An Introduction to Higher Categories

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Bibliography: J. Lurie's HTT & HA

Platonic Form

Ideally, an ∞-category is a mathematical object that assembles:

- *objects*,
- *1-morphisms between objects*,
- for every $n \geq 2$, a notion of *n-morphisms* between $n-1$-*morphisms*;
- for every $n \geq 1$, (weak) composition laws of n-morphisms only well-defined up to the data of higher morphisms.
- Associativity of compositions up to homotopy:

Proto-Example (Fundamental ∞-groupoid) X a CW-complex.

- Objects = points,
- 1-morphisms=paths,
- 2-morphisms=homotopy of paths (2-cells);
- 3-morphisms= homotopies between homotopies of paths (3-cells)
- ...

Models:

Problem: No direct definition of higher categories simultaneously operational and close to this platonic form (infinite axioms!).

First Breakthrough : Avoid the problem by modeling the platonic form of ∞-categories using "exaggerated" templates/models that contain more structure than what the platonic form requires.

Formally: Find a model category whose objects serve as models for ∞-categories (Quasi-categories, Segal Categories, Simplicial Categories, etc).

Modeling is a common practice:

- Homotopy Theory of Spaces (Homotopy Types) $\rightsquigarrow$ Modeled by topological spaces, simplicial sets, categories, etc
- Homotopy Theory of homotopy-commutative algebras over $\mathbb{Q}$: $\rightsquigarrow$ Modeled by simplicial algebras, diff. graded algebras.
- Derived and Higher Stacks (Modeled by simplicial presheaves);

Models: why important?

Question: What is the fundamental role of models and why are there are so many for the same theory?

Second Breakthrough: Every model category has an associated ∞-category which captures all the important information of the model structure (Dwyer-Kan Localization)

Models play a double role: (operational) Need ambient model to shape ∞-categories; (fundamental) Every other model incarnates as an object of this ambient model.

Consequence: Plenty of examples of wannabe ∞-categories:

- $\exists$ ∞-category of spaces $\mathcal{S}$: Model structure on topological spaces/simplicial sets;
- ∞-category of derived affine schemes and derived stacks;
- ∞-category of chain complexes up to quasi-iso..

Models: drawbacks

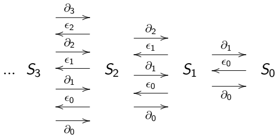
Question: If we already have the explicit models, why do we care about their associated ∞-categories?

Answer:

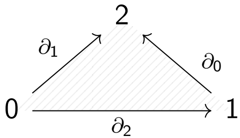
- Not all ∞-categories have a practical model presentation (Typical Examples: ∞-categories of algebra-objects in a $\otimes - \infty$-category: ring spectra);
- There is no sufficiently refined notion of functor to relate different models. The relevant notion is that of ∞-functor between the ∞-categories associated to the models.
- Models for diagrams are not in general given by diagrams of models.

Modeling with Simplicial Sets: Notations

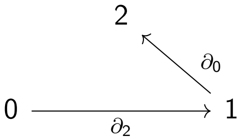
Category Δ: objects = finite ordered sets $[n] = \{0 < 1 < ... < n\}$; morphisms = order-preserving maps; SSets $:= \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Sets})$.



$\Delta[n]$:= Yoneda $([n])$; $S_n = \mathrm{Hom}(\Delta[n], S)$; $\Delta[2] :=$



$\Lambda_n^j :=$ $\Delta[n]$ - the jth face and the interior ; $\Lambda_2^1 =$



S simplicial set $\rightsquigarrow |S|$ its topological realization; Equivalences of simplicial sets := weak-homotopy equivalences ($\pi_\bullet(|-|)$-eq.)

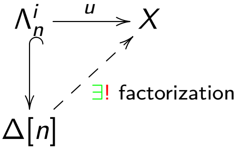
Modeling with Simplicial Sets

Modeling usual categories: $\mathcal{C}$ usual category $\rightsquigarrow$ simplicial set $\mathrm{N}(\mathcal{C})$ (Nerve) with

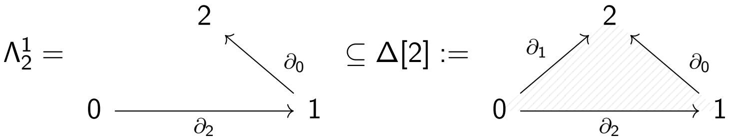
$$\{\text{n-simplexes}\} := \{\textit{strings}\ X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \ldots. \xrightarrow{f_{n-1}} X_n\ \}$$

Boundary maps ∂_i encode composition law. Degeneracy maps ϵ_i encode identity maps.

