Revisiting the filtered circle

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Object of this talk:

The role of the filtered circle in the HKR theorem in positive and mixed characteristic (Moulinos-R-Toën)

The ideas that go in the construction of the filtered circle started long ago...

In Pursuing Stacks, Grothendieck sets up a program to capture homotopy types in algebraic geometry over $\mathbb{Z}$, generalizing the study of rational and p-adic homotopy types.

Around 2001, Toën suggested an approach using the theory of *affine stacks*, ie stacks determined by their cosimplicial cohomology ring. In this framework, every homotopy type X has an algebraic model given by its *affinization* over $\mathbb{Z}$

$$\mathrm{Aff}(X) := \mathrm{coSpec}(\mathrm{C}^*_{\mathrm{co}\boldsymbol{\Delta}}(X, \mathbb{Z}))$$

A question that remains open in Toën's work is the computation of the affinization of the spaces $K(\mathbb{Z}, n)$. But a concrete proposal was put forward: use the ring of Witt vectors $\mathbb{W}$ to construct an integral version of the pro-unipotent completion of $\mathbb{Z}$.

20 years late that same proposal lead to the filtered circle and to solution to the schematization problem. This is the story of that proposal.

1. HKR-Filtration and the naive circle action
2. Reformulation via derived geometry
3. The filtered circle and Witt magic
4. Two applications

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Theorem (Hochschild-Kostant-Rosenberg)

$X = \mathrm{Spec}(R)$ a smooth affine scheme over a ring k. Then the anti-symmetrization map induces an isomorphism

$$\Omega^n_{R/k} \to \mathrm{HH}_n(R/k) = Tor^n_{R\otimes_k R}(R, R)$$

$$r_0.dr_1 \wedge \cdots \wedge dr_n \mapsto \sum_{\sigma \in \Sigma_n} (-1)^{\mathrm{sign}(\sigma)}[r_0 \otimes r_{\sigma(1)} \otimes \cdots \otimes r_{\sigma(n)}]$$

Remark

HH *is a chain complex:*

$$\mathrm{HH}(R/\mathrm{k}) := R \underset{R \overset{\mathbb{L}}{\underset{\mathrm{k}}{\otimes}} R}{\overset{\mathbb{L}}{\otimes}} R$$

Construction (Whitehead filtration)

HH *has a descending filtration* $\mathrm{HH}_{\mathrm{Fil}} :=$

$$\cdots \to \tau_{\geq 2}\,\mathrm{HH}(R/\mathsf{k}) \longrightarrow \tau_{\geq 1}\,\mathrm{HH}(R/\mathsf{k}) \longrightarrow \tau_{\geq 0}\,\mathrm{HH}(R/\mathsf{k}) = \mathrm{HH}(R/\mathsf{k})$$

$$\cdots \quad \mathrm{gr}^2 = \Omega^2_{R/\mathsf{k}}[2] \qquad \mathrm{gr}^1 = \Omega^1_{R/\mathsf{k}}[1] \qquad \mathrm{gr}^0 = \Omega^0_{R/\mathsf{k}}[0]$$

Remark

(chain level) HKR isomorphisms $\leftrightarrow$ *splittings of the filtration*

$$\mathrm{HH}_{\mathrm{Fil}} \simeq \bigoplus_{i\geq} \mathrm{gr}^i$$

Proposition

If $\mathrm{char}(\mathsf{k}) = 0$, *the anti-symmetrization map induces such a splitting.*

Remark

Algebra structure

$$\mathrm{HH}(R/\mathsf{k}) = R \underset{R \overset{\mathbb{L}}{\underset{\mathsf{k}}{\otimes}} R}{\overset{\mathbb{L}}{\otimes}} R \simeq R \underset{\mathsf{k}}{\overset{\mathbb{L}}{\otimes}} \mathrm{S}^1$$

Remark

Group structure $\mathrm{S}^1 \times \mathrm{S}^1 \to \mathrm{S}^1$

Theorem (Universal Property)

$\mathrm{S}^1 \circlearrowright \mathrm{HH}(R/\mathsf{k})$ *is the universal* R*-algebra with a* S^1*-action.*

Remark

Chain Level:

$$\mathrm{H}_1(\mathrm{S}^1, \mathsf{k}) \ni B : \mathrm{HH}(R/\mathsf{k}) \to \mathrm{HH}(R/\mathsf{k})[1]$$

Proposition

$$\begin{array}{ccc} \mathrm{HH}_n(R/\mathsf{k}) & \xrightarrow{B} & \mathrm{HH}_{n+1}(R/\mathsf{k}) \\ \downarrow \sim & & \downarrow \sim \\ \Omega^n_{R/\mathsf{k}} & \xrightarrow{\mathrm{d}_{\mathrm{dR}}} & \Omega^{n+1}_{R/\mathsf{k}} \end{array}$$

