M2 COURSE: RANDOM NOTES ON SPECTRAL SEQUENCES (UNFINISHED AND UNCHECKED!)

ABSTRACT. This notes form an introduction to computations with spectral sequences. We review Lurie's construction of a spectral sequence for a filtered object in a stable ∞-category and explain how all other more familiar spectral sequences can be recovered as an example.

CONTENTS

By default, and unless mentioned otherwise,all notations are homological.

1. LURIE'S SPECTRAL SEQUENCE OF A FILTERED OBJECT

Goal 1.1. Let C be a stable ∞-category with a t-structure and let $X : \mathbb{Z}_{\geq 0} \to \mathsf{C}$ be a sequence

$$0 = X_{-1} \to X_0 \to X_1 \to X_2 \to \cdots \tag{1}$$

We will denote by $X(\infty)$ the colimit of the diagram. We will be interested in computing the objects $\pi_n^\infty := \pi_n(X(\infty))$ in the heart of C. This object has a natural descending filtration given by the successive images of the maps $\pi_n(X_p) \to \pi_n(X(\infty))$. Namely, if we set

$$\mathrm{F}_n^p := \mathrm{Im}\ (\pi_n(X_p) \to \pi_n(X(\infty)))$$

we have a filtered object in the heart of C

$$0 \subseteq \mathrm{F}_n^0 \subseteq \mathrm{F}_n^1 \subseteq \cdots \subseteq \mathrm{F}_n^p \subseteq \cdots \subseteq \pi_n^\infty \tag{2}$$

If we suppose that the t-structure is compatible with sequencial colimits (such as the standard t-structure in Mod_k), then we find

$$\operatorname*{colim}_p \mathrm{F}_n^p \simeq \bigcup_p \mathrm{F}_n^p = \pi_n^\infty$$

meaning that the filtration of (2) is exaustive.

As we will see next, a spectral sequence is a scheme designed to compute the graded pieces $\mathrm{F}_n^p/\mathrm{F}_n^{p-1}$ of the filtration (2) out of the filtered object (1).

Remark 1.2. For instance, if $\mathsf{C} = \mathsf{Mod}_k$ with k a field, then every short exact sequence splits and therefore knowing the associated graded pieces gives us π_n^∞ by direct sums. More generally, to lift elements of E_∞ to the filtered pieces we will need to solve an extension problem.

Remark 1.3. What are the graded pieces of (2)? Let us consider for each p, the cofiber Y_p

$$\begin{array}{ccc} X_{p-1} & \longrightarrow & X(\infty) \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & Y_p \end{array} \tag{3}$$

Notice that Y_p can also be described as the colimit of the sequence

$$\operatorname*{colim}_{r\geq 0} \left(0 \to X_p/X_{p-1} \to X_{p+1}/X_{p-1} \to \cdots \to X_{p+r}/X_p \to \cdots\right) \simeq$$

$$\simeq \operatorname*{colim}_{r\geq 0} \left(X_{p+r}/X_{p-1}\right) \simeq Y_p$$

and by construction we have commutative diagrams

$$\begin{array}{ccccc} X_{p-1} & \longrightarrow & X_p & \longrightarrow & X(\infty) \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & X_p/X_{p-1} & \longrightarrow & Y_p \end{array} \tag{4}$$

from where we extract a short exact sequence of objects in the heart of C

$$0 \to \mathrm{F}_n^{p-1} \subseteq \mathrm{F}_n^p \to \mathrm{Im}\,(\pi_n(X_p) \to \pi_n(Y_p)) \to 0$$

Indeed, the surjectivity of the last map is automatic since we are taking the image. The fact that it is exact in the middle follows from the commutativity of the outer diagram in (4) and the associated long exact sequence of homotopy groups.

In summary we want to compute

$$\mathrm{F}_n^p/\mathrm{F}_n^{p-1} \simeq \mathrm{Im}\,(\pi_n(X_p) \to \pi_n(Y_p)) \tag{5}$$

Construction 1.4. Given that Y_p can be obtained as a filtered colimit $\underset{r\geq 0}{\mathsf{colim}}\,(X_{p+r}/X_{p-1})$, one can look at the commutative diagram

$$\begin{array}{ccccccccc} X_{p-1} & \longrightarrow & X_p & \longrightarrow \cdots \longrightarrow & X_{p+r} & \longrightarrow \cdots \longrightarrow & X(\infty) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & X_p/X_{p-1} & \longrightarrow \cdots \longrightarrow & X_{p+r}/X_{p-1} & \longrightarrow \cdots \longrightarrow & Y_p \end{array} \tag{6}$$

and consider for each $r \geq 0$ the sequence of images

$$r \mapsto \mathrm{Im}\,(\pi_n(X_p) \to \pi_n(X_{p+r}/X_{p-1})) \tag{7}$$

The main idea is that these images have enough structure (the structure of a spectral sequence) that allows them to be computed by induction in r.

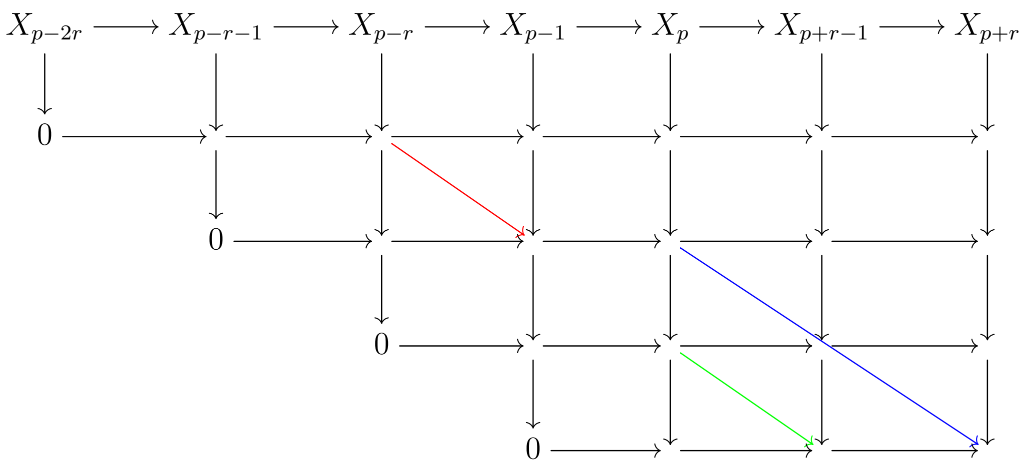
Since are are interested in what happens in the limit $r \mapsto \infty$, we can allow ourselves to modify the first terms of the sequence (7). For instance, since $X_{-1} \simeq X_{-2} \simeq \cdots 0$, for $r > p$ we have $X_p \simeq X_p/X_{p-r}$ so that (7) agrees with

$$r \mapsto \mathrm{Im}\,(\pi_n(X_p/X_{p-r}) \to \pi_n(X_{p+r-1}/X_{p-1})) \tag{8}$$

where now $r \geq 1^{(*)}$. For $r \leq p$ the two definitions don't have to agree, but again, this won't be a problem for the purpose of computing the large limit $r \to \infty$. Why is this trick relevant? Because this new formula satisfies a better recursive behavior that allows us to compute the pieces by induction in r:

Consider the commutative diagram obtained by iterated pushouts:

$^{(*)}$Notice the change of variables $(\mathit{previous}\, r) = (\mathit{new}\, r) + 1$



- In green we represent the map in (8), ie,

$$r \mapsto \operatorname{Im}\left(\pi_n(X_p/X_{p-r}) \to \pi_n(X_{p+r-1}/X_{p-1})\right) \tag{9}$$

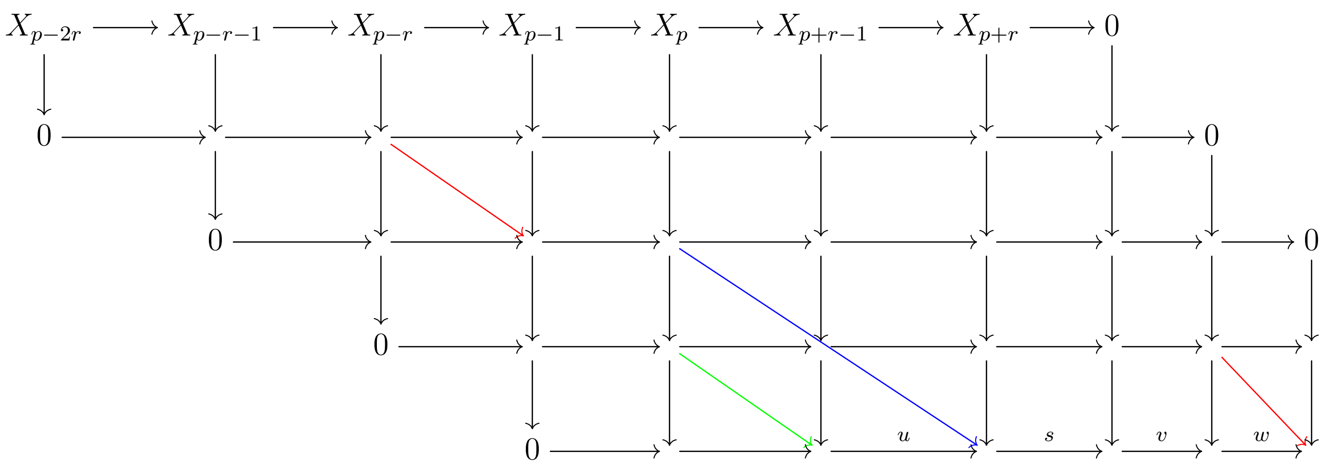
- In red we represent (8) replacing p by $p-1$, ie,

$$r \mapsto \operatorname{Im}\left(\pi_n(X_{p-r}/X_{p-2r}) \to \pi_n(X_{p-1}/X_{p-r-1})\right) \tag{10}$$

- In blue we represent we represent (8) replacing r by $r+1$, ie,

$$r \mapsto \operatorname{Im}\left(\pi_n(X_p/X_{p-r-1}) \to \pi_n(X_{p+r}/X_{p-1})\right) \tag{11}$$

By construction we see that the suspension of the map in red re-appears in front of the map in green: a few pushouts later:



providing us with a map

$$d_r : \mathrm{Im}\,(\pi_n(X_p/X_{p-r}) \to \pi_n(X_{p+r}/X_{p-1})) \to \mathrm{Im}\,(\pi_n(X_{p-r}/X_{p-2r}[1]) \to \pi_n(X_p/X_{p-r-1}[1]))$$

obtained by composition with $w \circ v \circ s \circ u$.

