its complex of endomorphisms;
- Let $C(k',n)$ be the dg-category with two objects a and b, with k in degree zero for their respective endomorphisms, $C(k',n)(b,a) = 0$ and $C(k',n)(a,b)$ given by k' as a complex concentrated in degree $n-1$. At the same time, let $P(k',n)$ be the dg-category with two objects a and b, with k in degree zero for their respective endomorphisms, $P(k',n)(b,a) = 0$ and $P(k',n)(a,b)$ given by the complex having a copy of k in degree n and another copy in degree $n-1$, with differential given by the identity map. The second kind of generating cofibrations is given by the family of dg-functors $S(k',n) : C(k',n) \to P(k',n)$, $n \in \mathbb{Z}$, corresponding to the inclusion of k in degree $n-1$.

In particular, it follows from this description that the dg-category T_0 in our initial sequence has to be $1_{k'}$ which of course by definition is cofibrant. Moreover, we can also check that all the domains $C(k',n)$ and $P(k',n)$ are cofibrant objects so that all the pushouts (9.2.19) are in fact homotopy pushouts.

Let us now proceed. We want to show that $F_f^{idem}(T)$ remains a dg-category of finite type, now over k. By the Lemma.9.2.7, the sequence

$$F_f^{idem}(1_{k'}) \to F_f^{idem}(T_1) \to ... \to F_f^{idem}(T_n) = F_f^{idem}(T) \tag{9.2.18}$$

can still be obtained as a sequence of homotopy pushouts

$$\begin{tikzcd} F_f^{idem}(T_i) \arrow[r] & F_f^{idem}(T_{i+1}) \\ F_f^{idem}(A) \arrow[u] \arrow[r] & F_f^{idem}(B) \arrow[u] \end{tikzcd} \tag{9.2.19}$$

for each $1 \leq i < n$. We now observe that as homotopy pullbacks commute with filtered colimits in the $(\infty,1)$-category of spaces, the homotopy pushout of dg-categories of finite type remains of finite type, so that using induction, to conclude that $F_f^{idem}(T)$ is of finite type over k it will be enough to check that $F_f^{idem}(1_{k'})$ is of finite type over k and that for any generating cofibration $C(k',n) \to P(k',n)$ over k', the dg-categories $F_f^{idem}(C(k',n))$ and $F_f^{idem}(P(k',n))$ are of finite type over k.

Following the Remarks 9.1.3 and 9.2.4 and as k' is by hypothesis flat over k, we have

$$F_f^{idem}(1_{k'}) \simeq F_f(\widehat{(1_{k'})}_{pe,k'}) \simeq \widehat{(1'_k)}_{pe,k} \tag{9.2.20}$$

where see $1_{k'}$ as a k-dg-category via the standard forgetful functor.

The same arguments, together with the strong equivalences $C(k',n) \simeq C(k,n) \otimes_k k'$ and $P(k',n) \simeq P(k,n) \otimes_k k'$, the projection formula formula from the Remark 9.2.5 , the fact that k' is flat over k and finally, the compatibility of the derived tensor product with idempotent completions, give

$$F_f^{idem}(C(k',n)) \simeq (\widehat{C(k,n)})_{pe,k} \otimes_k^{\mathbb{L}} \widehat{(1'_k)}_{pe,k} \tag{9.2.21}$$

and

$$F_f^{idem}(P(k',n)) \simeq (\widehat{P(k,n)})_{pe,k}) \otimes_k^{\mathbb{L}} \widehat{(1'_k)}_{pe,k} \tag{9.2.22}$$

Our course now both $(\widehat{C(k,n)})_{pe,k}$ and $(\widehat{P(k,n)})_{pe,k}$ are of finite type as k-dg-categories. As the derived tensor product of dg-categories of finite type is of finite type [35, 4.3] the conclusion will follow if $\widehat{(1'_k)}_{pe,k}$ is of finite type over k. But this is exactly what happens when we put smoothness into action: as proved in 6.3.8, k' being smooth over k as an commutative algebra implies that $\widehat{(1_{k'})}_{pe,k}$ is of finite type as a k-dg-category.

This concludes the proof.

□

Using this result we find a new adjunction

$$\begin{tikzcd} \mathcal{N}cS(k') \arrow[r, shift left, "(F^{idem})^{op}"] & \mathcal{N}cS(k) \arrow[l, shift left, "f^{*,nc}"] \end{tikzcd} \tag{9.2.23}$$

with $(F_f^{idem})^{op}$ now a left adjoint. We now observe that the projection formulas of the Remark 9.2.5 are in fact part of a higher set of data corresponding to all the coherences defining a map of modules, or in other words, that F_f^{idem} defines a 1-simplex in $Mod_{\mathcal{N}cS(k)^{\otimes}}(Cat_\infty)$ where $\mathcal{D}g^{idem}(k)^{\otimes}$ is a module over itself via the action given by the tensor product and $\mathcal{D}g^{idem}(k')$ is a module over $\mathcal{D}g^{idem}(k)^{\otimes}$ via the action induced by the monoidal functor $(-\otimes_k k')$. As we shall only need this in the case k' is smooth over k our task to deduce this higher structure on F_f^{idem} is simplified. Firstly, we know that the standard forgetful functor f is a map of modules between the strict theories of dg-categories. This is provided by formula (9.2.8) together with the usual coherence theorems whose necessary conditions one can verify by hand. We now want to say that this module structure induces a module structure on F_f. The problem is that as a derived functor, F_f does not preserve locally cofibrant dg-categories so that that in order to bring the image of F_f to the locally-cofibrant context with need to perform a cofibrant replacement which destroys the strict module structure. Luckily for us there exists a class of dg-categories which remain stable under tensor products and base change, compatible with the Dwyer-Kan equivalences and, more importantly, that under our smoothness hypothesis, remains stable under the forgetful functor: namely, the class of dg-categories enriched in complexes of flat modules. The verification of this claim is immediate by the definition of flat and the fact that if M is flat over k' and k' is flat over k then M is flat over k. Therefore, as the localization functor is monoidal and as f is compatible with weak-equivalences, it can be considered as a map of modules in pairs and its image along the localization

$$Mod_{(Cat^{flat}_{Ch(k)}, W_{DK})}(\mathcal{W}Cat^{vbig}) \subseteq Mod_{(N(Cat^{flat}_{Ch(k)}), W_{DK})}(\mathcal{W}Cat^{vbig}_\infty) \to Mod_{\mathcal{D}g(k)^{\otimes}}(Cat_\infty) \tag{9.2.24}$$

provides a structure of map of modules on F_f. We now use this to deduce the result for F_f^{idem}. It follows the Remark 9.2.4 that F_f preserves Morita equivalences, in the sense that if a map $T \to T'$ in $\mathcal{D}g(k')$ becomes an equivalence in $\mathcal{D}g^{idem}(k')$ after idempotent completion, then its image through F_f satisfies the same property. Therefore, F_f can be understood as a map in $Mod_{(\mathcal{D}g(k),W_{Mor})}(\mathcal{WP}r^L)$ so that its image by the localization functor Loc^{pr} provides a map of $\mathcal{D}g^{idem}(k)$-modules

$$\mathcal{D}g^{idem}(k') \simeq \mathcal{D}g(k')[W_{Mor'}^{-1}] \to \mathcal{D}g(k)[W_{Mor}^{-1}] \simeq \mathcal{D}g^{idem}(k) \tag{9.2.25}$$

By the universal property of the localization and the commutativity of (9.2.12) this map can be canonically identified with F_f^{idem} and the condition of being a map of modules implies the projection formula. Further explanations about this methodology to deduce projection formulas will be given in the proof of the Prop. 9.2.11 ahead.

Finally, the previous proposition is telling us that in fact F_f^{idem} lives as a 1-simplex in $Mod_{\mathcal{D}g^{idem}k^{\otimes}}(\mathcal{P}r_{\omega}^L)$ so that, as the restriction to compact objects is a monoidal functor (see [100, 5.3.2.11]), it induces a 1-simplex in $Mod_{\mathcal{D}g^{idem}k^{\omega,\otimes}}(Cat_{\infty})$ given by the restriction of F_f^{idem} to compact objects. To conclude we use the fact that $(-)^{op} : Cat_{\infty} \to Cat_{\infty}$ is monoidal for the cartesian product to deduce that the map $(F_f^{idem})^{op}$ in the diagram (9.2.23) is a map of modules and therefore satisfies the projection formula.

We will now use this to prove the following result:

Proposition 9.2.11. *Let $f : Spec(k') \to Spec(k)$ be a smooth morphism of finite type (assume k and k' Noetherian of finite Krull dimension for what concerns the commutative case). Then, both the induced pullback functors $f^* : \mathcal{SH}(k) \to \mathcal{SH}(k')$ and $f^{*.nc} : \mathcal{SH}_{nc}(k) \to \mathcal{SH}_{nc}(k')$ admit left adjoints, respectively denoted as $f_\sharp$ and $f_\sharp^{nc}$ such that:*

1. *The diagram*

$$\begin{tikzcd}
 & N(Affsm(k)) \arrow[rr, "L_{pe,k}"] \arrow[dd] \arrow[dr, phantom, "\tau"] & & \mathcal{N}cS(k) \arrow[dd] \\
N(Affsm(k')) \arrow[ur, "f\circ -"] \arrow[rr, "L_{pe,k'}"] \arrow[dd] \arrow[dr, phantom, "\sigma"] & & \mathcal{N}cS(k') \arrow[ur, "(F^{idem})^{op}"'] \arrow[dd] \arrow[dr, phantom, "\delta"] & \\
 & \mathcal{SH}(k) \arrow[rr, "\mathcal{L}_k"] \arrow[dr, phantom, "\tau'"] & & \mathcal{SH}_{nc}(k) \\
\mathcal{SH}(k') \arrow[ur, "f_\sharp"] \arrow[rr, "\mathcal{L}_{k'}"] & & \mathcal{SH}_{nc}(k') \arrow[ur, "f_\sharp^{nc}"'] &
\end{tikzcd} \tag{9.2.26}$$

commutes;

2. *$f_\sharp$ (respectively $f_\sharp^{nc}$) is a map of $\mathcal{SH}^{\otimes}(k)$-module objects (resp. $\mathcal{SH}_{nc}^{\otimes}(k)$-modules) in $\mathcal{P}r_{Stb}^L$ with $\mathcal{SH}(k')$ (resp. $\mathcal{SH}_{nc}(k')$) considered with the module-structure induced by the monoidal map f^* (resp. $f^{*,nc}$) and $\mathcal{SH}(k)$ (resp. $\mathcal{SH}_{nc}(k)$) seen as a module over itself via its tensor product.*

3. *For any pullback square of affine schemes*

$$\begin{tikzcd}
Spec(k' \otimes_k A) \arrow[r, "f'"] \arrow[d, "g'"] & Spec(A) \arrow[d, "g"] \\
Spec(k') \arrow[r, "f"] & Spec(k)
\end{tikzcd} \tag{9.2.27}$$

with f smooth, the diagrams

$$\begin{tikzcd} \mathcal{SH}(k' \otimes_k A) \arrow[r, "(f')_\sharp"] & \mathcal{SH}(A) \\ \mathcal{SH}(k') \arrow[u, "(g')^*"] \arrow[r, "f_\sharp"] & \mathcal{SH}(k) \arrow[u, "g^*"'] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{SH}_{nc}(k' \otimes_k A) \arrow[r, "(f')^{nc}_\sharp"] & \mathcal{SH}_{nc}(A) \\ \mathcal{SH}_{nc}(k') \arrow[u, "(g')^*"] \arrow[r, "f^{nc}_\sharp"] & \mathcal{SH}_{nc}(k) \arrow[u, "g^*"'] \end{tikzcd} \tag{9.2.28}$$

commute by means of the natural transformations

$$(f')_\sharp \circ (g')^* \to (f')_\sharp \circ (g')^* \circ f^* \circ f_\sharp \simeq (f')_\sharp \circ (f')^* \circ g^* \circ f_\sharp \to g^* \circ f_\sharp \tag{9.2.29}$$

respectively,

$$(f')^{nc}_\sharp \circ (g')^* \to (f')^{nc}_\sharp \circ (g')^* \circ f^* \circ f^{nc}_\sharp \simeq (f')^{nc}_\sharp \circ (f')^* \circ g^* \circ f^{nc}_\sharp \to g^* \circ f^{nc}_\sharp \tag{9.2.30}$$

Proof. As Cat^{big}_∞ is a quasi-category, a standard argument using the definition of quasi-categories allows us to reduce the proof that the diagram (9.2.26) commutes to the commutativity of the exterior faces. The commutativity of τ follows immediately from the fact the forgetful functor is compatible with idempotent completions (see the Remark 9.2.4.) Now we deal with the commutativity of the faces σ and δ. Using the universal property of presheaves, the adjunctions (9.2.5) and (9.2.23) induce new adjunctions between the associated ∞-categories of presheaves thus rendering the diagrams commutative

$$\begin{tikzcd} N(\mathit{Affsm}(k')) \arrow[r, "f\circ -"] \arrow[d, hook] & N(\mathit{Affsm}(k)) \arrow[d, hook] \\ \mathcal{P}(\mathit{Affsm}(k')) \arrow[r, dashed, "\widetilde{f_\sharp}"] & \mathcal{P}(\mathit{Affsm}(k)) \end{tikzcd} \qquad \begin{tikzcd} \mathcal{N}cS(k') \arrow[r, "(F^{idem})^{op}"] \arrow[d, hook] & \mathcal{N}cS(k) \arrow[d, hook] \\ \mathcal{P}(\mathcal{N}cS(k')) \arrow[r, dashed, "\widetilde{f^{nc}_\sharp}"] & \mathcal{P}(\mathcal{N}cS(k)) \end{tikzcd}$$

$$\begin{tikzcd} N(\mathit{Affsm}(k')) \arrow[d, hook] & N(\mathit{Affsm}(k)) \arrow[l, "f^*"'] \arrow[d, hook] \\ \mathcal{P}(N(\mathit{Affsm}(k'))) & \mathcal{P}(N(\mathit{Affsm}(k))) \arrow[l, dashed, "f^*"] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{N}cS(k') \arrow[d, hook] & \mathcal{N}cS(k) \arrow[l, "f^*"'] \arrow[d, hook] \\ \mathcal{P}(\mathcal{N}cS(k')) & \mathcal{P}(\mathcal{N}cS(k)) \arrow[l, dashed, "f^*"] \end{tikzcd} \tag{9.2.31}$$

The fact that $\widetilde{f_\sharp}$ (resp. $\widetilde{f^{nc}_\sharp}$) is a left adjoint to the pullback follows from Yoneda's Lemma. The important point here is that as the assignment of ∞-presheaves is a monoidal functor [100, 4.8.1.10], the maps $\widetilde{f_\sharp}$ and $\widetilde{f^{nc}_\sharp}$ will again be maps of modules. Indeed, they can be identified with the image of the 1-simplexes $(f \circ -)$ and $(F^{idem}_f)^{op}$ along the transport of modules

$$Mod_{N(\mathit{Affsm}(k))^\times}(Cat_\infty) \xrightarrow{\mathcal{P}(-)} Mod_{\mathcal{P}(N(\mathit{Affsm}(k)))^\times}(\mathcal{P}r^L) \tag{9.2.32}$$

respectively,

$$Mod_{\mathcal{N}cS(k)^\otimes}(Cat_\infty) \xrightarrow{\mathcal{P}(-)} Mod_{\mathcal{P}(\mathcal{N}cS(k))^\otimes}(\mathcal{P}r^L) \tag{9.2.33}$$

(see our discussion in 3.3.9).

We now deal with the Nisnevich localizations. In both cases we know that the pullback functors are compatible with Nisnevich local equivalences. We observe now that the same holds for both $\widetilde{f_\sharp}$

and $\widetilde{f_\sharp^{nc}}$. In the commutative case this follows because if X is a smooth affine scheme over $Spec(k')$ and $\{V_i \to X\}_{i\in I}$ is a Nisnevich covering of X over $Spec(k')$ then it will of course remain a Nisnevich covering of X seen as a scheme over $Spec(k)$ via the composition with f (the notion of Nisnevich covering is independent of the base). In the noncommutative case we can also show that $(F_f^{idem})^{op}$ preserves Nisnevich squares of dg-categories. Indeed, if

$$\begin{tikzcd} K_{\mathcal{X}-\mathcal{U}} \arrow[r] \arrow[d] & * \arrow[d] \\ T_{\mathcal{X}} \arrow[r] \arrow[d] & T_{\mathcal{U}} \arrow[d] \\ T_{\mathcal{V}} \arrow[r] & T_{\mathcal{W}} \\ K_{\mathcal{V}-\mathcal{W}} \arrow[r] \arrow[u] & * \arrow[u] \end{tikzcd} \tag{9.2.34}$$

is a Nisnevich square of dg-categories over k' (see the Definition 6.4.7) with associated close complements K_{X-U} and K_{V-W}, then its image

$$\begin{tikzcd} F_f^{idem}(K_{\mathcal{X}-\mathcal{U}}) \arrow[r] \arrow[d] & F_f^{idem}(*) \simeq * \arrow[d] \\ F_f^{idem}(T_{\mathcal{X}}) \arrow[r] \arrow[d] & F_f^{idem}(T_{\mathcal{U}}) \arrow[d] \\ F_f^{idem}(T_{\mathcal{V}}) \arrow[r] & F_f^{idem}(T_{\mathcal{W}}) \\ F_f^{idem}(K_{\mathcal{V}-\mathcal{W}}) \arrow[r] \arrow[u] & F_f^{idem}(*) \simeq * \arrow[u] \end{tikzcd} \tag{9.2.35}$$

in $\mathcal{D}g^{idem}(k)$ remains a Nisnevich square. Being a right adjoint F_f^{idem} preserves pullbacks and as f is smooth it preserves also dg-categories of finite type (Prop.9.2.10). Moreover, the functoriality ensures that $F_f^{idem}(K_{X-U}) \simeq F_f^{idem}(K_{V-W})$. We are reduced to showing that it preserves "open immersions of dg-categories", or in other words that the image of a pushout diagram in $\mathcal{D}g^{idem}(k')$ of the form

$$\begin{tikzcd} K \arrow[r] \arrow[d] & * \arrow[d] \\ T_{\mathcal{X}} \arrow[r] & T_{\mathcal{U}} \end{tikzcd} \tag{9.2.36}$$

with T_X and T_U of finite type and K having a compact generator, keeps the same properties. For that we use again that F_f^{idem} commutes with colimits (Lemma.9.2.7) and that as by assumption K is idempotent complete, $F_f^{idem}(K)$ can be identified with the image of K along the standard forgetful functor (see the Remark 9.1.3) so that its homotopy category is equivalent to the homotopy category of K and therefore has a compact generator.

The main conclusion is that both $\widetilde{f_\sharp}$ and $\widetilde{f_\sharp^{nc}}$ preserve Nisnevich local equivalences so that by the universal property of localization [99, 5.5.4.15] both admit canonical unique colimit preserving extensions $\widetilde{f_{\sharp,Nis}}$ and $\widetilde{f_{\sharp,Nis}^{nc}}$ rendering the diagrams

$$\begin{tikzcd}
\mathcal{P}(Affsm(k')) \arrow[r, "\widetilde{f_\sharp}"] \arrow[d] & \mathcal{P}(Affsm(k)) \arrow[d] \\
\mathcal{H}_{Nis}(k') \arrow[r, dashed, "\widetilde{f_{\sharp,Nis}}"] & \mathcal{H}_{Nis}(k)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{P}(\mathcal{N}cS(k')) \arrow[r, "\widetilde{f^{nc}_\sharp}"] \arrow[d] & \mathcal{P}(\mathcal{N}cS(k)) \arrow[d] \\
\mathcal{H}^{nc}_{Nis}(k') \arrow[r, dashed, "\widetilde{f^{nc}_{\sharp,Nis}}"] & \mathcal{H}^{nc}_{Nis}(k)
\end{tikzcd}
\tag{9.2.37}$$

commutative.

We now remark two important properties of these maps. First we observe that $\widetilde{f_{\sharp,Nis}}$ and $\widetilde{f^{nc}_{\sharp,Nis}}$ are left adjoints to the respective Nisnevich pullback functors. This follows because in this case the pullback functors preserve Nisnevich local objects: this is because at the level of presheaves they admit left adjoints which, as seen above, preserve Nisnevich local equivalences. Now we observe that both $\widetilde{f_{\sharp,Nis}}$ and $\widetilde{f^{nc}_{\sharp,Nis}}$ satisfy the projection formula with respect to pullbacks. We deal with the commutative case. The noncommutative one follows using similar arguments. As seen above, $\widetilde{f_\sharp}$ can be identified with a 1-simplex in the $(\infty,1)$-category $Mod_{\mathcal{P}(N(Affsm(k)))^\times}(\mathcal{P}r^L)$. The fact that it preserves Nisnevich local equivalences means that it lifts to a 1-simplex in $Mod_{(\mathcal{P}(N(Affsm(k)))^\times, W_{Nis})}(\mathcal{W}\mathcal{P}r^L)$ so that its image along the localization functor Loc^{pr} defines a 1-simplex in $Mod_{\mathcal{H}^\otimes_{Nis}(k)}(\mathcal{P}r^L)$. Finally, the universal property of the localization tells us that this 1-simplex can be canonically identified with $\widetilde{f_{\sharp,Nis}}$.

We proceed to the $\mathbb{A}^1$-localizations. Thanks to the fact that both $\widetilde{f_{\sharp,Nis}}$ and $\widetilde{f^{nc}_{\sharp,Nis}}$ are maps of modules with respect to the algebra structures induced by the respective pullback functors, we have formulas

$$\widetilde{f_{\sharp,Nis}}(j(X)\times j(\mathbb{A}^1_{k'})) \simeq \widetilde{f_{\sharp,Nis}}(j(X)\times f^*(j(\mathbb{A}^1_k))) \simeq \widetilde{f_{\sharp,Nis}}(j(X))\times j(\mathbb{A}^1_k) \tag{9.2.38}$$

and

$$\widetilde{f^{nc}_{\sharp,Nis}}(j_{nc}(\mathcal{X})\otimes j_{nc}(L_{pe}(\mathbb{A}^1_{k'}))) \simeq \widetilde{f^{nc}_{\sharp,Nis}}(j_{nc}(\mathcal{X})\otimes f^*(j_{nc}(L_{pe}(\mathbb{A}^1_k)))) \simeq \widetilde{f^{nc}_{\sharp,Nis}}(j_{nc}(\mathcal{X}))\otimes j_{nc}(L_{pe}(\mathbb{A}^1_k)) \tag{9.2.39}$$

so that both $\widetilde{f_{\sharp,Nis}}$ and $\widetilde{f^{nc}_{\sharp,Nis}}$ preserve $\mathbb{A}^1$-generating local equivalences and we deduce the existence of canonical colimit preserving extensions $\widetilde{f_{\sharp,Nis,\mathbb{A}^1}}$ and $\widetilde{f^{nc}_{\sharp,Nis,\mathbb{A}^1}}$ rendering the diagrams

$$\begin{tikzcd}
\mathcal{H}_{Nis}(k') \arrow[r, "\widetilde{f_{\sharp,Nis}}"] \arrow[d] & \mathcal{H}_{Nis}(k) \arrow[d] \\
\mathcal{H}_{Nis,\mathbb{A}^1}(k') \arrow[r, dashed, "\widetilde{f_{\sharp,Nis,\mathbb{A}^1}}"] & \mathcal{H}_{Nis,\mathbb{A}^1}(k)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{H}^{nc}_{Nis}(k') \arrow[r, "\widetilde{f^{nc}_{\sharp,Nis}}"] \arrow[d] & \mathcal{H}^{nc}_{Nis}(k) \arrow[d] \\
\mathcal{H}^{nc}_{Nis,\mathbb{A}^1}(k') \arrow[r, dashed, "\widetilde{f^{nc}_{\sharp,Nis,\mathbb{A}^1}}"] & \mathcal{H}^{nc}_{Nis,\mathbb{A}^1}(k)
\end{tikzcd}
\tag{9.2.40}$$

commutative, where the vertical arrows are the $\mathbb{A}^1$-localization functors. Using exactly the same arguments as in the Nisnevich case we deduce that 1) these functors are left adjoints to the pullback functors at the level of the Nisnevich-$\mathbb{A}^1$- theory and 2) define maps of modules in the current context.

In what concerns the noncommutative context and thanks to 6.4.19, we are done: $\widetilde{f^{nc}_{\sharp,Nis,\mathbb{A}^1}}$ is the left adjoint we were looking for. We are left with the remaining steps of the commutative case.

For the pointing step we start by recalling that for presentable $(\infty,1)$-categories the pointing map $\mathcal{C}\mapsto\mathcal{C}_*$ can be identified with the base-change along the monoidal functor $(-)_+ : \mathcal{S}^\times \to \mathcal{S}^\wedge_*$. This follows from the general theory of idempotent objects in [100, Section 4.8.2], more precisely [100, 4.8.2.11]. See also our compact summon of these results in 5.2.1. The universal properties involved provide a colimit preserving functor $\widetilde{f_{\sharp,Nis,\mathbb{A}^1,*}}$ rendering the diagram

$$\begin{tikzcd}
\mathcal{H}_{Nis,\mathbb{A}^1}(k') \arrow[rr, "\widetilde{f_{\sharp,Nis,\mathbb{A}^1}}"] \arrow[d, "(-)_*"] & & \mathcal{H}_{Nis,\mathbb{A}^1}(k) \arrow[d, "(-)_*"] \\
\mathcal{H}_{Nis,\mathbb{A}^1}(k')_* \simeq \mathcal{H}_{Nis,\mathbb{A}^1}(k') \otimes_{\mathcal{S}^\times} \mathcal{S}^\wedge_* \arrow[rr, dashed, "\widetilde{f_{\sharp,Nis,\mathbb{A}^1,*}}"] & & \mathcal{H}_{Nis,\mathbb{A}^1}(k)_* \simeq \mathcal{H}_{Nis,\mathbb{A}^1}(k) \otimes_{\mathcal{S}^\times} \mathcal{S}^\wedge_*
\end{tikzcd} \tag{9.2.41}$$

commutative. Moreover, as $\widetilde{f_{\sharp,Nis,\mathbb{A}^1}}$ defines a 1-simplex in $Mod_{\mathcal{H}_{Nis,\mathbb{A}^1}(k)^\times}(\mathcal{P}r^L)$ and as the base change $(-\otimes_{\mathcal{S}^\times} \mathcal{S}^\wedge_*)$ is a monoidal, its image along the induced transport map

$$Mod_{\mathcal{H}_{Nis,\mathbb{A}^1}(k)^\times}(\mathcal{P}r^L) \to Mod_{\mathcal{H}_{Nis,\mathbb{A}^1}(k)^\wedge_*}(\mathcal{P}r^L_{pt}) \tag{9.2.42}$$

is a map of modules which by the universal properties involved is enriching $\widetilde{f_{\sharp,Nis,\mathbb{A}^1,*}}$.