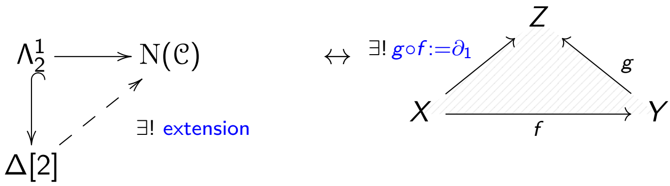
- $\{\text{Functors } \mathcal{C} \to \mathcal{D}\} \simeq \{\text{Simplicial Maps } \mathrm{N}(\mathcal{C}) \to \mathrm{N}(\mathcal{D})\}$
- $X \simeq \mathrm{N}(\mathcal{C}) \leftrightarrow \forall n \geq 2,\ \forall 0 < i < n,\ \forall\, u : \Lambda^i_n \to X$



Generating compositions: Take diagram shapes



$$\Lambda^1_2 \to \mathrm{N}(\mathcal{C}) \leftrightarrow \text{string of morphisms } X \xrightarrow{f} Y \xrightarrow{g} Z \text{ in } \mathcal{C}$$



∃! extension $\leftrightarrow$ ∃! compositions $\leftrightarrow$ shaded faces

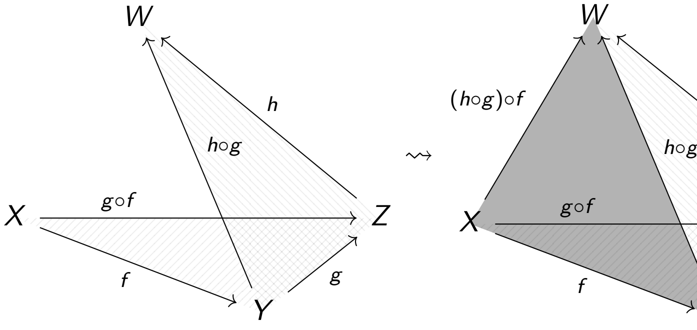
Modeling with Simplicial Sets

Processing Associativity:

$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$ in $\mathcal{C} \leftrightarrow \Lambda^1_2 \xrightarrow{(g,f)} \mathrm{N}(\mathcal{C}) + \Lambda^1_2 \xrightarrow{(h,g)} \mathrm{N}(\mathcal{C})$

Generate Compositions: $\Delta[2] \xrightarrow{(g,f)} \mathrm{N}(\mathcal{C}) + \Delta[2] \xrightarrow{(h,g)} \mathrm{N}(\mathcal{C})$

Glue

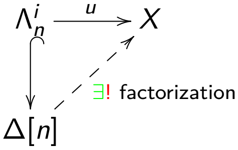


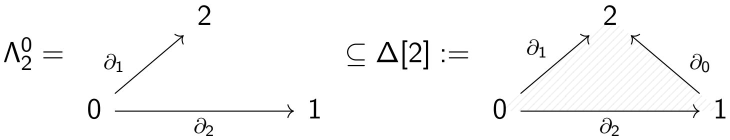
$\Lambda^1_3 \longrightarrow \mathrm{N}\mathcal{C}$

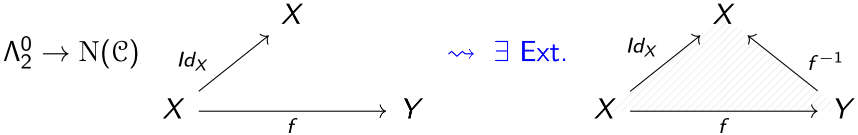
$(h \circ g) \circ f = h \circ (g \circ f)$ $\leftrightarrow$ Filling back face $\leftrightarrow$ Extend along $\Lambda^1_3 \subseteq \Delta[3]$

Modeling with Simplicial Sets

Groupoids: $X \simeq \mathrm{N}(\mathcal{C} \text{ groupoid }) \leftrightarrow \forall n \geq 2,\ \forall 0 \leq i \leq n,\ \forall\, u : \Lambda_n^i \to X$







Summary: Allowing lifting property for extremes $i = 0, n$ gives inverses. For terms in the middle $0 < i < n$ gives compositions.

Modeling with Simplicial Sets

Modeling spaces: T topological space $\rightsquigarrow$ simplicial set $\mathrm{Sing}(T)$ with n-simplexes given by continuous maps from the topological n-simplex Δ^n to T

$X \simeq Sing(T) \rightarrow \forall n \geq 2,\ \forall 0 \leq i \leq n,\ \forall\, u : \Lambda_n^i \rightarrow X\ \exists$ factorization not necessarily unique:

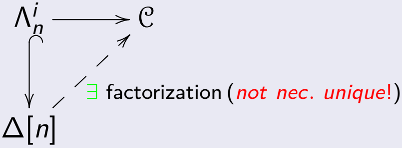
$$\begin{array}{ccc} \Lambda_n^i & \longrightarrow & X \\ \big\downarrow & \nearrow & \\ \Delta[n] & & \end{array}$$

Slogan: Kan-complexes are those simplicial sets where the direction of the arrows is irrelevant.

Modeling ∞-categories: A Quasi-category is a simplicial set which shares simultaneously features of categories and spaces:

Definition

A Quasi-category is a simplicial set $\mathcal{C}$ with the following property: $\forall n \geq 2,\ \forall\, 0 < i < n,\ \forall\, u : \Lambda^i_n \to \mathcal{C}$



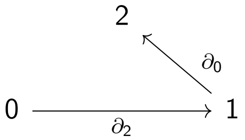
Examples:

- $\mathcal{C}$ category $\implies$ $\mathrm{N}(\mathcal{C})$ quasi-category with uniquely-defined compositions;
- X Kan complex $\implies$ X quasi-category;

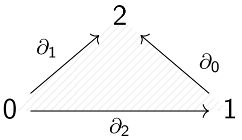
$\mathcal{C}$ quasi-category. How far can we go?