We introduce the following notation:

Notation 1.5. Let us set

$$q := n - p$$

With this choice, we can introduce a new notation $\mathrm{E}_r^{p,q}$:

$$\mathrm{Im}\,(\pi_n(X_p/X_{p-r}) \to \pi_n(X_{p+r}/X_{p-1})) = \underbrace{\mathrm{Im}\,(\pi_{p+q}(X_p/X_{p-r}) \to \pi_{p+q}(X_{p+r-1}/X_{p-1}))}_{\mathrm{E}_r^{p,q}}$$

This choice of indices is particularly convenient to write the maps d_r as

$$\mathrm{Im}\,(\pi_{p+q}(X_p/X_{p-r}) \to \pi_{p+q}(X_{p+r}/X_{p-1})) \to \mathrm{Im}\,(\pi_{p+q-1}(X_{p-r}/X_{p-2r}) \to \pi_{p+q-1}(X_{p+q-1}/X_{p-r-1}))$$

simply as

$$d_r : \mathrm{E}_r^{p,q} \to \mathrm{E}_r^{p-r,q+r-1}$$

Let us also denote as

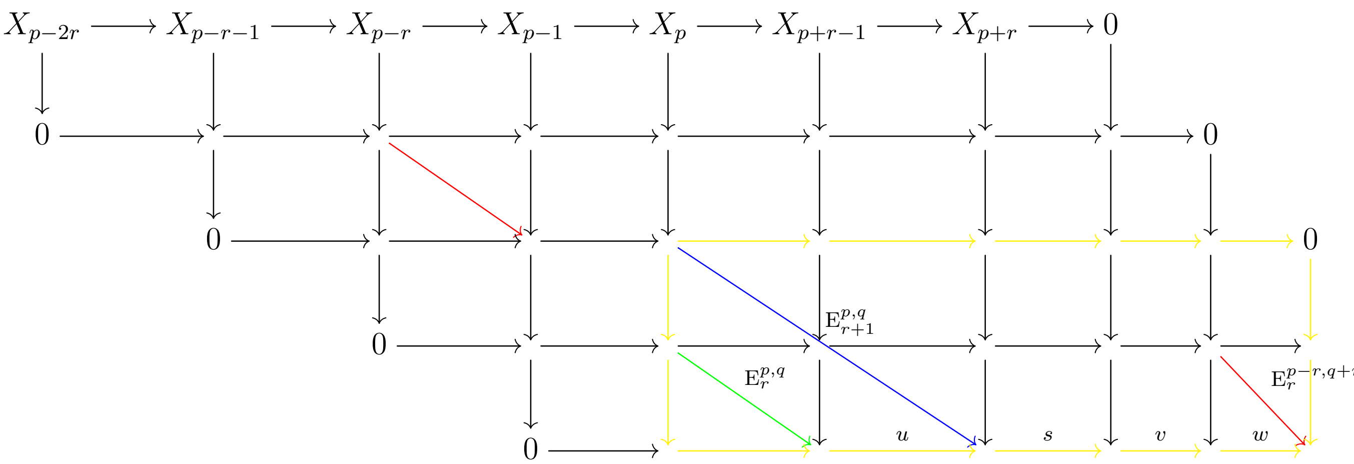
$$\mathrm{E}_\infty^{p,q} := \mathrm{Im}\,(\pi_{p+q}(X_p) \to \pi_{p+q}(Y_p))$$

the graded piece of (5).

Proposition 1.6. *We have $d_r \circ d_r = 0$ in the homotopy category of C. Moreover, we find canonical isomorphisms*

$$\mathrm{E}_{r+1}^{p,q} \simeq \frac{\mathrm{Ker}(d_r : \mathrm{E}_r^{p,q} \to \mathrm{E}_r^{p-r,q+r-1})}{\mathrm{Im}(d_r : \mathrm{E}_r^{p+r,q-r+1} \to \mathrm{E}_r^{p,q})} \tag{12}$$

Proof. We consider again the diagram constructed above



Since d_r is defined by composition with $w \circ v \circ s \circ u$, One can easily check then using the cofiber sequences that $d_r^2 = 0$. We can then use the long exact sequence associated to the pushout square in yellow to establish the isomorphism (12).

□

Definition 1.7. An homological spectral sequence is a family of objects $\{E_r^{p,q}\}_{r\geq 1}$ together with differentials $d_r : E_r^{p,q} \to E_r^{p-r,q+r-1}$ with $d_r^2 = 0$ and verifying the formula (12). We write

$$E_r^{p,q} \implies \pi_{p+q}$$

to say that there exists $R \geq 1$ such that for every $r \geq R$ the maps d_r vanish. In this case we write $E_\infty^{p,q}$ to denote the stable value

$$E_R^{p,q} \simeq E_{R+1}^{p,q} \simeq \cdots \simeq E_\infty^{p,q}$$

giving us the graded pieces of a filtration on π_{p+q}.

Remark 1.8. For $r = 1$ we have

$$E_1^{p,q} = \pi_n(X_p/X_{p-1}) = \pi_q(X_p/X_{p-1}[-p])$$

In this case the maps d_r are obtained as boundary maps

$$\cdots \to X_p/X_{p-1}[-p] \to \cdots \to X_2/X_1[-2] \to X_1/X_0[-1] \to X_0$$

and applying π_q we obtain a chain complex in the heart of C

$$\cdots \to \pi_{p+q}(X_p/X_{p-1}) \to \cdots \to \pi_{2+q}(X_2/X_1) \to \pi_{1+q}(X_1/X_0) \to \pi_{0+q}(X_0)$$

This is the usual presentation of the first page of the spectral sequence

$$\begin{array}{cccccc}
q & \pi_{0+q}(X_0) \longleftarrow & \pi_{1+q}(X_1/X_0) \longleftarrow & & \cdots \longleftarrow & \pi_{p+q}(X_p/X_{p-1}) \longleftarrow \cdots \\
\cdots & & & & & \\
q=2 & \pi_{0+2}(X_0) \longleftarrow & \pi_{1+2}(X_1/X_0) \longleftarrow & & \cdots \longleftarrow & \pi_{p+2}(X_p/X_{p-1}) \longleftarrow \cdots \\
q=1 & \pi_{0+1}(X_0) \longleftarrow & \pi_{1+1}(X_1/X_0) \longleftarrow & & \cdots \longleftarrow & \pi_{p+1}(X_p/X_{p-1}) \longleftarrow \cdots \\
q=0 & \pi_{0+0}(X_0) \longleftarrow & \pi_{1+0}(X_1/X_0) \longleftarrow & & \cdots \longleftarrow & \pi_{p+0}(X_p/X_{p-1}) \longleftarrow \cdots \\
q=-1 & \pi_{0+(-1)}(X_0) \longleftarrow & \pi_{1+(-1)}(X_1/X_0) \longleftarrow & & \cdots \longleftarrow & \pi_{p-1}(X_p/X_{p-1}) \longleftarrow \cdots \\
 & \vdots & \vdots & \vdots & & \vdots \\
 & p=0 & p=1 & \cdots & & \cdots
\end{array}$$

Remark 1.9. The hypothesis that the t-structure is compatible with filtered colimits can be replaced by the requirement that for every n the sequence of homotopy groups

$$\pi_n(X_0) \to \pi_n(X_1) \to \cdots \to \pi_n(X_p) \to \cdots$$

stabilizes after a finite number of steps, ie, for every n, there exists P such that $\forall$ $p \geq P$, we have

$$\pi_n(X_{P-1}) \to \pi_n(X_P) \simeq \pi_n(X_{P+1}) \simeq \pi_n(X_{P+2}) \simeq \cdots \simeq \pi_n(X(\infty))$$

Indeed, this condition allows us to drop the assumption that the t-structure is compatible with filtered colimits, since in this case the filtration F^p_n is automatically finite: since $\pi_n(X_P) \simeq \pi_n(X(\infty))$ we can define for $0 \leq p \leq P$

$$\mathrm{F}^p_n := \mathrm{Im}\ (\pi_n(X_p) \to \pi_n(X_P))$$

Remark 1.10. Instead of a colimit of a sequence (1) we could try to compute the limit of a sequence in C

$$\cdots \to X_2 \to X_1 \to X_0 \to 0 = X_{-1} \tag{13}$$

For this purpose we observe that (13) can be seen as a ascending sequence as (1) but in C^{op}, and the limit of (13) in C is the colimit in C^{op}. Therefore, the discussion above, together with the fact that the opposite of an abelian category remains abelian (with $\mathrm{Im}\, f^{\mathsf{op}} = \mathrm{Im}\, f$), implies the existence of a cohomological spectral sequence $\mathrm{E}^{p,q}_r$ with

$$\mathrm{E}^{p,q}_1 = \pi_{p+q}(C_p)$$

where $C_p := \mathrm{fiber}\,(X_p \to X_{p-1})$ and differentials

$$\mathrm{E}^{p,q}_r \to \mathrm{E}^{p+r,q-r+1}_r$$

obtained from the boundary maps

$$X_0 \to C_1[1] \to C_2[2] \to \cdots$$

converging to the graded pieces of the filtration of $\pi_n(\lim X_p)$ given by

$$\mathrm{F}^p_n := \mathrm{Ker}\,(\pi_n(\lim X_p) \to \pi_n(X_p))$$

Moreover, the higher pages of the spectral sequence are given by

$$\mathrm{E}^{p,q}_r = \mathrm{Ker}\,(\,\mathrm{fiber}(X_{p+r} \to X_{p-1}) \to \mathrm{fiber}(X_p \to X_{p-r}))$$

Example 1.11. Let us apply the machinery of the last section to compute the cohomology of $\mathbb{CP}^\infty$. The only thing we will need to know about this space is that admits a canonical filtration given by its CW-decomposition

$$\mathbb{CP}^0 \subseteq \mathbb{CP}^1 \subseteq \mathbb{CP}^2 \subseteq \cdots \subseteq \mathbb{CP}^\infty$$

where the successive pieces are obtained by homotopy pushouts by attaching a single cell

$$\begin{array}{ccc} \mathsf{S}^{2n+1} & \longrightarrow & \mathsf{D}^{2n+2} \simeq * \\ \downarrow & & \downarrow \\ \mathbb{CP}^n & \longrightarrow & \mathbb{CP}^{n+1} \end{array}$$