We are left to check that this extension is a left adjoint to the induced pointed pullback f^*_{pt} also arising from the universial property of pointing. As $\widetilde{f_{\sharp,Nis,\mathbb{A}^1,*}}$ commutes with colimits, it has a right adjoint u fitting in a commutative diagram

$$\begin{tikzcd}
\mathcal{H}_{Nis,\mathbb{A}^1}(k') & \mathcal{H}_{Nis,\mathbb{A}^1}(k) \arrow[l, "f^*"'] \\
\mathcal{H}_{Nis,\mathbb{A}^1}(k')_* \arrow[u] & \mathcal{H}_{Nis,\mathbb{A}^1}(k)_* \arrow[u] \arrow[l, dashed, "u"']
\end{tikzcd} \tag{9.2.43}$$

where the vertical maps are the forgetful functors - right adjoints to the pointing maps. We want to identify u with the map f^*_{pt}. As the forgetful functors are obviously conservative, it will be enough to check that the diagram (9.2.43) still commutes if we replace u by f^*_{pt}. But this follows from the explicit description of f^*_{pt} given in the Remark 5.2.2, together with the fact that as a result of the preceding steps, the pullback f^* is a right adjoint to $\widetilde{f_{\sharp,Nis,\mathbb{A}^1}}$ and therefore preserves the final object.

We now come to the last step in the construction of the left adjoints: the $\otimes$-inversion of the projective line pointed at infinity. The key step is a canonical identification of $\mathcal{SH}(k')$ with the base change of $\mathcal{H}_{Nis,\mathbb{A}^1}(k')_*$ along the monoidal functor $\mathcal{H}_{Nis,\mathbb{A}^1}(k)^\wedge_* \to \mathcal{SH}^{\otimes}(k)$. To understand this identification, recall that the inversion of $(\mathbb{P}^1_k,\infty)$ (see 4.1.8) is obtained by means of a pushout

$$\begin{tikzcd}
free^{\otimes}(\Delta[0]) \arrow[r] \arrow[d, "(\mathbb{P}^1_k{,}\infty)"] & \mathcal{L}^{\otimes}_{(free^{\otimes}(\Delta[0]),*)}(free^{\otimes}(\Delta[0])) \arrow[d] \\
\mathcal{H}_{Nis,\mathbb{A}^1}(k)^\wedge_* \arrow[r] & \mathcal{SH}^{\otimes}(k)
\end{tikzcd} \tag{9.2.44}$$

in $CAlg(\mathcal{P}r^L)$. In our case, as $(\mathbb{P}^1_{k'},\infty) \simeq f^*((\mathbb{P}^1_k,\infty))$ we can extend this to a new pushout

$$\begin{tikzcd}
free^{\otimes}(\Delta[0]) \arrow[r] \arrow[d, "(\mathbb{P}^1_k{,}\infty)"] & \mathcal{L}^{\otimes}_{(free^{\otimes}(\Delta[0]),*)}(free^{\otimes}(\Delta[0])) \arrow[d] \\
\mathcal{H}_{Nis,\mathbb{A}^1}(k)^\wedge_* \arrow[r] \arrow[d, "f^*"] & \mathcal{SH}^{\otimes}(k) \arrow[d] \\
\mathcal{H}_{Nis,\mathbb{A}^1}(k')^\wedge_* \arrow[r] & \mathcal{H}_{Nis,\mathbb{A}^1}(k')^\wedge_* \otimes_{\mathcal{H}_{Nis,\mathbb{A}^1}(k)^\wedge_*} \mathcal{SH}^{\otimes}(k)
\end{tikzcd} \tag{9.2.45}$$

and therefore, by the universal properties envolved, obtain the desired identification. Recall that the monoidal structure on commutative algebra objects is cocartesian - [100, 3.2.4.7] . Following this identification we can use the induced base-change along $\mathcal{H}_{Nis,\mathbb{A}^1}(k)^\wedge_* \to \mathcal{SH}^{\otimes}(k)$

$$Mod_{\mathcal{H}_{Nis,\mathbb{A}^1}(k)^\wedge_*}(\mathcal{P}r^L) \to Mod_{\mathcal{SH}^{\otimes}(k)}(\mathcal{P}r^L) \tag{9.2.46}$$

to transport the 1-simplex $\widetilde{f_{\sharp,Nis,\mathbb{A}^1,*}}$ to a map of modules

$$f_\sharp : \mathcal{SH}(k') \simeq \mathcal{H}_{Nis,\mathbb{A}^1}(k')_* \otimes_{\mathcal{H}_{Nis,\mathbb{A}^1}(k)^\wedge_*} \mathcal{SH}^{\otimes}(k) \to \mathcal{SH}(k) \tag{9.2.47}$$

and the unit natural transformation of the base-change adjunction produces commutative squares

$$\begin{tikzcd} \mathcal{H}_{Nis,\mathbb{A}^1}(k')_* \arrow[r, "\widetilde{f_{\sharp,Nis,\mathbb{A}^1,*}}"] \arrow[d] & \mathcal{H}_{Nis,\mathbb{A}^1}(k)_* \arrow[d] \\ \mathcal{SH}(k') \arrow[r, "f_\sharp"] & \mathcal{SH}(k) \end{tikzcd} \tag{9.2.48}$$

Finally, it follows from the functoriality of the base-change procedure that $f_\sharp$ is a left adjoint to the pullback $f^* : \mathcal{SH}(k) \to \mathcal{SH}(k)$ which, as the diagram (9.2.45) indicates, is the image of $f^* : \mathcal{H}_{Nis,\mathbb{A}^1}(k) \to \mathcal{H}_{Nis,\mathbb{A}^1}(k')$ along the same base-change. This proves 2). To conclude 1) the only thing left is the commutativity of the face τ'. For this purpose we use the existence of a family of compact generators in $\mathcal{SH}(k')$ given by Proposition 5.3.3. Thanks to the Prop. 2.1.2 and the fact all the maps in τ' preserve small colimits, we are left to check that τ' commutes for objects of the form $(\mathbb{P}^1_{k'})^{-n} \otimes V$ where V is a smooth affine scheme over k', identified with its image in $\mathcal{SH}(k')$. Therefore, the commutativity of τ' follows from the equivalence $\mathbb{P}^1_{k'} \simeq f^*(\mathbb{P}^1_k)$, together with the fact that both $\mathcal{L}_k$ and $\mathcal{L}_{k'}$ are monoidal, that thanks to 2), both $f_\sharp$ and $f_\sharp^{nc}$ satisfy the projection formula and finally, from the commutativity of τ, δ and σ. Finally, to prove 3) we can again make use of the existence of families of compact generators on both $\mathcal{SH}_{nc}(k')$ and $\mathcal{SH}(k')$: As the functors involved commute with all colimits, it will be enough to check that the diagrams commute for the compact generators. Again, these follows from the commutativity of the diagrams before passing to motives as a result of the transitivity of the fiber product of schemes and of the tensor product of dg-categories, together with the projection formulas proved in 2).

□

Remark 9.2.12. It follows from the base change property given in the Proposition 9.2.11 that for any open immersion of affine schemes $j : U \hookrightarrow X$, the functors $j_\sharp$ and $j_\sharp^{nc}$ are fully faithful. This is because the canonical maps $U \times_X U \to U$ are isomorphisms whenever j is a monomorphism. By adjuction, the same holds for j_* and j_*^{nc}.

Remark 9.2.13. One important observation is that if $f : Spec(k') \to Spec(k)$ is smooth and proper then the functors

$$f_\sharp, f_* : \mathcal{SH}_{nc}(k') \to \mathcal{SH}_{nc}(k) \tag{9.2.49}$$

are canonically equivalent. Indeed, recall from the Lemmas 9.2.6 and 9.2.7 the existence of a right adjoint $\tilde{\phi}_f$ to the forgetful functor

$$F_f^{idem} : \mathcal{D}g^{idem}(k') \to \mathcal{D}g^{idem}(k) \tag{9.2.50}$$

where by construction we have $F_f^{idem}(\tilde{\phi}_f(T)) \simeq \mathbb{R}\underline{Hom}_k(k', T)$. By [139, Thm 6.1] the last is given by $(\widehat{k' \otimes_k T})_{pspe}$ and as k' is smooth and proper over k it is equivalent to $(\widehat{k' \otimes_k T})_{pe}$ so that, as F_f^{idem} is conservative and satisfies the conjecture formula with respect to base-change $(k' \otimes_k -)$, the two functors f^* and $\tilde{\phi}_f$ agree on the class of dg-categories of finite type. In other words, the pullback $f^* : \mathcal{N}cS(k) \to \mathcal{N}cS(k')$ is simultaneously a left and a right adjoint to the forgetful functor $f_\sharp$. At the level of presheaves this implies that $\tilde{f}_\sharp : \mathcal{P}(\mathcal{N}cS(k')) \to \mathcal{P}(\mathcal{N}cS(k))$ is also a right adjoint to f^* and by the unicity of adjoints, equivalent to f_*. As both f^* and $\tilde{f}_\sharp$ are compatible with the localizations (Prop. 9.2.11) so that again by the unicity of adjoints, the motivic versions of $f_\sharp$ and f_* are necessarily equivelent.

We can now prove the descent property.

Proof of the Proposition (9.2.1):
After the results in the Proposition 9.2.11 we have commutative diagrams

$$\begin{tikzcd} \mathcal{SH}^{\otimes}(k) & \mathcal{SH}^{\otimes}(A) \arrow[l, "j_\sharp"'] \\ \mathcal{SH}^{\otimes}(V) \arrow[u, "p_\sharp"] & \mathcal{SH}^{\otimes}(A\otimes_k R) \arrow[l, "(j')_\sharp"] \arrow[u, "(p')_\sharp"'] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{SH}^{\otimes}_{nc}(k) & \mathcal{SH}^{\otimes}_{nc}(A) \arrow[l, "j^{nc}_\sharp"'] \\ \mathcal{SH}^{\otimes}_{nc}(R) \arrow[u, "p^{nc}_\sharp"] & \mathcal{SH}^{\otimes}_{nc}(A\otimes_k R) \arrow[l, "(j')^{nc}_\sharp"] \arrow[u, "(p')^{nc}_\sharp"'] \end{tikzcd} \tag{9.2.51}$$

left adjoints to the diagrams in (9.2.2). In particular, the natural maps in (9.2.3) and (9.2.4) both admit left adjoints

$$\mathcal{SH}(A)\times_{\mathcal{SH}(A\otimes_k R)}\mathcal{SH}(R)\to\mathcal{SH}(k) \tag{9.2.52}$$

and

$$\mathcal{SH}_{nc}(A)\times_{\mathcal{SH}_{nc}(A\otimes_k R)}\mathcal{SH}_{nc}(R)\to\mathcal{SH}_{nc}(k) \tag{9.2.53}$$

Objects in $\mathcal{SH}(A)\times_{\mathcal{SH}(A\otimes_k R)}\mathcal{SH}_{nc}(R)$ can of course be described as triples $(a\in\mathcal{SH}(A), r\in\mathcal{SH}(R), w\in\mathcal{SH}(A\otimes_k R))$ together with equivalences $(p')^*(a)\simeq w\simeq (j')^*(r)$, and the left adjoint in (9.2.52) can be informally described by the assignment sending such a pair to a pushout $P(a,b,w)$ of the diagram

$$\begin{tikzcd} j_\sharp((p')_\sharp\circ(p')^*)(a))) \arrow[r] \arrow[d] & p_\sharp(r) \arrow[d] \\ j_\sharp(a) \arrow[r] & P(a,b,w) \end{tikzcd} \tag{9.2.54}$$

in $\mathcal{SH}(k)$ induced by the counits of the adjunctions $((p')_\sharp,(p')^*)$ and $((j')_\sharp,(j')^*)$. Here we have

$$j_\sharp((p')_\sharp\circ(p')^*)(a)))\simeq j_\sharp\circ(p')_\sharp(w)\simeq p_\sharp\circ(j')_\sharp(w)\simeq p_\sharp((j')_\sharp\circ(j')^*)(r))$$

Mutatis-mutandis for the left adjoint (9.2.53). We will now check this construction defines an equivalence. Following the existence and form of compact generators in $\mathcal{SH}(k)$ (see Prop. 5.3.3) and as all the maps involved preserve all colimits preserve generators, we are left to check the units and co-units of this adjunction are equivalences when restricted to smooth schemes.

Let S be a smooth affine scheme over k seen as an object in $\mathcal{SH}(k)$ via the canonical map. As the induced pullback in motives is compatible with the canonical map, its image along (9.2.3) is the triple $(S\times_X U, S\times_X V, S\times_X W)$ where again we understand these schemes as objects in motives via their canonical maps to k. Its image along (9.2.52) corresponds to the pushout diagram

$$\begin{tikzcd} j_\sharp\circ(p')_\sharp(S\times_X W) \arrow[r] \arrow[d] & p_\sharp(S\times_X V) \arrow[d] \\ j_\sharp(S\times_X U) \arrow[r] & P(S\times_X U, S\times_X V, S\times_X W) \end{tikzcd} \tag{9.2.55}$$

which, using the Prop. (9.2.11-1.) can be identified with

$$\begin{tikzcd} S \times_X W \arrow[r] \arrow[d] & S \times_X V \arrow[d] \\ S \times_X U \arrow[r] & P(S \times_X U, S \times_X V, S \times_X W) \end{tikzcd} \tag{9.2.56}$$

where we understand these schemes as defined over X by composing their structural maps with j, p and $p \circ (j') = j \circ (p')$.

The universal property of the pushout gives us a canonical map

$$P(S \times_X U, S \times_X V, S \times_X W) \to S \simeq S \times_X X \tag{9.2.57}$$

that corresponds to the counit of the adjunction (9.2.52). We are left to justify why this map is an equivalence. The reason is obvious: the diagram (U, V, W) forms a Nisnevich covering of X and as Nisnevich coverings are stable by pullbacks the diagram (9.2.56) is Nisnevich and therefore, by construction of $\mathcal{SH}(k)$, a pushout.
For the co-unit of the noncommutative case we use exactly the same arguments together with the Prop.6.4.16 that ensures $L_{pe,k}$ sends classical Nisnevich squares to nisnevich squares in the noncommutative setting and the Prop. 6.4.14 ensuring that (noncommutative) Nisnevich squares are stable under tensor products.

This proves that the maps (9.2.3) and (9.2.4) both are fully faithful. To show that they are equivalences we show that the maps in (9.2.52) and (9.2.53) are conservative. As all the $(\infty, 1)$-categories involves are stable, it will be enough to show that if the image of an object is zero then the object itself is zero. Again, we are reduced to work with the compact generators. Let Y be a smooth scheme over U and Y' be a smooth scheme over V, seen as objects, respectively, in $\mathcal{SH}(A)$ and $\mathcal{SH}(R)$, together with an equivalence $(j')^*Y' \simeq (p')^*Y$ in $\mathcal{SH}(A \otimes_k R)$ and suppose that $P(Y, Y', (j')^*Y'))$ is a zero object in $\mathcal{SH}(X)$. We want to show that both Y and Y' are zero objects. In what concerns Y we show that the map

$$j^*(j_\sharp(Y)) \to j^*(P(Y, Y', (j')^*Y'))) \simeq j^*(0) \simeq 0 \tag{9.2.58}$$

is an equivalence.
Indeed, as j^* is exact we have a pushout square

$$\begin{tikzcd} j^*(j_\sharp((p')_\sharp \circ (p')^*)(Y)))) \simeq j^*(p_\sharp((j')_\sharp \circ (j')^*)(Y'))) \arrow[r] \arrow[d] & j^*(p_\sharp(Y')) \arrow[d] \\ j^*(j_\sharp(Y)) \arrow[r] & j^*(P(Y, Y', (j')^*Y'))) \end{tikzcd} \tag{9.2.59}$$

in $\mathcal{SH}(U)$ and by the Remark 9.2.12 we have $(j')^* \circ (j')_\sharp \simeq Id$ and $j^* \circ j_\sharp \simeq Id$ so that we are reduced to showing that the top row in (9.2.59) is an equivalence. This follows from the equivalence $(j')^* \circ (j')_\sharp \simeq Id$ together with the Smooth base change of the Prop. 9.2.11-3.

We are now reduced to showing the following property: if Y' is a smooth scheme over V such that both $(j')^*(Y')$ and $p_\sharp(Y')$ are zero objects, then Y' is also a zero object in $\mathcal{SH}(V)$. This follows because for a Zariski cover, p is also an open immersion so that again by the Remark 9.2.12 $p_\sharp$ is fully faithful.
The proof for the noncommutative case follows exactly by the same arguments.

□

Remark 9.2.14. All the results in this section apply mutatis-mutandis to the further localization $\mathcal{SH}_{nc}^{Loc}$ discussed in Chapter 7.

9.3 Motives and Noncommutative Motives over a Scheme

In this section we use the results of the previous two sections to extend the definitions of motives and noncommutative motives to a commutative base not necessarily affine. One possibility is the following: as $CAlg(\mathcal{P}r^L_{Stb})$ admits all limits and the (non-full!) inclusion $CAlg(\mathcal{P}r^L_{Stb}) \hookrightarrow CAlg(Cat^{vbig}_\infty)$ preserves them (recall that limits of algebras are computed by means of the forgetful functor [100, 3.2.2.1, 3.2.2.5] and the non-full inclusion $\mathcal{P}r^L_{Stb} \subseteq Cat^{vbig}_\infty$ preserves limits [99, 5.5.3.13]), the universal property of ∞-presheaves [99, Thm. 5.1.5.6] provides an equivalence

$$Fun(N(BAff), CAlg(\mathcal{P}r^L_{Stb})^{op}) \simeq Fun^L(\mathcal{P}(N(BAff)), CAlg(\mathcal{P}r^L_{Stb})^{op}) \tag{9.3.1}$$

through which we can Kan-extend the ∞-functors $\mathcal{SH}^\otimes$ and $\mathcal{SH}^\otimes_{nc}$ to obtain new ∞-functors

$$Kan(\mathcal{SH})^\otimes, Kan(\mathcal{SH}_{nc})^\otimes : \mathcal{P}(N(BAff))^{op} \to CAlg(\mathcal{P}r^L_{Stb}) \tag{9.3.2}$$

sending colimits in $\mathcal{P}(Aff)$ to limits. The same universal property transports also the original natural transformation $\mathcal{L}^\otimes$ to a natural transformation between the extensions. As a result, given a stack $F \in \mathcal{P}(N(BAff))$, $\mathcal{SH}(F)^\otimes$ is canonically equivalent to the limit of the $\mathcal{SH}(k)^\otimes$ indexed by all $Spec(k) \to F$. Mutatis-Mutandis for $\mathcal{SH}^\otimes_{nc}$. In particular, as the category of schemes can be identified with a full subcategory of stacks (via the functor that assigns to a scheme its functor of points), the restriction provides ∞-functors $Kan(\mathcal{SH})^\otimes, Kan(\mathcal{SH}_{nc})^\otimes : BSch^{op} \to CAlg(\mathcal{P}r^L_{Stb})$ which we can now informally describe as

$$S \mapsto Kan(\mathcal{SH})^\otimes(X) := lim_{u:Spec(A)\to S}\mathcal{SH}^\otimes(A) \tag{9.3.3}$$

$$S \mapsto Kan(\mathcal{SH}_{nc})^\otimes(X) := lim_{u:Spec(A)\to S}\mathcal{SH}^\otimes_{nc}(A) \tag{9.3.4}$$

An important consequence of the Zariski descent property for both $\mathcal{SH}^\otimes$ and $\mathcal{SH}^\otimes_{nc}$ is that both $Kan(\mathcal{SH})^\otimes$ and $Kan(\mathcal{SH}_{nc})^\otimes$ send Zariski local equivalences to equivalences and therefore, factor naturally through the Zariski localization $Sh_{Zar}(BSch) \subseteq \mathcal{P}(BSch)$ and in particular are well-defined for schemes.

Remark 9.3.1. In the commutative case we can follow a more concrete approach. Let us discuss first the commutative case. As in Chapter 5, the theory of Morel-Voevodsky $\mathcal{SH}^\otimes$ can be defined directly for any Noetherian scheme of finite Krull dimension S. So far we restricted to affine schemes because until now we were only looking for a comparison with non-commutative motives over a ring but we can easily see use the same arguments of section 9.1-Step 1), to give sense to an ∞-functor $\mathcal{SH}^\otimes : BSch^{op} \to CAlg(\mathcal{P}r^L)$ that extends the map 9.1.15 to all schemes. Moreover, for smooth morphisms we can use the same arguments of the section 9.2 to prove the existence of the adjoints $(-)_\sharp$ verifying the Prop. 9.2.11 and the statement of Zariski descent. Finally, as both the natural extension $\mathcal{SH}^\otimes$ and the Kan extension $Kan(\mathcal{SH})^\otimes$ satisfy Zariski descent, the canonical map $\mathcal{SH}^\otimes \to Kan(\mathcal{SH})^\otimes$ induced by the universal property of limits is an equivalence.

Notation 9.3.2. For the rest of this thesis we will simply write $\mathcal{SH}^\otimes_{nc}$ to denote the Kan extension $Kan(\mathcal{SH}_{nc})^\otimes$

Remark 9.3.3. In the commutative case the previous remark tells us that the results of the Prop. 9.2.11 hold over a general basis. One can also confirm this in the non-commutative context. More precisely, we can check that for every smooth map of base schemes $f : X \to Y$ the base change functor $f^* : \mathcal{SH}_{nc}(Y) \to \mathcal{SH}_{nc}(X)$ has a left adjoint $f_\sharp$ satisfying the compatibility properties of the Prop. 9.2.11. One can see this in a directy way by describing the formula for this adjoint: if $\{u_\alpha : U_\alpha \hookrightarrow Y\}$ is a Zariski covering of Y by affine, we consider the fiber product

$$\begin{tikzcd} X_\alpha \arrow[r, hook, "v_\alpha"] \arrow[d, "f_\alpha"] & X \arrow[d, "f"] \\ U_\alpha \arrow[r, hook, "u_\alpha"] & Y \end{tikzcd} \tag{9.3.5}$$

and the induced pullback diagram

$$\begin{tikzcd} \mathcal{SH}_{nc}(X) \arrow[r, "v_\alpha^*"] & \mathcal{SH}_{nc}(X_\alpha) \\ \mathcal{SH}_{nc}(Y) \arrow[u, "f^*"] \arrow[r, "u_\alpha^*"] & \mathcal{SH}_{nc}(U_\alpha) \arrow[u, "f_\alpha^*"] \end{tikzcd} \tag{9.3.6}$$

One can then prescribe a formula for $f_\sharp$ using the Zariski descent of $\mathcal{SH}_{nc}$: given a Zariski covering $\{t_{\alpha,i} : W_{\alpha,i} \hookrightarrow X_\alpha\}$ by affine and $E \in \mathcal{SH}_{nc}(X)$ we consider the object

$$((f_\alpha \circ t_{\alpha,i})_\sharp (v_\alpha \circ t_{\alpha,i})^*(E))_{(\alpha)} \in \prod_\alpha \mathcal{SH}_{nc}(U_\alpha) \tag{9.3.7}$$

Notice that $(f_\alpha \circ t_{\alpha,i})_\sharp$ exists because the source and target are now affine. We now observe that this family agrees in the intersections $U_\alpha \times_Y U_\beta$ (these are affine because by assumption Y is quasi-separated). To see this we are reduced to contemplate the commutativity of the diagram

$$\begin{tikzcd} \mathcal{SH}_{nc}(W_{\alpha,i}) \arrow[r] \arrow[d, "(f_\alpha \circ t_{\alpha,i})_\sharp"] & \mathcal{SH}_{nc}(W_{\alpha,i} \times_{U_\alpha} (U_\alpha \times_Y U_\beta)) \arrow[d] \\ \mathcal{SH}_{nc}(U_\alpha) \arrow[r] & \mathcal{SH}_{nc}(U_\alpha \times_Y U_\beta) \end{tikzcd} \tag{9.3.8}$$

for every α, β and i: this follows from the results of the Prop. 9.2.11 for affine schemes.

In other words this means that the family (9.3.7) lives in the equalizer K of the two restrictions $\prod_\alpha \mathcal{SH}_{nc}(U_\alpha) \to \prod_{\alpha,\beta} \mathcal{SH}_{nc}(U_\alpha \times_Y U_\beta)$ and by Zariski descent there is an equivalence $K \xrightarrow[\sim]{\psi} \mathcal{SH}_{nc}(Y)$ which allows us to glue it. One can now easily manipulate all these adjunctions to check that this procedure provides a left adjoint to f^*. Moreover, by construction, it satisfies all the properties listed in the Prop. 9.2.11.

As a consequence of this discussion we have the following corollary:

Corollary 9.3.4. *For any scheme X, the image of the tensor unit $1_X^{nc} \in \mathcal{SH}_{nc}^{\otimes}(X)$ through the right adjoint $\mathcal{M}_X : \mathcal{SH}_{nc}(X) \to \mathcal{SH}(X)$ is equivalent to the object KH_X in $\mathcal{SH}(X)$ representing homotopy invariant algebraic K-theory. Moreover, by the same arguments explained in the introduction of Chapter 7 the natural transformation $\mathcal{L} : \mathcal{SH} \to \mathcal{SH}_{nc}$ factors through the theory of KH-modules.*

Proof. The Corollary 7.0.35 tells us that this is true if X is affine. In the non-affine case we start by considering a a Zariski covering of S, $\{f_i : U_i \to X\}_{i \in I}$ where each U_i is affine. As the homotopy invariant algebraic K-theory of schemes satisfies Nisnevich (therefore Zariski descent) we are reduced to check that for every $i \in I$, the commutative diagram

$$\begin{tikzcd} \mathcal{SH}(U_i) \arrow[r, "\mathcal{L}_{U_i}"] & \mathcal{SH}_{nc}(U_i) \\ \mathcal{SH}(X) \arrow[u, "u_i^*"] \arrow[r, "\mathcal{L}_X"] & \mathcal{SH}_{nc}(X) \arrow[u, "(u_i^{nc})*"] \end{tikzcd} \tag{9.3.9}$$

is horizontally right adjointable. This follows because each as u_i is smooth, by the discussion preceeding this Corollary, the results of the Prop. 9.2.11 work in the non-affine case, so that the diagram is vertically left-adjointable, and therefore by adjunction, horizontally right adjointable.

□

Remark 9.3.5. By Zariski descent and the Prop. 6.4.20, $\mathcal{SH}(X) \to \mathcal{SH}_{nc}(X)$ sends $(\mathbb{P}^1, \infty)$ to the unit non-commutative motive.

Remark 9.3.6. One can also consider the Kan extension of the assignment $k \mapsto \mathcal{D}g^{idem,\otimes}(k)$ to introduce a Morita theory of dg-categories over a general base scheme X, in the spirit of [140, Def. 2.6], [147, Section 4.4] and [55]. By definition, an object in $T \in \mathcal{D}g^{idem}(X)$ is then a family of dg-categories $T_A \in \mathcal{D}g^{idem}(A)$, one for each morphism $Spec(A) \to X$, compatible with base change. In the same way we can also Kan extend the theory of non-commutative spaces $\mathcal{N}cS^{\otimes}$. By the universal property of the Kan extension this definition comes naturaly equipped with a natural transformation $\mathcal{N}cS^{\otimes} \to \mathcal{SH}_{nc}^{\otimes}$ that allows us to pass from dg-categories of finite type over X to non-commutative motives over X.

Remark 9.3.7. The results of this section apply mutatis-mutadis to the localizing version of non-commutative versions studied in Chapter 7. In particular, we have also the smooth base change properties of the Prop. 9.2.11.