Generating "Compositions":

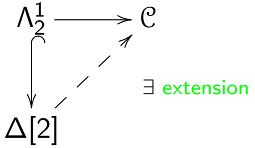
$\Lambda_2^1 =$



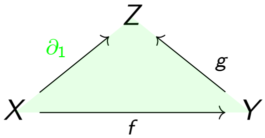
$\subseteq \Delta[2] :=$



$\Lambda_2^1 \to \mathcal{C} \leftrightarrow$ string of 1-simplexes $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{C}$



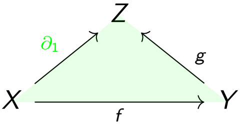
$\leftrightarrow$



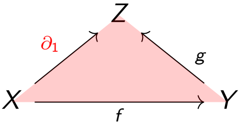
Each 2-simplex in $\mathcal{C}$ "makes a form of commutativity". There can be many!

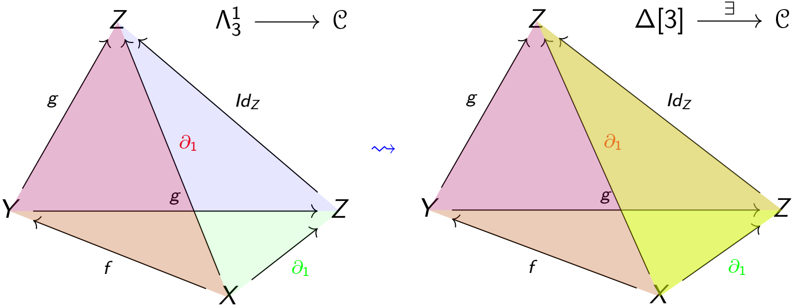
Control of non-uniqueness of "compositions":

$u_1 : \Delta[2] \to \mathcal{C}$



$u_2 : \Delta[2] \to \mathcal{C}$





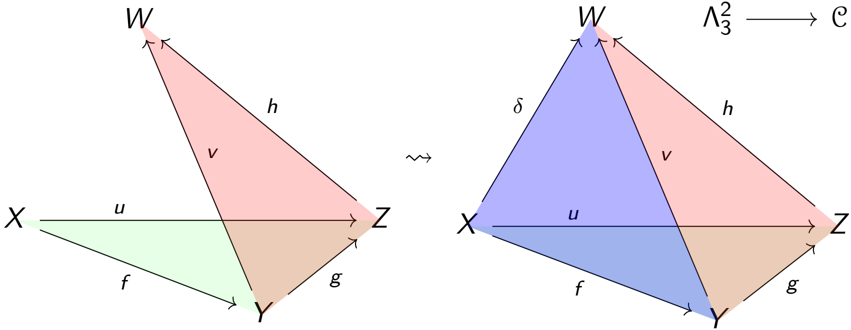
2-simplex $\leftrightarrow$ homotopy of compositions $\partial_1 \sim \partial_1$. Lifting Mechanism provides compatibility between compositions.

More Generally: compatibility of n different compositions $\rightsquigarrow$ n-simplexes.

Processing "Associativity":

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W \text{ in } \mathcal{C} \leftrightarrow \Lambda^1_2 \xrightarrow{(g,f)} \mathcal{C} + \Lambda^1_2 \xrightarrow{(h,g)} \mathcal{C}$$

Exhibit compositions: $\Delta[2] \xrightarrow{(g,f)} \mathcal{C} + \Delta[2] \xrightarrow{(h,g)} \mathcal{C}$



$\delta \sim$ "$h \circ u$" $\leftrightarrow$ Filling back face $\leftrightarrow$ Extend along $\Lambda^1_3 \subseteq \Delta[3]$

Quasi-categories as ∞-categories: ∞-functors

Definition: An ∞-category is a Quasi-category. Objects= 0-simplexes. Morphisms = 1-simplexes.

Definition: An ∞-functor is a map of simplicial sets between quasi-categories.

Explanation: Functors $\leftrightarrow$ functions that preserve commutative diagrams $\leftrightarrow$ send n-simplexes to n-simplexes and preserve boundaries.

Prop.(∞-category of functors) $\mathcal{C}$ ∞-category, K simplicial set (diagram shape):

$$\mathrm{Fun}(K, \mathcal{C}) := \underline{\mathrm{Hom}_{\Delta}}(K, \mathcal{C}) \quad \text{(internal-hom)}$$

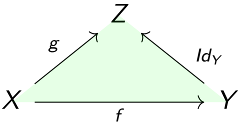
is a ∞-category.

Prop.(Products) $\mathcal{C}, \mathcal{D}$ ∞-categories $\implies$ $\mathcal{C} \times \mathcal{D}$ ∞-category.