Therefore, the complex of singular cochains $\mathsf{C}^*(\mathbb{CP}^\infty, \mathbb{Z})$ can be obtained as a limit of the sequence

$$\mathsf{C}^*(\mathbb{CP}^\infty, \mathbb{Z}) \simeq \operatorname{\mathsf{lim}} \left(\cdots \to \mathsf{C}^*(\mathbb{CP}^2, \mathbb{Z}) \to \mathsf{C}^*(\mathbb{CP}^1, \mathbb{Z}) \to \mathsf{C}^*(\mathbb{CP}^0, \mathbb{Z})\right)$$

where we have cofiber-sequences

$$\begin{array}{ccc} \mathsf{C}^*(\mathbb{CP}^{n+1}, \mathbb{Z}) & \longrightarrow & \mathsf{C}^*(\mathbb{CP}^n, \mathbb{Z}) \\ \downarrow & & \downarrow \\ \mathbb{Z} & \longrightarrow & \mathsf{C}^*(\mathsf{S}^{2n+1}, \mathbb{Z}) \end{array}$$

where $\mathsf{C}^*(\mathsf{S}^{2n+1}, \mathbb{Z}) \simeq \mathbb{Z} \oplus \mathbb{Z}[-2n-1])$. We have then the limit spectral sequence of Remark **1.10**, yielding

$$\mathrm{E}_1^{p,q} := \mathsf{H}_{p+q}(\text{fiber}\left(\mathsf{C}^*(\mathbb{CP}^p, \mathbb{Z}) \to \mathsf{C}^*(\mathbb{CP}^{p-1}, \mathbb{Z})\right)) \implies \mathsf{H}_{p+q}\mathsf{C}^*(\mathbb{CP}^\infty, \mathbb{Z})$$

with

$$\text{fiber}\left(\mathsf{C}^*(\mathbb{CP}^p, \mathbb{Z}) \to \mathsf{C}^*(\mathbb{CP}^{p-1}, \mathbb{Z})\right) \simeq \mathbb{Z}[-2p]$$

Therefore, the first page writes as

$$\mathrm{E}_1^{p,q} = \mathsf{H}_{p+q}(\mathbb{Z}[-2p]) = \begin{cases} \mathbb{Z} & \text{if } q = -3p \\ 0 & \text{otherwise} \end{cases}$$

that we can picture as

q	$p=0$	$p=1$	$p=2$	$p=3$	$\cdots$
$q=0$	$\mathbb{Z}$	0	0	0	$\cdots$
$q=-1$	0	0	0	0	$\cdots$
$q=-2$	0	0	0	0	$\cdots$
$q=-3$	0	$\mathbb{Z}$	0	0	$\cdots$
$q=-4$	0	0	0	0	$\cdots$
$q=-5$	0	0	0	0	$\cdots$
$q=-6$	0	0	$\mathbb{Z}$	0	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

It follows that all boundary maps d_1 vanish and the spectral sequence degenerates already at page 1. It follows that for $n \leq 0$

$$\mathsf{H}_{2n}(\mathsf{C}^*(\mathbb{CP}^\infty, \mathbb{Z})) \simeq k$$

and vanishes otherwise. It follows that as a complex we have

$$\mathsf{C}^*(\mathbb{CP}^\infty, \mathbb{Z}) \simeq \bigoplus_{n \leq 0} \mathbb{Z}[2n]$$

2. Serre Spectral Sequence

We will start with the spectral sequence associated to a Serre fibration $p : X \to Y$. Let us fix a base point $y : * \to Y$ and denote by F_y the homotopy fiber product

$$\begin{array}{ccc} F_y & \longrightarrow & X \\ \downarrow & & \downarrow p \\ * & \longrightarrow & Y \end{array} \tag{14}$$

Since Serre fibrations are in particular right fibrations [Lur09, Chapter 2], we can use the Grothendieck construction for spaces to present p as an ∞-functor

$$\mathsf{P} : Y \to \mathsf{S}$$

sending $y \mapsto F_y$.

Remark 2.1. By taking fiber products in (14) we obtain a canonical action of the loop space $\Omega_y Y \simeq * \underset{Y}{\times} *$ on the fiber F_y. Let us assume that Y is connected. Then this action completely determines the original Serre fibration since this case by the Boardman-Vogt theorem [Lur17, 5.2.6.10], Y is equivalent to the bar-construction on the $\mathsf{E}_1^{\otimes}$-loop group $\Omega_y Y$ so that the functor $Y \to \mathsf{S}$ can be written as

$$\mathsf{B}(\Omega_y Y) \to \mathsf{S}$$

It follows from [Lur09, 3.3.4.3] that

$$X \simeq \underset{Y}{\mathsf{colim}}\, \mathsf{P} \tag{15}$$

In order to understand this colimit we will use the technique for decomposing diagrams in [Lur09, 4.2.3]. More precisely, we consider the diagram

$$F : \mathbb{N} \to \mathsf{S}/Y$$

given by the different skeletal decompositions of Y, ie,

$$n \mapsto \mathrm{Sk}_n(Y)$$

with

$$\mathrm{Sk}_0(Y) \subseteq \mathrm{Sk}_1(Y) \subseteq \cdots \subseteq Y$$

It follows from [Lur09, 4.2.3.4, 4.2.3.9, 4.2.3.10] that

$$X \simeq \underset{n}{\mathsf{colim}} \left(\underset{\mathrm{Sk}_n(Y)}{\mathsf{colim}}\, \mathsf{P}_{|_{\mathrm{Sk}_n(Y)}} \right) \simeq \underset{n}{\mathsf{colim}}\, X_n \tag{16}$$

where X_n is the fiber product

$$\begin{array}{ccc} X_n & \longrightarrow & X \\ \big\downarrow & & \big\downarrow{\scriptstyle p} \\ \mathrm{Sk}_n(Y) & \longrightarrow & Y \end{array} \tag{17}$$

We now consider the ∞-functor $\Sigma^\infty : \mathsf{S} \to \mathsf{Sp}$ and its composition with $- \underset{\mathbb{S}}{\otimes} \mathsf{H}\mathbb{Z} : \mathsf{Sp} \to \mathsf{Mod}_{\mathbb{Z}}$. This composition coincides with the functor of singular chains $\mathsf{C}_*(-, \mathbb{Z})$ and commutes with all homotopy colimits. We obtain a colimit preserving functor

$$Y \simeq \mathsf{B}(\Omega_y Y) \to \mathsf{Mod}_{\mathbb{Z}}$$

sending $y \mapsto \mathsf{C}_*(F_y, \mathbb{Z})$. It follows then from (15) that

$$\mathsf{C}_*(X, \mathbb{Z}) \simeq \operatorname*{\mathsf{colim}}_{y \in Y} \mathsf{C}_*(F_y, \mathbb{Z}) \tag{18}$$

We now apply the discussion in Section **1** to the sequence

$$\mathsf{C}_*(X_0, \mathbb{Z}) \to \mathsf{C}_*(X_1, \mathbb{Z}) \to \mathsf{C}_*(X_2, \mathbb{Z}) \to \cdots \to \mathsf{colim} = \mathsf{C}_*(X, \mathbb{Z})$$

to deduce a (homological) spectral sequence

$$\mathrm{E}_1^{p,q} := \mathsf{H}_{p+q}(X_p, X_{p-1}, \mathbb{Z}) \simeq \pi_{p+q}(\mathsf{C}_*(X_p, X_{p-1}, \mathbb{Z})) \implies \pi_{p+q}(\mathsf{C}_*(X, \mathbb{Z})) = \mathsf{H}_{p+q}(X, \mathbb{Z})$$

where $\mathsf{C}_*(X_p, X_{p-1}, \mathbb{Z})$ is the cofiber of $\mathsf{C}_*(X_{p-1}, \mathbb{Z}) \to \mathsf{C}_*(X_p, \mathbb{Z})$ in $\mathsf{Mod}_{\mathbb{Z}}$ and by definition computes the relative homology of the pair (X_p, X_{p-1}).

We can finally establish the main theorem of this section:

Proposition 2.2. *Suppose the action of $\pi_1(Y)$ on F_y is trivial. Then there is an homological spectral sequence with*

$$\mathrm{E}_2^{p,q} \simeq \mathsf{H}_p(Y, \mathsf{H}_q(F_y, \mathbb{Z})) \implies \mathsf{H}_{p+q}(X, \mathbb{Z}))$$

Moreover, in the particular case where the groups $\mathsf{H}_q(F_y, \mathbb{Z})$ are finitely generated free $\mathbb{Z}$-modules, we have

$$\mathsf{H}_p(Y, \mathsf{H}_q(F_y, \mathbb{Z})) \simeq \mathsf{H}_p(Y, \mathbb{Z}) \otimes \mathsf{H}_q(F_y, \mathbb{Z}))$$

Proof. As explained in the Remark **1.8**, the first page of the spectral sequence is given by

$$\cdots \to \pi_q(C_*(X_p, X_{p-1}, \mathbb{Z})[-p]) \cdots \to \pi_q(C_*(X_1, X_0, \mathbb{Z})[-1]) \to \pi_q(C_*(X_0, \mathbb{Z}))$$

We claim that we can identify the homology groups H_p of this complex with

$$\mathsf{H}_p(Y, \mathsf{H}_q(F_y, \mathbb{Z}))$$

$\square$

Example 2.3. Let us use the homological Serre spectral sequence associated to the fibration

$$\begin{array}{ccc} \mathsf{S}^1 & \longrightarrow & \mathsf{S}^5 \\ \downarrow & & \downarrow \\ * & \longrightarrow & \mathbb{CP}^2 \end{array}$$

to compute the singular homology of $\mathbb{CP}^2$. Since $\pi_1(\mathbb{CP}^2) = 0$ and the homology groups of the circle are free $\mathbb{Z}$-modules, the spectral sequence gives

$$\mathrm{E}_2^{p,q} := \mathsf{H}_p(\mathbb{CP}^2, \mathbb{Z}) \otimes \mathsf{H}_q(\mathsf{S}^1, \mathbb{Z}) \implies \mathsf{H}_{p+q}(\mathsf{S}^5)$$