9.4 The Grothendieck six operations in the commutative world - A higher categorical framework

This section is mostly expository. Our main goal is to describe a general framework from which we can deduce the existence of a formalism of six operations in the world of higher categories and explain how to apply it to the motivic stable homotopy theory of schemes. None of this is new. Recently, in [93, 94] the authors introduce a technique to describe the existence of a formalism of six operations for the ∞-categorical enrichment of the derived categories of etale sheaves. At the same time, the existence of six operations for the motivic theory of schemes is also well know at the level of the classical associated homotopy theories: it has been establish in J. Ayoub's thesis [6, 7] inspired by the insights of Voevodsky [41]. More recently, D-C.Cisinski and F. Déglise [30] gave a more distilled presentation of the arguments in the proofs. In this sense, this section is the mere pedagogical exercise of explaining how the techniques of [93, 94] can be used in a general context and applied to provide a higher categorical enhancement of Ayoub's results. Moreover, we prepare the path for the next chapter where we explain our attempts to establish a similar behavior in the noncommutative world.

Throughout this section we will fix a category of base schemes $BSch$ consisting of all small Noetherian schemes of finite Krull dimension. One could also fix a scheme S and set $BSch$ to be the category of quasi-projective schemes over S. An important aspect is that whatever definition we choose, it should verify Nagata's compactification [39, Thm 4.1] (see below).

Let us start by recalling what is meant by a formalism of six operations. Re-writting the presentation in [30] in the language of higher categories, it be presented as follows:

i) For any base scheme X, one is supposed to provide the data of a stable presentable symmetric monoidal $(\infty, 1)$-category $\mathcal{T}(X)^{\otimes}$. By the Adjoint Functor Theorem $\mathcal{T}(X)^{\otimes}$ will necessarily be closed. A classical example is to take $\mathcal{T}(X)$ as some ∞-categorical enrichment of the derived category of étale sheaves on X. Another example, which will be of primary interest to us in this paper, are the stable presentable symmetric monoidal $(\infty, 1)$-category $\mathcal{SH}(X)^{\otimes}$ and $\mathcal{SH}_{nc}(X)^{\otimes}$ encoding the theories of motives, resp. noncommutative motives;

ii) For any morphism of base schemes $f : Y \to X$, a colimit preserving monoidal functor $f^* : \mathcal{T}(X)^{\otimes} \to \mathcal{T}(Y)^{\otimes}$ called *pullback along* f. For any composition $f \circ g : Z \to Y \to X$ we also ask for natural equivalences between the compositions $(f \circ g)^* \simeq g^* \circ f^*$. As a consequence of the Adjoint Functor Theorem the compatibility with colimits forces f^* to have a right-adjoint f_* which is necessarily lax monoidal [100, 7.3.2.7].

iii) For any morphism $f : Y \to X$ which is separated and of finite type, a colimit preserving functor $f_! : \mathcal{T}(Y) \to \mathcal{T}(X)$. Again, this assignment should be functorial in the sense that for any composition of separated morphisms of finite type $f \circ g : Z \to Y \to X$ we should have a natural equivalence of functors $f_! \circ g_! \simeq (f \circ g)_!$. Moreover, again by the Adjoint functor theorem, $f_!$ will have a right-adjoint $f^!$.

iv) Recall from [100, 4.8.1.3, 4.8.1.4, 4.8.1.14] (or from our survey in Chapter 3) the existence of a natural symmetric monoidal structure $\mathcal{P}r^{L,\otimes}$ in the $(\infty,1)$-category $\mathcal{P}r^L$ of presentable $(\infty,1)$-categories together with colimit preserving maps as morphisms. Recall also that stable presentable $(\infty,1)$-categories are stable under this tensor product and that stable presentable symmetric monoidal $(\infty,1)$-categories can be understood as commutative algebras in $\mathcal{P}r^{L,\otimes}_{Stb}$. Under this identification, the condition (ii) above says that for any morphism of schemes $f : Y \to X$, the monoidal functor f^* endows $\mathcal{T}(Y)^{\otimes}$ with the structure of a commutative algebra over $\mathcal{T}(X)^{\otimes}$. In particular, this algebra-structure restricts to a module structure $\mathcal{T}(X) \otimes \mathcal{T}(Y) \to \mathcal{T}(Y)$ where the module action can be intuitively described by the formula $(X, Y) \mapsto f^*(X) \otimes Y$ the last tensor being product taken in $\mathcal{T}(Y)^{\otimes}$. If f happens to be separated of finite type, condition (iii) provides a new functor $f_! : \mathcal{T}(Y) \to \mathcal{T}(X)$. As part of the framework of the six operations we ask for $f_!$ to be a map of $\mathcal{T}(X)^{\otimes}$-modules in $\mathcal{P}r^{L,\otimes}_{Stb}$. Intuitively this means that we have formulas like

$$f_!(f^*(F) \otimes E) \simeq F \otimes f_!(E) \tag{9.4.1}$$

for any object $E \in \mathcal{T}(Y)$ and $F \in \mathcal{T}(X)$. These are known as *Projection Formulas.* By adjunction, these imply also other formulas such as

$$f^! \underline{Hom}_X(F, G) \simeq \underline{Hom}_Y(f^*(F), f^!(G)) \tag{9.4.2}$$

with both F and G in $\mathcal{T}(X)$ and

$$\underline{Hom}_X(f_!(E), F) \simeq f_* \underline{Hom}_Y(E, f^!(F)) \tag{9.4.3}$$

v) For any morphism $f : Y \to X$ separated and of finite type, we want to have a natural transformation of ∞-functors $f_! \to f_*$ which we ask to be an equivalence whenever f is *proper.*

vi) For any *cartesian square* of schemes

$$\begin{tikzcd} Y' \arrow[r, "p'"] \arrow[d, "f'"] & X' \arrow[d, "f"] \\ Y \arrow[r, "p"] & X \end{tikzcd} \tag{9.4.4}$$

with f *separated of finite type*, we ask for natural equivalences of ∞-functors

$$p^* \circ f_! \simeq (f')_! \circ (p')^* \tag{9.4.5}$$

and

$$f^! \circ p_* \simeq (p')_* \circ (f')^! \tag{9.4.6}$$

Of course, together with the condition (vi), when f is proper the first formula is equivalent to proper base change.

vii) For any smooth morphism of relative dimension d, $f : Y \to X$ the adjunctions $(f_!, f^!)$ and (f^*, f_*) are expected to be related by an auto-equivalence Σ_f of $\mathcal{T}(Y)$ - the so-called *tangent Thom transformation of* f, by means of an equivalence

$$f^* \simeq \Sigma_f^{-1} \circ f^! \tag{9.4.7}$$

If $\mathcal{T}$ is equipped with the choice of an *orientation*, Σ_f is equivalent to the operation of tensoring with $1_X(d)[2d]$ where $1_X(d)$ is the d-iterated tensor power of the Tate twist.

These axioms form the core of the formalism. In some examples the previous formalism is deeply related to the duality form studied in [66]:

D1) Absolute Purity: For any closed immersion $Z \to S$ between regular schemes of constant codimension c, one asks for canonical equivalences

$$1_Z(-c)[-2c] \xrightarrow{\sim} i^!(1_S)$$

with 1_Z and 1_S, respectively, the units of $\mathcal{T}(Z)^{\otimes}$ and $\mathcal{T}(S)^{\otimes}$.

D2) If S is a regular base scheme and K_S is a $\otimes$-invertible object in $\mathcal{T}(S)^{\otimes}$, then for any separated finite type morphism $f : X \to S$ we set $K_X := f^!(K_S)$ and define the *duality functor with respect to* K_S as

$$D_X := \underline{Hom}_X(-, K_X) : \mathcal{T}(X)^{op} \to \mathcal{T}(X) \tag{9.4.8}$$

and we ask for the following properties:

- The canonical natural transformation $Id \to D_X \circ D_X$ is an equivalence;
- For any objects M, $N \in \mathcal{T}(X)$, there are canonical equivalences

$$D_X(M \otimes D_X(N)) \simeq \underline{Hom}_X(M, N)$$

- For any morphism between separated S-schemes of finite type u: $Y \to X$, the duality functors D_X and D_Y interchange u^* and $u^!$ (resp. u_* and $u_!$).

These duality properties are not known for motivic stable homotopy theory of schemes.

Remark 9.4.1. Such a duality phenomenon provides an alternative fashion to define the adjunction $(f_!, f^!)$ because the combination of the properties above yields to $f^! \simeq D_Y \circ f^* \circ D_X$. This is useful for instance to establish the framework in the context of D-modules (see [26, Chapter VII, 10.2] and [42, Def. 3.1.5]). In a different context, in [56, 58] the authors use this form of duality to explain the interplay between the two assignments

$$X \mapsto D(X) := \infty\text{-derived category quasi-coherent sheaves on } X$$

and

$$X \mapsto IndCoh(X) := \text{Ind completion of the } (\infty, 1)\text{-category of coherent sheaves on } X$$

The existence of such a formalism has deep consequences:

Example 9.4.2. (Grothendieck-Serre Duality) The usual statement of Grothendieck-Serre duality (as presented for instance in [66]) can be seen as a consequence of a formalism of six operations: if $f : X \to S$ is a proper morphism we have $f_! \simeq f_*$ and the formula (9.4.3) can be written as

$$D_S(f_*(E)) \simeq f_*(D_X(E)) \tag{9.4.9}$$

with $K_S := 1_S$ and the dualizing complex is given by $K_X = f^!(1_S)$. This has the form of Grothendieck-Serre dualty.

Example 9.4.3. (Spanier-Whithead Duality) If $f : X \to S$ is a smooth proper map, separated of finite type and of relative dimension c then $f_\sharp(1_X)$ and $f_!(1_X)$ are duals in $\mathcal{T}(S)^{\otimes}$. Indeed, for any $E \in \mathcal{T}(S)$, as $f_! \simeq f_*$, we have

$$\underline{Hom}_S(f_!(1_X), E) \simeq f_! \underline{Hom}_X(1_X, f^!(E)) \simeq f_! f^!(E) \tag{9.4.10}$$

$$\simeq f_\sharp(1_X(c)[2c] \otimes 1_X(-c)[-2c] \otimes f^*(E)) \simeq f_\sharp(1_X) \otimes_S E \tag{9.4.11}$$

where the last equivalence follows from the projection formula.

The main result we want to emphasize in this chapter (see Thm 9.4.36) is that $\mathcal{T}^{\otimes} = \mathcal{SH}^{\otimes}$ satisfies the Grothendieck formalism of six operations (as a functor valued in stable presentable symmetric monoidal $(\infty, 1)$-categories).

Remark 9.4.4. Again, and as mentioned in the introduction, the fact the homotopy categories of $\mathcal{SH}^{\otimes}$ satisfy the six operations is well-known after the results of Ayoub [6, 7]. Our only contribution is to present the functoriality of the assignment $S \mapsto \mathcal{SH}^{\otimes}(S)$ in the framework of $(\infty, 1)$-categories. As we shall see in this chapter, this follows from the techniques recently introduced in [93, 94] and from the fact all the $(\infty, 1)$-categories involved are stable so that all the proofs can be reduced to the level of the homotopy categories.

9.4.1 A higher categorical framework for the six operations - following Liu-Zheng

In this section we explain how to built up a machine that encodes what we described in the introduction as the formalism of six operations. There are rather complicated technical issues, mainly the fact that all the higher coherences need to be considered as part of the data. In this section we review the methods of [93, 94] that allow us to encode these higher coherences in a clean and systematic way. Nothing in this section is new apart from the exposition.

First of all we would like to recall an important construction given in [100] (which we already used in the previous section). If $\mathcal{C}^{\otimes}$ is a symmetric monoidal $(\infty, 1)$-category, we can construct a generalized ∞-operad $Mod(\mathcal{C})^{\otimes}$ whose objects in the underlying $(\infty, 1)$-category $Mod(\mathcal{C})$ can be understood as pairs (A, M) where A is a commutative algebra object in $\mathcal{C}$ and M is an object in $\mathcal{C}$ equipped with a structure of a left A-module and a morphism of pairs $(A, M) \to (B, N)$ can be understood as a map of commutative algebra objects $u : A \to B$ together with a map of objects $M \to N$ in $\mathcal{C}$, which is A-linear with N endowed with the A-module structure induced by u - see [100, Constuction 3.3.3.1, Definition 3.3.3.8, Theorem. 3.3.3.9 , Corollary 3.4.3.4, Definition 4.2.1.13, Example 4.2.1.18, Proposition 4.4.1.4 and Theorem 4.5.3.1]. This $(\infty, 1)$-category is a key figure in the framework of the six operations. Let us give a brief explanation of why this is so. Assume for the moment that we have an assignment $X \to \mathcal{T}(X)^{\otimes}$ satisfying the form of six operations presented in the introduction of the paper. For any morphism of schemes $f : Y \to X$ we have a monoidal pullback functor $f^* : \mathcal{T}(X)^{\otimes} \to \mathcal{T}(Y)^{\otimes}$ which makes $\mathcal{T}(Y)^{\otimes}$ a commutative algebra object in $\mathcal{P}r^L_{Stb}$ defined over $\mathcal{T}(X)^{\otimes}$. In particular, this action makes $\mathcal{T}(Y)$ a $\mathcal{T}(X)^{\otimes}$-module so that we can now understand the pair $(\mathcal{T}(X)^{\otimes}, \mathcal{T}(Y))$ as an object in $Mod(\mathcal{P}r^L_{Stb})$. Moreover, if f is separated of finite type then we have also a direct image $f_! : \mathcal{T}(Y) \to \mathcal{T}(X)$ satisfying the projection formula. The important point is that the projection formula is nothing but saying that $f_!$ is a map of $\mathcal{T}(X)^{\otimes}$-modules. In this case, an equivalent and more compact way to formulate the projection formula is to say that $f_!$ should be understood as map $(\mathcal{T}(X)^{\otimes}, \mathcal{T}(Y)) \to (\mathcal{T}(X)^{\otimes}, \mathcal{T}(X))$ in $Mod(\mathcal{P}r^L_{Stb})$. At the same time, for any morphism of schemes $f : Y \to X$, the induced pullback f^* can itself also be seen as a morphism of pairs $(\mathcal{T}(X)^{\otimes}, \mathcal{T}(X)) \to (\mathcal{T}(X)^{\otimes}, \mathcal{T}(Y))$ given by the identity of $\mathcal{T}(X)^{\otimes}$ and f^* seen as a map of $\mathcal{T}(X)^{\otimes}$-modules. Our goal with this discussion is to motivate $Mod(\mathcal{P}r^L_{Stb})$ as a natural target place to formulate the six operations.

We can now ask what properties or additional structure should an ∞-functor $\mathcal{T} : BSch^{op} \to CAlg(\mathcal{P}r^L_{Stb})$ have in order to satisfy the formalism of the six operations, or in different words, how can we extract from $\mathcal{T}$ the adjunctions $(f_!, f^!)$ and when will they satisfy the expected properties. The first basic ingredients are the following assumptions under which we shall work for the rest of this section:

(I) For any smooth morphism of finite type $f : Y \to X$, $f^* := \mathcal{T}(f)$ has a left adjoint $f_\sharp : \mathcal{T}(Y) \to \mathcal{T}(X)$ such that:

  (a) (Smooth Projection Formula) As f^* is monoidal, by adjunction, for any $E \in \mathcal{T}(Y)$ and $B \in \mathcal{T}(X)$, there is a natural map

$$f_\sharp(E \otimes f^*(B)) \to f_\sharp(E) \otimes B$$

  We ask for this map to be an equivalence;

  (b) (Smooth base change) For any cartesian square of base schemes

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \tag{9.4.12}$$

  with f smooth of finite type, the commutative diagram

$$\begin{tikzcd} \mathcal{T}(X') & \mathcal{T}(Y') \arrow[l, "(f')^*"'] \\ \mathcal{T}(X) \arrow[u, "(g')^*"] & \mathcal{T}(Y) \arrow[l, "f^*"] \arrow[u, "g^*"'] \end{tikzcd} \tag{9.4.13}$$

  is horizontally left-adjointable. In other words, if the diagram

$$\begin{tikzcd} \mathcal{T}(X') \arrow[r, "(f')_\sharp"] & \mathcal{T}(Y') \\ \mathcal{T}(X) \arrow[u, "(g')^*"] \arrow[r, "f_\sharp"] & \mathcal{T}(Y) \arrow[u, "g^*"'] \end{tikzcd} \tag{9.4.14}$$

  commutes by means of the natural transformation

$$(f')_\sharp \circ (g')^* \simeq (f')_\sharp \circ (g')^* \circ Id \to (f')_\sharp \circ (g')^* \circ f^* \circ f_\sharp \simeq (f')_\sharp \circ (f')^* \circ g^* \circ f_\sharp \to g^* \circ f_\sharp \tag{9.4.15}$$

(II) For any proper morphism of base schemes $f : Y \to X$ the pullback f^* has a right adjoint f_* given by the Adjoint functor Theorem [99, 5.5.2.9]. We assume that

(a) this right adjoint itself also has a right adjoint;

(b) (Proper Projection Formula) As f^* is monoidal, by adjunction, for any $E \in \mathcal{T}(Y)$ and $B \in \mathcal{T}(X)$, there is a natural map

$$f_*(E) \otimes B \to f_*(E \otimes f^*(B))$$

We ask for this map to be an equivalence;

(c) (Proper base change) For any cartesian square of base schemes (9.4.12) this time with f proper, the induced pullback diagram (9.4.13) is right adjointable. In other words, the diagram

$$\begin{tikzcd} \mathcal{T}(X') \arrow[r, "(f')_*"] & \mathcal{T}(Y') \\ \mathcal{T}(X) \arrow[u, "(g')^*"] \arrow[r, "f_*"] & \mathcal{T}(Y) \arrow[u, "g^*"'] \end{tikzcd} \tag{9.4.16}$$

commutes by means of the natural transformation

$$g^* \circ f_* \to (f')_* \circ (f')^* \circ g^* \circ f_* \simeq (f')_* \circ (g')^* \circ f^* \circ f_* \to (f')_* \circ (g')^* \tag{9.4.17}$$

Remark 9.4.5. It is important to remark that the two projections formulas I-a) and II-b) can be written in terms of the adjointable condition of certain commutative diagrams. If f^* has a left adjoint $f_\sharp$ then $id \otimes f_\sharp$ is a left adjoint to the top horizontal arrow in the commutative diagram

$$\begin{tikzcd} \mathcal{T}(X) \otimes \mathcal{T}(X) \arrow[r, "id\otimes f^*"] \arrow[d, "\otimes_X"] & \mathcal{T}(X) \otimes \mathcal{T}(Y) \arrow[d, "m"] \\ \mathcal{T}(X) \arrow[r, "f^*"] & \mathcal{T}(Y) \end{tikzcd} \tag{9.4.18}$$

so that, to say that $f_\sharp$ satisfies I-a) is equivalent to say that this diagram is left adjointable. One can easily establish a similar statement for II-b) in terms of a right-adjointable diagram.

Example 9.4.6. After our discussion in section 9.3, both the examples $\mathcal{T} = \mathcal{SH}^{\otimes}$, $\mathcal{T} = \mathcal{SH}_{nc}^{\otimes}$ and $\mathcal{T} = \mathcal{SH}_{nc}^{Loc,\otimes}$ satisfy all assumptions in (I). Assumption (II)-a) is also satisfied. Indeed as f_* is a functor between stable presentable $(\infty, 1)$-categories, for it to admit a right adjoint, by the Adjoint Functor Theorem, it is enough to check that it preserves filtered colimits. Let $d : I \to \mathcal{SH}(Y)$ be a filtered diagram. We need to show the canonical map

$$colim_{i\in I} f_* \circ d \to f_*(colim_{i\in I}(d))$$

is an equivalence. For this purpose we use the existence of a nice family of compact generators in $\mathcal{SH}(X)$ given by the Prop. 5.3.3. It is enough to check the generators see this map as an equivalence. But this follows from their form, the fact they are compact and the fact f^* preserves them. Similar arguments imply that both $\mathcal{T} = \mathcal{SH}_{nc}^{\otimes}$ and $\mathcal{T} = \mathcal{SH}_{nc}^{Loc,\otimes}$ also verify (II)-a).

We will see in the last section of this chapter that in the example $\mathcal{T} = \mathcal{SH}^{\otimes}$ all the assumptions left in (II) are also satisfied. This follows from the fundamental results of J. Ayoub.

Under these assumptions a first answer to the problem of constructing the six operations was given by P. Deligne in [2, XVII - 3.3.2] in the classical case where $\mathcal{T}$ takes values in triangulated strict 1-categories. The first ingredient is the Nagata's compactification theorem (see [39, Thm 4.1]) which tells us that any separated morphism of finite type $f : Y \to X$ between quasi-compact separated schemes admits a factorization

$$X \xhookrightarrow{j} \overline{X} \xrightarrow{p} Y \tag{9.4.19}$$

with j an open immersion and p proper. Using this result, Deligne proposes a definition of $f_!$ (f separated of finite type) as a colimit in the category of functors from $\mathcal{T}(Y)$ to $\mathcal{T}(X)$ of the diagram $(j,p) \mapsto p_* \circ j_\sharp$ indexed by all the possible factorizations of f in the form of the compactification theorem. If f is proper, the factorization with j the identity is an initial object in the category of factorizations, so that $f_! \simeq f_*$. In the same spirit we have also $f_! \simeq f_\sharp$ if f is an open immersion. It is an immediate observation that in order for the assignment $f \mapsto f_!$ constructed this way to be functorial we need to make another assumption on $\mathcal{T}$:

Definition 9.4.7. *(Deligne). Let $\mathcal{T} : BSch^{op} \to CAlg(\mathcal{P}r^L_{Stb})$ satisfy (I) and (II). We say that $\mathcal{T}$ satisfies the* Support Property *(Supp) if for any cartesian diagram of schemes (9.4.12) with g* proper *and f an* open immersion*, the commutative diagram (9.4.14) written as*

$$\begin{tikzcd} \mathcal{T}(X) \arrow[r, "(g')^*"] \arrow[d, "f_\sharp"'] & \mathcal{T}(X') \arrow[d, "(f')_\sharp"] \\ \mathcal{T}(Y) \arrow[r, "g^*"] & \mathcal{T}(Y') \end{tikzcd} \tag{9.4.20}$$

is right adjointable. In order words, if the diagram

$$\begin{tikzcd} \mathcal{T}(X) \arrow[d, "f_\sharp"'] & \mathcal{T}(X') \arrow[l, "(g')_*"'] \arrow[d, "(f')_\sharp"] \\ \mathcal{T}(Y) & \mathcal{T}(Y') \arrow[l, "g_*"'] \end{tikzcd} \tag{9.4.21}$$

commutes by means of the adjoint natural transformation

$$f_\sharp \circ (g')_* \simeq Id \circ f_\sharp \circ (g')_* \to g_* \circ g^* \circ f_\sharp \circ (g')_* \simeq g_* \circ (f')_\sharp \circ (g')^*(g')_* \to g_* \circ (f')_\sharp \tag{9.4.22}$$

In Deligne's theory, since the target of $(-)_!$ is a (2,1)-category in order to verify that functoriality holds, one only needs to consider coherence up to 2-cells (see [30, Section 2.2.9]). Back to our context where $\mathcal{T}$ takes values in a $(\infty,1)$-category, repeating the strategy of Deligne by hand is virtually impossible for we need to verify an infinite amount of coherences. As a solution to this problem, in [93, 94] the authors propose a technical procedure based on the language of multi-simplicial sets that provides a machine that encodes and ensures all these coherences. Their main result can be stated as follows:

Theorem 9.4.8. *([93, 94]) Let $\mathcal{T} : BSch^{op} \to CAlg(\mathcal{P}r^L_{Stb})$ satisfy (I) and (II) and (Supp). Then there exists an ∞-functor $(-)_!$ defined on the subcategory spanned by the separated morphisms of finite type, sending a scheme Y to $\mathcal{T}(Y)$ and such that if f is an open immersion we have a natural equivalence $f_! \simeq j_\sharp$ and if f is proper we have $f_! \simeq p_*$. Together with $\mathcal{T}$, $(-)_!$ satisfies the formalism of the six operations described in (i)-(vi) in the introduction.*

Remark 9.4.9. The result in the theorem 9.4.8 can also be proved in a much compreensible way, avoiding the rather technical aspects of multi-simplicial sets. It follows from a universal characterization of the $(\infty,2)$-category of correspondences built out of a fixed class of maps. This approach is currently being worked by D. Gaitsgory and N. Rozenblyum and will soon appear in literature. A preliminary version is already available at [59]. Their results provide a much clear understanding of the results in [93, 94]. The key idea is that out of an $(\infty,1)$-category $\mathcal{C}$ together with two classes of morphisms $\mathcal{E}_1$ and $\mathcal{E}_2$ we can produce $(\infty,2)$-categories $\mathcal{C}^{corr}_{\mathcal{E}_1}$ (resp. $\mathcal{C}^{corr}_{\mathcal{E}_1}$) having the same objects of $\mathcal{C}$ and morphisms given by correspondences $X \leftarrow W \to Y$ where the second arrow $W \to Y$ is in $\mathcal{E}_i$. The good behavior of compositions in these categories is equivalent to the base-change conditions described above. If $\mathcal{E}_1$ and $\mathcal{E}_2$ are contained in a bigger class of morphisms $\mathcal{E}_0$ such as in Nagata's lemma, there is a procedure to glue the $(\infty,2)$-categories $\mathcal{C}^{corr}_{\mathcal{E}_1}$ and $\mathcal{C}^{corr}_{\mathcal{E}_2}$. See [58, Thm. 5.2.2 and Section 5.4.3] for a more detailed explanation.

As the tools mentioned in the Remark 9.4.9 are not yet available we will use the technical steps of [93, 94]. In order to give the text a more definitive form we isolate the two main technical steps - Lemmas 9.4.10 and 9.4.14. Once the results of Gaitsgory-Rozenblyum become available in the literature these are exactly the two main lemmas whose content will become much clear. The rest of the proof should then follow by the very same steps presented here.

The goal of this section is then to review the techniques of [93, 94] and explain the proof of the theorem. In 9.4.1.1 we review the theory of multi-simplicial sets, multi-marked simplicial sets and multi-bimarked simplicial sets and explain how to relate them. We recall the notion of Cartesian restricted nerve. In 9.4.1.3 we follow the steps of loc.cit to give a precise form and proof to the theorem.

This result solves the problem of constructing $(-)_!$ in a coherent way. The question whether this construction satisfies the phenomenon in (vii) is of independent nature. More precisely, and following [30] it is possible to give conditions on $\mathcal{T}$ such that the output of the theorem satisfies (vii). We shall briefly review this in section 9.4.2.4.