Quasi-categories as ∞-categories: Homotopy Category

$\mathcal{C}$ ∞-category $\rightsquigarrow$ truncation (forgets higher cells) produces usual category $\mathrm{h}\mathcal{C}$ (homotopy category):

- Objects of $\mathrm{h}\mathcal{C}$: 0-simplexes of $\mathcal{C}$;
- Morphisms of $\mathrm{h}\mathcal{C}$:= homotopy classes of 1-morphisms: f, g are equivalent iff there exists a 2-morphism $u : \Delta[2] \to \mathcal{C}$



rendering f homotopic to g

- Compositions: Well-defined using the lifting property.

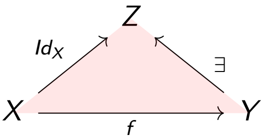
Definition: Subcategory of a quasi-category $\mathcal{C}$ is a sub-simplicial set $\mathcal{C}'$ obtained as a fiber product in simplicial sets

$$\begin{array}{ccc} \mathcal{C}' & \longrightarrow & \mathcal{C} \\ \downarrow & & \downarrow \\ \mathrm{N}(\mathrm{D}) & \longrightarrow & \mathrm{N}(\mathrm{h}\mathcal{C})) \end{array}$$

with D a subcategory of $\mathrm{h}\mathcal{C}$.

Quasi-categories as ∞-categories: Equivalences

Definition: A 1-morphisms f of $\mathcal{C}$ is said to be an equivalence if its homotopy class $[f]$ in $\mathrm{h}\mathcal{C}$ is an isomorphism, ie,$\exists$



rendering f invertible up to a 2-cell

Definition: An ∞-category is an ∞-groupoid if all its 1-morphisms are invertible.

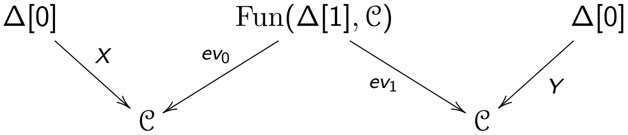
Prop: An $\mathcal{C}$ ∞-category is an ∞-groupoid if and only if it is a Kan-complex.

Example: T topological Space $\rightsquigarrow$ (Proto-Example) Fundamental ∞-groupoid of $T := \mathrm{Sing}(T)$.

Quasi-categories as ∞-categories: Mapping Spaces

$\mathcal{C}$ ∞-category. X, Y objects $\rightsquigarrow$ $\exists$ "space" of morphisms $X \to Y$,

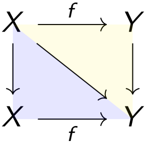
Definition: The mapping space between X and Y is the simplicial set $\mathrm{Map}_{\mathcal{C}}(X, Y)$ defined as the fiber product



Proposition: Let $\mathcal{C}$ be an ∞-category. Then $\mathrm{Map}_{\mathcal{C}}(X, Y)$ is a Kan-complex. ($\leftrightarrow$"for $n \geq 2$, n-morphisms are invertible").

0-simpl.$:=$ $f : \Delta[1] \to \mathcal{C}$ with $\partial_1(f) = X$, $\partial_0(f) = Y$.

1-simpl.$:=$ $\Delta[1] \times \Delta[1] \to \mathcal{C}$



, n$:=$ $\Delta[n] \times \Delta[1] \to \mathcal{C}$

Remark: $\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X, Y) = \pi_0(\mathrm{Map}_{\mathcal{C}}(X, Y))$

Quasi-categories as ∞-categories: Mapping Spaces & Compositions

Warning: There is no strict manifestation of the "composition law" in $\mathcal{C}$, $\mathrm{Map}_{\mathcal{C}}(X, Y) \times \mathrm{Map}_{\mathcal{C}}(Y, Z) -\times- > \mathrm{Map}_{\mathcal{C}}(X, Z)$

Rectification (Lurie) $\exists$ alternative simplicial sets $\widetilde{\mathrm{Map}}_S(X, Y)$, together with a canonical zig-zags of weak-equivalences of SSets

$$\widetilde{\mathrm{Map}}_{\mathcal{C}}(X, Y) \xrightarrow{\sim} \bullet \xleftarrow{\sim} \mathrm{Map}_{\mathcal{C}}(X, Y)$$

such that the "composition law" of $\mathcal{C}$ is translated by concrete strict maps

$$\widetilde{\mathrm{Map}}_{\mathcal{C}}(X, Y) \times \widetilde{\mathrm{Map}}_{\mathcal{C}}(Y, Z) \to \widetilde{\mathrm{Map}}_{\mathcal{C}}(X, Z)$$

Good $\mathcal{C}$ is converted into a concrete **simplicial category** (notation: $\mathfrak{C}[\mathcal{C}]$) with strictly associative compositions;

Bad $\widetilde{\mathrm{Map}}_{\mathcal{C}}(X, Y)$ are not really "spaces" , ie, not Kan-complexes. Very complicated.

Quasi-categories as ∞-categories: Simplicial Categories

Rectification $\mathcal{C} \rightsquigarrow$ Simplicial category $\mathfrak{C}[\mathcal{C}]$

Question: Simplicial Categories —-? → Quasi-categories?

Construction. There exists a non-trivial extension of the nerve construction N from usual categories to simplicial enriched categories that takes into account the simplicial structure. $\mathrm{N}_{\Delta}(-)$ (Simplicial Nerve).