Writing this page explicitely, and using the fact that $\mathbb{CP}^2$ is a 4-dimension real manifold (see Lemma 2.34-(b).)

$$\begin{array}{cccccc} p=0 & p=1 & p=2 & p=3 & p=4 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ \mathbb{Z} & \mathsf{H}_1(\mathbb{CP}^2) & \mathsf{H}_2(\mathbb{CP}^2) & \mathsf{H}_3(\mathbb{CP}^2) & \mathsf{H}_4(\mathbb{CP}^2) & 0 \\ \mathbb{Z} & \mathsf{H}_1(\mathbb{CP}^2) & \mathsf{H}_2(\mathbb{CP}^2) & \mathsf{H}_3(\mathbb{CP}^2) & \mathsf{H}_4(\mathbb{CP}^2) & 0 \end{array}$$

At the same time, since this spectral sequence is concentraded in rows one and two, it follows that it degenerates at the page 3, with $d_3 = 0$. Since $\mathsf{H}_0(\mathsf{S}^5) = \mathbb{Z}$, $\mathsf{H}_5(\mathsf{S}^5) = \mathbb{Z}$ and all other homology groups vanish, this page 3 is

q	$p=0$	$p=1$	$p=2$	$p=3$	$p=4$	$p=5$	$\cdots$
$q=2$	0	0	0	0	0	0	$\cdots$
$q=1$	0	0	0	0	$\mathbb{Z}$	0	$\cdots$
$q=0$	$\mathbb{Z}$	0	0	0	0	0	$\cdots$

By degree reasons, this implies that in page 2 we must have

$$\mathsf{H}_1(\mathbb{CP}^2) = 0\,, \;\; \mathsf{H}_2(\mathbb{CP}^2) \to \mathbb{Z} \text{ is an iso, and}$$

$$\mathsf{H}_4(\mathbb{CP}^2) \to \mathsf{H}_2(\mathbb{CP}^2)\,, \quad \mathsf{H}_3(\mathbb{CP}^2) \to \mathsf{H}_1(\mathbb{CP}^2)$$

are isomorphisms. Combining these remarks, we conclude that H_p vanish for $p \geq 5$ and we get $\mathbb{Z}$ if p is even and zero otherwise.

Remark 2.4. Under the same hypothesis, the homological spectral sequence of Proposition **2.2** has a cohomological version with

$$\mathrm{E}_2^{p,q} \simeq \mathsf{H}^p(Y, \mathsf{H}^q(F_y, \mathbb{Z})) \implies \mathsf{H}^{p+q}(X, \mathbb{Z}))$$

Example 2.5. Let us use the cohomological spectral sequences associated to the fibrations

$$\begin{array}{ccc} \mathsf{S}^1 & \longrightarrow & \mathsf{S}^{2n+1} \\ \downarrow & & \downarrow \\ * & \longrightarrow & \mathbb{CP}^n \end{array}$$

to compute the cohomology of the projective space. From the long exact sequence of homotopy groups associate to the fibration, we see that that $\pi_1(\mathbb{CP}^n) = 0$ so that we can apply the existence result for the spectral sequence. Moreover, since the cohomology groups of the circle are free $\mathbb{Z}$-modules, the E_2 page reads as

$$\mathrm{E}_2^{p,q} = \mathsf{H}^p(\mathbb{CP}^n, \mathbb{Z}) \otimes \mathsf{H}^q(\mathsf{S}^1, \mathbb{Z})$$

q	$p=0$	$p=1$	$p=2$	$p=3$	$p=4$	$\cdots$
$q=2$	0	0	0	0	0	$\cdots$
$q=1$	$\mathbb{Z}$	H^1	H^2	H^3	H^4	$\cdots$
$q=0$	$\mathbb{Z}$	H^1	H^2	H^3	H^4	$\cdots$

By degree reasons, it follows that no differential reaches $\mathsf{H}^1 = \mathrm{E}_2^{1,0}$ and since it cannot persist to infinity (as it does not appear in the cohomology of S^{2n+1}), it must vanish. By degree reasons, to must vanish all cohomology groups for odd p. The second page therefore reads as

$p=0$	$p=1$	$p=2$	$p=3$	$p=4$	$\cdots$	$p=2n$	$\cdots$
0	0	0	0	0	$\cdots$		
$\mathbb{Z}$	0	H^2	0	H^4	$\cdots$	H^{2n}	$\cdots$
$\mathbb{Z}$	0	H^2	0	H^4	$\cdots$	H^{2n}	$\cdots$

and again by inspection of the cohomology of S^{2n+1} we see that all the differentials

$$\mathsf{H}^{2k} \to \mathsf{H}^{2k+2}$$

have to be isomorphisms for $0 \leq k \leq n-1$ and that for $k = n$, the map

$$\mathsf{H}^{2n} \to \mathsf{H}^{2n+2} = 0$$

has to be the zero map (since $\mathbb{CP}^n$ is a $2n$-topological manifold), so that the copy of $\mathsf{H}^{2n+1}(\mathsf{S}^{2n+1}, \mathbb{Z}) \simeq \mathbb{Z}$ is possible at infinity in position $p = 2n$, $q = 1$. It follows that $\mathsf{H}^p = 0$ for $p > 2n$.

Example 2.6. Let use the cohomological Serre spectral sequence to compute the cohomology of the space $\mathsf{K}(\mathbb{Z},2)$ knowing that we have a homotopy fiber sequence

$$\begin{array}{ccc} \mathsf{S}^1 \simeq \mathsf{K}(\mathbb{Z},1) & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & \mathsf{K}(\mathbb{Z},2) \end{array}$$

Since $\pi_1(\mathsf{K}(\mathbb{Z},2)) = 0$ by construction, we find a cohomological spectral sequence

$$\mathrm{E}_2^{p,q} := \mathsf{H}^p(\mathsf{K}(\mathbb{Z},2)) \otimes \mathsf{H}^q(\mathsf{S}^1) \implies \mathsf{H}^{p+q}(*)$$

Writing this page explicitely in cohomological notation, we find

q	$p=-1$	$p=0$	$p=1$	$p=2$	$p=3$	$\cdots$
$q=2$	0	0	0	0	0	$\cdots$
$q=1$	0	$\mathbb{Z}$	$\mathsf{H}^1(\mathsf{K}(\mathbb{Z},2))$	$\mathsf{H}^2(\mathsf{K}(\mathbb{Z},2))$	$\mathsf{H}^3(\mathsf{K}(\mathbb{Z},2))$	$\cdots$
$q=0$	0	$\mathbb{Z}$	$\mathsf{H}^1(\mathsf{K}(\mathbb{Z},2))$	$\mathsf{H}^2(\mathsf{K}(\mathbb{Z},2))$	$\mathsf{H}^3(\mathsf{K}(\mathbb{Z},2))$	$\cdots$

Snce this spectral sequence is concentraded in rows one and two, it follows that it degenerates at the page 3, with $d_3 = 0$. Since $\mathsf{H}^0(*) = \mathbb{Z}$, and all other homology groups vanish of the point vanish, this page 3 is

q	$p=0$	$p=1$	$p=2$	$p=3$	$p=4$	$p=5$	$\cdots$
$q=2$	0	0	0	0	0	0	$\cdots$
$q=1$	0	0	0	0	0	0	$\cdots$
$q=0$	$\mathbb{Z}$	0	0	0	0	0	$\cdots$

It follows that all the maps in the E_2-page are isomorphisms. Therefore, we find

$$\mathsf{H}^{2n}\mathsf{K}(\mathbb{Z},2) = \mathbb{Z}$$

and zero otherwise. In particular, this shows that the class $1 \in \mathbb{Z} \simeq \mathsf{H}^2(\mathbb{CP}^\infty, \mathbb{Z})$ computed in Example **1.11**, seen as a homotopy class of maps

$$\mathbb{CP}^\infty \to \mathsf{K}(\mathbb{Z}, 2)$$

induces an isomorphism on singular cochains. In particular, the Hurewicz theorem implies that it is an equivalence.

Example 2.7. Let us illustrate how sometimes the data of the spectral sequence alone is not enough to compute the cohomology of the total space of a fibration. Let us consider the Hopf fibration

$$\begin{array}{ccc} \mathsf{S}^1 & \longrightarrow & \mathsf{S}^3 \\ \downarrow & & \downarrow \\ * & \longrightarrow & \mathsf{S}^2 \end{array}$$

The first page of the associated cohomological spectral sequence is

q	$p=0$	$p=1$	$p=2$	$p=3$	$\cdots$
$q=2$	0	0	0	0	$\cdots$
$q=1$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\cdots$
$q=0$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\cdots$

A priori, the only thing we know about the map d_2 is that it is a map of $\mathbb{Z}$-modules,

and therefore it is given by multiplication by some $n \in \mathbb{Z}$. In fact, we can show that n=1, because we know by other means that the cohomology of S^3 is zero in degrees 1 and 2, so that both the kernel and cokernl of d_2 have to vanish, meaning that d_2 is an isomorphism. But this observation is extrinsic to the spectral sequence. In the next section we will see that it is possible to endow the spectral sequence with extra structure that allows us to have more explicit knowledge of the boundary maps.