Problem

$B = \mathrm{d}_{\mathrm{dR}}$ *is not compatible with HKR-filtration:*

$$\cdots \to \underset{\mathrm{S}^1\circlearrowright}{\tau_{\geq 2}\,\mathrm{HH}(R/\mathsf{k})} \longrightarrow \underset{\mathrm{S}^1\circlearrowright}{\tau_{\geq 1}\,\mathrm{HH}(R/\mathsf{k})} \longrightarrow \underset{\mathrm{S}^1\circlearrowright}{\tau_{\geq 0}\,\mathrm{HH}(R/\mathsf{k})}$$

$$\mathrm{S}^1 \circlearrowright \mathrm{gr}^n = \Omega^n_{R/\mathsf{k}}[n], \qquad \Omega^n_{R/\mathsf{k}}[n] \underset{0}{\to} \Omega^n_{R/\mathsf{k}}[n+1]$$

Levelwise circle action on the filtration is too naive. Does not capture graded weights.

First step: Consider $\bigoplus_{n\geq 0} \Omega^n_{R/\mathsf{k}}[n]$ as a graded module and d_{dR} as an extra operator that increases the weight.

Construction (Graded modules)

$$\mathrm{Ch}^{\mathrm{gr}}_{\mathsf{k}} := \prod_{n\in\mathbb{Z}} \mathrm{Ch}_{\mathsf{k}} \quad \ni E = (E_n)_{n\in\mathbb{Z}}, \qquad \bigoplus : \mathrm{Ch}^{\mathrm{gr}}_{\mathsf{k}} \to \mathrm{Ch}_{\mathsf{k}}$$

$$E \overset{\mathbb{L}}{\underset{\mathsf{k}}{\otimes}} F = \Big(\bigoplus_{n+m=\ell} E_n \overset{\mathbb{L}}{\underset{\mathsf{k}}{\otimes}} F_m \Big)_{\ell\in\mathbb{Z}}$$

Construction

$\mathsf{k}[\epsilon]_{\mathrm{gr}}$ = *graded strictly associative dg-algebra freely generated by an element* ϵ *in homological degree* 1 *and weight* 1, *and strictly verifying* $\epsilon^2 = 0$.

Remark

A left-$\mathsf{k}[\epsilon]_{\mathrm{gr}}$*-module in* $\mathrm{Ch}_{\mathsf{k}}^{\mathrm{gr}}$ $\Leftrightarrow$ $E = (E_n)_{n\in\mathbb{Z}}$ + *operator*

$$\epsilon : E(1)[1] \to E$$

with $\epsilon \circ \epsilon = 0$ *(strict).*

Construction

$\mathsf{k}[\epsilon]_{\mathrm{gr}}$ *carries a strictly commutative graded Hopf structure*

$$\mathsf{k}[\epsilon]_{\mathrm{gr}} \to \mathsf{k}[\epsilon]_{\mathrm{gr}} \otimes_{\mathrm{gr}} \mathsf{k}[\epsilon]_{\mathrm{gr}}$$

determined by

$$\epsilon \mapsto \epsilon \otimes 1 + 1 \otimes \epsilon$$

$\Rightarrow$ *tensor product of* $\mathsf{k}[\epsilon]_{\mathrm{gr}}$*-modules makes sense.*

Definition

- *Mixed graded modules* = left-modules over $\mathsf{k}[\epsilon]_{\mathrm{gr}}$.
- *Mixed graded algebras* = commutative algebra objects in the symmetric monoidal ∞-category of mixed graded modules.

Proposition

Over any ring k, *setting* $\epsilon := \mathrm{d}_{\mathrm{dR}}$ *endows* $\bigoplus_{n\in\mathbb{Z}} \Omega^n_{R/\mathsf{k}}[n]$ *with a structure of mixed graded algebra.*

However: So far, no relation between mixed graded structure and HKR filtration.