9.4.1.1 Multi-simplicial sets, Multi-marked simplicial sets and Multi-bimarked simplicial sets

We now review the main ingredients in the theory of multi-simplicial sets and following [93, 94] explain how to use them to achieve our goals. Let I be a finite set. We denote by Δ^I the category of functors from I (seen as a discrete category) to Δ. By definition, an I-simplicial set is an object in $\widehat{\Delta^I}$, meaning, a presheaf of sets over Δ^I. Following the notations of [93, 94] we let $\Delta^{([n_i])_{i\in I}}$ denote the image by Yoneda of the object $([n_i])_{i\in I} \in \Delta^I$. For any map of sets $f : I \to J$ the composition with f induces a functor $(\Delta^f)^* : \Delta^J \to \Delta^I$ and the composition with this functor produces a new functor $\widehat{\Delta^I} \to \widehat{\Delta^J}$. It admits a right adjoint $(\Delta^f)_*$. There are two cases of special interest to us:

- $I = \{1, ..., k\}$, $J = \{*\}$ and the unique map. In this case the induced functor from $\Delta = \Delta^J$ to $\Delta^k := \underbrace{\Delta \times ... \times \Delta}_{k} = \Delta^I$ is the diagonal functor (which we shall denote as δ_k) and the induced functor $\delta_k^* := (\Delta^f)^* : \widehat{\Delta^k} \to \widehat{\Delta}$ is determined by the formula $\delta_k^*(S)_n := S(\underbrace{[n], ..., [n]}_{k})$ with the boundary and degeneracy maps given by the compositions with the image of the boundaries and degeneracies in Δ along the diagonal. In particular, the image of the representable $\Delta^{([n_i])_{i\in I}}$ can be canonically identified with the product $\prod_{i\in I} \Delta[n_i]$ in $\widehat{\Delta}$ where $\Delta[n_i]$ is the representable. This follows directly from the definition of δ_k^* and the Yoneda's lemma. This observation allows us to prescribe a right adjoint $\delta_*^k := (\Delta^f)_*$ to δ_k^*

$$\begin{tikzcd} \widehat{\Delta^k} \arrow[r, bend left, "\delta_k^*"] & \widehat{\Delta} \arrow[l, bend left, "\delta_*^k"] \end{tikzcd}$$

  by means of the following universal property: for any simplicial set X,

$$Hom_{\widehat{\Delta^k}}(\Delta^{([n_i])_{i\in I}}, \delta_*^k(X)) \simeq Hom_{\widehat{\Delta}}(\prod_{i\in I} \Delta[n_i], X) \tag{9.4.23}$$

  The following terminology will be useful later: if S is a k-simplicial set we say that a cell $\tau : \Delta^{([n_i])_{i\in\{1,...,k\}}} \to S$ is *of direction* $i \in I$ if the map τ factors by the degeneracy maps everywhere except in position i where it factors as the identify map:

$$\Delta^{([n_i])_{i\in\{1,...,k\}}} \to \Delta^{([0],...,[n_i],...,[0])} \to S$$

- $J = \{1, ..., k\}$ and $I \subseteq J$ with f the inclusion. In this case the composition map $\Delta^k = \Delta^J \to \Delta^I$ corresponds to the product of the projections in the coordinates indexed by the elements in I. As $[0]$ is a final object in Δ, this functor admits a right adjoint $\Delta^I \to \Delta^k$ sending a family $([n_i])_{i\in I}$ to the family $([m_j])_{j\in\{1,...,k\}}$ with $m_j = n_j$ if $j \in I$ and $[0]$ otherwise. We let $\epsilon^k_I : \widehat{\Delta^k} \to \widehat{\Delta^I}$ denote the functor induced by composition with this last functor. In particular we set the notation ϵ^k_i when I consists of one element $i \in J$. By definition, an n-cell of $\epsilon^k_i(\Delta^{([n_j])_{j\in\{1,...,k\}}})$ is an element in the product

$$\Delta[n_1]_0 \times ... \times \Delta[n_{i-1}]_0 \times \Delta[n_i]_n \times \Delta[n_{i+1}]_0 \times ... \times \Delta[n_k]_0 \tag{9.4.24}$$

Another important construction is that of taking partial opposites. Recall that Δ admits an endofunctor $op : \Delta \to \Delta$ defined by the identity on the objects and sending boundary maps $\partial^n_i : [n-1] \to [n]$ to ∂^n_{n-i} and degeneracy maps $\epsilon^n_i : [n+1] \to [n]$ to ϵ^n_{n-i}. The opposite of a simplicial set X is then defined to be the simplicial set obtained by composing X with op. If now S is an I-simplicial set and $K \subseteq I$, we define the partial opposite of S with respect to K, denoted as $op^I_K(S)$, as the composition of S with the functor $\Delta^I \to \Delta^I$ defined by applying op to the copies of Δ indexed by K and the identity functor in the remaining ones. If $I = \{1, ..., k\}$ we set $\delta^*_{k,K}$ as the composition $\delta^*_k \circ op^k_K$.

To conclude, let us remark that for any $k, k' \geq 0$, there is a functor

$$\boxtimes : \widehat{\Delta^k} \times \widehat{\Delta^{k'}} \to \widehat{\Delta^{k+k'}} \tag{9.4.25}$$

defined by the formula

$$(S \boxtimes S')([n_1], ..., [n_k], [n_{k+1}], ..., [n_{k+k'}]) := S([n_1], ..., [n_k]) \times S'([n_{k+1}], ..., [n_{k+k'}])$$

Moreover, it follows immediately from the definitions that the diagrams

$$\begin{tikzcd} \widehat{\Delta^k} \times \widehat{\Delta^{k'}} \arrow[r, "\boxtimes"] \arrow[d, "\delta^*_k \times \delta^*_{k'}"] & \widehat{\Delta^{k+k'}} \arrow[d, "\delta^*_{k+k'}"] \\ \widehat{\Delta} \times \widehat{\Delta} \arrow[r, "\times"] & \widehat{\Delta} \end{tikzcd} \tag{9.4.26}$$

commute.

Th first key technical result from [93] is the following lemma

Lemma 9.4.10. *(see [93, Prop. 1.4.3]) Let $I = \{1, .., k\}$ and let S be a k-simplicial set. Let $J \subseteq I$ be a subset and $f : \delta^*_k S \to Cat_\infty$ be a map of simplicial sets. Assume the following conditions:*

1. *For every $j \in J$ and every edge $e : \Delta[1] \to \delta^*_k S$ of direction j, the functor $f(e) : \Delta[1] \to \delta^*_k S \to Cat_\infty$ has a left adjoint;*

2. *For all $i \in I - J$, $j \in J$ and every $\tau \in \epsilon^k_{i,j}(S)$, the induced square*

$$f(\tau) : \Delta[1] \times \Delta[1] \xrightarrow{\delta^*_k(\tau)} \delta^*_k(S) \xrightarrow{f} Cat_\infty \tag{9.4.27}$$

   is left adjointable.

Then, there exists a map of simplicial sets $f_J : \delta^*_{I,J} S \to Cat_\infty$ satisfying the following properties:

1. *f and f_J are equal when restricted to those cells in any direction in $I - J$. More precisely, we have $f_J|\delta^*_{I-J}(\Delta^u)_* S) = f|\delta^*_{I-J}(\Delta^u)_* S)$ with u the inclusion map $I - J \subseteq I$.*

2. *For every $j \in J$ and every edge $e : \Delta[1] \to \delta^*_{I,J} S$, the functor $f_J(e)$ is a left adjoint to $f(e)$.*

3. *For all $i \in I - J$, $j \in J$ and every $\tau \in \epsilon^k_{i,j}(S) = \epsilon^k_{i,j}(op^k_J(S))$ the induced square $f_J(\tau)$ is left adjoint to $f(\tau)$ (in the sense of 2.1.10).*

Remark 9.4.11. This lemma is in fact a shadow of a very beautiful result to appear in the works of Gaitsgory-Rozenblyum related to the Gray tensor product of $(\infty, 2)$-categories: If S, T and X and $(\infty, 2)$-categories then there is an equivalence between the space of 2-functors from $S \otimes T \to X$ sending 1-morphisms in S to 1-morphisms in X having a right-adjoint and the space of 2-functors from $T \otimes S^{op} \to X$ sending 1-morphisms in S^{op} to 1-morphisms in X with a left-adjoint. I'm very grateful to Nick Rozenblyum for explaning me this very elegant result.

We now move to the second main technical result. The second notion that we will need is that of a *Multi-marked simplicial set*. Recall that a marked simplicial set is a simplicial set X together with a collection $\mathcal{E}$ of edges in X containg all the degenerated edges. As a natural extension of this notion, an *I-marked simplicial set* (I again a finite set) consists of a pair $(X, \mathcal{E} = \{\mathcal{E}_i\}_{i\in I})$ with X a simplicial set and $\mathcal{E}$ a family of collections of edges in X, all containing the degenerated edges. We let $\widehat{\Delta}_{I+}$ denote the category of I-marked simplicial sets with the obvious morphisms.

The theories of I-marked simplicial sets and I-simplicial sets are naturally related: if $(X, \mathcal{E} = \{\mathcal{E}_i\}_{i\in I})$ is an I-marked simplicial set we define an I-simplicial $\Psi(X, \mathcal{E})$ as the full sub-I-simplicial set of $\delta^k_*(X)$ spanned by those $([n_i])_{i\in I}$-cells corresponding via (9.4.23) to those maps in $\widehat{\Delta}$

$$\prod_{i\in I} \Delta[n_i] \to X$$

such that for every $i \in I$ and every map $\sigma : \Delta[1] \to \epsilon^k_i(\Delta^{([n_j])_{j\in\{1,\ldots,k\}}})$, the composition

$$\Delta[1] \simeq \Delta[0] \times \ldots \times \underbrace{\Delta[1]}_{\text{ith-position}} \times \ldots \times \Delta[0] \xrightarrow{\sigma} \textstyle\prod_{i\in I} \Delta[n_i] \longrightarrow X \tag{9.4.28}$$

is an edge in $\mathcal{E}_i$. This construction is functorial $\Psi : \widehat{\Delta}_{I+} \to \widehat{\Delta^I}$ and admits a left adjoint sending an I-simplicial set S to the I-marked simplicial set defined by the pair $(\delta^*_k(S), \{\mathcal{E}_i\}_{i\in I})$ with $\mathcal{E}_i$ is the collection of edges $\Delta[1] \to \delta^*_k(S)$ corresponding to those cells $\Delta^{([1])_{i\in I}} \to S$ which factor as $\Delta^{([1])_{i\in I}} \to \Delta^{([0],\ldots,[1],\ldots,[0])} \to S$.

The last ingredient is the notion of a *Multi-bimarked simplicial set*. It is an extension of the previous notion: instead of considering a simplicial set with collections of edges we consider collections of commutative squares. More precisely, an I-bimarked simplicial set is a pair $(X, \mathcal{B} = \{\mathcal{B}_{ij}\}_{i,j\in I})$ where X is a simplicial set and $\mathcal{B}$ is an I-indexed family of collections $\mathcal{B}_{ij}$ of maps $\Delta[1] \times \Delta[1] \to X$ such that for any i, j in I, $\mathcal{B}_{ij}$ contains the degenerated squares $\Delta[1] \times \Delta[1] \to \Delta[0] \to X$ and the $\mathcal{B}_{ij}$ and $\mathcal{B}_{ji}$ are in bijection by means of the permutation of the two factors in $\Delta[1] \times \Delta[1]$. We set $\widehat{\Delta}_{I++}$ the category of I-bimarked simplicial sets with the natural morphisms compatible with the markings.

We shall now see how Ψ factors through the theory of Multi-bimarked simplicial sets. To start with, there is an adjunction between the theories of I-bimarked simplicial sets and that of I-marked simplicial sets, obtained by the following procedure: if $(X, \mathcal{E} = \{\mathcal{E}_i\}_{i\in I})$ is an I-marked simplicial set we consider the I-bimarked simplicial set $U(X, \mathcal{E}) := (X, \{\mathcal{E}_i *_X \mathcal{E}_j\}_{i,j\in I})$ where $\mathcal{E}_i *_X \mathcal{E}_j$ is defined as the collection of maps $\Delta[1] \times \Delta[1] \to X$ such that

- for any map $\sigma : \Delta[1] \to \epsilon^2_1(\Delta^{([1],[1])})$ the composition

$$\Delta[1] \simeq \Delta[1] \times \Delta[0] \xrightarrow{\sigma} \Delta[1] \times \Delta[1] \longrightarrow X \tag{9.4.29}$$

  is in $\mathcal{E}_i$;

- for any map $\sigma' : \Delta[1] \to \epsilon_2^2(\Delta^{([1],[1])})$ the composition

$$\Delta[1] \simeq \Delta[0] \times \Delta[1] \xrightarrow{\sigma'} \Delta[1] \times \Delta[1] \longrightarrow X \tag{9.4.30}$$

is in $\mathcal{E}_j$;

One can show that this construction admits a left adjoint (see [93, 94]).

At the same time there is also an adjunction between the theories of I-bimarked simplicial sets and that of I-simplicial sets: if $(X, \mathcal{B} = \{\mathcal{B}_{ij}\}_{i,j\in I})$ is an I-bimarked simplicial set we consider the I-simplicial set $\Phi(X, \mathcal{B})$ obtained as a full sub-I-simplicial set of $\delta_*^k(X)$ spanned by those cells

$$\prod_{i\in\{1,\dots,k\}} \Delta[n_i] \to X$$

such that for every $i, j \in I$ and every map $u : \Delta[1] \times \Delta[1] \to \epsilon^k_{\{i,j\}}(\Delta^{([n_i])_{i\in\{1,..,k\}}})$, the composition

$$\Delta[1] \times \Delta[1] \simeq \Delta[0] \times \dots \times \underbrace{\Delta[1]}_{\text{ith-position}} \times \dots \times \underbrace{\Delta[1]}_{\text{jth-position}} \times \dots \times \Delta[0] \xrightarrow{\quad u \quad} \prod_{i\in I} \Delta[n_i] \longrightarrow X \tag{9.4.31}$$

Again, it is possible to show the existence of a left adjoint to Φ.

The commutativity of the diagram

$$\begin{tikzcd} \widehat{\Delta^I} & \widehat{\Delta}_{I++} \arrow[l, "\Phi"'] \\ & \widehat{\Delta}_{I+} \arrow[ul, "\Psi"] \arrow[u, "U"'] \end{tikzcd} \tag{9.4.32}$$

is obvious from the definitions. We can now modify a bit the constructions and establish to the following definition:

Definition 9.4.12. *Let $I = \{1, ..., k\}$ and let $\mathcal{C}$ be an $(\infty, 1)$-category with finite limits, together with collections $\mathcal{E}_1, ..., \mathcal{E}_k$ of edges in $\mathcal{C}$. The* Cartesian retricted nerve *of $\mathcal{C}$, denote as $\mathcal{C}^{Cart}_{\mathcal{E}_1,...,\mathcal{E}_k}$, is the k-simplicial set obtained by applying Φ to the sub-I-bimarked simplicial set of $U(\mathcal{C}, \{\mathcal{E}_i\}_{i\in\{1,...,k\}}) = (\mathcal{C}, \{\mathcal{E}_i *_X \mathcal{E}_j\}_{i,j\in\{1,...,k\}})$ consisting of the same underlying simplicial set $\mathcal{C}$ but this time equipped with the sub-collections $\mathcal{E}_i *^{cart}_X \mathcal{E}_j$ defined as follows:*

- *if $i = j$, $\mathcal{E}_i *^{cart}_X \mathcal{E}_j = \mathcal{E}_i *_X \mathcal{E}_j$;*
- *if $i \neq j$, $\mathcal{E}_i *^{cart}_X \mathcal{E}_j$ is the sub-collection of $\mathcal{E}_i *_X \mathcal{E}_j$ spanned by the elements that are pullback diagrams in $\mathcal{C}$.*

The simplicial sets of the form $\delta^*_k \mathcal{C}^{Cart}_{\mathcal{E}_1,...,\mathcal{E}_k}$ are the key players in the methods developed in [93, 94] to encode the formalism of the six operations. In the case $k = 2$ and assuming the conditions in Def. 9.4.12 we have

- The 0-cells of $\delta^*_2 \mathcal{C}^{Cart}_{\mathcal{E}_1,\mathcal{E}_2}$ are the objects in $\mathcal{C}$;
- 1-cells are given by commutative squares in $\mathcal{C}$

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\sigma"] & Y \arrow[d] \\ X' \arrow[r] & Y' \end{tikzcd} \tag{9.4.33}$$

such that σ is a pullback in $\mathcal{C}$, the horizontal maps are in $\mathcal{E}_1$ and the vertical maps are in $\mathcal{E}_2$. The source of this 1-cell is X and the target is Y'. Moreover, σ is of direction 1 (resp. 2) if the vertical (resp. horizontal) maps are identities;

- the 2-cells are already complicated to visualize;
- etc;

Remark 9.4.13. In the case $\mathcal{C}$ is the nerve of an strict 1-category this description can be improved: the 0-cells and 1-cells admit the same description and because of the definition of the nerve functor, the n-cells can be identified with commutative diagrams in $\mathcal{C}$

$$\begin{tikzcd}
X_{0,0} \arrow[r] \arrow[d] & X_{1,0} \arrow[r] \arrow[d] & X_{2,0} \arrow[r] \arrow[d] & .... \arrow[r] \arrow[d] & X_{n,0} \arrow[d] \\
X_{0,1} \arrow[r] \arrow[d] & X_{1,1} \arrow[r] \arrow[d] & X_{2,1} \arrow[r] \arrow[d] & .... \arrow[r] \arrow[d] & X_{n,1} \arrow[d] \\
... \arrow[r] \arrow[d] & ... \arrow[r] \arrow[d] & ... \arrow[r] \arrow[d] & .... \arrow[r] \arrow[d] & ... \arrow[d] \\
X_{0,n} \arrow[r] & X_{1,n} \arrow[r] & X_{2,n} \arrow[r] & .... \arrow[r] & X_{n,n}
\end{tikzcd} \tag{9.4.34}$$

where every horizontal map is in $\mathcal{E}_1$, every vertical map is in $\mathcal{E}_2$ and every square is a pullback. Such a cell is of direction 1 (resp. 2) if the vertical (resp.horizontal) maps are identities. In this case every 1-cell

$$\begin{tikzcd}
X \arrow[r, "p'"] \arrow[d, "f'"] \arrow[dr, phantom, "\sigma"] & Y \arrow[d, "f"] \\
X' \arrow[r, "p"] & Y'
\end{tikzcd} \tag{9.4.35}$$

can be tautologically described as an edge in the ∂^1-boundary of two different 2-cells, namely,

$$A = \begin{tikzcd}
X \arrow[r, "p'"] \arrow[d, equal] \arrow[dr, phantom, "\sigma(p')"] & Y \arrow[r, equal] \arrow[d, equal] & Y \arrow[d, equal] \\
X \arrow[r, "p'"] \arrow[d, "f'"] & Y \arrow[r, equal] \arrow[d, "f"] \arrow[dr, phantom, "\sigma(f)"] & Y \arrow[d, "f"] \\
X' \arrow[r, "p"] & Y' \arrow[r, equal] & Y'
\end{tikzcd} \tag{9.4.36}$$

and

$$B = \begin{tikzcd}
X \arrow[r, equal] \arrow[d, "f'"'] \arrow[dr, phantom, "\sigma(f')"] & X \arrow[r, "p'"] \arrow[d, "f'"] & Y \arrow[d, "f"] \\
X' \arrow[r, equal] \arrow[d, equal] & X' \arrow[r, "p"] \arrow[d, equal] \arrow[dr, phantom, "\sigma(p)"] & Y' \arrow[d, equal] \\
X' \arrow[r, equal] & X' \arrow[r, "p"] & Y'
\end{tikzcd} \tag{9.4.37}$$

where $\sigma(f)$ and $\sigma(f')$ are 1-cells of direction 1 and $\sigma(p')$ and $\sigma(p)$ are of direction 2. In both cases the ∂^1-boundary is given by the outer commutative square. In this case, for any map of simplicial sets $\phi : \delta_2^* \mathcal{C}^{Cart}_{\mathcal{E}_1,\mathcal{E}_2} \to \mathcal{D}$ with $\mathcal{D}$ an $(\infty, 1)$-category, will have (by the definition of the composition operation in a quasi-category and the fact ϕ is a map of simplicial sets, therefore compatible with boundary maps)

$$\phi(\sigma) = \phi(\partial^1(A)) = \partial^1(\phi(A)) \simeq \partial^0(\phi(A)) \circ \partial^2(\phi(A)) \simeq \phi(\partial^0(A)) \circ \phi(\partial^2(A)) \simeq \phi(\sigma(f)) \circ \phi(\sigma(p'))$$

and by the same steps

$$\phi(\sigma) = \phi(\partial^1(B)) \simeq \phi(\sigma(p)) \circ \phi(\sigma(f'))$$

This will be the key feature the encoding base-change properties.

We are almost done with the preliminaries. As a last step we compare the functorial behavior of multi-marked simplicials, multi-bimarked simplicials sets and multi-simplicial sets. Let $f : I \to J$ be a map of finite sets. In the beginning of this section we saw that f produces an adjunction $((\Delta^f)^*, (\Delta^f)_*)$ between I-marked simplicial sets and J-marked simplicial sets. We now contemplate a similar phenomenon for multi-marked simplicial sets and multi-bimarked simplicial sets. For the first, f defines a natural adjunction

$$\begin{tikzcd} \widehat{\Delta}_{I_+} \arrow[r, bend left, "(f_+)^*"] & \widehat{\Delta}_{J_+} \arrow[l, bend left, "(f_+)_*"] \end{tikzcd} \tag{9.4.38}$$

defined by the formulas

- Given $(\mathcal{C}, \{\mathcal{E}_i\}_{i\in I}) \in \widehat{\Delta}_{I_+}$ we set $(f_+)^*((\mathcal{C}, \{\mathcal{E}_i\}_{i\in I})) := (\mathcal{C}, \{\tilde{\mathcal{E}}_j\}_{j\in J})$ where by definition we set $\tilde{\mathcal{E}}_j := \bigcup_{i\in f^{-1}(\{j\})})\mathcal{E}_i$ if $f^{-1}(\{j\})$ is non-empty and $\tilde{\mathcal{E}}_j =$ collection of degenerated simplices in $\mathcal{C}$ if $f^{-1}(\{j\})$ is empty;
- At the same time, given $(\mathcal{C}, \{\mathcal{O}_j\}_{j\in J}) \in \widehat{\Delta}_{J_+}$ we set $(f_+)_*((\mathcal{C}, \{\mathcal{O}_j\}_{j\in J})) := (\mathcal{C}, \{\tilde{\mathcal{O}}_i\}_{i\in I})$ where by definition we set $\tilde{\mathcal{O}}_i := \mathcal{O}_{f(i)}$;

One sees easily that these constructions are adjoint.

In the same spirit, one can also establish adjunctions relating the theories of multi-bimarked simplicial sets:

$$\begin{tikzcd} \widehat{\Delta}_{I_++} \arrow[r, bend left, "(f_++)^*"] & \widehat{\Delta}_{J_++} \arrow[l, bend left, "(f_++)_*"] \end{tikzcd} \tag{9.4.39}$$

To conclude, by a direct manipulation of the definitions, we contemplate the commutativity of the diagram of right adjoints

$$\begin{tikzcd} & \widehat{\Delta}_{I++} \arrow[dl, "\Phi_I"'] & \\ \widehat{\Delta^I} & & \widehat{\Delta}_{I+} \arrow[ul, "\Psi_I"'] \arrow[ll, "U_I"'] \\ & \widehat{\Delta}_{J++} \arrow[uu, "(f_++)_*"'] \arrow[dl, "\Phi_J"'] & \\ \widehat{\Delta^J} \arrow[uu, "(\Delta^f)_*"] & & \widehat{\Delta}_{J+} \arrow[ul, "\Psi_J"'] \arrow[ll, "U_J"'] \arrow[uu, "(f_+)_*"'] \end{tikzcd} \tag{9.4.40}$$

We can finally present the second key technical result from [94]. It is the heart of the mechanism through which it will be possible to implement the same gluing strategy of Deligne. We consider the previous diagram in the case where f corresponds to the map $I = \{1, 2, ..., k\} \to J = \{1, 2, ..., k-1\}$ defined by $f(1) = f(2) = 1$ and $f(i) = i-1$ for $i \geq 3$. In this case, if $\mathcal{C}$ is an $(\infty, 1)$-category and $\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2, ..., \mathcal{E}_k$ are collections of edges in $\mathcal{C}$ with both $\mathcal{E}_1$ and $\mathcal{E}_2$ contained in $\mathcal{E}_0$, then, using the commutativity of the diagram (9.4.40) and adjunctions, we find a natural map

$$(\Delta^f)^*(\mathcal{C}^{Cart}_{\mathcal{E}_1,\mathcal{E}_2,...,\mathcal{E}_k}) \to \mathcal{C}^{Cart}_{\mathcal{E}_0,\mathcal{E}_3,...,\mathcal{E}_k} \tag{9.4.41}$$

As the composition of f with the projection $J \to \{1\}$ is the projection $I \to \{1\}$, using the definitions, we find $\delta^*_{k-1} \circ (\Delta^f)^* = \delta^*_k$. In particular, applying δ^*_{k-1} to the previous map we find a map

$$\delta^*_k(\mathcal{C}^{Cart}_{\mathcal{E}_1,\mathcal{E}_2,...,\mathcal{E}_k}) \to \delta^*_{k-1}\mathcal{C}^{Cart}_{\mathcal{E}_0,\mathcal{E}_3,...,\mathcal{E}_k} \tag{9.4.42}$$

Lemma 9.4.14. *([94, Theorem 0.1, Corollary 5.3 and Remark 5.4]) Let $\mathcal{C}$ be an $(\infty, 1)$-category given by the nerve of a strict 1-category and consider $\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2, ..., \mathcal{E}_k$ collections od edges in $\mathcal{C}$. Assume the following*

1. *$\mathcal{C}$ admits pullbacks;*
2. *For every $0 \leq i \leq k$, the collection $\mathcal{E}_i$ contains every isomorphism of $\mathcal{C}$ and is stable under pullbacks;*
3. *Both $\mathcal{E}_1$ and $\mathcal{E}_2$ are contained in $\mathcal{E}_0$ and satisfy the following condition: for any pair of composable arrows u, v in $\mathcal{C}$, with $v \in \mathcal{E}_i$, the composition $v \circ u$ is $\mathcal{E}_i$ if and only if u is;*
4. *Every morphism $p \in \mathcal{E}_0$ is of the form $p = v \circ u$ with $v \in \mathcal{E}_1$ and $u \in \mathcal{E}_2$*

Then, the map (9.4.42) is a weak-equivalence of simplicial sets with respect to the Joyal model structure.

9.4.1.2 The ∞-operad $\mathcal{P}f^{\otimes}$

In this section we analyse the properties of an ∞-operad introduced in [93] as a means to study module objects in the commutative case. We let **Pf** denote the coloured operad with two colors $\{\mathbf{a}, \mathbf{b}\}$ and operations given by:

$$Hom_{\mathbf{Pf}}(\{X_i\}_{i\in I}, Y) := \begin{cases} \{*\} & \text{if } X_i = \mathbf{a} \text{ for all } i \in I \text{ and } Y = \mathbf{a} \\ \{*\}, & \text{if } \exists j \in I \text{ with } X_j = \mathbf{m}, X_i = \mathbf{a} \text{ for all } i \in I - \{j\} \text{ and } Y = \mathbf{m} \\ \emptyset & \text{otherwise} \end{cases} \tag{9.4.43}$$

We let $\mathcal{P}f^{\otimes}$ denote its operadic nerve (see [100, 2.1.1.23]). It follows immediately from this definition that the full subcategory of $\mathcal{P}f^{\otimes}$ spanned by the operations involving only the colour **a** is a copy of the commutative operad. As explained in [93, Remark 1.5.7] this ∞-operad is a simplified version of the ∞-operad $\mathcal{LM}^{\otimes}$ of [100, Def. 4.2.1.7] (see also our survey in Chapter 3.3.6) in the sense that for any symmetric monoidal $(\infty, 1)$-category $\mathcal{C}^{\otimes}$ we have a natural equivalence

$$Alg_{\mathcal{P}f^{\otimes}}(\mathcal{C}) \simeq Mod^{Comm}(\mathcal{C}) \tag{9.4.44}$$

or in other words, an algebra over $\mathcal{P}f^{\otimes}$ consists of a commutative algebra object given by the evaluation at **a** and a module over this algebra given by the evaluation at **m**.