3. Multiplicative structure on the cohomological Serre spectral sequence

As seen in the previous example of the last section, sometimes the data of the spectral sequence is not enough to have a satisfactory description of the boundary maps. In this section we will discuss an extra structure on the cohomological Serre spectral sequence that allows us to solve this problem in some cases. It is motivated by the following example:

Example 3.1. We can now try to compute the cohomology of $\mathsf{K}(\mathbb{Z},3)$ with $\mathbb{Z}$ and $\mathbb{Q}$ coefficients, using the multiplicative structure on the cohomological Serre spectral sequence for the fiber product

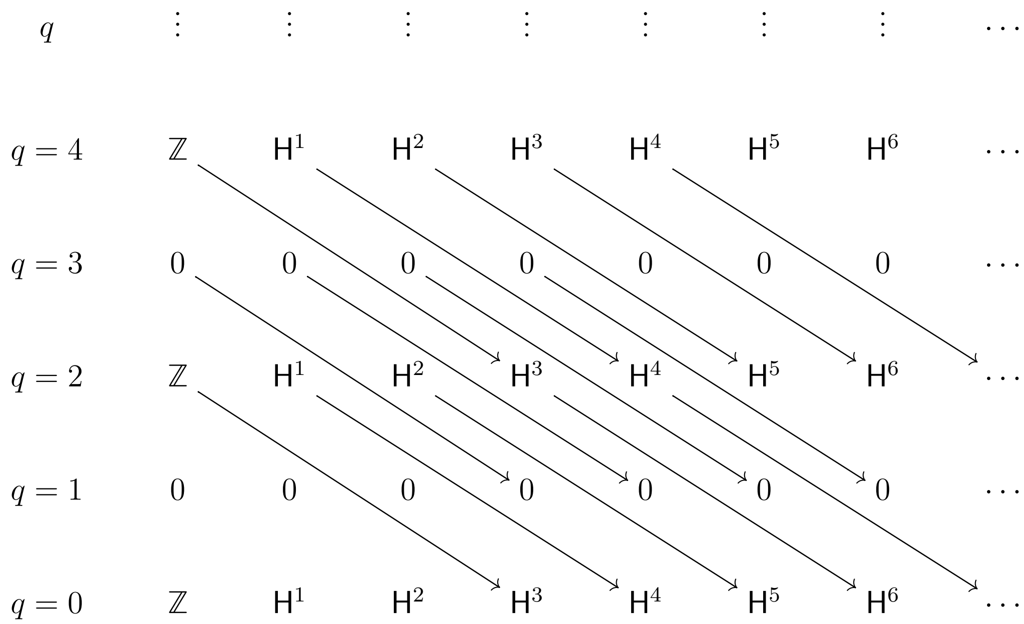
$$\begin{array}{ccc} \mathsf{K}(\mathbb{Z},2) \simeq \mathsf{K}(\mathbb{Z},1) & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & \mathsf{K}(\mathbb{Z},3) \end{array}$$

$$\mathrm{E}_2^{p,q} := \mathsf{H}^p(\mathsf{K}(\mathbb{Z},3),\mathbb{Z}) \otimes \mathsf{H}^q(\mathsf{K}(\mathbb{Z},2),\mathbb{Z}) \implies \mathsf{H}^{p+q}(*,\mathbb{Z})$$

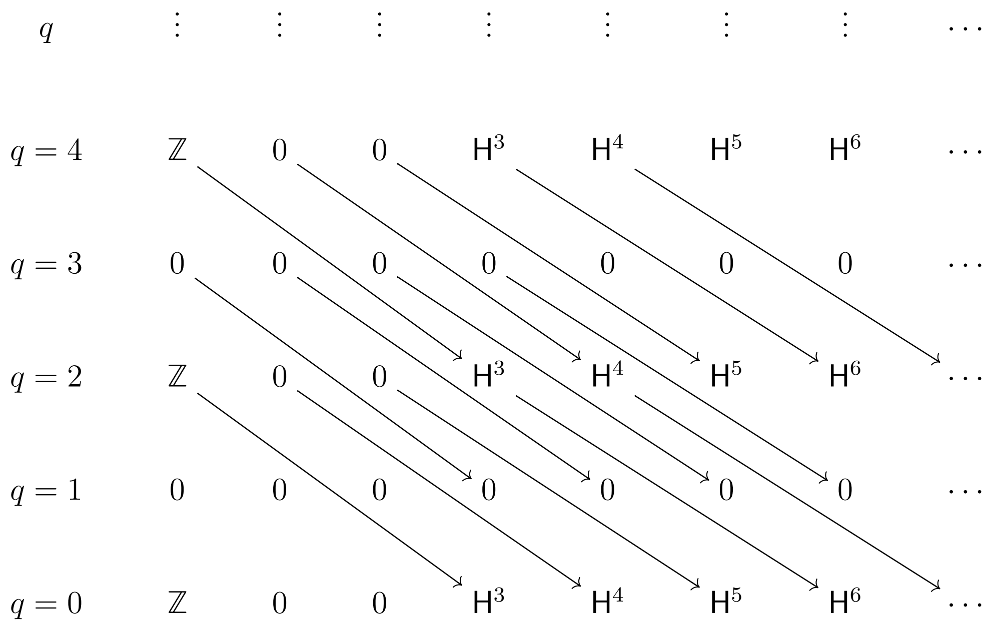
Writing $\mathsf{H}^p := \mathsf{H}^p(\mathsf{K}(\mathbb{Z},3),\mathbb{Z})$, we get a second page where all boundary maps vanish.

q	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$
$q=5$	0	0	0	0	0	0	0	$\cdots$
$q=4$	$\mathbb{Z}$	H^1	H^2	H^3	H^4	H^5	H^6	$\cdots$
$q=3$	0	0	0	0	0	0	0	$\cdots$
$q=2$	$\mathbb{Z}$	H^1	H^2	H^3	H^4	H^5	H^6	$\cdots$
$q=1$	0	0	0	0	0	0	0	$\cdots$
$q=0$	$\mathbb{Z}$	H^1	H^2	H^3	H^4	H^5	H^6	$\cdots$

Therefore, for the third page we find



Notice that this spectral sequence converges to $\mathsf{H}^{p+q}(*)$ which is concentrated in $p = q = 0$. Since the copies of $\mathsf{H}^1 = \mathrm{E}_2^{1,0}$ and $\mathsf{H}^2\mathrm{E}_2^{2,0}$, by the nature of their positions are not touched by any non-zero differentials for the higher pages of the spectral sequence, it follows that $\mathsf{H}^1 = \mathsf{H}^2 = 0$. We can re-write the spectral sequence as



The same discussion as for H^1 and H^2 above, the copy of $\mathsf{H}^4 = \mathrm{E}_2^{4,0}$ by the nature its position position is not touched by any non-zero differentials for the higher pages of the spectral sequence. Therefore it persist to the limit $r \to \infty$. It follows that

$\mathsf{H}^4 = 0$. Therefore the page E_3 is actually given by

q	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$
$q=4$	$\mathbb{Z}$	0	0	H^3	0	H^5	H^6	$\cdots$
$q=3$	0	0	0	0	0	0	0	$\cdots$
$q=2$	$\mathbb{Z}$	0	0	H^3	0	H^5	H^6	$\cdots$
$q=1$	0	0	0	0	0	0	0	$\cdots$
$q=0$	$\mathbb{Z}$	0	0	H^3	0	H^5	H^6	$\cdots$

Let us look at the first map $d : \mathbb{Z} \to \mathsf{H}^3$. We claim that this map is an isomorphism. Indeed, in the next page of the spectral sequence, we find

q	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$
$q=4$	?	0	0	?	0	H^5	?	$\cdots$
$q=3$	0	0	0	0	0	0	0	$\cdots$
$q=2$	$\ker d$	0	0	?	0	H^5	?	$\cdots$
$q=1$	0	0	0	0	0	0	0	$\cdots$
$q=0$	$\mathbb{Z}$	0	0	$\operatorname{coker} d$	0	H^5	?	$\cdots$

By the same arguments as above, we see that both $\operatorname{coker} d$ and $\ker d$ survive to the limit page and therefore have to vanish. This proves that the map d is an isomorphism was we wanted. Therefore, the E_3-page can be written as

q	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$
$q=4$	$\mathbb{Z}$	0	0	$\mathbb{Z}$	0	H^5	H^6	$\cdots$
$q=3$	0	0	0	0	0	0	0	$\cdots$
$q=2$	$\mathbb{Z}$	0	0	$\mathbb{Z}$	0	H^5	H^6	$\cdots$
$q=1$	0	0	0	0	0	0	0	$\cdots$
$q=0$	$\mathbb{Z}$	0	0	$\mathbb{Z}$	0	H^5	H^6	$\cdots$

Unfortunately, there is not much one can do at this point without further input.

Definition 3.2. A multiplicative structure on a cohomological spectral sequence E is the data of a commutative bigraded ring structure on the bigraded objects E_r (ie, if $x \in \mathrm{E}_r^{p,q}$ and $y \in \mathrm{E}_r^{p',q'}$ then $x.y \in \mathrm{E}_r^{p+p',q+q'}$), such that d_r is a graded derivation, ie, if $x \in \mathrm{E}_r^{p,q}$ and $y \in \mathrm{E}_r^{p',q'}$

$$d_r(x.y) = d_r(x).y + (-1)^{p+q} x.d_r(y)$$

As a result the cohomology groups with respect to d_r, $\mathsf{H}(\mathrm{E}_r^{p,q})$, are bigraded groups. We further ask that the isomorphisms

$$\mathsf{H}(\mathrm{E}_r^{p,q}) \simeq \mathrm{E}_{r+1}^{p,q}$$

are isomorphisms of bigraded groups.

Example 3.3. Let us compute the ring structure on the graded module $\mathsf{H}^*(\mathbb{CP}^n, \mathbb{Z})$ using the work already done in the Example **2.6**. Let us revisit the E_2 page

$$
\begin{array}{cccccccc}
p=0 & p=1 & p=2 & p=3 & p=4 & \cdots & p=2n & \cdots \\
 & & & & & & & \\
0 & 0 & 0 & 0 & 0 & \cdots & & \\
\mathbb{Z} \searrow & 0 & \mathsf{H}^2 \searrow & 0 & \mathsf{H}^4 & \cdots & \mathsf{H}^{2n} & \cdots \\
\mathbb{Z} & 0 & \mathsf{H}^2 & 0 & \mathsf{H}^4 & \cdots & \mathsf{H}^{2n} & \cdots
\end{array}
$$

We now know that it admits the structure of a bigraded-ring. Lets name ϵ the generator of $\mathsf{H}^1(\mathsf{S}^1, \mathbb{Z})$ and $s := d_2(\epsilon)$. Then the second page reads as

$$
\begin{array}{cccccccc}
\mathbb{Z}.\epsilon \searrow & 0 & \mathbb{Z} \searrow & 0 & \mathbb{Z} & \cdots & \mathbb{Z} & \cdots \\
\mathbb{Z} & 0 & \mathbb{Z}.s & 0 & \mathbb{Z} & \cdots & \mathbb{Z} & \cdots
\end{array}
$$

It follows from the bigraded rule that $\epsilon.s$ is in the copy of $\mathbb{Z} = \mathrm{E}_E^{2,1}$. We claim that this element is non-zero. Indeed, since multiplication by ϵ in $\mathsf{H}^*(\mathsf{S}^1, \mathbb{Z})$ induces an isomorphism between $\mathsf{H}^0(\mathsf{S}^1, \mathbb{Z})$ and $\mathsf{H}^1(\mathsf{S}^1, \mathbb{Z})$, it induces an isomorphism between the bottom line and the top line of the spectral sequence. Finally, using the fact that d_2 is a derivation we have

$$d_2(\epsilon.s) = d_2(\epsilon).s + (-1)^{1+0}.\epsilon.d_2(s) = s^2 - \epsilon.0 = s^2$$