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Proposition (Toën and Vezzosi, Ben-Zvi and Nadler)

Let $X = \mathrm{Spec}(R)$*:*

- $\mathrm{HH}(R/\mathsf{k}) = \mathcal{O}_{\mathbb{R}\mathrm{Map}(\mathrm{S}^1, X)}$.
- $\bigoplus_{n\geq 0} \Omega^n_{X/\mathsf{k}}[n] \simeq \mathrm{Sym}^{\Delta}(\Omega^1_{X/\mathsf{k}}[1]) \simeq \mathcal{O}_{\mathrm{T}[-1]X}$

Remark

$\mathbb{R}\mathrm{Map}(\mathrm{S}^1, X)$ *has an* S^1*- action.*

$$\mathcal{O}_{\mathbb{R}\mathrm{Map}(\mathrm{S}^1, X)/\mathrm{S}^1} = \mathrm{HH}(R/\mathsf{k}) \ + \ \mathrm{S}^1 - \textit{action}.$$

Remark

$\mathrm{T}[-1]X$ *has a* $\mathbb{G}_m$*-scaling along the fibers. This is responsible for the grading.*

Lemma

$$\{\textit{Chain complexes with additional } \mathbb{Z}\textit{-grading}\}^{\otimes} \simeq \underbrace{\mathrm{Qcoh}_{\infty}(\mathrm{B}\mathbb{G}_m)^{\otimes}}_{\mathbb{G}_m-\textit{representations}}$$

Corollary

$$\pi : (\mathrm{T}[-1]X)/\mathbb{G}_m \to \mathrm{Spec}(\mathsf{k})/\mathbb{G}_m = \mathrm{B}\mathbb{G}_m$$

$$\mathcal{O}_{(\mathrm{T}[-1]X)/\mathbb{G}_m} = \pi_*\mathcal{O} \simeq \bigoplus_{n \geq 0} \Omega^n_{X/\mathsf{k}}[n] \textit{ with grading}$$

What about d_{dR}?

Construction

$$\mathsf{k}[\eta] := \mathsf{k} \oplus \underbrace{\mathsf{k}[-1]}_{\textit{weight } -1}$$

with the trivial square zero multiplication as a cosimplicial graded commutative algebra.

$$\underbrace{\mathrm{coSpec}(\mathsf{k}[\eta])}_{\textit{Affine stack}} \circlearrowleft \mathbb{G}_m$$

.

Proposition

$\mathrm{coSpec}(\mathsf{k}[\eta])$ *admits a unique abelian group structure, compatible with the grading.*

Definition

$$\underbrace{\mathrm{S}^1_{\epsilon-\mathrm{gr}}}_{\text{Mixed graded circle}} := [\mathrm{coSpec}(\mathsf{k}[\eta])/\mathbb{G}_m] \to \mathrm{B}\mathbb{G}_m$$

Lemma

$$\mathrm{Qcoh}(\mathrm{B}_{\mathrm{B}\mathbb{G}_m}(\mathrm{S}^1_{\epsilon-\mathrm{gr}}))^{\otimes} \simeq \{\textit{Mixed graded modules}\}^{\otimes}$$

Remark

When $\mathrm{char}(\mathsf{k}) = 0$

$$\mathrm{coSpec}(\mathsf{k}[\eta]) \underbrace{\simeq}_{\mathbb{G}_m-eq.} \mathrm{B}\mathbb{G}_a$$

Proposition

Equivalence of $\mathbb{G}_m$-stacks (independent of $\mathrm{char}(\mathsf{k})$)

$$\mathbb{R}\mathrm{Map}(\mathrm{coSpec}(\mathsf{k}[\eta]), X) \simeq \mathrm{T}[-1]X$$

Construction

$$\mathrm{S}^1_{\epsilon-\mathrm{gr}} \circlearrowright (\mathrm{T}[-1]X)/\mathbb{G}_m$$

$$[(\mathrm{T}[-1]X)/\mathbb{G}_m]/\mathrm{S}^1_{\epsilon-\mathrm{gr}} \to \mathrm{B}_{\mathrm{B}\mathbb{G}_m}(\mathrm{S}^1_{\epsilon-\mathrm{gr}})$$

Proposition

$$\mathcal{O}_{[(\mathrm{T}[-1]X)/\mathbb{G}_m]/\mathrm{S}^1_{\epsilon-\mathrm{gr}}} = \bigoplus_{n\geq 0} \Omega^n_{X/\mathsf{k}}[n] \textit{ with grading } + \mathrm{d}_{\mathrm{dR}}$$

Geometrization of the filtration

Need to explain

$$\mathrm{S}^1 \circlearrowright \mathrm{L}X \Leftarrow \underbrace{\text{geometrization of } \mathrm{HH}_{\mathrm{Fil}}}_{\underset{?}{\circlearrowright}} \Rightarrow \mathrm{S}^1_{\epsilon-\mathrm{gr}} \circlearrowright \mathrm{T}[-1]X$$

Bridge between the underlying object and the associated graded.