Application:. Model a simplicial cat. E by a simplicial set $\mathrm{N}_{\Delta}(\mathrm{E})$

Prop. E enriched by Kan-complexes $\Longrightarrow$ $\mathrm{N}_{\Delta}(\mathrm{E})$ is a quasi-category with

- Objects of $\mathrm{N}_{\Delta}(\mathrm{E})$= Objects of E
- Weak-equivalences of simplicial sets

$$\mathrm{Map}_{\mathcal{C}}(X, Y) \to \mathrm{E}(X, Y) \text{ (enrichement)}$$

Question: How rich is this dictionary Simplicial categories $\leftrightarrow$ Quasi-categories?

Quasi-categories as ∞-categories: Equivalence of Models

$\mathfrak{C}$

Theorem (Joyal-Lurie)

There exists a model structure on the category SSets *with*

- *cofibrant-fibrant objects = Quasi-categories*
- *weak-equivalences = essentially surjective + weak-equivalences of mapping spaces*

Theorem (Bergner)

There exists a model structure on the category of simpl. cats.

- *fibrant objects = simp. cats. enriched in Kan-complexes*
- *weak-equivalences = essentially surjective + weak-equivalences of mapping spaces*

N_Δ

Theorem (Lurie)

$(\mathfrak{C}, \mathrm{N}_\Delta)$ *forms a Quillen equivalence.*

Quasi-categories as ∞-categories: Plug in Examples.

Use the dictionary Simplicial Cats $\leftrightarrow$ Quasi-categories to produce examples:

Machine to produce Examples:

$\mathcal{M}$ simplicial model category (model structure + compatible simplicial enrichement). $\rightsquigarrow$

$\mathrm{E}_{\mathcal{M}} := \mathcal{M}^{\circ}$ full simplicial subcategory of cofibrant-fibrant objects in $\mathcal{M}$ is a simplicial category enriched in Kan-complexes

$$\rightsquigarrow \mathrm{N}_{\Delta}(\mathrm{E}_{\mathcal{M}}) \text{ is a quasi-category}$$

.

Quasi-categories as ∞-categories: Plug in Examples.

First Non-trivial Example: ∞-category of spaces $\mathcal{S} := \mathrm{N}_\Delta(\mathrm{E}_\mathcal{M})$:

- $\mathcal{M} = \mathrm{SSets}$; model structure to study weak-homotopy equivalences;
- Cofibrant-Fibrant objects are Kan-complexes X, Y, etc
- Simplicial structure = simplicial set of maps $\underline{\mathrm{Hom}}_\Delta(X, Y)$ (also a Kan-complex).
- Objects of $\mathcal{S}$= Kan complexes; $\mathrm{Map}_\mathcal{S}(X, Y) \sim \underline{\mathrm{Hom}}_\Delta(X, Y)$
- Equivalences = weak-homotopy eq. of Kan-complexes

Example: ∞-category of ∞-categories $\mathrm{Cat}_\infty := \mathrm{N}_\Delta(\mathrm{E}_\mathcal{M})$:

- $\mathcal{M} = \mathrm{SSets}$; model structure of Lurie-Joyal is NOT a *simplicial model category* $\rightsquigarrow$ can be modifed to become one (*marked simplicial sets*).
- Objects of Cat_∞= Quasi-cat., $\mathrm{Map}_{\mathrm{Cat}_\infty}(\mathcal{C}, \mathcal{D}) \sim$ space of ∞-functors.
- Equivalences = surjective + homotopy equivalences of maps;

Quasi-categories as ∞-categories: Plug in Examples.

Rectification of Diagrams: $\mathcal{M}$ a simplicial model category *combinatorial*. D be a usual category. Then there is an equivalence of quasi-categories

$$\mathrm{Fun}(\mathrm{N}(\mathrm{D}), \mathrm{N}_{\boldsymbol{\Delta}}(\mathrm{E}_{\mathcal{M}})) \simeq \mathrm{N}_{\boldsymbol{\Delta}}(E_{\mathcal{M}^{\mathrm{D}}})$$

where $\mathcal{M}^{\mathrm{D}} :=$ category of D-diagrams in $\mathcal{M}$ equipped with the projective model structure to study the homotopy theory of diagrams.

Corollary: Third Non-trivial Example: D a usual category. $\mathcal{M} = \mathrm{SSets} +$ model structure for weak-homotopy equivalences;

$$\mathrm{Fun}(\mathrm{N}(\mathrm{D}^{\mathrm{op}}), \mathcal{S}) \simeq \mathrm{Fun}(\mathrm{N}(\mathrm{D}^{\mathrm{op}}), \mathrm{N}_{\boldsymbol{\Delta}}(\mathrm{E}_{\mathcal{M}})) \simeq \mathrm{N}_{\boldsymbol{\Delta}}(E_{\mathcal{M}^{\mathrm{D}^{\mathrm{op}}}})$$

l.h.s $= \mathcal{P}(\mathrm{N}(\mathrm{D})) :=$ presheaves of spaces;
r.h.s $=$ (Projective) Model structure on Simplicial Presheaves

Quasi-categories as ∞-categories: Limits and Colimits.

$\mathcal{C}$ ∞-category $\rightsquigarrow$ theory of limits and colimits internal to the language of quasi-categories.