Moreover the fact that d_2 is also an isomorphism, s^2 is non-zero. We can therefore write the second page as

$$
\begin{array}{cccccccc}
\mathbb{Z}.\epsilon \searrow & 0 & \mathbb{Z}.\epsilon.s \searrow & 0 & \mathbb{Z} & \cdots & \mathbb{Z} & \cdots \\
\mathbb{Z} & 0 & \mathbb{Z}.s & 0 & \mathbb{Z}.s^2 & \cdots & \mathbb{Z} & \cdots
\end{array}
$$

We can iterate this argument to show all the way until we reach $p = 2n$, where we find

$$
\begin{array}{cccccccc}
\mathbb{Z}.\epsilon \searrow & 0 & \mathbb{Z}.\epsilon.s \searrow & 0 & \mathbb{Z} & \cdots & \mathbb{Z}.\epsilon.s^n & \cdots \\
\mathbb{Z} & 0 & \mathbb{Z}.s & 0 & \mathbb{Z}.s^2 & \cdots & \mathbb{Z}.s^n & \cdots
\end{array}
$$

and running it only one last time, we deduce that $s^{n+1} = 0$. In other words, we proved that as graded algebras we have

$$\mathsf{H}^*(\mathbb{CP}^n) \simeq \mathbb{Z}[s]/(s^{n+1})$$

where s is a generator in degree 2. The same computation applied to the Example **2.6** to get that

$$\mathsf{H}^*(\mathbb{CP}^\infty) \simeq \mathbb{Z}[s]$$

is a polynomial algebra with a free generator in degree 2.

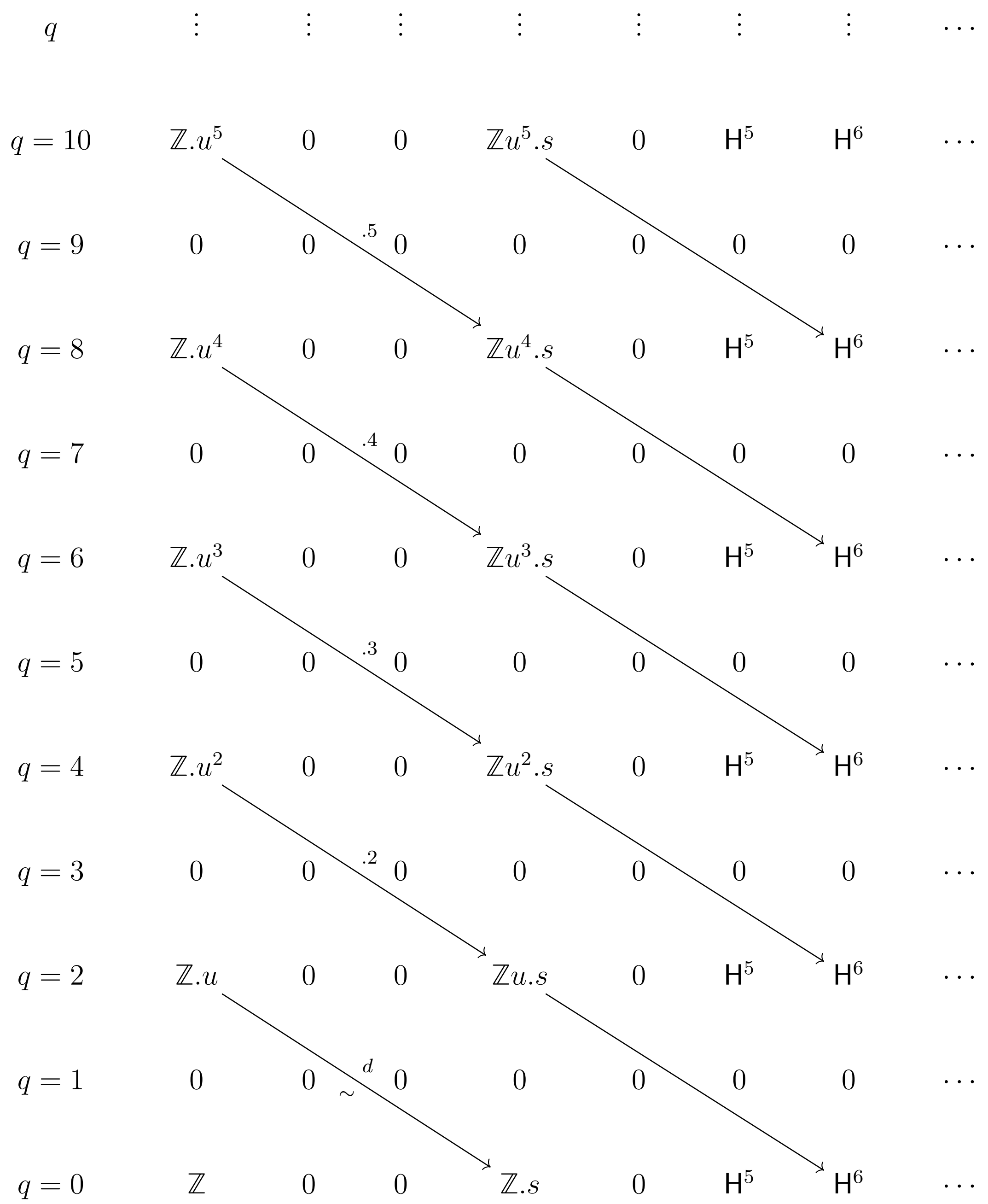
Example 3.4. Now that we are aware of the multiplicative structure, let us continue the Example **3.1** above. The cup-product operation on the graded algebra $\mathbb{H}^*(\mathsf{K}(\mathbb{Z},3),\mathbb{Z})$ manifests itself as an operation between the different pieces of the spectral sequence. For this purpose we use the computation of the ring structure on $\mathbb{H}^*(\mathsf{K}(\mathbb{Z},2),\mathbb{Z}) = \mathbb{H}^*(\mathbb{CP}^\infty,\mathbb{Z}) = \mathbb{Z}[u]$ of the Example **3.3** the free polynomial algebra with a generator in coh. degree 2. Moreover, the second page of the spectral sequence is obtained as a tensor product of two graded rings

$$\mathbb{H}^*(\mathsf{K}(\mathbb{Z},2),\mathbb{Z}) \otimes \mathbb{H}^*(\mathsf{K}(\mathbb{Z},3),\mathbb{Z})$$

and the differentials d_3 are derivations with respect to the bigraded ring structure on the tensor product. Let us write $s := d(u)$. It follows from the formula for derivations

$$d_3(u^n) = n.u^{n-1}.du = n.u^{n-1}.s$$

that we can the E_3-page can be written as

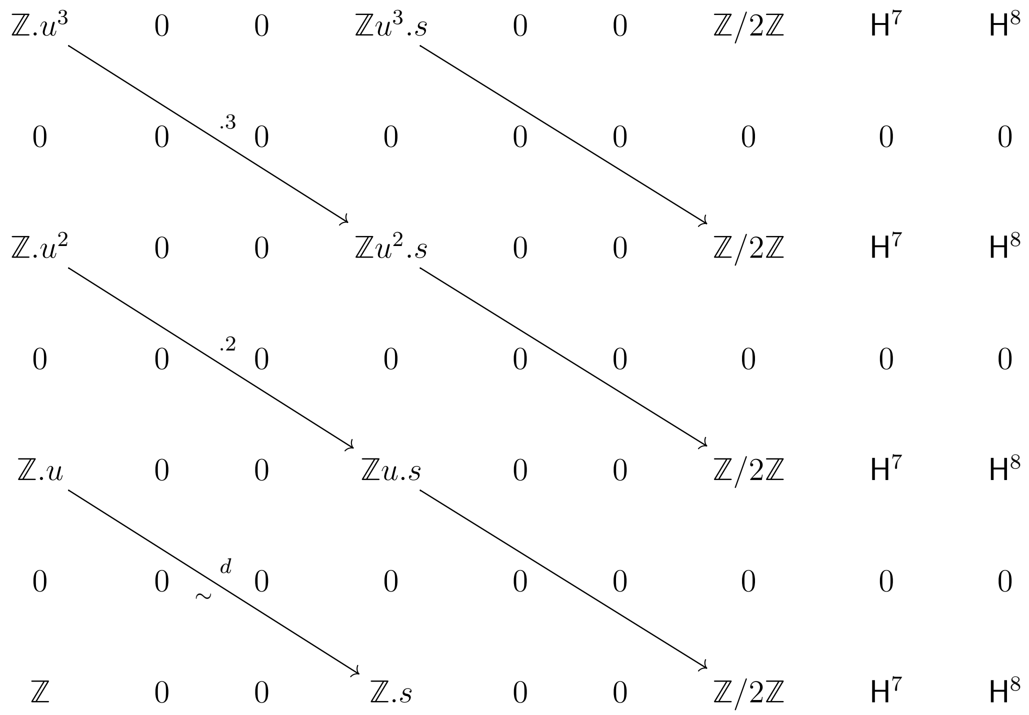


Since the multiplication by 2, is injective, we deduce that the copy of $\mathsf{H}^5 = E_2^{5,0}$ is

persistant in the spectral sequence, so that it must be zero. Also for degree reasons, and taking into account the shape of the last page, we deduce that the sequence