In [93, Notation 1.5.8] the authors remark a very important description of $\mathcal{P}f^{\otimes}$ which we shall now explain. Let $\Delta[1]^{\amalg}$ be the ∞-operad associated to $\Delta[1]$ by means of the universal property of cocartesian monoidal structures [100, Thm 2.4.3.18] (consult also our fast survey in 3.1.8). By construction (see [100, 2.4.3.1]), the objects of $\Delta[1]^{\amalg}$ can be identified with sequences $(\langle n\rangle; (X_1, ..., X_n))$ with

$\langle n\rangle \in N(Fin_*)$ and X_1,..., X_n objects in $\Delta[1]$. Morphisms $(\langle n\rangle;(X_1,...,X_n)) \to (\langle m\rangle;(Y_1,...,Y_m))$ correspond to pairs $(\alpha,(f_i)_{i\in\alpha^{-1}(\langle m\rangle^+)})$ with $\alpha:\langle n\rangle\to\langle m\rangle$ and $f_i: X_i \to Y_{\alpha(i)}$ in $\Delta[1]$. As $\Delta[1]$ only has two objects a unique non-trivial morphism, which we denote here as $\mathbf{0}\to\mathbf{1}$, objects in $\Delta[1]^{\amalg}$ can be identified with pairs $(\langle n\rangle;S)$ where S is the subset of $\langle n\rangle^+$ consisting of those indexes i with $X_i=\mathbf{1}$. In this notation, a morphism $(\langle n\rangle,S)\to(\langle m\rangle,T)$ is just the data of a morphism $\alpha:\langle n\rangle\to\langle m\rangle$ in $N(Fin_*)$ with $\alpha(S)\subseteq T\bigcup\{0\}$. It follows from this description that the ∞-operad $\mathcal{P}f^{\otimes}$ can be identified with the sub-simplicial set of $\Delta[1]^{\amalg}$: we identify the colour $\mathbf{a}$ with $\mathbf{0}$, the colour $\mathbf{m}$ with $\mathbf{1}$ and consider only those morphism $\alpha:\langle n\rangle\to\langle m\rangle$ where the map $S\cap\alpha^{-1}(T)\to T$ is a bijection. in other words, we discard the operations generated by the unique arrow $\mathbf{0}\to\mathbf{1}$. This inclusion is a map of ∞-operads $\mathcal{P}f^{\otimes}\to\Delta[1]^{\amalg}$.

Let us now explain the relevance of this discussion. Let $\mathcal{C}^{\otimes}$ be a symmetric monoidal $(\infty,1)$-category and let $f:A\to B$ be a morphism of commutative algebra-objects in $\mathcal{C}$. Then we expect to be able to say that B aqcuires the structure of an A-module via f. The map of ∞-operads $\mathcal{P}f^{\otimes}\to\Delta[1]^{\amalg}$ encodes exactly this possibility. More precisely, we can see f as an object in $Fun(\Delta[1],CAlg(\mathcal{C}))$ so that by the universal property of $\Delta[1]^{\amalg}$, f corresponds to a uniquely defined map of ∞-operads $\Delta[1]^{\amalg}\to\mathcal{C}^{\otimes}$. Composing this map with the inclusion $\mathcal{P}f^{\otimes}\to\Delta[1]^{\amalg}$ we obtain a map of ∞-operads $\mathcal{P}f^{\otimes}\to\mathcal{C}^{\otimes}$ which as explained in the formula (9.4.44) corresponds to a uniquely determined object in $Mod(\mathcal{C})$. It follows from the nature of the inclusion $\mathcal{P}f^{\otimes}\to\Delta[1]^{\amalg}$ that this procedure returns the pair (A,B) with B considered with the A-module structure induced by f.

To conclude this section we prove a very useful result that combines the key Lemma 9.4.10 with the discussion of this section, in a essential way

Proposition 9.4.15. *Let $f^{\otimes}:\mathcal{C}^{\otimes}\to\mathcal{D}^{\otimes}$ be a colimit preserving monoidal functor between presentable symmetric monoidal $(\infty,1)$-categories. Assume that the underlying functor f of $f^{\otimes}$ has a left adjoint $g:\mathcal{D}\to\mathcal{C}$ and that g satisfies the projection formula in the condition I-a). Then g lifts to a map of module-objects in $\mathcal{P}r^{L,\otimes}$, where $\mathcal{D}$ is equipped with the structure of $\mathcal{C}^{\otimes}$-module induced by $f^{\otimes}$ and $\mathcal{C}$ has its natural $\mathcal{C}^{\otimes}$-module structure induced by the tensor product.*

Proof. We start by writing the map of module objects produced by $f^{\otimes}$, $(\mathcal{C}^{\otimes},\mathcal{C})\to(\mathcal{C}^{\otimes},\mathcal{D})$. For that purpose we consider the commutative square $\Delta[1]\times\Delta[1]\to CAlg(\mathcal{P}r^L)$ given by

$$\begin{tikzcd} \mathcal{C}^{\otimes} \arrow[r, "id"] \arrow[d, "id"] & \mathcal{C}^{\otimes} \arrow[d, "f^{\otimes}"] \\ \mathcal{C}^{\otimes} \arrow[r, "f^{\otimes}"] & \mathcal{D}^{\otimes} \end{tikzcd} \tag{9.4.45}$$

We can see this diagram as an object in $Fun(\Delta[1],Fun(\Delta[1],CAlg(\mathcal{P}r^L)))$ and thanks to the procedure explained in the paragraph above, by composing with the inclusion $\mathcal{P}f^{\otimes}\to\Delta[1]^{\amalg}$, the diagram provides an object in $\sigma_0\in Fun(\Delta[1],Mod(\mathcal{P}r^L))$ which we can naturally identify with the map of modules $(\mathcal{C}^{\otimes},\mathcal{C})\to(\mathcal{C}^{\otimes},\mathcal{D})$ given by the identity on $\mathcal{C}^{\otimes}$ and f as a map of modules between $\mathcal{C}$ and $\mathcal{D}$. Our goal is to, using the existence of $g:\mathcal{D}\to\mathcal{C}$ satisfying the projection formula, say that g is a map of modules $(\mathcal{C}^{\otimes},\mathcal{D})\to(\mathcal{C}^{\otimes},\mathcal{C})$. For that purpose we use the Lemma 9.4.10. We consider the fact that $\mathcal{P}r^{L,\otimes}$ has a lax inclusion in $Cat_{\infty}^{vbig,\times}$ and consider the composition

$$\Delta[1]\xrightarrow{\sigma} Mod(\mathcal{P}r^L)\simeq Alg_{\mathcal{P}f^{\otimes}}(\mathcal{P}r^L)\longrightarrow Alg_{\mathcal{P}f^{\otimes}}(Cat_{\infty}^{vbig}) \tag{9.4.46}$$

so that as $Cat_{\infty}^{vbig,\times}$ is cartesian, the $(\infty,1)$-category $Alg_{\mathcal{P}f^{\otimes}}(Cat_{\infty}^{vbig})$ is equivalent to the $(\infty,1)$-category of $\mathcal{P}f^{\otimes}$-monoid objects $Mon_{\mathcal{P}f^{\otimes}}(Cat_{\infty}^{vbig})$ - see [100, Def. 2.4.2.1]). As defined in loc. cit, these are objects in $Fun(\mathcal{P}f^{\otimes},Cat_{\infty}^{vbig})$ satisfying a standard Segal-like condition. In particular, σ_0 establishes an ∞-functor $\sigma_1:\Delta[1]\times\mathcal{P}f^{\otimes}\to Cat_{\infty}^{vbig}$ satisfying the Segal condition in the second variable. Now we introduce the technology of multi-simplicial sets. After the formula (9.4.26) we have

$$\delta_2^*(\Delta[1]\boxtimes \mathcal{P}f^{\otimes}) \simeq \Delta[1]\times \mathcal{P}f^{\otimes} \tag{9.4.47}$$

and we apply Lemma 9.4.10 to the $I = \{1,2\}$-simplicial set $S = \Delta[1]\boxtimes \mathcal{P}f^{\otimes}$ with the map $\sigma_1 : \delta_2^*(\Delta[1]\boxtimes \mathcal{P}f^{\otimes}) \to Cat_\infty^{vbig}$ and $J = \{1\}$:

1. The assumption that f has a left adjoint g ensures that for any edge e of direction 1, the functor $\sigma_1(e)$ has a left adjoint.
2. The assumption that the square in Cat_∞^{vbig} induced by any commutative square in $\epsilon_{1,2}^k(\Delta[1]\boxtimes \mathcal{P}f^{\otimes})$ is left adjointable follows from the assumption that g satisfies the projection formula by mimicking the argument explained in the Remark 9.4.5.

Then the Lemma 9.4.10 provides us with a new map of simplicial sets

$$\sigma_2 : \delta_{2,\{1\}}^*(\Delta[1]\boxtimes \mathcal{P}f^{\otimes}) \simeq \Delta[1]^{op}\times \mathcal{P}f^{\otimes} \to Cat_\infty^{vbig} \tag{9.4.48}$$

which equals σ_1 in direction 2. In particular, it continues to verify the Segal-like conditions. Moreover, as $\Delta[1]\simeq\Delta[1]^{op}$, σ_2 can again be written as a map

$$\Delta[1] \to Alg_{\mathcal{P}f^{\otimes}}(Cat_\infty^{vbig}) \tag{9.4.49}$$

and again becasue σ_2 equals σ_1 in direction 2 and as left adjoints commute with colimits, σ_2 factors as

$$\Delta[1] \to Mod(\mathcal{P}r^L) \to Alg_{\mathcal{P}f^{\otimes}}(Cat_\infty^{vbig}) \tag{9.4.50}$$

It follows from the results of the Lemma that σ_2 is the module enrichment of g we were looking for.

□

9.4.1.3 Gluing restricted Nerves and the formalism of the six operations

In this section we explain the proof of the Theorem 9.4.8. To illustrate how the Lemma 9.4.14 encodes the mechanism that allows us to implement the strategy of Deligne, let us consider $BSch'$ the subcategory of $BSch$ containing all objects together only with those morphisms that are separated and of finite type and consider also P the class of proper maps and J the class of open immersions. Then, the combination of this Lemma with the Nagata's compactification theorem gives the following result

Proposition 9.4.16. *([94, Corollary 0.3]) The natural map of the Lemma 9.4.14*

$$\delta_2^* N(BSch')_{P,J}^{Cart} \to N(BSch') \tag{9.4.51}$$

is a weak-equivalence of simplicial sets with respect to the Joyal's model structure.

The potential of this result is obvious: if $\mathcal{D}$ is an $(\infty,1)$-category, the data of an ∞-functor $F : N(BSch') \to \mathcal{D}$ is equivalent to the data provided by a map of simplicial sets $\delta_2^* N(BSch')_{P,J}^{Cart} \to \mathcal{D}$ which is essentially the same as specifying the value of F on open immersions and proper maps. The gluing is made by means of a choice of a quasi-inverse to the composition with (9.4.51).

Let us now explain how to use this gluing strategy to construct the six operations. Let $\mathcal{T} : BSch^{op} \to CAlg(\mathcal{P}r^L_{Stb})$ be an ∞-functor satisfying the assumptions (I), (II) and (Supp). Everything will be extracted from the map induced by the composition with $\mathcal{T}$:

$$Fun(\Delta[1], BSch^{op}) \to Fun(\Delta[1], CAlg(\mathcal{P}r^L_{Stb})) \tag{9.4.52}$$

By the universal property of $\Delta[1]^{\amalg}$ and composing with the inclusion $\mathcal{P}f^{\otimes}\to\Delta[1]^{\amalg}$ as explained in the previous section, we obtain a map

$$Fun(\Delta[1], BSch^{op}) \to Fun(\Delta[1], CAlg(\mathcal{P}r^L_{Stb})) \to Mod(\mathcal{P}r^L_{Stb}) \tag{9.4.53}$$

that corresponds exactly to the assignment sending a morphism of base schemes $f : Y \to X$ to the pair $(\mathcal{T}(X)^{\otimes}, \mathcal{T}(Y)) \in Mod(\mathcal{P}r^L_{Stb})$ where $\mathcal{T}(X)^{\otimes}$ is a commutative algebra object in $\mathcal{P}r^L_{Stb}$ by means of its monoidal structure and $\mathcal{T}(Y)$ is seen as an module over $\mathcal{T}(X)$ by means of the pullback functor f^*.

Let us now manipulate the source of this map. It is obviously equivalent to $Fun(\Delta[1], BSch)^{op}$ and therefore, by the properties of multi-marked simplicial sets to $\delta^*_{1,\{1\}}(Fun(\Delta[1], BSch))$. Let

$$\begin{tikzcd} Y_0 \arrow[r, "u"] \arrow[d, "f_0"] & Y_1 \arrow[d, "f_1"] \\ X_0 \arrow[r, "v"] & X_1 \end{tikzcd} \tag{9.4.54}$$

be an arrow in $Fun(\Delta[1], BSch)$. We consider four different collections of arrows:

- $\mathcal{E}_3 := ALL$ is the collection of all arrows with u and v arbitrary;
- $\mathcal{E}_0 := F$ is the collection of all arrows with u and v separated of finite type;
- $\mathcal{E}_1 := P$ is the collection of arrows with u and v proper;
- $\mathcal{E}_2 := J$ is the collection of arrows with u and v open immersions;

In particular we have both P and J contained in F and all contained in ALL, so that using the properties of multi-marked simplicial sets and the natural adjunctions, we have canonical map of simplicial sets

$$\begin{tikzcd} \delta^*_{3,\{1,2,3\}}(Fun(\Delta[1], BSch)^{Cart}_{P,J,ALL}) \arrow[d] \\ \delta^*_{2,\{1,2\}}(Fun(\Delta[1], BSch)^{Cart}_{F,ALL}) \arrow[d] \\ \delta^*_{1,\{1\}}(Fun(\Delta[1], BSch)^{Cart}_{ALL}) \arrow[d] \\ \delta^*_{1,\{1\}}(Fun(\Delta[1], BSch)) \end{tikzcd} \tag{9.4.55}$$

We now focus on the composition map

$$\sigma_0 : \delta^*_{3,\{1,2,3\}}(Fun(\Delta[1], BSch)^{Cart}_{P,J,ALL}) \to Mod(\mathcal{P}r^L_{Stb}) \tag{9.4.56}$$

Using the lax inclusion $\mathcal{P}r^{L,\otimes}_{Stb} \hookrightarrow Cat^{vbig,\times}_{\infty}$ and same strategy as in the proof of the Prop. 9.4.15, we establish a map of simplicial sets

$$\sigma_1 : \delta^*_{3,\{1,2,3\}}(Fun(\Delta[1], BSch)^{Cart}_{P,J,ALL}) \times \mathcal{P}f^{\otimes} \to Cat^{vbig}_{\infty} \tag{9.4.57}$$

satisfying the standard Segal-like conditions in the second variable. Moreover, the formula in 9.4.26 implies that the source of σ_1 is in equivalent to $\sigma_1 : \delta^*_{4,\{1,2,3\}}(Fun(\Delta[1], BSch)^{Cart}_{P,J,ALL} \boxtimes \mathcal{P}f^{\otimes})$. We now come to the main step of the construction. We apply the dual version of the key Lemma 9.4.10 to the $I = \{1, 2, 3, 4\}$-simplicial set $op^4_{\{1,2,3\}}(Fun(\Delta[1], BSch)^{Cart}_{P,J,ALL} \boxtimes \mathcal{P}f^{\otimes})$ together with the map σ_1 and $J = \{1\}$. This is possible because:

1. The existence of right adjoints for morphisms in direction 1 follows from the assumption II)-a);
2. The right adjointable condition for both directions (1,2) and (1,3) results from the assumption II-c);
3. As in the proof of the Prop.9.4.15 the right adjointable condition for direction (1,4) is given by the proper base change formula in II-b).

The output of 9.4.10 is a new map of simplicial sets

$$\sigma_2 : \delta^*_{4,\{2,3\}}(Fun(\Delta[1], BSch)^{Cart}_{P,J,ALL} \boxtimes \mathcal{P}f^{\otimes}) \to Cat^{vbig}_{\infty} \tag{9.4.58}$$

which is equal to σ_1 in all directions except the first where it now sends a proper map of schemes p to the induced pushfoward p_*.

We still need to correct the assignment for open immersions. For this purpose we apply now the Lemma 9.4.10 to the $I = \{1,2,3,4\}$-simplicial set $op^4_{\{2,3\}}(Fun(\Delta[1], BSch)^{Cart}_{P,J,ALL} \boxtimes \mathcal{P}f^{\otimes})$ together with the map σ_1 and $J = \{1\}$. Again, this is possible because

1. The existence of left adjoints for morphisms of direction 2 follows from the assumption I);
2. The left adjointable condition for direction (2,1) is a consequence of (Supp);
3. The left adjointable condition for direction (2,3) results from the assumption I-c);
4. As in the proof of the Prop.9.4.15 the left adjointable condition for direction (2,4) is given by the smooth base change formula in I-a).

Again, the output is a new map of simplicial sets

$$\sigma_3 : \delta^*_{4,\{3\}}(Fun(\Delta[1], BSch)^{Cart}_{P,J,ALL} \boxtimes \mathcal{P}f^{\otimes}) \to Cat^{vbig}_{\infty} \tag{9.4.59}$$

which is equal to σ_2 in all directions except the second where it now sends a open immersion j to the left adjoint $j_\sharp$. In particular, it is equal to σ_1 in what concerns the last direction so that it still satisfies the Segal conditions and therefore can be re-written as a map

$$\sigma_3 : \delta^*_{3,\{3\}}(Fun(\Delta[1], BSch)^{Cart}_{P,J,ALL}) \to Alg_{\mathcal{P}f^{\otimes}}(Cat^{vbig}_{\infty}) \tag{9.4.60}$$

and again because it equals σ_1 in what concerns the module-structures, it factors again as

$$\sigma_3 : \delta^*_{3,\{3\}}(Fun(\Delta[1], BSch)^{Cart}_{P,J,ALL}) \to Mod(\mathcal{P}r^L) \tag{9.4.61}$$

The final ingredient is where Deligne's strategy of using Nagata's compactification theorem is implemented. It results from the combination of the key Lemma 9.4.14 with the classical Nagata's compactification theorem [39, Thm 4.1]

Lemma 9.4.17. *[93, Lemma 3.2.4] The natural map*

$$\delta^*_{3,\{3\}}(Fun(\Delta[1], BSch)^{Cart}_{P,J,ALL}) \to \delta^*_{2,\{2\}}(Fun(\Delta[1], BSch)^{Cart}_{F,ALL}) \tag{9.4.62}$$

is a weak-equivalence of simplicial sets with respect to the Joyal model structure.

By means of a choice of a quasi-inverse to the composition with (9.4.62), σ_3 determines a uniquely defined map of simplicial sets

$$\sigma_4 : \delta^*_{2,\{2\}}(Fun(\Delta[1], BSch)^{Cart}_{F,ALL}) \to Mod(\mathcal{P}r^L) \tag{9.4.63}$$

This concludes the procedure. Let us now read the information encoded by σ_4. Following the Remark 9.4.13, we have the following description of the simplicial set $\delta^*_{2,\{2\}}(Fun(\Delta[1], BSch)^{Cart}_{F,ALL})$

- its 0-cells are morphisms of schemes $f : Y \to X$.
- a 1-cell from $f_0 : Y_0 \to X_0$ to $f_3 : Y_3 \to X_3$ is a commutative diagrams of schemes

$$\begin{tikzcd} & Y_0 \arrow[rr] \arrow[dl] \arrow[dd, "f_0"] & & Y_1 \arrow[dl] \arrow[dd, "f_1"] \\ Y_2 \arrow[rr] \arrow[dd, "f_2"] & & Y_3 \arrow[dd, "f_3"] & \\ & X_0 \arrow[rr] \arrow[dl] & & X_1 \arrow[dl] \\ X_2 \arrow[rr] & & X_3 & \end{tikzcd} \tag{9.4.64}$$

where the diagonal arrows can be any morphism of schemes and the horizontal arrows are separated morphisms of finite type. Moreover, both the lower and upper faces of the cube are pullback squares in the category of schemes. In particular, a 1-cell of direction 1 is the data of a commutative square of schemes

$$\begin{tikzcd} Y_0 \arrow[r] \arrow[d, "f_0"] & Y_3 \arrow[d, "f_3"] \\ X_0 \arrow[r] & X_3 \end{tikzcd} \tag{9.4.65}$$

where both horizontal arrows are separated morphisms of finite type.

Using this descrition we have:

- By construction, σ_4 of a 0-cell $f : Y \to X$ is the pair $(\mathcal{T}(X)^{\otimes}, \mathcal{T}(Y)) \in Mod(\mathcal{P}r^L)$, with $\mathcal{T}(X)$ considered with its monoidal structure and $\mathcal{T}(Y)$ seen as a module-object over $\mathcal{T}(X)$ via f;
- Let $f : Y \to X$ be a separated morphism of finite type. Then the commutative square τ:=

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "f"] & X \arrow[d, "Id_X"] \\ X \arrow[r, "Id_X"] & X \end{tikzcd} \tag{9.4.66}$$

is a 1-cell of direction 1. It follows the construction that $\sigma_4(\tau) : (\mathcal{T}(X)^{\otimes}, \mathcal{T}(Y)) \to (\mathcal{T}(X)^{\otimes}, \mathcal{T}(X))$ encodes the !-pushfoward $f_!$ as a map of modules.

- To observe that the assignment $(-)_!$ satisfies the required base-changed formulas, one considers the 1-cell τ:=

$$\begin{tikzcd}
& Y_0 \arrow[rr, "f"] \arrow[dl, "u"] \arrow[dd] & & Y_1 \arrow[dl, "v"] \arrow[dd] \\
Y_2 \arrow[rr, "g"] \arrow[dd] & & Y_3 \arrow[dd] & \\
& Spec(\mathbb{Z}) \arrow[rr, "Id"] \arrow[dl, "Id"] & & Spec(\mathbb{Z}) \arrow[dl, "Id"] \\
Spec(\mathbb{Z}) \arrow[rr, "Id"] & & Spec(\mathbb{Z}) &
\end{tikzcd} \tag{9.4.67}$$

where the vertical maps are the canonical projections, u and v are arbitrary and f and g are separated of finite type. The base-change formula between $(-)_!$ and $(-)^*$ follows now from the mechanism explained in the Remark 9.4.13.

To conclude, we explain how to extract the assigment $f \to f_!$ as a functor out of the category of schemes with separated morphisms between them. For that purpose, we consider the functor $BSch \to Fun(\Delta[1], BSch)$ sending a scheme X to the structure morphism $X \to Spec(\mathbb{Z})$. It induces a map

$$\delta^*_{2,\{2\}} BSch^{Cart}_{F,ALL} \to \delta^*_{2,\{2\}}(Fun(\Delta[1], BSch)^{Cart}_{F,ALL}) \tag{9.4.68}$$

which, when composed with σ_4 and restricted to the first direction, produces an ∞-functor

$$(-)_! : BSch^{sep} \to Mod(\mathcal{P}r^L) \tag{9.4.69}$$

where $BSch^{sep}$ is the nerve of the category of base schemes and separated morphisms between them. This concludes the proof of the Theorem 9.4.8.

Notation 9.4.18. For future notations we will denote the map σ_4 resulting from this procedure as $EO(\mathcal{T})$

9.4.2 Ayoub-Voevodsky fundamental properties

In this section we review the basic properties understood by Voevodsky in [41] and studied by Ayoub in [6]. As we shall review in the next section, together they imply the whole formalism of the six operations.

Throughout this section we fix an ∞-functor $\mathcal{T}^{\otimes} : BSch^{op} \to CAlg(\mathcal{P}r^L_{Stb})$ satisfying the assumptions (I) and (II)-a).

9.4.2.1 Localization Property

Let $i : Z \to X$ be a closed immersion of base schemes. Let $U := X - Z$ be the open complementary of i and let $j : U \subseteq X$ be the associated open immersion. As j is smooth, by I)-b) and because the intersection of U and Z in X is empty, we find $j^* \circ i_* \simeq 0$. In this case, for every object $E \in \mathcal{T}(X)$ we have a commutative diagram produced by the units and counits of the adjunctions

$$\begin{tikzcd}
j_\sharp \circ j^*(E) \arrow[r] \arrow[d] & E \arrow[d] \\
0 \arrow[r] & i_* \circ i^*(E)
\end{tikzcd} \tag{9.4.70}$$

We say that $\mathcal{T}^{\otimes}$ satisfies the *Localization Property with respect to i* if the following two conditions hold:

1. The pushfoward i_* is conservative;
2. For every $E \in \mathcal{T}(X)$, the square (9.4.70) is a cofiber/fiber sequence.

One says that $\mathcal{T}^{\otimes}$ satisfies the *Localization Property* if it satisfies the Localization property with respect to every closed immersion i.

Remark 9.4.19. This property has powerful consequences:

A) An obvious consequence of the fiber/cofiber sequence is that a morphism $u : E \to F$ in $\mathcal{T}(S)$ is an equivalence if and only if both the restrictions $j^*(u)$ and $i^*(u)$ are equivalences.

B) i_* is fully faithful. Indeed, from the exact sequence (9.4.70) applied to $E = i_*M$ with $M \in \mathcal{T}(Z)$, as $j^* \circ i_* \simeq 0$ we find $i_*M \simeq i_* \circ i^*(i_*(M))$. But as i_* is conservative we deduce $M \simeq i^* \circ i_*(M)$ so that i_* is fully faithful.

C) If $\mathcal{T}$ satisfies Localization with respect to a closed immersion $i : Z \to X$ then $\mathcal{T}$ will also satisfy conditions II-b) with respect to i. It follows from the combination of A) ,B) and the fact i^* is monoidal.

One can also describe this property in the following terms:

Proposition 9.4.20. *Let $\mathcal{T}$ be as above and $i : Z \to X$ be a closed immersion. The following are equivalent:*

1. *$\mathcal{T}$ satisfies the localization property with respect to i.*
2. *The commutative square of $(\infty, 1)$-categories*

$$\begin{tikzcd} \mathcal{T}(Z) \arrow[r, "i_*"] \arrow[d] & \mathcal{T}(X) \arrow[d, "j^*"] \\ * \arrow[r, "0"] & \mathcal{T}(U) \end{tikzcd} \tag{9.4.71}$$

   is an homotopy fiber sequence.
3. *The following conditions hold:*
   a) *i_* is conservative;*
   b) *i_* satisfies the projection formula II-b)*
   c) *the commutative square*

$$\begin{tikzcd} j_\sharp \circ j^*(1_X) \arrow[r] \arrow[d] & 1_X \arrow[d] \\ 0 \arrow[r] & i_* \circ i^*(1_X) \end{tikzcd} \tag{9.4.72}$$

   is a cofiber/fiber sequence in $\mathcal{T}(X)$.