Definition: $X \in \mathcal{C}$ initial object iff for every object $Z \in \mathcal{C}$ the Kan-complex $\mathrm{Map}_{\mathcal{C}}(X, Z)$ is contractible.

Slogan: Universal properties are defined only up to a contractible space of choices.

Cones: K simplicial set (diagram shape), $d : K \to \mathcal{C}$ ∞-functor (diagram). Construct a new simplicial set $K^{\triangleright}$ by formally adding an exterior vertex to K. A "cone under d" is a map of simplicial sets $\tilde{d} : K^{\triangleright} \to \mathcal{C}$ whose restriction to K is d.

Prop.: There exists a quasi-category of "cones under d", $\mathcal{C}_{d/}$.

Definition: A Colimit of d is an initial object of $\mathcal{C}_{d/}$. In particular, by definition the collection of candidates for a colimit form a contractible space.

Similar for Limits.

Quasi-categories as ∞-categories: Interpreting Homotopy Colimits.

Prop: $F : J \to E$ simplicial functor between simplicial categories enriched in Kan complexes. Let C be an object of E together with a compatible family of maps $\{\eta_j : F(j) \to C\}_{j \in J}$. Then

C is a homotopy colimit of F iff the induced map of simplicial sets $\mathrm{N}_\Delta(J)^\triangleright \to \mathrm{N}_\Delta(E)$ is a colimit diagram.

Example: $Coeq^{\mathcal{S}}(* \overset{Id}{\underset{Id}{\rightrightarrows}} *) = colim^{\mathcal{S}} \begin{array}{ccc} * \coprod * & \longrightarrow & * \\ \downarrow & & \\ * & & \end{array} =$

$$= \text{hcolim.} \begin{array}{ccc} * \coprod * & \longrightarrow & * \\ \downarrow & & \\ * & & \end{array} = \text{colim.} \begin{array}{ccc} * \coprod * & \longrightarrow & \Delta^1 \\ \downarrow & & \\ \Delta^1 & & \end{array} = S^1$$

Example: $x : * \to X$ in $\mathcal{S}$. $* \times^h_X * \simeq \Omega_x X$

Quasi-categories as ∞-categories: Dwyer-Kan Localization

Now: We have Cat_∞ and can talk about limits and colimits inside.

A usual category $\rightsquigarrow$ $\mathrm{N}(A)$ ∞-category with unique compositions.

Definition: W class of morphisms in A. An ∞-localization of A along W, is a quasi-category $\mathrm{N}(A)[W^{-1}]_\infty$ + a map of simplicial sets $\ell : \mathrm{N}(A) \to \mathrm{N}(A)[W^{-1}]_\infty$ such that for any quasi-category $\mathcal{C}$,

$$\mathrm{Fun}(\mathrm{N}(A)[W^{-1}]_\infty, \mathcal{C}) \to \mathrm{Fun}(\mathrm{N}(A), \mathcal{C})$$

is fully faithful with essential image = ∞-functors sending W to equivalences in $\mathcal{C}$.

Remark: $\mathrm{N}(A)[W^{-1}]_\infty$ can be obtained as a pushout in Cat_∞.

$$\begin{array}{ccc} \coprod_{w \in W} \mathrm{N}([1]) & \longrightarrow & \mathrm{N}(A) \\ \big\downarrow & & \big\downarrow \ell \\ \coprod_{w \in W} \mathrm{N}(J) & \longrightarrow & \mathrm{N}(A)[W^{-1}]_\infty \end{array}$$

$J =$ groupoid 2 objects, 1 iso.

Quasi-categories as ∞-categories: Dwyer-Kan Localization

Quillen, Dwyer-Kan: $\mathcal{M}$ simplicial model category with W weak-equivalences. There there is a chain of equivalences of ∞-categories

$$\mathrm{N}(\mathcal{M})[W^{-1}]_\infty \underset{cof.repl.}{\underset{\sqcup}{\simeq}} \mathrm{N}(\mathcal{M}^c)[W_c^{-1}]_\infty \simeq \mathrm{N}_\Delta(\mathrm{E}_\mathcal{M})$$

Definition: We use the terminology underlying ∞-category of $\mathcal{M}$ to address one of these equivalent quasi-categories.

Remark: $\mathrm{h}(\mathrm{N}(\mathcal{M})[W^{-1}]_\infty)$ is the usual homotopy category of $\mathcal{M}$ (Gabriel-Zisman localization).

Examples

- ∞-derived category of a ring R, $D_\infty(R) := \mathrm{N}(\mathcal{M})[W^{-1}]_\infty$ with $\mathcal{M}=$ strict category of chain complexes of R-modules. $W=$quasi-iso's.
- ∞-category of cdga's/$\mathbb{C}$, $\mathrm{cdga}_\mathbb{C}^\infty := \mathrm{N}(\mathcal{M})[W^{-1}]_\infty$ with $\mathcal{M}=$ strict cdga's, $W=$quasi-iso's.