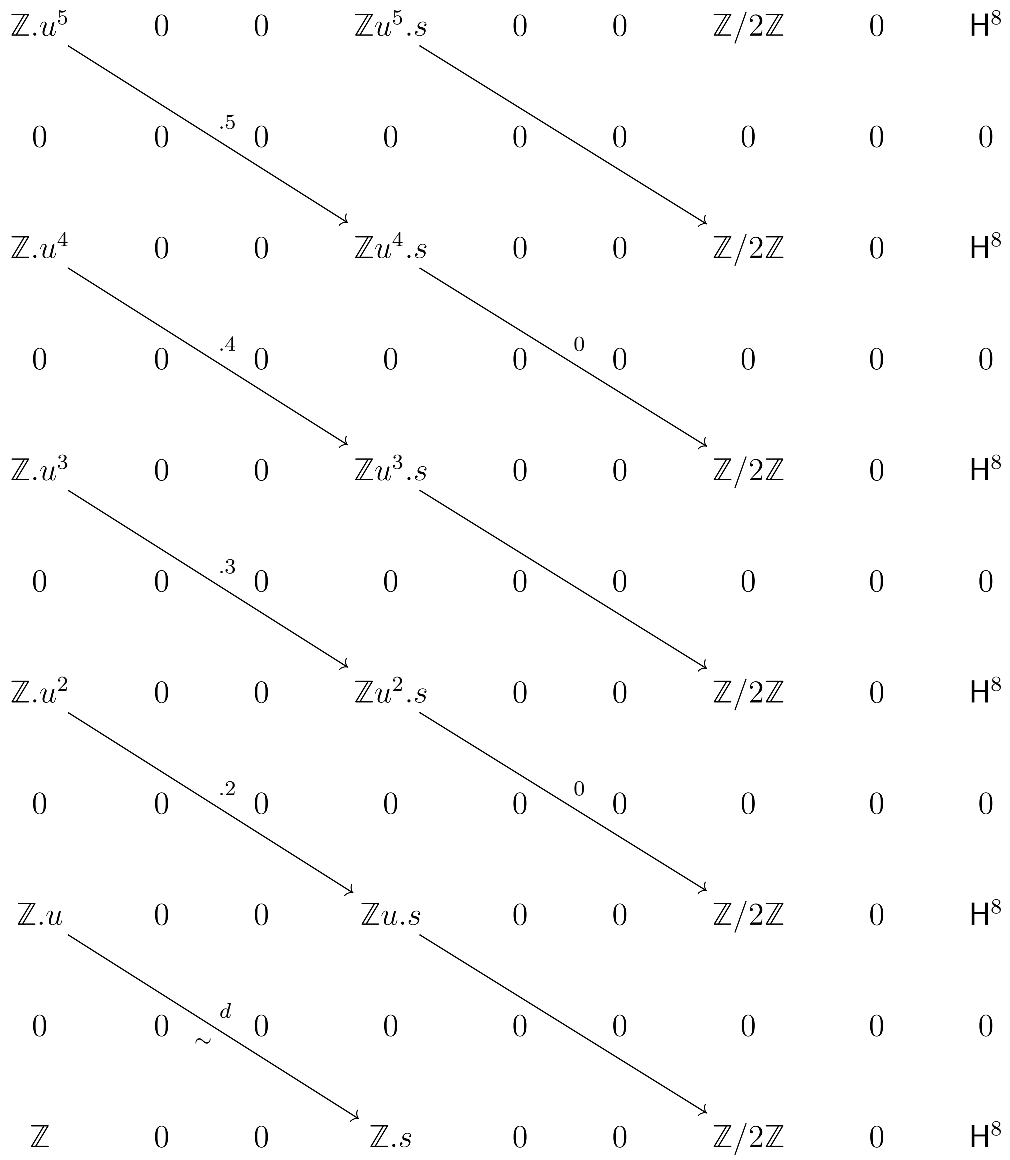
$$\mathbb{Z} \to_{.2} \mathbb{Z} \to \mathsf{H}^6 \to 0$$

is exact on the right. It follows that $\mathsf{H}^6 \simeq \mathbb{Z}/2\mathbb{Z}$. Finally, the E_3-page reads as



Notice that $d_3(u^2.s) = 2.ud_3(u).s = 2.u.s.s = 2u.s^2 = u.2.s^2$ and $2.s^2 = 0$ in H^6.

Similarly, we have $d_3(u^n.s) = n.u^{n-1}d_3(u).s = n.u^{n-1}.s.s = n.u.s^2 = u.n.s^2$ which is 0 if n is even.
Also, an element in $\mathsf{H}^7 = E_2^{7,0}$ can only be killed by the element u^3 in the page E_7. But since multiplication by 3 is injective, this element is already killed in page 4. We conclude that $\mathsf{H}^7 = 0$.
We have:

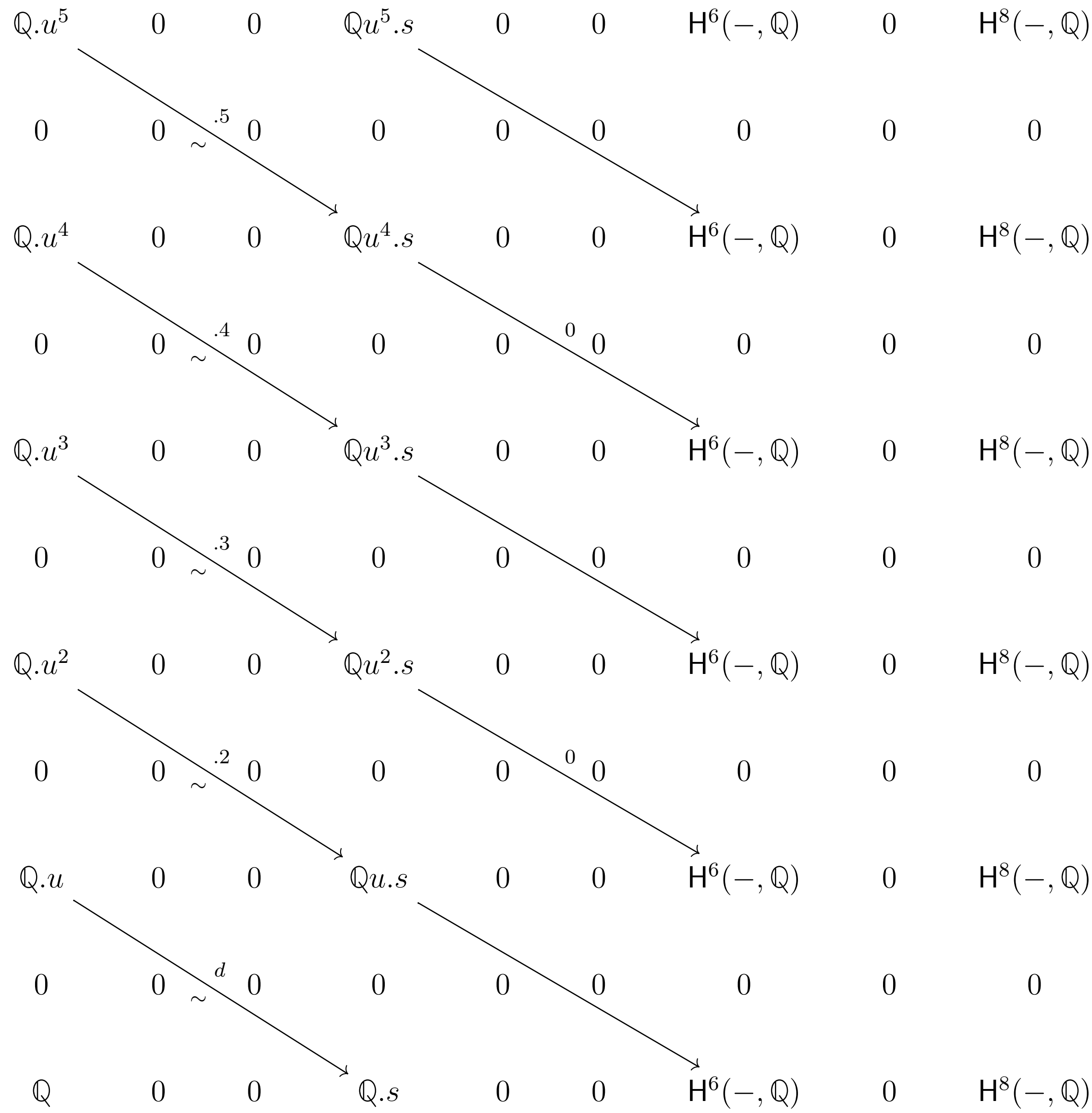


We see that in the place of $u^2.s$ in page $4 =$ page 5, appears a copy of $\mathbb{Z}/3\mathbb{Z}$. But

we know that at this copy is not present at infinity. The only way this can happen is if the map $d_5 : \mathbb{Z}/3\mathbb{Z} \to \mathsf{H}^8$, determined by the element $t := d_5(u^2.s)$, is an isomorphism. So far we have computed all the following cohomology groups:

$p=0$	$p=1$	$p=2$	$p=3$	$p=4$	$p=5$	$p=6$	$p=7$	$p=8$
$\mathbb{Z}$	0	0	$\mathbb{Z}.s$	0	0	$\mathbb{Z}/2\mathbb{Z}$	0	$\mathbb{Z}/3\mathbb{Z}$

We could try to continue the computation (see this seminar of Cartan) but we it would be impossible to compute H^{14} without further input. Instead let us observe that what we have so far is enough to deduce the cohomology groups of $\mathsf{K}(\mathbb{Z},3)$ with coefficients in $\mathbb{Q}$. Indeed, in this case, the maps in page 3 given by multiplication by n become isomorphisms and the E_3-page with rational coefficients becomes



Therefore, for $E_4 = E_5$, we find

$$\begin{array}{ccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & \mathsf{H}^6(-,\mathbb{Q}) & 0 & \mathsf{H}^8(-,\mathbb{Q}) \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \mathsf{H}^6(-,\mathbb{Q}) & 0 & \mathsf{H}^8(-,\mathbb{Q}) \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \mathsf{H}^6(-,\mathbb{Q}) & 0 & \mathsf{H}^8(-,\mathbb{Q}) \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \mathsf{H}^6(-,\mathbb{Q}) & 0 & \mathsf{H}^8(-,\mathbb{Q}) \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \mathsf{H}^6(-,\mathbb{Q}) & 0 & \mathsf{H}^8(-,\mathbb{Q}) \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\mathbb{Q} & 0 & 0 & 0 & 0 & 0 & \mathsf{H}^6(-,\mathbb{Q}) & 0 & \mathsf{H}^8(-,\mathbb{Q})
\end{array}$$

We see that for $\mathsf{H}^p(-,\mathbb{Q}) = 0$ for $p \geq 3$ and that $\mathsf{H}^3(-,\mathbb{Q}) = \mathbb{Q}$. Finally, to compute

the graded ring structure $\mathsf{H}^*(\mathsf{K}(\mathbb{Z},3),\mathbb{Q})$ we see that

$$d_3(u.s) = d_3(u).s + (-1)^{2+0}.u.d_3(s) = s.s + 0 = s^2$$

that has to be zero because as we just saw, $\mathsf{H}^6(-,\mathbb{Q}) = E_3^{6,0} = 0$. It follows that $\mathsf{H}^*(\mathsf{K}(\mathbb{Z},3),\mathbb{Q})$ is the graded exterior algebra with a generator in degree 3.