Proof. The fact that 1) and 3) are equivalent follows immediately from the assumption II-b). Let us explain the equivalence between 1) and 2). Denote by $\mathcal{T}_Z(X)$ the homotopy fiber of the pullback functor $j^* : \mathcal{T}(X) \to \mathcal{T}(U)$. It is naturally identified with the full subcategory of $\mathcal{T}(X)$ spanned by those objects E with $j^*(E) \simeq 0$ in $\mathcal{T}(U)$. As we have $j^* \circ i_* \simeq 0$ by the universal property of the fiber there is a unique (up to homotopy) factorization of i_* as

$$\mathcal{T}(Z) \to \mathcal{T}_Z(X) \subseteq \mathcal{T}(X) \tag{9.4.73}$$

so that 3) is equivalent to say that this map $\mathcal{T}(Z) \to \mathcal{T}_Z(X)$ is an equivalence. We first observe it is fully faithful. This is because i_* itself is fully fathful as explained in the previous remark.

We now prove that $\mathcal{T}(Z) \to \mathcal{T}_Z(X)$ is essentially surjective. To see this we observe that the inclusion $\mathcal{T}_Z(X) \subseteq \mathcal{T}(X)$ has a natural left adjoint: For any $E \in \mathcal{T}(X)$, the homotopy cofiber

$$\begin{tikzcd} j_\sharp \circ j^*(E) \arrow[r] \arrow[d] & E \arrow[d] \\ 0 \arrow[r] & c(E) \end{tikzcd} \tag{9.4.74}$$

lives in the subcategory $\mathcal{T}_Z(X)$ because j^* is an exact functor and because, as j is smooth, the smooth base change property implies $j_\sharp$ is fully faithful so that the image of the top horizontal map in (9.4.74) is the identify map and therefore $j^*(c(E)) \simeq 0$. It follows from the fact this sequence is exact that the assignment $E \mapsto c(E)$ is left adjoint to the inclusion $\mathcal{T}_Z(X) \subseteq \mathcal{T}(X)$. This makes $\mathcal{T}_Z(X)$ a reflexive localization of $\mathcal{T}(X)$. In particular, c is essentially surjective. Finally, assumption 1) is equivalent to say that as an object in $\mathcal{T}(X)$, $c(E)$ is canonically equivalent to $i_* \circ i^*(E)$. Therefore, if 1) holds, the map is essentially surjective. If 3) holds then the sequence in (9.4.70) is exact due to the definition of c.

□

We now describe the local behavior of this property. Assume that now $\mathcal{T}$ satisfies Zariski descent (like in the Prop. 9.2.1). We say that $\mathcal{T}$ satisfies the *Nisnevich (resp. Zariski) separation property* if for every Nisnevich (resp. Zariski) morphism of schemes $X' \to X$ the pullback map $\mathcal{T}(X) \to \mathcal{T}(X')$ is conservative. Of couse, if $\mathcal{T}$ satisfies Nisnevich descent it satisfies Zariski separation.

Example 9.4.21. All $\mathcal{T} = \mathcal{SH}$, $\mathcal{T} = \mathcal{SH}_{nc}$ and $\mathcal{T} = \mathcal{SH}_{nc}^{Loc}$ satisfy the Nisnevich separation properties. Indeed, let $f : X' \to X$ be a Nisnevich morphism. Then, as we are working with stable $(\infty, 1)$-categories and exact functors, it is enough to check that the image of an object $E \in \mathcal{T}(X)$ in $\mathcal{T}(X')$ is a zero object, then E is itself a zero object. Let us first deal with the commutative case. Thanks to the existence and description of compact generators for $\mathcal{SH}$ (Prop. 5.3.3) it is enough to consider E the image in motives of a smooth variety over V over X. In this case, the pullback $f^* : \mathcal{SH}(X) \to \mathcal{SH}(X')$ is given by the fiber product $V \mapsto V \times_X X'$. Moreover, as f is smooth, f^* has a left adjoint $f_\sharp$. Moreover, as f is Nisnevich and as Nisnevich morphisms are stable under pullbacks, the projection $V \times_X X' \to V$ is a Nisnevich covering of V in the category of smooth schemes over X. Moreover, as $V \times_X X'$ seen as a smooth scheme over X is canonically equivalent to $f_\sharp \circ f^*(V)$, so that as a result of forcing Nisnevich localization, we have $f_\sharp \circ f^*(V) \simeq V$ in $\mathcal{SH}(X)$. The conclusion is that if $f^*(V)$ is zero, so will V be. For the non-commutative case $\mathcal{SH}_{nc}$ one observe first that by zariski descent it is enough to consider the case where V and X are affine. We can now proceed by the same arguments as in the commutative case using the Prop. 6.4.14-2). Similarly for $\mathcal{SH}_{nc}^{Loc}$.

Proposition 9.4.22. *If $\mathcal{T}^{\otimes}$ satisfies the Nisnevich separation property then the Localization property is local for the Nisnevich topology.*

Proof. Let $\{u_\alpha : X_\alpha \to X\}$ be a Nisnevich covering of X and let $i : Z \to X$ be closed immersion. Let $j : U \to X$ be its open complementary. We set

$$\begin{tikzcd} Z_\alpha := Z \times_X X_\alpha \arrow[r, "i_\alpha"] \arrow[d, "u_\alpha^Z"] & X_\alpha \arrow[d, "u_\alpha"] & U_\alpha := U \times_X X_\alpha \arrow[l, "j_\alpha"'] \arrow[d, "u_\alpha^U"] \\ Z \arrow[r, "i"] & X & U \arrow[l, "j"'] \end{tikzcd} \tag{9.4.75}$$

and show that $\mathcal{T}$ satisfies localization with respect to $i : Z \to X$ if and only if for every α, $\mathcal{T}$ satisfies localization with respect to i_α.

The morphism $\coprod_\alpha X_\alpha \to X$ is a Nisnevich morphism and as $\mathcal{T}$ is assumed to satisfy the Nisnevich separation property, the map $\mathcal{T}(X) \to \mathcal{T}(\coprod_\alpha X_\alpha)$ is conservative. Moreover, as by assumption $\mathcal{T}$ satisfies Nisnevich separation, we have $\mathcal{T}(\coprod_\alpha X_\alpha) \simeq \prod_\alpha \mathcal{T}(X_\alpha)$ so that the previous pullback is the product of the pullbacks $\mathcal{T}(X) \to \mathcal{T}(X_\alpha)$. The same for Z and U. Therefore, we have a commutative diagram

$$\begin{tikzcd}
\prod_\alpha \mathcal{T}(Z_\alpha) & \prod_\alpha \mathcal{T}(X_\alpha) \arrow[l, "\prod_\alpha (i_\alpha)^*"'] \arrow[r, "\prod_\alpha (j_\alpha)^*"'] & \prod_\alpha \mathcal{T}(U_\alpha) \\
\mathcal{T}(Z) \arrow[u, "\prod_\alpha (u_\alpha^Z)^*"] & \mathcal{T}(X) \arrow[l, "i^*"] \arrow[r, "j^*"'] \arrow[u, "\prod_\alpha (u_\alpha)^*"] & \mathcal{T}(U) \arrow[u, "\prod_\alpha (u_\alpha^U)^*"]
\end{tikzcd} \tag{9.4.76}$$

and the key observation is that the diagram

$$\begin{tikzcd}
\prod_\alpha \mathcal{T}(Z_\alpha) \arrow[r, "\prod_\alpha (i_\alpha)_*"] & \prod_\alpha \mathcal{T}(X_\alpha) & \prod_\alpha \mathcal{T}(U_\alpha) \arrow[l, "\prod_\alpha (j_\alpha)_\sharp"'] \\
\mathcal{T}(Z) \arrow[u, "\prod_\alpha (u_\alpha^Z)^*"] \arrow[r, "i_*"] & \mathcal{T}(X) \arrow[u, "\prod_\alpha (u_\alpha)^*"] & \mathcal{T}(U) \arrow[l, "j_\sharp"'] \arrow[u, "\prod_\alpha (u_\alpha^U)^*"]
\end{tikzcd} \tag{9.4.77}$$

also commutes. Indeed, as each u_α is a Nisnevich map, in particular, it is étale, therefore smooth. Smooth base change implies that the right square commutes. It also implies that the left square commutes: as $i^* \circ (u_\alpha)_\sharp \simeq (u_\alpha^Z)_\sharp \circ (i_\alpha)^*$, by adjunction we have $(u_\alpha)^* \circ i_* \simeq (i_\alpha)_* \circ (u_\alpha^Z)^*$.

The conclusion now follows from the fact the maps $\prod_\alpha (u_\alpha^U)^*$, $\prod_\alpha (u_\alpha^Z)^*$ and $\prod_\alpha (u_\alpha)^*$ are conservative.

□

Remark 9.4.23. The previous result is of course true if we replace Nisnevich by Zariski.

Assume now our base schemes are S-schemes for some scheme S. Let us introduce some more terminology. We say that $\mathcal{T}^\otimes$ satisfies the *Weak Localization property* if it satisfies the localization property with respect to every closed immersion $i : Z \to X$ with Z and X smooth over the basis S.

Proposition 9.4.24. *Assume that $\mathcal{T}$ satisfies the Nisnevich separation property. Then, $\mathcal{T}$ satisfies the Weak Localization property if and only if it satisfies localization with respect to every closed immersion $i : Z \to X$ which admits a smooth retract.*

Proof. This follows from the previous result because locally for the Nisnevich topology i (with Z and X smooth) admits a smooth retract (see [40, 4.5.11]). □

To conclude we recall a crucial theorem of Morel-Voevodsky

Theorem 9.4.25. *(Morel-Voevodsky [105, Thm 2.21 pag. 114] and [7, Section 4.5.3]) The motivic stable homotopy theory of schemes $\mathcal{SH}^\otimes$ satisfies the Localization property.*

Although we won't provide here the proof of this theorem it is useful to identify the three key ingredients:

- If $i : Z \hookrightarrow S$ is a closed immersion of schemes, then locally for the Nisnevich topology every smooth scheme over Z is the pullback of a smooth scheme over S [6, Lemma 2.2.11];
- In the commutative world the Nisnevich localization is topological in the sense that Nisnevich coverings form a Grothendieck topology. This is not the case in the noncommutative setting. In particular, in the commutative world, to prove that a map of presheaves is Nisnevich equivalence it is enough to test its values on Henselian local rings [105, Lemma 3.1.11];

- In the commutative world a scheme is smooth iff locally for the Zariski topology it admits an etale map to some affine space [63, 17.11.4]

In the next chapter we will explain our attempts to establish a similar result in the non-commutative world, where unfortunately, none of these phenomenom are known for dg-categories of finite type.

9.4.2.2 Homotopy Invariance

$\mathcal{T}$ is said to verify *homotopy invariance* if for any base scheme X, the pullback π^* along the canonical projection $\pi : \mathbb{A}^1_X \to X$ is fully faithful. In other words, if the natural transformation $\pi_\sharp \circ \pi^* \to Id$ is an equivalence. As $\pi_\sharp$ satisfies the projection formula, this is equivalent to ask for the canonical map $\pi_\sharp(\pi^*(1_X)) \to 1_X$ to be an equivalence.

Example 9.4.26. Both the examples $\mathcal{T} = \mathcal{SH}$ and $\mathcal{T} = \mathcal{SH}_{nc}$ satisfy the homotopy invariance property just described, as an immediate consequence of forcing $\mathbb{A}^1$-invariance.

9.4.2.3 Stability

Let $f : X \to S$ be a smooth separated morphism of finite type between base schemes and suppose it admits a section $s : S \to X$ (which in this case will necessarily be a closed immersion). Following [30] one defines the *Thom transformation* associated to the pair (f, s) as the endofunctor $Th(f, s)$ of $\mathcal{T}(S)$ defined by the composition

$$\mathcal{T}(S) \xrightarrow{s_*} \mathcal{T}(X) \xrightarrow{f_\sharp} \mathcal{T}(S) \tag{9.4.78}$$

As s is closed and $\mathcal{T}$ satisfies II-a), it has a right adjoint

$$\mathcal{T}(S) \xleftarrow[s^!]{} \mathcal{T}(X) \xleftarrow[f^*]{} \mathcal{T}(S) \tag{9.4.79}$$

Definition 9.4.27. *We say that $\mathcal{T}$ (under the hypothesis of this section) satisfies the Stability property if for any pair (f, s) as above, the Thom transformation $Th(f, s)$ is an equivalence of $(\infty, 1)$-categories.*

Remark 9.4.28. Let $(f : X \to S, s)$ and $(g : Y \to S, t)$ be two pairs as above and let $p : Y \to X$ be a morphism compatible with both f and g and s and t. In this case by the universal property of pullbacks we have a diagram

$$\begin{tikzcd} S \arrow[d, "t'"] \arrow[dr, "t"] & & \\ S \times_X Y \arrow[r, "s'"] \arrow[d, "p'"] & Y \arrow[d, "p"] \arrow[dr, "g"] & \\ S \arrow[r, "s"] & X \arrow[r, "f"] & S \end{tikzcd} \tag{9.4.80}$$

and using the assumed base change properties we find a canonical map

$$Th(g, t) \to Th(f, s) \circ Th(p', t') \tag{9.4.81}$$

We now recall a result of [30] re-writting this Stability condition in terms of the Tate motive being $\otimes$-invertible, whenever $\mathcal{T}$ satisfies the homotopy property. We start with a small remark

Remark 9.4.29. Let $\mathcal{T}$ satisfy the hypothesis of this section. Then, if $\mathcal{T}$ satisfies the weak localization property, the Thom transformation associated to a pair (f, s) as above is equivalent to the endofunctor of $\mathcal{T}(S)$ given by tensoring with the object $Th(f, s)(1_S)$. Indeed, as s is a closed embedding admitting a smooth retract, the assumption of weak localization implies localization with respect to s (see Prop.

9.4.24) so that by the Remark 9.4.19-C), s_* satisfies the projection formula. As f is smooth, $f_\sharp$ satisfies the projection formula (assumption I-a) and the result follows from the equivalence $s^* \circ f^* \simeq Id$.

In particular, in the situation of the Remark 9.4.28 the canonical map in 9.4.81 produces a map

$$Th(g,t)(1_S) \to Th(f,s)(1_S) \otimes_S Th(p',t')(1_S) \tag{9.4.82}$$

Notice also that if $j : U \subseteq X$ is the open complementary of S in X, we have an exact sequence in $\mathcal{T}(X)$

$$\begin{tikzcd} j_\sharp \circ j^*(1_X) \arrow[r] \arrow[d] & 1_X \arrow[d] \\ 0 \arrow[r] & s_*(1_S) \end{tikzcd} \tag{9.4.83}$$

and its image through $f_\sharp$ remains exact.

Proposition 9.4.30. *Let $\mathcal{T}$ satisfy the hypothesis of this section. Then, if $\mathcal{T}$ satisfies also the weak localization property and the Zariski separation, the following become equivalent:*

1. *$\mathcal{T}$ satisfies the Stability property;*
2. *The Thom transformation associated to the canonical projection $\mathbb{A}^1_S \to S$ (together with the zero section) is an equivalence;*
3. *The image of the tensor unit $1_S \in \mathcal{T}(S)^\otimes$ along the Thom transformation associated to the projection $\mathbb{A}^1_S \to S$ is a $\otimes$-invertible object in $\mathcal{T}(S)^\otimes$.*

Proof. The equivalence between 1) and 2) is [30, Prop. 2.4.11] and the equivalence with 3) follows from the Remark 9.4.29. □

Let S be a base scheme and let $p : \mathbb{P}^1_S \to S$ be the projection. Following the standard conventions, we define the *Tate object over S* as $1_X(1) := \Omega^2(K)$ where Ω is the inverse of the suspension functor and K is fiber of the map $p_\sharp(p^*(1_S)) \to 1_S$ in $\mathcal{T}(S)$. As p admits a section s_∞ given by the point at infinity, K can also be described as the cofiber of the map $s_\infty : 1_S \to p_\sharp \circ p^*(1_S)$, which we usually denote as $(\mathbb{P}^1_S, \infty)$ (see also the Remark 5.2.2). We denote as $1_X(d)$ the d-iterated tensor product of the Tate object. As in section 5.3 we consider the standard covering of $\mathbb{P}^1_S$ by two copies of the line $\mathbb{A}^1_S$ intersecting at $\mathbb{G}m_S$, all considered pointed at 1, and combined with the Remark 9.4.29, deduce the existence of pushout/pullback squares in $\mathcal{T}(S)$

$$\begin{tikzcd} (\mathbb{G}m_S, 1) \arrow[r] \arrow[d] & (\mathbb{A}^1_S, 1) \arrow[d] \\ (\mathbb{A}^1_S, 1) \arrow[r] \arrow[d] & (\mathbb{P}^1_S, 1) \arrow[d] \\ * \arrow[r] & Th(\pi, s)(1_S) \end{tikzcd} \tag{9.4.84}$$

where $Th(\pi, s)(1_S)$ is the image of 1_S along the Thom transformation associated to the natural projection $\pi : \mathbb{A}^1_S \to S$. In particular, if $\mathcal{T}$ satisfies the homotopy property, the choice of pointing at 1 or 0 becomes equivalent and the map $(\mathbb{A}^1_S, 1) \to *$ becomes an equivalence. So does $(\mathbb{P}^1_S, 1)$ and $(\mathbb{P}^1_S, \infty)$ become equivalent and the induced map $(\mathbb{P}^1_S, \infty) \to Th(\pi, s)(1_S)$ an equivalence. In particular, as the tensor product in $\mathcal{T}(S)$ is exact in each variable, we have the following conclusion of the previous proposition:

Proposition 9.4.31. *Let $\mathcal{T}$ satisfy the general hypothesis of this section. Then if $\mathcal{T}$ satisfies the Homotopy property $\mathcal{T}$ satisfies the Stability property if and only if the Tate object is $\otimes$-invertible.*

9.4.2.4 Purity and Orientations

In this section we recall the discussion in [30, Sections 2.4.b and 2.4.c] concerning the notions of purity and orientation.

Let $f : X \to S$ be a smooth and proper morphism of base schemes. By considering the pullback diagram

$$\begin{tikzcd} X \times_S X \arrow[r, "f'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ X \arrow[r, "f"] & S \end{tikzcd} \tag{9.4.85}$$

together with the diagonal $\delta : X \to X \times_S X$, the pair (f', δ) is in the conditions of the previous section and it makes sense to consider its Thom transformation $\Sigma_f := Th(f', \delta) : \mathcal{T}(X) \to \mathcal{T}(X)$. We will call it the *tangent Thom transformation of f*. As f' is a retract of δ, we have $\delta^* \circ (f')^* \simeq Id$ and as the diagram (9.4.85) is cartesian and $\mathcal{T}$ satisfies I-b), manpulating the adjunctions we find a canonical natural transformation $f_\sharp \to f_* \circ \Sigma_f$.

Definition 9.4.32. *We say that f is $\mathcal{T}$-pure if both the Thom transformation Σ_f and the natural transformation $f_\sharp \to f_* \circ \Sigma_f$ are equivalences. We say that $\mathcal{T}$ satisfies the Purity property for smooth proper morphism if every smooth and proper morphism $X \to S$ is $\mathcal{T}$-pure. We say that $\mathcal{T}$ satisfies the Weak Purity property if for every $n \geq 0$ the canonical projection $\mathbb{P}^n_S \to S$ is $\mathcal{T}$-pure.*

Remark 9.4.33. Assume now that $\mathcal{T}$ satisfies I, II and (Supp). Then by the Theorem 9.4.8 there is a well-defined ∞-functor $(-)_!$, equivalent to $(-)_*$ on proper morphisms. In this case the notion of Purity described in the previous definition can be extended to any smooth separated morphisms of finite type $f : X \to S$ not necessarily proper: using the adjunctions and base change properties we can now produce a natural transformation $f_\sharp \to f_! \circ \Sigma_f$

Let us now recall the following crucial result

Theorem 9.4.34. *(Homotopy Purity - see [30, Thm 2.4.35]) Let $\mathcal{T}$ satisfy the hypothesis in the beginning of this section and assume it satisfies also the weak localization property, Nisnevich separation and the homotopy property. Then for every closed immersion of smooth S-schemes*

$$\begin{tikzcd} Z \arrow[r, hook, "i"] \arrow[dr, "g"'] & X \arrow[d, "p"] \\ & S \end{tikzcd} \tag{9.4.86}$$

the deformation to the normal cone produces a canonical equivalence in $\mathcal{T}(S)$

$$p_\sharp(i_*(1_Z)) \simeq (g \circ q)_\sharp(e_*(1_Z)) =: Th_S(N_Z X) \tag{9.4.87}$$

where $N_Z X$ is the normal bundle of Z in X, $q : N_Z X \to Z$ is the canonical projection and $e : Z \to N_Z X$ is the zero section.

Let $\mathcal{T}$ be as in the theorem and let now $f : X \to S$ be a smooth separated morphism of finite type. In this case, combining the Remark 9.4.29 with the theorem we have

$$\Sigma_f := (f')_\sharp \circ \delta_* \simeq ((f')_\sharp \circ \delta_*(1_X) \otimes_X -) \simeq (Th_X(T_f) \otimes_X -) \tag{9.4.88}$$

where where we define T_f as the normal bundle of X in $X \times_S X$.

To conclude this section we recall the notion of an orientation on $\mathcal{T}$. Assume $\mathcal{T}$ satisfies the weak localization property. Let X be a base scheme and let $E \to X$ be a vector bundle on X. Together with its zero section it has an associated Thom transformation which we will denote here as $Th_X(E) : \mathcal{T}(X) \to \mathcal{T}(X)$. As explained in the Remark 9.4.29, it is given by tensoring with the object $Th_X(E)(1_X)$. By definition an *orientation t* on $\mathcal{T}$ consists of the following data:

- For every base scheme X and every vector bundle $E \to X$ of rank n, an equivalence in $\mathcal{T}(X)$

$$t_{X,E} : Th_X(E)(1_X) \to 1_X(n)[2n] \tag{9.4.89}$$

satisfying the following conditions:

1. compatibiity with isomorphism of vector bundles;
2. coherence under pullbacks of vector bundles on X along morphisms $Y \to X$;
3. for any exact sequence of vector bundles on X, $E' \to E \to E''$, the Thom equivalence $t_{X,E}$ is equivalent (via the map in 9.4.82) to the tensor product $t_{X,E'} \otimes_X t_{X,E''}$

Thanks to the formula (9.4.88) and to the considerations in the Remark 9.4.33 we conclude that if $\mathcal{T}$ satisfies I), II), (Supp), weak localization and comes equipped with an orientation, for every morphism $f : X \to S$ smooth separated of finite type of relative dimension d we will have $\Sigma_f \simeq 1_X(d)[2d] \otimes_X -$. In particular, if f is $\mathcal{T}$-pure we have

$$f_\sharp \simeq f_!(1_X(d)[2d] \otimes_X -) \tag{9.4.90}$$

Remark 9.4.35. Let $\mathcal{T} = \mathcal{SH}^\otimes$. Given a base scheme X there is a notion of *orientation* on a commutative algebra object $A \in CAlg(\mathcal{SH}(X))$ (see [30, Def. 12.2.2]). Thanks to [149, Thm 4.3] orientations in A are in one-to-one correspondence with morphisms of commutative algebras $MGL \to A$ where $MGL \in \mathcal{SH}(X)$ is the algebraic cobordism spectrum of [150]. As explained in [30, Example 12.2.3] if we have a collection of commutative algebra objects $\{A_X\}_{X \in BSch}$ compatible under base-change, the data of a compatible system of orientations for the algebras is in one-to-one correspondence to the data of a compatible system of orientations of the system of symmetric monoidal $(\infty, 1)$-categories $X \mapsto Mod_{A_X}(\mathcal{SH}(X))^\otimes$ in the sense of the preceeding discussion.

More generally, given $\mathcal{T}$ together with monoidal colimit preserving natural transformation $\psi : \mathcal{SH}^\otimes \to \mathcal{T}$ with right adjoint λ, if $\mathcal{T}$ satisfies the weak localization property then one can check that the data of a orientation on $\mathcal{T}$ is equivalent to the data of an orientation for each algebra object $\lambda(1_{\mathcal{T}(X)})$.

9.4.3 The formalism of six operations for $\mathcal{SH}^\otimes$

We now come to the main result of this chapter:

Theorem 9.4.36. *The ∞-functor $\mathcal{SH}^\otimes$ satisfies the assumptions (I), (II) , (Supp) so that the methods of the previous section can be applied to construct an enhanced map*

$$EO(\mathcal{SH}^\otimes) : \delta^*_{2,\{2\}} Fun(\Delta[1], BSch)^{Cart}_{F,All} \longrightarrow Mod(\mathcal{P}r^L_{Stb}) \tag{9.4.91}$$

satisfying all the properties (i)-(vi). Moreover, every smooth separated morphism of finite type is $\mathcal{SH}$-pure and together with the orientation we have (vii).

Both the assumptions (I) and (II)-a) were already confirmed (see the Example 9.4.6). The main results of Ayoub and Cisinski- Déglise can now be stated as follows:

Theorem 9.4.37. *(Ayoub-Voevodsky and Cisinski- Déglise) Let $\mathcal{T} : BSch^{op} \to CAlg(\mathcal{P}r^L_{Stb})$ satisfy the assumptions (I) and (II)-a). Then,*

i) if $\mathcal{T}$ satisfies the localization property, homotopy invariance and stability, then it satisfies weak purity.

ii) if $\mathcal{T}$ satisfies the localization property and weak purity then is satisfies the assumptions II-b), II-c), (Supp). In particular the Theorem 9.4.8 can be applied to produce a formalism of six operations. Moreover, it satisfies (Purity) for any smooth separated morphism of finite type in the sense explained in the Remark 9.4.33.

Proof. As all the $(\infty,1)$-categories involved are stable, we are reduced to check the statement at the level of the associated homotopy categories. For i) this is [6, 1.7.9] (see also [30, 2.4.2.8]). For ii) this is [30, 2.3.13 and 2.4.23] for II-b), II-c) and (Supp) and [30, 2.4.26-(3)] for (Purity). □

Proof of the Theorem 9.4.36: As $\mathcal{T}^{\otimes} = \mathcal{SH}^{\otimes}$ satisfies the basic assumptions I and II-a) (as explained in the Remark 9.4.6) and as it also satisfies the Nisnevich separation property (see 9.4.21), the fundamental results of Ayoub and Cisinski- Déglise recorded in the previous theorem allow us to reduce the first claim to the verification of the Localization property, Homotopy invariance and Stability. The assumptions of homotopy invariance and stability are clear after the Prop. 9.4.31. The localization property was already recalled in Thm 9.4.25.