Yoneda and Presheaves

Definition: quasi-category of presheaves of spaces on quasi-category $\mathcal{C}$

$$\mathcal{P}(\mathcal{C}) := \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S})$$

Seen before: A usual category, $\mathcal{C} := \mathrm{N}(A)$ then $\mathcal{P}(\mathcal{C}) \simeq \mathrm{N}_{\Delta}(E_{\mathcal{M}})$, $\mathcal{M} :=$ simplicial presheaves on A.

Prop (Yoneda): For any quasi-category $\mathcal{C}$ there exists a fully faithfull ∞-functor

$$j : \mathcal{C} \to \mathcal{P}(\mathcal{C})$$

with the following **universal property**: If $\mathcal{D}$ has all colimits, then the composition

$$\underbrace{\mathrm{Fun}^{\mathrm{L}}}_{\text{Left-adjoints}}(\mathcal{P}(\mathcal{C}), \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

is an equivalence of ∞-categories

Remark: To construct j one needs to exhibit a cocartesian fibration $\mathcal{N} \to \mathcal{C}^{op} \times \mathcal{C}$. $\mathcal{N}$ is given by the ∞-category of *Twisted arrows* in $\mathcal{C}$.

How to construct Yoneda?

More generally: How to construct ∞-functors? Not enough to say what happens on objects and 1-morphisms. Need to explain the action on n-simplexes.

Idea: Re-organize an ∞-functor $F : \mathcal{C} \to \mathcal{D}$ as a "*family with a connection data*".

$$\begin{array}{ccccc} \mathcal{N} & & \mathcal{C} = p^{-1}(0) & & \mathcal{D} = p^{-1}(1) \\ \downarrow p & & & & \\ \Delta^1 & & 0 & \longrightarrow & 1 \end{array}$$

with $\mathrm{Map}_{\mathcal{N}}((0,c),(1,d)) \simeq \mathrm{Map}_{\mathcal{D}}(F(c),d)$

Key point: Not all families over Δ^1 define ∞-functors. Need connection data to jump between fibers.

coCartesian fibration:= family + appropriate connection data.

Grothendieck construction

Previous example: Single ∞-functor $\leftrightarrow$ Family over Δ^1. This dictionary can be explained by an ∞-categories

$$\mathrm{coCart}/\Delta^1 \simeq \mathrm{Fun}(\Delta^1, \mathrm{Cat}_\infty)$$

More generally: Replacing Δ^1 by any simplicial set S, it is still true that diagrams of ∞-categories indexed by S can be exhibited as families over S with connection:

$$\mathrm{coCart}/S \simeq \mathrm{Fun}(S, \mathrm{Cat}_\infty)$$

The way the connection data is implemented in the definition of cocartesian fibration ensures the functoriality up to homotopy.

Back to Yoneda: Need to construct ∞-functor $\mathcal{C} \times \mathcal{C}^{op} \to \mathcal{S} \subseteq \mathrm{Cat}_\infty$ sending $(X, Y) \to \mathrm{Map}_{\mathcal{C}}(X, Y)$. Instead, construct the appropriate family over $\mathcal{C} \times \mathcal{C}^{op}$.

Exercise: Formulate the notion of pair of adjoint functors as a family over Δ^1 where the connection data works in both directions- bicartesian fibration.

Presentable Categories

κ strongly inaccessible card. $\leftrightarrow \mathcal{U} :=$ sets of rank $< \kappa$, universe

$$\mathcal{U} \subseteq \mathcal{V} \text{ universes} \rightsquigarrow \mathcal{U}\text{-small SSets} \leftrightarrow \mathcal{V}\text{-small SSets} := \mathcal{U} - \textit{big}$$

$$\mathrm{Cat}_\infty^{\mathrm{small}} \leftrightarrow \mathrm{Cat}_\infty^{\mathrm{big}}$$

Presentable data: $\mathcal{U}$-Big Data determined by $\mathcal{U}$-Small Data.

Definition: $\mathcal{C}_0$ small ∞-category. $\kappa' < \kappa$. $\mathcal{C}_0$ is κ' -filtered if for every κ'-small simplicial set K and diagram $F : K \to \mathcal{C}_0$ there exists an extension $K^{\triangleright} \to \mathcal{C}_0$.

Definition: $\mathcal{C}_0$ small ∞-category. $\kappa' < \kappa$. $\mathrm{Ind}_{\kappa'}(\mathcal{C}_0) :=$ full subcategory of $\mathcal{P}(\mathcal{C}_0)$ containing representables + stable under κ'-filtered colimits.

Definition: $\mathcal{C}$ ∞-category is presentable if $\mathcal{C}$ has all small colimits and is of the form $\mathrm{Ind}_{\kappa'}(\mathcal{C}_0)$ for some small ∞-category $\mathcal{C}_0$.

Examples: $\mathcal{S}$ is presentable, of the form Ind_ω of (homotopy) finite CW-complexes. All categories of presheaves are presentable.

Presentable Categories: Adjoint Functor Theorem

Prop. (Adjoint Functor Theorem). $F : \mathcal{C} \to \mathcal{D}$ ∞- functor between presentable ∞-categories. Suppose F commutes with all small colimits. Then F admits a right adjoint.

Definition: Pr^{L} is the (non-full) subcategory of $\mathrm{Cat}_{\infty}^{\mathrm{big}}$ spanned by presentable ∞-categories and colimit preserving functors.