Remark 3.5. More generally, the argument in the previous example can be used by induction to show that the cohomology ring of $\mathsf{H}^*(\mathsf{K}(\mathbb{Z}, n), \mathbb{Q})$ is the free graded algebra with a generator in degree n (see Here, Prop. 5.21).

4. Spectral sequence of the geometric realization of a simplicial object

This section is a quick summary of the results in [Lur17, Section 1.2.4].

Let us consider a simplicial object $X : \Delta^{\mathsf{op}} \to \mathsf{C}$ with C a stable ∞-category with a t-structure. Let us assume that $|X| := \underset{\Delta^{\mathsf{op}}}{\mathsf{colim}} X$- its geometric realization exists in C. We will construct a spectral sequence converging to the homotopy groups $\pi_n|X|$.

Let us denote by $\Delta^{\mathsf{op}}_{\leq n} \subseteq \Delta^{\mathsf{op}}$ the full subcategory of Δ^{op} spanned by those objects $[m]$ with $m \leq n$. Let us denote by $X_{\leq n}$ the restriction

$$\Delta^{\mathsf{op}}_{\leq n} \subseteq \Delta^{\mathsf{op}} \to \mathsf{C}$$

Then using the decomposition of Δ^{op} given by the functor

$$\mathbb{N} \to \mathsf{SSets}/\Delta^{\mathsf{op}} \qquad\qquad n \mapsto \Delta^{\mathsf{op}}_{\leq n}$$

and [Lur09, 4.2.3.4, 4.2.3.9, 4.2.3.10], we obtain a new formula for the geometric realization as a colimit

$$|X| \simeq \underset{n}{\mathsf{colim}} \left(\underset{\Delta^{\mathsf{op}}_{\leq n}}{\mathsf{colim}}\, X_{\leq n} \right) \simeq \underset{n}{\mathsf{colim}} \left(\underset{\Delta^{\mathsf{op}}_{\leq 0}}{\mathsf{colim}}\, X_{\leq 0} = X([0]) \to \underset{\Delta^{\mathsf{op}}_{\leq 1}}{\mathsf{colim}}\, X_{\leq 1} \to \cdots \right)$$

Let us write

$$D_n := \underset{\Delta^{\mathsf{op}}_{\leq n}}{\mathsf{colim}}\, X_{\leq n}$$

$$|X| \simeq \underset{n}{\mathsf{colim}}\, (D_0 \to D_1 \to D_2 \to \cdots)$$

and therefore, we are in the situation of the Section **1** where we can produce a spectral sequence that converges to the homotopy groups of the colimit

$$\mathrm{E}_1^{p,q} := \pi_{p+q}(D_p/D_{p-1}) \implies \pi_{p+q}|X|$$

if we assume that the t-structure in C is compatible with filtered colimits.

We can provide a more explicit description of the object D_p/D_{p-1}, ie, of the cofiber of

$$\operatorname{colim}_{\Delta^{\mathsf{op}}_{\leq p-1}} X_{\leq p-1} \to \operatorname{colim}_{\Delta^{\mathsf{op}}_{\leq p}} X_{\leq p}$$

Indeed, let us write $\mathsf{LKE}_{(n-1)\to n}(X_{\leq n-1})$ for the left Kan extension of $X_{\leq n-1}$ along the inclusion

$$\Delta^{\mathsf{op}}_{\leq n-1} \subseteq \Delta^{\mathsf{op}}_{\leq}$$

It follows from the definition of colimits as a left Kan extension that

$$\operatorname{colim}_{\Delta^{\mathsf{op}}_{\leq p-1}} X_{\leq p-1} \simeq \operatorname{colim}_{\Delta^{\mathsf{op}}_{\leq p}} \mathsf{LKE}_{(p-1)\to p}(X_{\leq p-1})$$

It follows from the definitions that for each p we have a natural transformation of functors

$$\Delta^{\mathsf{op}}_{\leq p} \to \mathsf{C} \qquad\qquad \mathsf{LKE}_{p-1\to p}(X_{\leq p-1}) \to X_{\leq p}$$

which induces equivalences for

$$\mathsf{LKE}_{n-1\to n}(X_{\leq n-1})([m]) \to X_{\leq n}([m]) \quad \forall\ m \leq n-1$$

Let us write C_n for the cofiber

$$\begin{array}{ccc} \mathsf{LKE}_{n-1\to n}(X_{\leq n-1})([n]) & \longrightarrow & X_{\leq n}([n]) \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & C_n \end{array}$$

Since cofibers of diagrams are computed objectwise, the cofiber of $LKE_{p-1\to p}(X_{\leq p-1}) \to X_{\leq p}$ is the functor $\Delta^{\mathsf{op}}_{\leq p} \to \mathsf{C}$ given by C_p in level p and zero everywhere else. It follows from [Lur17, 1.2.4.18] that the colimit of this functor $\Delta^{\mathsf{op}}_{\leq p} \to \mathsf{C}$ is given by $C_p[p]$. Therefore, we find

$$D_p/D_{p-1} \simeq C_p[p]$$

As in the Remark **1.8**, the first page is given by applying π_q to the sequence of boundary maps

$$\cdots \to C_p[p][-p] \cdots \to C_2[2][-2] \to C_1[1][-1] \to D_0 = X_0$$

$$\mathrm{E}_1^{p,q} = \pi_q(C_p)$$

If we now assume that the simplicial object X takes values in $\mathsf{C}_{\geq 0}$ we can drop assumption that the t-structure is compatible with filtered colimits and still get a stronger notion of convergence for the spectral sequence. Indeed, in this case since $\mathsf{C}_{\geq 0} \subseteq \mathsf{C}$ is stable under all colimits, we find that all D_p, C_p and $|X|$ are $\mathsf{C}_{\geq 0}$. It follows that the first page of the spectral sequence $\mathrm{E}_1^{p,q}$ is concentrated in the first quadrant $(p, q \geq 0)$ and so are all the higher pages. In particular, we also see that since C_p is in $\mathsf{C}_{\geq 0}$, $\mathsf{C}_p[p] \simeq D_p/D_{p-1}$ is in $\mathsf{C}_{\geq p}$. It follows from the long exact sequence of homotopy groups associated to the cofiber-sequence

$$\begin{array}{ccc} D_{p-1} & \longrightarrow & D_p \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & C_p[p] \end{array}$$

that

$$\begin{array}{ccccc} \cdots & & \cdots & \longrightarrow & \pi_p(C_p[p]) \\ \pi_{p-1}D_{p-1} & \longrightarrow & \pi_{p-1}D_p & \longrightarrow & \pi_{p-1}(C_p[p]) = 0 \\ \pi_{p-2}D_{p-1} & \longrightarrow & \pi_{p-2}D_p & \longrightarrow & \pi_{p-2}(C_p[p]) = 0 \cdots \end{array}$$

showing that

$$\pi_{p-1}D_{p-1} \to \pi_{p-1}D_p$$

is surjective and that

$$\pi_{p-2}D_{p-1} \to \pi_{p-2}D_p$$

is an isomorphism. By induction, this shows that

$$\pi_n(D_{n+1}) \simeq \pi_n(D_{n+2}) \simeq \pi_n(D_{n+3}) \simeq \cdots \simeq \pi_n(|X|)$$

We are therefore in the context of the Remark **1.9**.

Remark 4.1. We can get reasonable connectivity estimates for C_p even without the hypothesis of a t-structure on C. This is what is done in [Lur17, 1.2.4.7]. Indeed, one can show that $X([n])$ is a split direct sum $L_n \oplus C_n$ using the usual Dold-Kan construction for additive idempotent complete categories. If we assume then that C has a t-structure and that $X([n]) \in \mathsf{C}_{\geq 0}$, it follows that C_n also is connective.

Remark 4.2. The complex $\pi_q(C_p)$ is the normalized chain complex associated to the simplicial object $\pi_q(X)$.

5. Descent Spectral Sequence

Proposition 5.1. *Let X be a topological space and $\{U_i\}_{i\in I}$ be an open cover. Let $f : \coprod_{i\in I} U_i \to X$ be the canonical map and $\mathsf{N}_\bullet(f)$ its nerve. Then the canonical map*

$$\mathsf{colim}_{\Delta^{\mathsf{op}}} \mathsf{N}_\bullet(f) \to X$$

is an equivalence in S.

Proof. See for instance Dugger Notes for the cech cover - Theorem 1.1 □

Remark 5.2. As a consequence of the previous proposition and of the discussion in the previous section, one can compute the cohomology of a space X by means of a spectral sequence. Namely, we have

$$\mathsf{C}^*(X, k) \simeq \mathsf{lim}_{\mathsf{N}(\Delta^{op})} \mathsf{C}^*(\mathsf{N}_\bullet(f), k)$$

and the first page of the spectral sequence is given by the normalized Dold-Kan complex of the co-simplicial abelian group $\pi_q(\mathsf{C}^*(\mathsf{N}_\bullet(f), k))$.

Proposition 5.3 (Leray)**.** *Let X be a topological space and $\{U_i\}_{i\in I}$ be a cover. Let C denote the category whose objects are the opens in the cover and their intersections and whose morphisms are given by inclusions and let $N(C)$ denote its nerve as a simplicial set. Then if $\{U_i\}_{i\in I}$ is a good cover, ie, all intersections are either empty or contractible, then $N(C)$ is homotopy equivalent to X, ie, if W denotes the class of all morphisms in $N(C)$ then $N(C)[W^{-1}] \simeq X$.*

Proof. Use Lemma 5.4.5.10 together with the fact that $\{U_i\}$ is cofinal in $Disk(X)$. The fact that the colimit of Disk(X) gives X follows from Proposition A.3.2 and A.3.1. Notice that the Dugger - Corollary 3.5 is the same as Lemma A.3.3. □

This section is a quick survey of the discussion in [PY16, §8] presenting the Leray spectral sequence to compute global sections of a sheaf using the discussion of the previous section.

References

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