□

To conclude this section we contemplate the extension of this formalism to module-objects. Let $\mathcal{T} : BSch^{op} \to CAlg(\mathcal{P}r^L)$ be an ∞-functor verifying the conditions I), II) and (Supp) so that by the Theorem 9.4.8 it can be enhanced to a map of simplicial sets

$$EO(\mathcal{T}) : \delta^*_{2,\{2\}} Fun(\Delta[1], BSch)^{Cart}_{F,All} \longrightarrow Mod(\mathcal{P}r^L_{Stb}) \tag{9.4.92}$$

Assume that for each base scheme X we are given a commutative algebra-object A_X in $\mathcal{T}(X)^{\otimes}$ and that his data is compatible with pullbacks. Then, by following the methods previously used in this chapter, we can present the system of stable presentable symmetric monoidal $(\infty,1)$-categories $Mod_{A_X}(\mathcal{T}(X))^{\otimes}$ as an ∞-functor $Mod_A(\mathcal{T})^{\otimes} : BSch^{op} \to CAlg(\mathcal{P}r^L)$, together with the data of a natural transformation of diagrams $\mathcal{T}^{\otimes} \to Mod_A(\mathcal{T})^{\otimes}$. Moreover, using the fact the forgetful functors $Mod_{A_X}(\mathcal{T}(X)) \to \mathcal{T}(X)$ are conservative, we can deduce that $Mod_A(\mathcal{T})$ continues to verify I), II) and (Supp) so that the Theorem 9.4.8 provides an enhanced $(\infty,1)$-functor

$$EO(Mod_A(\mathcal{T})) : \delta^*_{2,\{2\}} Fun(\Delta[1], BSch)^{Cart}_{F,All} \longrightarrow Mod(\mathcal{P}r^L_{Stb}) \tag{9.4.93}$$

codifying the formalism of six operations for modules.

Remark 9.4.38. Using the projection formulas and the fact the forgetful functors are conservative we can easily check that the transformation $\mathcal{T} \to Mod_A(\mathcal{T})$ is compatible with operations $()^*$, $()_*$ for any morphism of base schemes and $()_\sharp$ for smooth morphisms. In particular it is compatible with the Thom transformations. We leave this as a small exercise to the reader.

Remark 9.4.39. As the family of objects KH representing homotopy invariant algebraic K-theory is stable under base-change (see [29, Prop. 3.8]) this discussion applies to its theory of modules. One of the important features of KH is the Bott periodicity phenomenom that gives us an equivalence in $\mathcal{SH}$

$$KH \simeq \mathbb{R}\underline{Hom}((\mathbb{P}^1,\infty), KH) \tag{9.4.94}$$

compatible with base change. See [29]. As $(\mathbb{P}^1,\infty)$ is $\otimes$-invertible in $\mathcal{SH}^{\otimes}$, this is equivalent to say that $KH \simeq (\mathbb{P}^1,\infty) \otimes KH$. By the projection formula for the adjunction $\mathcal{SH} \to Mod_{KH}(\mathcal{SH})$ and the fact the forgetful functor is conservative, this is equivalent to say that the image of $(\mathbb{P}^1,\infty)$ in $Mod_K H(\mathcal{SH})$ is a unit for the monoidal structure. As KH is also known to be orientable in the sense of the Remark 9.4.35 (see [30, Remark 13.2.2]) the system $X \mapsto Mod_{KH_X}(\mathcal{SH}(X))$ has a canonical orientation, which, as the Tate motive is now trivial because of the preceeding discussion, makes all Thom spaces trivial. In particular, at the level of KH-modules, $()_\sharp$ and $()_!$ are equivalent.

This concludes this chapter.

Chapter **10**

Towards a formalism of six operations in the Noncommutative World and Fully-faithfulness over a general base

In this final chapter we explain our attempts to establish a formalism of six operations in the setting of non-commutative motives. More precisely, we fix a base field k of characteristic zero, set $BSch$ as the category of smooth quasi-projective schemes over k and explore the possibility to enhance the $(\infty,1)$-functor $\mathcal{SH}_{nc}^{\otimes} : BSch \to CAlg(\mathcal{P}r_{Stb}^{L})$ with the extra operation $(-)_!$ and the standard compatibilities. Thanks to Hironaka's theorem of resolution of singularities the output of Nagata's compactification can be made smooth and we can apply the machinery described in the last chapter. After the discussion in the previous chapter, as $\mathcal{SH}_{nc}^{\otimes}$ satisfies the basic assumptions I and II-a) (see the Remark 9.4.6), the Homotopy property (because of $\mathbb{A}^1$-invariance) and the Stability property (because the non-commutative image of $(\mathbb{P}^1,\infty)$ is the tensor unit - Prop. 6.4.20), if the localization property (Section 9.4.2.1) is verified, the existence of $(-)_!$ and the other standard properties will then follow automatically from the theorems 9.4.8 and 9.4.37. Let us briefly review what it means to prove this property. Following the discussion in Chapter 9 (more precisely, the Remarks 9.2.14, 9.3.7 and 9.4.23 and the Example 9.4.21), we know that this localization property is local for the Zariski topology and we are reduced to consider closed immersions between affine base schemes. Let $S = Spec(A)$ be a smooth quasi-projective k-scheme and let $i : Z = Spec(A/I) \hookrightarrow S$ be a closed immersion corresponding to an ideal $I \subseteq A$ and let $j : U \hookrightarrow S$ denote its open complementar. One first observation is that as our base schemes are smooth, i is then a regular embedding (see [63, 17.12.1]) and I can be assumed to be generated by a regular sequence. A second important observation is that as $\mathcal{SH}_{nc}$ satisfies the Nisnevich separation property (Example 9.4.21) we can assume that i admits a smooth retract (Prop. 9.4.24). According to the Prop. 9.4.20-(3) and to the description of the compact generators in non-commutative motives, we are reduced to showing that

A) $i_*^{nc} : \mathcal{SH}_{nc}(Z) \to \mathcal{SH}_{nc}(S)$ is conservative. This is immediately seen to be true because of the existence of a retract and because of the functoriality of $(-)_*$;

B) the commutative diagram

$$\begin{tikzcd} j_\sharp^{nc} \circ j_{nc}^*(1_S^{nc}) \arrow[rr] \arrow[d] & & 1_S^{nc} \arrow[d] \\ 0 \arrow[r] & i_*^{nc} \circ i_{nc}^*(1_S^{nc}) \simeq i_*^{nc}(1_Z^{nc}) & \end{tikzcd} \tag{10.0.1}$$

is a cofiber/fiber sequence in $\mathcal{SH}_{nc}(S)$;

C) For any A-dg-category of finite type T, the natural map in $\mathcal{SH}_{nc}(A)$

$$i_*^{nc}(1_Z^{nc}) \otimes T \to i_*^{nc}(i_{nc}^*(T)) \tag{10.0.2}$$

is an equivalence. Here T denotes the non-commutative motive obtained by localizing the presheaf co-represented by T.

Notation 10.0.40. All the tensor products used in this chapter are derived, unless otherwise specified.

Unfortunately, we were not able to accomplish the proof of this conditions for $\mathcal{SH}_{nc}$. We found two main difficulties. The first major problem concerns the use of dg-categories of finite type as the basic models for noncommutative spaces. These are not known to satisfy any of the properties used in the proof of the localization property in the commutative world (see the list following the Thm 9.4.25).

The goal of this chapter is to explain our attempts to solve this question and explore some of its consequences. The first main difficulty that we found concerns the proof of a very basic statement in the theory of dg-categories, which we could confirm in the case of smooth and proper dg-categories and also for dg-categories of commutative origin. The general case of a dg-category of finite type remains unknown.

Conjecture 10.0.41. *Let k be a field of characteristic zero and let $f : X \to S = Spec(A)$ be a proper morphism between smooth k-schemes. Thanks to the results of [141] (see also the formula 6.3.1) we know that for any idempotent complete A-dg-category T of finite type, there is a natural inclusion of A-dg-categories of A-dg-modules*

$$(L_{pe,A}\widehat{(X) \otimes_A} T)_c \subseteq (L_{pe,A}\widehat{(X) \otimes_A} T)_{pspe} \tag{10.0.3}$$

We claim that this inclusion is an isomorphism.

Remark 10.0.42. As any dg-category of finite type T over A is of the form $\widehat{U}_c$ for some A-dg-algebra U which is homotopically finitely presented, the conjecture can also be formulated as follows:

Let k be a field of characteristic zero and let A be a smooth k-algebra. For any proper commutative A-scheme X smooth over k and any A-dg-algebra U homotopically finitely presented, the forgeful functor

$$_{L_{pe,A}(X)}BiMod_U(A) \to LMod_U(A) \tag{10.0.4}$$

detects compact objects.

Remark 10.0.43. Another useful equivalent formulation is the following: As in the Remark 9.2.13, using the existence of a right adjoint $\tilde{\phi}_f$ to the forgetful functor[1]

$$F_f^{idem} : \mathcal{D}g^{idem}(X) \to \mathcal{D}g^{idem}(A) \tag{10.0.5}$$

satisfying $F_f^{idem}(\tilde{\phi}_f(T)) \simeq (L_{pe,A}\widehat{(X) \otimes_A} T)_{pspe}$, together with the fact F_f^{idem} is conservative and satisfies the projection formula with respect to base-change f^*, we can formulate the conjecture by saying that the two functors f^* and $\tilde{\phi}_f$ agree on the class of dg-categories of finite type.

As base-change preserves dg-categories of finite type it is obvious from this description that the conjecture is stable under compositions.

Let us now comment the contents of the conjecture itself. To start with, it is clearly not true if we allow it for all A-dg-categories T not necessarily of finite type. Indeed, as X is quasi-compact and quasi-separated $L_{pe,A}(X)$ is of the form $\widehat{B}_c^A$ for some proper A-dg-algebra B. We observe that the conjecture cannot be true for B itself unless f is also a smooth morphism. Indeed, if the inclusion

[1] here $\mathcal{D}g^{idem}(X)$ is defined as the limit of $\mathcal{D}g^{idem}(R)$ for all R affine over X.

$$(\widehat{B \otimes_A B})_c \subseteq (\widehat{B \otimes_A B})_{pspe} \tag{10.0.6}$$

were an equivalence this would imply, in particular, that, as B is compact as an B dg-module (over A), it would also be compact as an $B \otimes_A B$ A-dg-module and therefore (as discussed in section 6.3.2) it would be smooth over A, which is the case if and only if f is smooth. This follows from the results of [141] (see our survey in section 6.3.2).

To prove the weak localization property we will need to know the conjecture in the case of a regular closed immersion $i : Z = Spec(A/I) \hookrightarrow S = Spec(A)$. We can confirm it in the following cases:

Proposition 10.0.44. *Let k be a field of characteristic zero and let $i : Z = Spec(A/I) \hookrightarrow S = Spec(A)$ be a closed immersion of smooth k-schemes. Then the conjecture is true for the following dg-categories:*

(i) $T = A$ as an A-dg-category;

(ii) T smooth and proper over A;

(iii) A-dg-categories coming from the commutative world in the form of perfect complexes.

Proof. (i) As explained in the Remark 10.0.42 the conjecture is equivalent to say that the forgetful functor

$$i_* : \widehat{A/I} \to \widehat{A} \tag{10.0.7}$$

detects perfect complexes. As we are in the smooth case, perfect complexes correspond to bounded coherent complexes (see for instance [137, Prop. 2.2.12]). It is obvious that the forgetful functor preserves the boundness condition (as it preserves the cohomology groups). The result now follows because an A/I-module is finitely generated if and only if it is finitely generated as an A-module.

(ii) The case of a smooth and proper A-dg-category T follows easly from (i). As such a T is dualizable we have

$$\mathbb{R}\underline{Hom}_A(\widehat{A/I}_c, T) \simeq \mathbb{R}\underline{Hom}_A(\widehat{A/I}_c \otimes_A T^\vee, A) \simeq \mathbb{R}\underline{Hom}_A(T^\vee, \mathbb{R}\underline{Hom}_A(\widehat{A/I}_c, A)) \tag{10.0.8}$$

which by (i) is equivalent to $\mathbb{R}\underline{Hom}_A(T^\vee, \widehat{A/I}_c)$ an again by duality to

$$\mathbb{R}\underline{Hom}_A(A, T \otimes_A \widehat{A/I}_c) \simeq T \otimes_A \widehat{A/I}_c \tag{10.0.9}$$

(iii) The last case now follows by the exact same arguments of (i): to say that T is of geometric origin is the same as saying that there exists a smooth map $f : V \to Spec(A)$ with $T \simeq L_{pe,A}(V)$. As in (i) the conjecture is equivalent to say that the forgetful functor

$$A/I \otimes_A \widehat{L_{pe,A}(V)}^A \to \widehat{L_{pe,A}(V)}^A \tag{10.0.10}$$

detects perfect complexes. Notice that this functor corresponds to the direct image along the closed immersion $Z \times_S V \hookrightarrow V$ obtained by the pullback of f along i. As the pullback of a smooth map is smooth this is again a closed immersion of smooth k-schemes and therefore locally it is given by a regular sequence. By Zariski descent we can now suppose that V is affine. The same arguments of (i) tell us that as V-dg-categories the direct image

$$\widehat{A/I \otimes_A V}^V \to \widehat{V}^V \tag{10.0.11}$$

detects perfect complexes. The conclusion now follows because of the Remark 9.2.3: the forgetful functor F_f from V-dg-categories to A-dg-categories is compatible with the theories of dg-modules, so that (10.0.11) is the image of (10.0.10). The notion of compact object does not depend on the enrichement.

□

Unfortunately, knowing these particular examples is not enough to deduce the localization property: one is required to know the conjecture for all dg-categories of finite type.

The second difficulty concerns the nature of the noncommutative Nisnevich localization: contrary to the geometric situation, non-ncommutative Nisnevich squares do not form a Grothendieck topology. In particular, we could not identify the corresponding analogue of points.

Open Problem 10.0.45. Let k be a field of characteristic zero and let $i : Z = Spec(A/I) \hookrightarrow S = Spec(A)$ be a closed immersion of smooth k-schemes. Does the induced direct image functor $i_* : \mathcal{P}^{big}(\mathcal{N}cS(A/I)) \to \mathcal{P}^{big}(\mathcal{N}cS(A))$ preserves (noncommutative) Nisnevich local equivalences?

The commutative analogue of this problem is easily solved by knowing the description of points in the geometric Nisnevich topology as henselian local rings (see [105, Prop. 1.27]).

Remark 10.0.46. We can use the Conjecture 10.0.41 to formulate this problem in a more explicit form. If we denote $j_{A/I} : \mathcal{N}cS(A/I) \hookrightarrow \mathcal{P}(\mathcal{N}cS(A/I))$ the Yoneda map, our goal is to describe the i_* image of a square

$$\begin{tikzcd} j_{A/I}(\mathcal{W}) \arrow[r] \arrow[d] & j_{A/I}(\mathcal{V}) \arrow[d] \\ j_{A/I}(\mathcal{U}) \arrow[r] & j_{A/I}(\mathcal{X}) \end{tikzcd} \tag{10.0.12}$$

associated to a Nisnevich square of A/I-dg-categories

$$\begin{tikzcd} T_{\mathcal{X}} \arrow[r] \arrow[d] & T_{\mathcal{U}} \arrow[d] \\ T_{\mathcal{V}} \arrow[r] & T_{\mathcal{W}} \end{tikzcd} \tag{10.0.13}$$

By defintion, we have

$$i_*(j_{A/I}(\mathcal{X})) \simeq Map_{\mathcal{N}cS(A/I)}(i^*(-), \mathcal{X}) \simeq Map_{\mathcal{D}g^{idem}(A/I)}(T_{\mathcal{X}}, i^*(-)) \tag{10.0.14}$$

and if the conjecture is true, as formulated in the Remark 10.0.43 we have $i^*(-) \simeq \tilde{\phi}_i$ so that using the adjunction $(F_i^{idem}, \tilde{\phi}_i)$ the mapping space in the previous formula is equivalent to

$$Map_{\mathcal{D}g^{idem}(A)}(F_i^{idem}(T_{\mathcal{X}}), -)$$

so that solving the problem is equivalent to understanding how far the square of A-dg-categories

$$\begin{tikzcd} F_i^{idem}(T_{\mathcal{X}}) \arrow[r] \arrow[d] & F_i^{idem}(T_{\mathcal{U}}) \arrow[d] \\ F_i^{idem}(T_{\mathcal{V}}) \arrow[r] & F_i^{idem}(T_{\mathcal{W}}) \end{tikzcd} \tag{10.0.15}$$

is from being Nisnevich. After the discussion in the proof of the Prop 9.2.11, the only property that is not preserved by F^{idem} in this case is the condition of being of finite type.

The last difficulty, less serious, concerns a tecnhnical issue with the definition of non-commutative motives. Essentially, the problem is that by localizing with respect to our notion of noncommutative Nisnevich squares we don't have sufficent control on what happens to exact sequences of dg-categories when we regard them in motives. Of course, this is exactly the kind of problem we can solve simply by forcing the extra localization $\mathcal{SH}_{nc}(S) \to \mathcal{SH}_{nc}^{Loc}(S)$ described in Chapter 8.

At the present moment our efforts can be summarized as follows:

Proposition 10.0.47. *Let k be a field of characteristic zero. Then:*

- *Assume that the Conjecture 10.0.41 holds for a closed immersion $f = i : Z = Spec(A/I) \hookrightarrow S = Spec(A)$ with both S and Z smooth over k. Then the ∞-functor $\mathcal{SH}_{nc}^{Loc,\otimes}$ satisfies the property $B)$ with respect to i.*
- *Assume that the Conjecture 10.0.41 holds for a closed immersion $f = i : Z = Spec(A/I) \hookrightarrow S = Spec(A)$ with both S and Z smooth over k and that the Open Problem 10.0.45 has a positive answer. Then the ∞-functor $\mathcal{SH}_{nc}^{Loc,\otimes}$ satisfies the property $C)$ with respect to i. In particular, by (i) it also satisfies $B)$ and as it satisfies the homotopy property and stability, by the Theorem 9.4.37 it verifies the full formalism of six operations over the category of smooth quasi-projective schemes over a field of characteristic zero.*

We will explain the proof of this proposition in the section 10.1 below.

As a consequence of the conjecture, and independently of the solution to the open problem. More importantly, we establish the following fully-faithfulness extension of the Corollary 8.0.16 to any smooth base S:

Proposition 10.0.48. *Let k be field of charecteristic zero and let S be a regular scheme over k. Assume the Conjecture 10.0.41. Then the canonical functor from motives to noncommutative motives*

$$\mathcal{L}_{KH_S} : Mod_{KH_S}(\mathcal{SH}(S)) \to \mathcal{SH}_{nc}(S) \to \mathcal{SH}_{nc}^{Loc}(S) \tag{10.0.16}$$

is fully-faithful.

The proof of this Proposition will be explained in section 10.2.

10.1 Towards the six operations

Our goal in this section is to explain the proof of the Proposition 10.0.47. Most of the contents in this section are independent of the Conjecture 10.0.41 and of the Open Problem 10.0.45. These will only be used, respectively, in the proofs of Prop. 10.1.8 and 10.1.10, respectively in the last paragraph of this section.

We start by exploring the conditon $B)$. Our first observation is that as the Localization property holds in the commutative world and as the natural transformation $\mathcal{L}^{\otimes} : \mathcal{SH}^{\otimes} \to \mathcal{SH}_{nc}^{Loc,\otimes}$ is monoidal, exact and commutes with $f_\sharp$ for f smooth (Prop 9.2.11), the square

$$\begin{tikzcd} L_S(j_\sharp \circ j^*(1_S)) \simeq j_\sharp^{nc} \circ j_{nc}^*(1_S^{nc}) \arrow[r] \arrow[d] & L_S(1_S) \simeq 1_S^{nc} \arrow[d] \\ 0 \arrow[r] & L_S(i_*(1_Z)) \end{tikzcd} \tag{10.1.1}$$

is a cofiber/fiber sequence in $\mathcal{SH}_{nc}(S)$. In particular, to show $B)$ we are reduced to showing that the natural map

$$\psi : L_S(i_*(1_Z)) \to i_*^{nc}(1_Z^{nc}) \tag{10.1.2}$$

is an equivalence. To understand this map we investigate first both its source and target. Let us start with the source. For that purpose we consider the exact sequence of A-dg-categories

$$L_{pe,A,Z}(S) \to L_{pe,A}(S) \to L_{pe,A}(U) \tag{10.1.3}$$

associated to the closed immersion $i : Z \hookrightarrow S$ (see the discussion in the section 6.4.1). Unfortunately we do not have any control over what happens to this sequence when seen in $\mathcal{SH}_{nc}(A)$ but we can correct this problem by working in the further localization $\mathcal{SH}_{nc}^{Loc}(A)$, where it becomes a cofiber/fiber sequence of motives. More precisely, its image along the composition map

$$\mathcal{D}g^{idem}(A)^{op} \xrightarrow{\chi_A^{(-)}} \mathcal{P}^{big}(\mathcal{N}c\mathcal{S}(A)) \xrightarrow{l^{nc}_{\mathbb{A}^1,Nis}} \mathcal{SH}_{nc}(A) \xrightarrow{l_{Loc}} \mathcal{SH}_{nc}^{Loc}(A) \tag{10.1.4}$$

given by

$$l_{Loc} l^{nc}_{\mathbb{A}^1,Nis} \chi_A^{L_{pe,A}(U)} \to l_{Loc} l^{nc}_{\mathbb{A}^1,Nis} \chi_A^{L_{pe,A}(S)} \to l_{Loc} l^{nc}_{\mathbb{A}^1,Nis} \chi_A^{L_{pe,A,Z}(S)} \tag{10.1.5}$$

is a cofiber/fiber sequence (consult our notations in Chapter 8). Moreover, as the $\sharp$-image along $j : U \hookrightarrow S$ commutes with the localizations (Remark 9.4.23), the non-commutative motive $l_{Loc} l^{nc}_{\mathbb{A}^1,Nis} \chi_A^{L_{pe,A}(U)}$ is naturally equivalent to $j^{nc}_{\sharp}(1^{nc}_U)$ and the last cofiber/fiber sequence is equivalent to the one in the diagram (10.1.1) and the unversal property of cofibers gives us a natural equivalence

$$L_S(i_*(1_Z)) \simeq l_{Loc} l^{nc}_{\mathbb{A}^1,Nis} \chi_A^{L_{pe,A,Z}(S)} \tag{10.1.6}$$

By definition, the right hand side of this equivalence is the localization of the presheaf of spaces $\chi_A^{L_{pe,A,Z}(S)} : \mathcal{D}g^{idem}(A)^{ft} \to \mathcal{S}$ given by the formula

$$T \mapsto Map_{\mathcal{D}g^{idem}(A)}(L_{pe,A,Z}(S), T) \tag{10.1.7}$$

It is already Nisnevich local (as it commutes with limits) and as explained for instance by the arguments in the proof of [140, Prop. 3.9]), the A-dg-category $L_{pe,A,Z}(S)$ has a compact generator, namely, the object A/I seen as an A-dg-module. In particular this implies that $L_{pe,A,Z}(S)$ is Morita equivalent to the A-dg-category $\widehat{End_A(A/I)}_c^A$ of perfect A-dg-modules over the A-dg-algebra $End_A(A/I)$ of endomorphisms of A/I in the derived category of A-dg-modules. As in this case we have a model structure this agrees with the classifying object of endomorphisms discussed in Section 3.4.

Proposition 10.1.1. *Let $i : Spec(A/I) \hookrightarrow Spec(A)$ be a closed immersion of smooth k-schemes. Then the natural map of A-dg-categories*

$$\widehat{A/I}_c^A \to \widehat{End_A(A/I)}_c^A \tag{10.1.8}$$

induced by the map of A-dg-algebras corresponding to the multiplication map

$$A/I \to End_A(A/I) \tag{10.1.9}$$

is an $\mathbb{A}^1_A$-homotopy equivalence of A-dg-categories in the naive sense.

Proof. This follows because $End_A(A/I)$ is equivalent as an A-dg-algebra to $End_{\widehat{A}}(A/I)$ where here $\widehat{A}$ denotes the formal completion of the ring A with respect to the ideal I, together with the fact that this formal completion is in this case isomorphic to $A/I[[I/I^2]]$. This implies that $End_A(A/I)$ is equivalent as an A-dg-algebra to the underlying A-dg-algebra of the free A/I-cdga on the A/I-module $(I/I^2)^\vee[-1]$. In this case $End_A(A/I)$ has a natural grading and the $A[t]$-action defining this grading is the required co-homotopy. ∎

An immediate consequence of this result is that $\chi_A^{L_{pe,A,Z}(S)}$ is homotopy equivalent as a presheaf of spaces to the presheaf $\chi_A^{(\widehat{A/I}_c^A)}$, which is also Nisnevich local. In particular

Corollary 10.1.2. *The non-commutative motive $L_S(i_*(1_Z)) \in \mathcal{SH}_{nc}^{Loc}(A)$ is naturally equivalent to the non-commutative motive obtained by localizing the presheaf of spaces defined by the formula*

$$T \mapsto Map_{\mathcal{D}g^{idem}(A)}(\widehat{A/I}_c^A, -) \tag{10.1.10}$$

We now investigate the source of the map ψ. By adjunction, for any dg-category T of finite type over A we have

$$Map^{Sp}_{\mathcal{SH}_{nc}^{Loc}(A)}(T, i_*^{nc}(1_Z)) \simeq Map^{Sp}_{\mathcal{SH}_{nc}^{Loc}(A/I)}(i^*(T), 1_Z) \simeq KH_Z(i_{nc}^*(T)) \tag{10.1.11}$$

where the last equivalence follows from the Corollary 7.0.35.

Notation 10.1.3. Let R be a commutative ring. Throughout this section, and as in the previous formula, we will adopt the notation KH_R for the $\mathbb{A}^1_R$-localization of non-connective K-theory K_R^S.

Proposition 10.1.4. *Let T be a dg-category of finite type over A. There is a natural equivalence*

$$KH_Z(i_{nc}^*(T)) \simeq KH_A(T \otimes_A A/I) \tag{10.1.12}$$

where $T \otimes_A A/I$ denotes the tensor product of T with the A-dg-category $\widehat{A/I}_c^A$ in $\mathcal{D}g^{idem}(A)$.

Proof. We ask the reader to remind the notations in Section 7.1.2. We start by comparing the connective K-theories of both sides. the formula for the Waldhausen's construction tells us that

$$K_Z^C(i_{nc}^*(T)) := \Omega\, colim_{[n]\in\Delta^{op}} Map_{\mathcal{D}g^{idem}(A/I)}((\widehat{[n-1]_{A/I}})_c^{A/I}, i_{nc}^*(T)) \tag{10.1.13}$$

and as we have a canonical equivalence

$$(\widehat{[n-1]_{A/I}})_c^{A/I} \simeq i_{nc}^*((\widehat{[n-1]_A})_c^A) \tag{10.1.14}$$

the formula becomes

$$\simeq \Omega\, colim_{[n]\in\Delta^{op}} Map_{\mathcal{D}g^{idem}(A/I)}(i_{nc}^*((\widehat{[n-1]_A})_c^A), i_{nc}^*(T)) \tag{10.1.15}$$

and by adjunction

$$\simeq \Omega\, colim_{[n]\in\Delta^{op}} Map_{\mathcal{D}g^{idem}(A)}((\widehat{[n-1]_A})_c^A, F_i^{idem}(i_{nc}^*(T)) \tag{10.1.16}$$

which, by means of the projection formula for the forgetful functor, becomes equivalent to $K_A^C(T \otimes_A A/I)$ (see our notations in section 9.1). This equivalence is valid not only for the K-theory spaces but also for their associated connective spectra. This can be seen using the explicit description of this spectra by iterating the Waldhausen S- construction. Finally, the result follows from the explicit formulas for both the B-construction of Thomason and the $\mathbb{A}^1$-localization - both are constructed by taking colimits/limits computed objectwise in spectra (see our discussion in Chapter 7)- together with the fact that both $(\mathbb{G}_m)_{A/I}$ and $(\mathbb{A}^1)_{A/I}$ come from $(\mathbb{G}_m)_A$ and $(\mathbb{A}^1)_A$ via base change, and finally, because base-change is monoidal and the forgetful functor satisfies the projection formula.