Remark: One can also define a (non-full) subcategory $\mathrm{Pr}^{\mathrm{R}} \subseteq \mathrm{Cat}_{\infty}^{\mathrm{big}}$ spanned by presentable categories and functors which preserve limits and κ's filtered colimits for some cardinal $\kappa' < \kappa$. We have $(\mathrm{Pr}^{\mathrm{R}})^{op} \simeq \mathrm{Pr}^{\mathrm{L}}$

Prop: The inclusions $\mathrm{Pr}^{\mathrm{R}}, \mathrm{Pr}^{\mathrm{L}} \subseteq \mathrm{Cat}_{\infty}^{\mathrm{big}}$ preserve all limits.

Prop: $\mathcal{C}$ and $\mathcal{D}$ presentable. Then $\mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathcal{D})$ is presentable.

Presentable Categories: Localizations

Prop. (Presentable Localizations): $\mathcal{C}$ presentable ∞-category, W small collection of 1-morphisms in $\mathcal{C}$. $\mathcal{C}^{W-\mathrm{local}} :=$ full subcategory of $\mathcal{C}$ spanned by X such that for every $S \to S'$ in W the map

$$\mathrm{Map}_{\mathcal{C}}(S', X) \to \mathrm{Map}_{\mathcal{C}}(S, X)$$

is an equivalence. Then $\mathcal{C}^{W-\mathrm{local}}$ is presentable and the inclusion $\mathcal{C}^{W-\mathrm{local}} \subseteq \mathcal{C}$ admits a left adjoint $\mathcal{C} \to \mathcal{C}^{W-\mathrm{local}}$ which exhibits $\mathcal{C}^{W-\mathrm{local}}$ as an ∞-localization of $\mathcal{C}$ along W internal to the theory of presentable ∞-categories.

Examples: Produce examples of presentable categories as presentable localizations of presheaves. In fact:

Prop: $\mathcal{C}$ is presentable iff it is a presentable localization of some $\mathcal{P}(\mathcal{C}_0)$.

Prop: A usual category. Presentable localizations of $\mathcal{P}(\mathrm{N}(A))$ correspond via the rectification result to Bousfield localizations of simplicial presheaves on A.

What is a symmetric monoidal structure?: A usual category + unit $* \to A$ + n-ary operations $\otimes^n : \underbrace{A \times ... \times A}_{n} \to A$ + associativity constrains + symmetry constrains under the action of the Σ_n.

Segal: This combinatorial pattern is described by the category of finite pointed sets Fin_* and the data of a symmetric monoidal category is a the data of a pseudo-functor

$$A^{\otimes} : \mathrm{Fin}_* \to \mathrm{Cat}, \text{ with property } A^{\otimes}(\{n\}_0) = \underbrace{A \times ... \times A}_{n}$$

Example: $\{0,1,2\} \to \{0,1\}$ $1,2 \mapsto 1$, $0 \mapsto 0$, Image under $A^{\otimes}$ gives operation $A \times A \to A$. Image of permutation of 1 by 2, $\{0,1,2\} \to \{0,1,2\}$ gives symmetry.

Symmetric monoidal ∞-category:= ∞-functor

$$\mathcal{C}^{\otimes} : \mathrm{N}(\mathrm{Fin}_*) \to \mathrm{Cat}_{\infty}, \text{ with } \mathcal{C}^{\otimes}(\{n\}_0) \to \prod_{i=1}^{n} \mathcal{C}^{\otimes}(\{0,1\}) \text{ eq.}$$

Some subtle points:

- **"The diagram commutes"**. The claim does not make sense unless the n-simplex providing the commutativity is exhibited. One should look for an explicit construction of the simplex or for a universal property or "machine" defining it.
- **"The ∞-functor $F : \mathcal{C} \to \mathcal{D}$ doing this on objects and that on 1-morphisms"**. Such a description should always be understood as informal. The only construction with mathematical sense is that of a map of simplicial sets $F : \mathcal{C} \to \mathcal{D}$ which specifies what happens with all simplexes. Typically we construct ∞-functors via some universal property involved, via Quillen adjunctions or via the exhibition of a cartesian/cocartesian fibration.
- **"The object has an (associative, commutative, Lie)-algebra structure"**. The exhibition and construction of such structures is in general very subtle.

Main ∞-categories in these lectures:

- Spaces $\mathcal{S}$, ∞-categories $\mathrm{Cat}_\infty^{\mathrm{small}}$, $\mathrm{Cat}_\infty^{\mathrm{big}}$, presentable ∞-categories Pr^{L}, stable presentable $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{Stb}}$
- cdga's $\mathrm{cdga}_{\mathbb{C}}^{\infty}$, derived schemes/manifolds/stacks/prestacks;
- Spectra Sp, derived category of a ring R, $\mathrm{D}_\infty(R)$, $\mathrm{Qcoh}_\infty(X)$, $\mathrm{Perf}(X)$ (quasi-coherent and perfect complexes),
- ∞-categories of algebra-objects and modules over them in all the ∞-categories above.
- (∞, n)-categories of correspondences, Lagrangian correspondences, cobordisms, etc.

Ready for action!