□

The following result extends the Theorem 7.0.32 and its Corollary 7.0.33

Proposition 10.1.5. *Let R be a commutative ring and let T be an object in $\mathcal{D}g^{idem}(R)$ having a compact generator. Then the presheaf $KH_R(T \otimes_R -)$ defines an object in $\mathcal{SH}_{nc}(R)$, naturally equivalent to the non-commutative motive obtained by localizing the presheaf of spaces*

$$Map_{\mathcal{D}g^{idem}(R)}(R, T \otimes_R -) : \mathcal{D}g^{idem}(R)^{ft} \to \mathcal{S} \tag{10.1.17}$$

In particular, when $T = R$ we recover the Theorem 7.0.32 and its Corollary 7.0.33.

Proof. The proof of this result is an exercise of re-writting the main steps of Chapter 7 carrying along the operation $T \otimes_R -$. We perform the crutial steps.

To start with we observe that $KH_R(T \otimes_R -)$ indeed defines an object in $\mathcal{SH}_{nc}(R)$. Indeed, as the tensor product of an exact sequence in $\mathcal{D}g^{idem}(R)$ with a dg-category T remains an exact sequence (see the arguments in the proof of Prop.6.4.14) $KH_R(T \otimes_R -)$ sends exact sequences to exact sequences of spectra. Moreover, by the same arguments as in the Prop 7.1.4, it is Nisnevich local. By construction it is also $\mathbb{A}^1$-local.

We now follow the same steps used in Chapter 7 to prove the Theorem 7.0.32. For a dg-category $T \in \mathcal{D}g^{idem}(R)$ we consider the simplicial object $Seq^T : N(\Delta^{op}) \to \mathcal{P}^{big}(\mathcal{N}cS(R))$ given by the formula

$$[n] \mapsto Map_{\mathcal{D}g^{idem}(R)}((\widehat{([n-1]_R)}_c^R, T \otimes_R -) \tag{10.1.18}$$

which, contrary to the situation in chapter 7 is not induced by a simplicial object in $\mathcal{N}cS(R)$. However, it remains Nisnevich local because of our assumption that T has a compact generato. This follows from the Prop. 6.4.14 together with the Remark 6.4.15. Moreover, the Waldhausen's construction gives us

$$K^C_{space}(T \otimes_R -) := \Omega\, colim_{[n]\in\Delta^{op}} Map_{\mathcal{D}g^{idem}(R)}((\widehat{([n-1]_R)}_c^R, T \otimes_R -) \tag{10.1.19}$$

This construction can also be iterated to construct a functor with values in connective spectra $K^C_{spectra}(T \otimes_R -)$ with $\Omega^\infty K^C_{spectra}(T \otimes_R -) \simeq K^C_{space}(T \otimes_R -)$. We now ask the reader to remind the commutative diagram (7.4.10) and the result in the Prop. 7.4.2. With these results in mind we start by observing that, as the B-construction of Chapter 7 is determined object-by-object, the Nisnevich localization of $K^C_{spectra}(T \otimes_R -)$ is the non-connective K-theory $K^S(T \otimes_R -)$. By the same reason its further $\mathbb{A}^1$-localization is the spectral presheaf $KH_R(T \otimes_R -)$. The corresponding diagram-chasing of (7.4.10) gives

$$\begin{tikzcd}
\Omega\, colim_{[n]\in\Delta^{op}} Seq^T_n \arrow[d, "l^{nc}_{0,\mathbb{A}^1}"'] & K^S(T \otimes_R -) \arrow[l, "\Omega^\infty_{Nis}"'] \arrow[d, "l^{nc}_{\mathbb{A}^1}"] \\
l^{nc}_{0,\mathbb{A}^1}(\Omega\, colim_{[n]\in\Delta^{op}} Seq^T_n) & KH_R(T \otimes_R -) \arrow[l, "\Omega^\infty_{Nis,\mathbb{A}^1}"', "\sim"]
\end{tikzcd} \tag{10.1.20}$$

and as in the Lemma 7.4.4 we find

$$l^{nc}_{0,\mathbb{A}^1}(\Omega\, colim_{[n]\in\Delta^{op}} Seq^T_n) \simeq \Omega\, l^{nc}_{0,\mathbb{A}^1}(colim_{[n]\in\Delta^{op}} Seq^T_n) \simeq \Omega\, colim_{[n]\in\Delta^{op}} l^{nc}_{0,\mathbb{A}^1}(Seq^T_n) \tag{10.1.21}$$

At last, by the same arguments as in the Prop. 7.4.5, the natural map

$$Map_{\mathcal{D}g^{idem}(R)}((\widehat{([n-1]_R)}_c^R, T \otimes_R -) \to Map_{\mathcal{D}g^{idem}(R)}(R^{\oplus n}, T \otimes_R -) \tag{10.1.22}$$

is an $\mathbb{A}^1$-homotopy equivalence for every $n \geq 0$ and the last colimit in the formula (10.1.21) becomes

$$\Omega\, colim_{[n]\in\Delta^{op}} l^{nc}_{0,\mathbb{A}^1}(Map_{\mathcal{D}g^{idem}(R)}(R^{\oplus n}, T \otimes_R -)) \tag{10.1.23}$$

To conclude, by the same arguments presented after the Corollary 7.4.6 we are reduced to check that this colimit is equivalent to

$$\Sigma\, l^{nc}_{0,\mathbb{A}^1}(Map_{\mathcal{D}g^{idem}(R)}(R, T \otimes_R -)) \tag{10.1.24}$$

But this follows because $\mathcal{SH}_{nc}(R)$ is stable and because the simplicial object

$$[n] \mapsto l^{nc}_{0,\mathbb{A}^1}(Map_{\mathcal{D}g^{idem}(R)}(R^{\oplus n}, T \otimes_R -)) \tag{10.1.25}$$

satisfies the Segal conditions for the same reasons that

$$[n] \mapsto l^{nc}_{0,\mathbb{A}^1}(Map_{\mathcal{D}g^{idem}(R)}(R^{\oplus n}, -)) \tag{10.1.26}$$

does. This was proved in the Lemma 7.4.7. This concludes the proof.

□

Remark 10.1.6. As $KH_R(T \otimes_R -)$ satisfies Localization, the statement of the Prop. 10.1.5 also remains valid after the localization $\mathcal{SH}_{nc}(R) \to \mathcal{SH}^{Loc}_{nc}(R)$.

Finally, combining the last remark with the Prop. 10.1.4 we conclude that

Corollary 10.1.7. *The non-commutative motive $i^{nc}_*(1^{nc}_Z) \in \mathcal{SH}^{Loc}_{nc}(A)$ is naturally equivalent to the non-commutative motive obtained by localizing the presheaf of spaces defined by the formula*

$$T \mapsto Map_{\mathcal{D}g^{idem}(A)}(A, A/I \otimes_A -) \tag{10.1.27}$$

This concludes our description of the source and target of ψ. This is the moment when the Conjecture 10.0.41 is needed: it allows us to contruct an inverse to ψ.

Proposition 10.1.8. *Let k be a field of characteristic zero and assume the Conjecture 10.0.41 for regular closed immersions. Then ψ is an equivalence.*

Proof. We construct an inverse to ψ. By the existence and description of internal-homs in $\mathcal{D}g^{idem}(R)$ (see [139] and our survey in section 6.1.1) for any A-dg-category of finite type T, we have natural equivalences

$$Map_{\mathcal{D}g^{idem}(A)}(\widehat{A/I}^A_c, T) \simeq (\widehat{A/I \otimes_A T}^A_{pspe})^{\simeq} \tag{10.1.28}$$

which, if the conjecture is true for the closed immersion $i : Spec(A/I) \hookrightarrow Spec(A)$, is then isomorphic to

$$\simeq (\widehat{A/I \otimes_A T}^A_c)^{\simeq} \simeq Map_{\mathcal{D}g^{idem}(A)}(A, A/I \otimes_A T) \tag{10.1.29}$$

In particular, when we pass to motives this equivalence becomes an inverse to ψ.

□

To conclude this section we are left to verify the condition C). Again we investigate both sides of the natural map

$$\theta_T : i^{nc}_*(1^{nc}_Z) \otimes T \to i^{nc}_*(i^*_{nc}(T)) \tag{10.1.30}$$

where T denotes the non-commutative motive $l^{nc}_{0,\mathbb{A}^1}\chi^T$.

Remark 10.1.9. The case when T is smooth and proper is trivial: from one side we have

$$Map_{\mathcal{SH}^{Loc}_{nc}(A)}(-, i^{nc}_*(1^{nc}_Z) \otimes T) \simeq Map_{\mathcal{SH}^{Loc}_{nc}(A)}(- \otimes_A T^\vee, i^{nc}_*(1^{nc}_Z)) \simeq KH_A(- \otimes_A T^\vee \otimes_A A/I) \tag{10.1.31}$$

where the last equivalence follows from the Prop. 10.1.5. For the right hand side we have

$$Map_{\mathcal{SH}^{Loc}_{nc}(A)}(-, i^{nc}_*(i^*_{nc}(T))) \simeq Map_{\mathcal{SH}^{Loc}_{nc}(A/I)}(i_*(-), i_*(T)) \simeq Map_{\mathcal{SH}^{Loc}_{nc}(A/I)}(i_*(-) \otimes_A {}_{1*}(T)^\vee, 1^{nc}_Z) \tag{10.1.32}$$

and the result follows again from the Prop. 10.1.5.

Proposition 10.1.10. *Assume the conjecture 10.0.41 is valid for a regular closed immersion $i : Spec(A/I) \hookrightarrow Spec(A)$. Then the non-commutative motive $i^{nc}_*(1^{nc}_Z) \otimes T \in \mathcal{SH}^{Loc}_{nc}(A)$ is equivalent to the $\mathbb{A}^1$-localization of presheaf $Map_{\mathcal{D}g^{idem}(A)}(T, A/I \otimes_A -)$.*

Proof. After the Cor. 10.1.7, the Conjecture provides an equivalence between $i_*^{nc}(1_Z^{nc})$ and $l_{0,\mathbb{A}^1}^{nc}\chi_A^{A/I}$ and as the $\mathbb{A}^1$-localization is monoidal we find an equivalence

$$i_*^{nc}(1_Z^{nc}) \otimes T \simeq l_{0,\mathbb{A}^1}^{nc}(\chi_A^T \otimes \chi_A^{A/I}) \tag{10.1.33}$$

We now remark that $\chi_A : \mathcal{D}g^{idem}(A)^{op} \to \mathcal{P}^{big}(\mathcal{N}cS(A))$ is also a monoidal functor with respect to the opposite of the monoidal structure in $\mathcal{D}g^{idem}(A)$ of the Prop. 6.1.20 and the canonical monoidal structure in $\mathcal{P}^{big}(\mathcal{N}cS(A))$ recalled in the Example 3.2.7. Indeed, as $\mathcal{P}^{big}(\mathcal{N}cS(A))$ as all limits χ_A^{op} can be described as the Left Kan extension

$$\begin{tikzcd} \mathcal{D}g^{ft}(A) \arrow[r, "j^{op}"] \arrow[d, hook] & \mathcal{P}^{big}(\mathcal{N}cS(A))^{op} \\ Ind(\mathcal{D}g^{ft}(A)) \simeq \mathcal{D}g^{idem}(A) \arrow[ur, dashed, "\chi_A^{op}"'] & \end{tikzcd} \tag{10.1.34}$$

and by the monoidal universal property of the Ind-completion (see again our survey in Section 3.2.8), χ_A is monoidal. This tells us that $i_*^{nc}(1_Z^{nc}) \otimes T$ is equivalent to the $\mathbb{A}^1$-localization of the presheaf $\chi_A^{T\otimes_A A/I}$ which, by the Conjecture 10.0.41, is equivalent to the presheaf $Map_{\mathcal{D}g^{idem}(A)}(T, A/I \otimes_A -)$. □

Considering the base-change adjunction with respect to i

$$\begin{tikzcd} \mathcal{D}g^{idem}(A/I)^{op} \arrow[r, shift left, "F_i^{idem}"] & \mathcal{D}g^{idem}(A)^{op} \arrow[l, shift left, "i^*"] \end{tikzcd} \tag{10.1.35}$$

we notice that because of the projection formula for the forgetful functor (see the Remark 9.2.5), $Map_{\mathcal{D}g^{idem}(A)}(T, A/I \otimes_A -)$ can also be written as the composition $\chi_A^T \circ F_i^{idem} \circ i^*$, or by defintion of $i_* : \mathcal{P}^{big}(\mathcal{N}cS(A/I)) \to \mathcal{P}^{big}(\mathcal{N}cS(A))$, as $i_*(\chi_A^T \circ F^{idem}) \simeq i_*\chi_Z^{i^*(T)}$.

Let us now describe the target of θ_T. By adjunction, for any dg-category T of finite type over A we can identify the space-valued presheaf $i_*^{nc}(i_{nc}^*(T))$ with the composition $l_{0,\mathbb{A}_Z^1}^{nc}\chi^{i^*(T)} \circ i^*$ where $l_{0,\mathbb{A}_Z^1}^{nc}\chi_Z^{i^*(T)}$ is now an object in $\mathcal{SH}_{nc}(A/I)$. Moreover, as this composition is $\mathbb{A}_X^1$-local, thanks to the Prop. 10.1.10 the map θ_T can be identified with the canonical map induced by the universal property of the $\mathbb{A}_X^1$-localization

$$l_{0,\mathbb{A}_X^1}^{nc}(i_*\chi_Z^{i^*(T)}) \to l_{0,\mathbb{A}_Z^1}^{nc}\chi_Z^{i^*(T)} \circ i^* \tag{10.1.36}$$

Unfortunately we were not able to prove that this map is an equivalence and this is the reason why the Open Problem 10.0.45 is relevant to us. Recall from the formula (7.4.8) that the $\mathbb{A}^1$-localization functor of space-valued Nisnevich sheaves iterates the naive $\mathbb{A}^1$-localization with the Nisnevich localization. Moreover, one can easily check that as $\mathbb{A}_X^1 \simeq i^*(\mathbb{A}_Z^1)$, the map in (10.1.36) is an equivalence if we just apply the naive $\mathbb{A}^1$-localization. The problem concerns the Nisnevich localization. The conclusion of this discussion can be stated as follows:

If the Open Problem 10.0.45 admits a positve answer and if the Conjecture 10.0.41 is valid for regular closed immersions, then the property C) holds for any dg-category of finite type.

The author will continue to investigate these questions in the future.

10.2 Towards fully-faithfulness over a smooth base

Let us now explain how to prove the Prop. 10.0.48. Let us start with a small observation:

Proposition 10.2.1. *Let $f^{\otimes}:\mathcal{C}^{\otimes}\to\mathcal{D}^{\otimes}$ be a monoidal functor between stable presentable symmetric monoidal $(\infty,1)$ categories compactly generated in the sense of section 2.1.23 and suppose the underlying functor $f:\mathcal{C}\to\mathcal{D}$ admits a right adjoint g which by [100, 7.3.2.7] we know is lax-monoidal. In particular, $g(1_{\mathcal{D}})$ has the structure of commutative algebra object in $\mathcal{C}^{\otimes}$. Then, the following are equivalent:*

1. *For every object $E\in\mathcal{D}$, the natural map given by adjunction*
$$g(1_{\mathcal{D}})\otimes_{\mathcal{C}}E\to g(f(E)) \tag{10.2.1}$$
is an equivalence.

2. *The composition*
$$\bar{f}:Mod_{g(1_{\mathcal{D}})}(\mathcal{C})\xrightarrow{f}Mod_{f\circ g(1_{\mathcal{D}})}(\mathcal{D})\xrightarrow{-\otimes_{f\circ g(1_{\mathcal{D}})}1_{\mathcal{D}}}Mod_{1_{\mathcal{D}}}(\mathcal{D})\simeq\mathcal{D} \tag{10.2.2}$$
is fully-faithful. Here the first map is the map induced by f at the level of modules (see section 3.3.9) and the second map is base-change with respect to the canonical map of algebras $f\circ g(1_{\mathcal{D}})\to 1_{\mathcal{D}}$.

Proof. Notice first that $\bar{f}$ admits a right adjoint $\bar{g}$ and that again by the discussion in section 3.3.9, the two diagrams

$$\begin{tikzcd} \mathcal{C} \arrow[r, "f"] \arrow[d, "-\otimes_{1_{\mathcal{C}}}g(1_{\mathcal{D}})"'] & \mathcal{D} \\ Mod_{g(1_{\mathcal{D}})}(\mathcal{C}) \arrow[ur, "\bar{f}"'] & \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C} & \mathcal{D} \arrow[l, "g"'] \arrow[dl, "\bar{g}"] \\ Mod_{g(1_{\mathcal{D}})}(\mathcal{C}) \arrow[u, "forget"] & \end{tikzcd} \tag{10.2.3}$$

commute. Moreover, under the assumption that $\mathcal{C}$ admits a family of compact generators, say $\{E_\alpha\}_{\alpha\in A}$, by the Prop. 3.8.3, the family $\{E_\alpha\otimes_{1_{\mathcal{C}}}g(1_{\mathcal{D}})\}_{\alpha\in A}$ is a family of compact generators in $Mod_{g(1_{\mathcal{D}})}(\mathcal{C})$.

In this case, $\bar{f}$ is fully faithful if and only if for every α the unit morphism
$$E_\alpha\otimes_{1_{\mathcal{C}}}g(1_{\mathcal{D}})\to\bar{g}(\bar{f}(E_\alpha\otimes_{1_{\mathcal{C}}}g(1_{\mathcal{D}}))) \tag{10.2.4}$$
is an equivalence in $Mod_{g(1_{\mathcal{D}})}(\mathcal{C})$. The commutativity of the diagram (10.2.3) and the fact $forget$ is conservative, tells us that this is true if and only if
$$forget(E_\alpha\otimes_{1_{\mathcal{C}}}g(1_{\mathcal{D}}))\to g(f(E_\alpha)) \tag{10.2.5}$$
is an equivalence. Finally, the projection formula for the forgetful functor (which results from construction of tensor products) concludes the equivalence between 1) and 2).

□

We will use this proposition to prove the theorem. Let S be a scheme. So far we know about the existence of a family of compact generators in $\mathcal{SH}(S)$ given by the Prop. 5.3.3: using the tools introduced in this chapter it can be described as the family of objects $(\mathbb{P}^1_S,\infty)^{\otimes n}\otimes_S f_\sharp(1_V)\in\mathcal{SH}(S)$ indexed by the collection of smooth morphisms $f:V\to S$ and $n\in\mathbb{Z}$. Notice that by the projection formula for $f_\sharp$ these can also be writen in the form $f_\sharp(f^*((\mathbb{P}^1_S,\infty)^{\otimes n}))\simeq f_\sharp((\mathbb{P}^1_X,\infty)^{\otimes n})$. As a consequence of the fact that $\mathcal{SH}^{\otimes}$ satisfies the six operations and localization, one can use Nagata's compactification theorem to find S-compactifications of f and use their associated localization sequences to prove the following result:

Proposition 10.2.2. *(Ayoub [6, Lemma 2.2.23] and Cisinski-Déglise [30, Prop. 4.2.13 and Cor. 4.4.3]) Let k be a field of characteristic zero. If S is a regular scheme over k then the collection of objects of the form $f_*((\mathbb{P}_X,\infty)^{\otimes n})$ for $f : X \to S$ a projective morphism over k, X regular as a k-scheme and $n \in \mathbb{Z}$, is a family of compact generators in $\mathcal{SH}(S)$ in the sense discussed in section 2.1.23.*

We will now show that the condition 1) in the Prop. 10.2.1 holds for composition of the adjunctions

$$\begin{tikzcd} \mathcal{SH}(S) \arrow[r, shift left, "\mathcal{L}_S"] & \mathcal{SH}_{nc}(S) \arrow[l, shift left, "\mathcal{M}_S"] \arrow[r, shift left, "l_{loc,S}"] & \mathcal{SH}_{nc}^{Loc}(S) \arrow[l, shift left] \end{tikzcd} \tag{10.2.6}$$

when S is smooth over a field k of characteristic zero.

Notation 10.2.3. To simplify the notations, we will again denote the composed adjunction by $(\mathcal{L}_S, \mathcal{M}_S)$.

As in the proof of loc. cit, it will be enough to check that the condition holds for the compact generators introduced in the Prop. 10.2.2. Let $f : X \to S$ be a projective morphism with X regular over k. We want to show that for every $n \in \mathbb{Z}$, the canonical morphism

$$\mathcal{M}_S(1_S^{nc}) \otimes_S f_*((\mathbb{P}_X,\infty)^{\otimes n}) \to \mathcal{M}_S(\mathcal{L}_S(f_*((\mathbb{P}_X,\infty)^{\otimes n}))) \tag{10.2.7}$$

is an equivalence. As in the commutative world all the six operations hold, f_* satisfies the projection formula and all we have to prove is that the map

$$\mathcal{M}_S(1_S^{nc}) \otimes_S f_*(1_X) \otimes_S (\mathbb{P}_S,\infty)^{\otimes n} \to \mathcal{M}_S(\mathcal{L}_S(f_*(1_X) \otimes_S (\mathbb{P}_S,\infty)^{\otimes n})) \tag{10.2.8}$$

is an equivalence. Using the Remark 9.3.5 the right hand side is immediately seen to be equivalent to $\mathcal{M}_S(\mathcal{L}_S(f_*(1_X)))$.

For the left hand side, by the Corollary 9.3.4, $\mathcal{M}_S(1_S^{nc})$ is equivalent to the object KH_S in $\mathcal{SH}(S)$ that represents homotopy invariant algebraic K-theory. Using the projection formula for f_* in the commutative setting, we find

$$(\mathbb{P}_S,\infty)^{\otimes n} \otimes_S KH_S \otimes_S f_*(1_X) \simeq f_*(f^*((\mathbb{P}_S,\infty)^{\otimes n} \otimes_S KH_S)) \tag{10.2.9}$$

where the last is equivalent to $f_*((\mathbb{P}_X,\infty)^{\otimes n}) \otimes_X KH_X)$ because by [29, Prop. 3.8] we have $f^*KH_S \simeq KH_X$. As it was already mentioned in the Remark 9.4.39 the Bott isomorphism in K-theory implies that the later term is equivalent to f_*KH_X. In this case, to prove the theorem we are reduced to check that the canonical map

$$f_*KH_X \to \mathcal{M}_S(\mathcal{L}_S(f_*(1_X))) \tag{10.2.10}$$

is an equivalence. The left hand side can immediately be identified with the spectral presheaf sending a smooth S-scheme V to the homotopy invariant K-theory $KH_S(V \times_S X)$. Let us now describe the right hand side. As f is a projective morphism between smooth k-schemes, it factors (over k) as

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[dr] & \mathbb{P}^n_S \arrow[d, "\pi"] & U \arrow[l, hook', "j"'] \arrow[dl] \\ & S & \end{tikzcd} \tag{10.2.11}$$

with i a closed embedding between smooth k-schemes, p is the canonical projection and j the open immersion of the complementary of i. The localization property in the commutative setting gives us an associated exact sequence in $\mathcal{SH}(\mathbb{P}^n_S)$

$$j_\sharp 1_U \to 1_{\mathbb{P}^n_S} \to i_* 1_X \tag{10.2.12}$$

which as π_* is exact produces a new exact sequence in $\mathcal{SH}(S)$

$$\pi_* j_\sharp 1_U \to \pi_* 1_{\mathbb{P}^n_S} \to f_* 1_X \tag{10.2.13}$$

Using the notations of the previous chapter, weak purity in the commutative world implies that π_* is equivalent to $\pi_\sharp(Th_{\mathbb{P}^n_S}(T_\pi) \otimes -)$ so that the previous sequence can be written as

$$\pi_\sharp(Th_{\mathbb{P}^n_S}(T_\pi) \otimes j_\sharp 1_U) \to \pi_\sharp(Th_{\mathbb{P}^n_S}(T_\pi) \otimes 1_{\mathbb{P}^n_S}) \to f_* 1_X \tag{10.2.14}$$

and as $\mathcal{L}_S$ is exact we find a new exact sequence in $\mathcal{SH}^{Loc}_{nc}(S)$

$$\mathcal{L}_S(\pi_\sharp(Th_{\mathbb{P}^n_S}(T_\pi) \otimes j_\sharp 1_U)) \to \mathcal{L}_S(\pi_\sharp(Th_{\mathbb{P}^n_S}(T_\pi) \otimes 1_{\mathbb{P}^n_S})) \to \mathcal{L}_S(f_* 1_X) \tag{10.2.15}$$

After the discussion in Chapter 9, we know that $\mathcal{L}$ and $()_\sharp$ commute and as $\mathcal{L}$ is monoidal the sequence can also be written as

$$\pi^{nc}_\sharp(\mathcal{L}_S(Th_{\mathbb{P}^n_S}(T_\pi)) \otimes j^{nc}_\sharp 1^{nc}_U) \to \pi^{nc}_\sharp(\mathcal{L}_S(Th_{\mathbb{P}^n_S}(T_\pi)) \otimes 1^{nc}_{\mathbb{P}^n_S}) \to \mathcal{L}_S(f_* 1_X) \tag{10.2.16}$$

The key point now is that as $\mathcal{L}$ factors through the theory of modules over K-theory (Cor. 9.3.4). As explained in the Remark 9.4.39 the later is orientable with a Tate object equivalent to a tensor unit and as this factorization is monoidal the object $\mathcal{L}_S(Th_{\mathbb{P}^n_S}(T_\pi))$ is the unit non-commutative motive. In this case the previous exact sequence is equivalent to

$$\pi^{nc}_\sharp j^{nc}_\sharp 1^{nc}_U \to \pi^{nc}_\sharp 1^{nc}_{\mathbb{P}^n_S} \to \mathcal{L}_S(f_* 1_X) \tag{10.2.17}$$

By Zariski descent, and as base change and $(-)_\sharp$ are compatible we can assume that S is affine. In this case the first term can be identified with the localization of the presheaf co-represented by $L_{pe}(U)$ as an S-dg-category, which we shall denote as $\chi_S^{L_{pe}(U)}$. In the same way the second term is the motive associated to $\chi_S^{L_{pe}(\mathbb{P}^n_S)}$ for $L_{pe}(\mathbb{P}^n_S)$ as an S-dg-category. Finally, by the definition of $\mathcal{SH}^{Loc}_{nc}(S)$ the sequence (10.2.17) is the motivic image of the exact sequence of S-dg-categories

$$\chi_S^{L_{pe}(U)} \to \chi_S^{L_{pe}(\mathbb{P}^n_S)} \to \chi_S^{L_{pe,X}(\mathbb{P}^n_S)} \tag{10.2.18}$$

where $L_{pe,X}(\mathbb{P}^n_S)$ is the full S-dg-category of $L_{pe}(\mathbb{P}^n_S)$ spanned by those perfect complexes supported on X. By the same arguments used to prove the Proposition 10.1.1, this dg-category is $\mathbb{A}^1_S$-homotopy equivalent to the S-dg-category $L_{pe}(X)$.

The conclusion now follows as in the Prop. 10.1.10 because of the Prop. 10.1.5

Corollary 10.2.4. *Let k be a field of characteristic zero and assume the Conjecture 10.0.41 is valid for proper morphisms of smooth k-schemes $X \to S$. Then the natural map θ_T is an equivalence for any S-dg-category of finite type T.*

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