Lemma (Simpson)

The quotient stack $[\mathbb{A}^1/\mathbb{G}_m]$ encodes filtrations, ie,

$$\{\textit{chain complexes with additional } \mathbb{Z}\textit{-filtration}\}^{\otimes} \underbrace{\simeq}_{\textit{Rees}} \mathrm{Qcoh}_\infty([\mathbb{A}^1/\mathbb{G}_m])^{\otimes}$$

Remark

$$\begin{array}{ccccc}
\mathrm{Qcoh}_\infty(\mathrm{Spec}(k)) & \xleftarrow{1^*} & \mathrm{Qcoh}_\infty([\mathbb{A}^1/\mathbb{G}_m]) & \xrightarrow{0^*} & \mathrm{Qcoh}_\infty(B\mathbb{G}_m) \\
\downarrow \sim & & \downarrow \sim & & \downarrow \sim \\
\{\textit{complexes}\} & \xleftarrow{\text{underlying}} & \{\textit{Filtered complexes}\} & \xrightarrow{\text{ass}-\text{gr}} & \{\textit{Graded complexes}\}
\end{array}$$

Definition

A *filtered (derived) stack* is a (derived) stack Z together with a map $Z \to [\mathbb{A}^1/\mathbb{G}_m]$.

$$\begin{array}{ccccc} \text{underlying} & \longrightarrow & Z & \longleftarrow & \text{associated graded} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Spec}(\mathsf{k}) & \xrightarrow{1} & [\mathbb{A}^1/\mathbb{G}_m] & \xleftarrow{0} & \mathrm{B}\mathbb{G}_m \end{array}$$

Theorem (Moulinos-R-Toën)

- *There exists a filtered abelian group stack*

$$\mathrm{S}^1_{\mathrm{Fil}} \to [\mathbb{A}^1/\mathbb{G}_m]$$

  that implements a filtration on the affinization of the topological circle with associated graded $\mathrm{S}^1_{\epsilon-\mathrm{gr}}$.

- *Universal property of the HKR-filtration:*

$$\mathrm{HH}_{\mathrm{Fil}}(R/k) \simeq \mathcal{O}_{\mathbb{R}\mathrm{Map}(\mathrm{S}^1_{\mathrm{Fil}}, X)}$$

  is the universal filtered R-algebra with an action of the filtered circle $\mathrm{S}^1_{\mathrm{Fil}}$.

Remark

A similar universal property has been obtained by Raksit.

Example

X quasi-smooth

$$\mathrm{HH}_{\mathrm{Fil}}(X)^{h\mathrm{S}^1_{\mathrm{Fil}}} = \mathrm{HC}^-(X/\mathsf{k}) \text{ with Antieau filtration}$$

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Witt vectors

Definition

$$\mathbb{W}(R) := \underbrace{1 + tR[[t]]}_{\text{invertible formal power series}}$$

$$(\mathbb{W}, +) : \{\text{Commutative rings}\} \to \{\text{Abelian groups}\}$$

Remark

Underlying scheme

$$\mathbb{W} \simeq \prod_{i \geq 0}^{\infty} \mathbb{A}^1$$

As a group, built from successive extensions by $\mathbb{G}_a$.

Construction

- *Ghost coordinates*

$$-\frac{d}{dt}\log : (1 + t.R[[t]])^{\times} \to R[[t]], \qquad f \mapsto -\frac{f'}{f}$$

transforms products of formal power series into sums.

$$Ghost : (\mathbb{W}, +) \to (\prod_{i=1}^{\infty} \mathbb{G}_a, +)$$

- *Frobenius endomorphisms*

$$\mathrm{Frob}_n : \mathbb{W} \to \mathbb{W}, \quad \forall n \in \mathbb{N}$$

determined by the shift maps on Ghost coordinates

$$\mathrm{Shift}_n : \prod_{i=1}^{\infty} \mathbb{G}_a \to \prod_{i=1}^{\infty} \mathbb{G}_a \quad (\omega_i) \mapsto (\omega_{ni})$$

- $\mathbb{G}_m \circlearrowright \mathbb{W}$ *given by multiplication by Teichmuller representative.*

Definition

$$\mathrm{Fix} := \bigcap_n (\mathrm{Frob}_n - \text{fixed points}) \subseteq \mathbb{W}$$

$$\mathrm{Ker} := \bigcap_n (\mathrm{Kernel\ Frob}_n) \subseteq \mathbb{W}$$

Remark

Ker *is closed under* $\mathbb{G}_m$*-action.*

Proposition (Moulinos-R-Toën over $\mathbb{Z}_{(p)}$, J. Tapia, J.Sauloy and Toën over $\mathbb{Z}$)

$$\mathrm{BFix} \simeq \mathrm{Aff}(\mathrm{S}^1)$$

$$(\mathrm{BKer})/\mathbb{G}_m \simeq \mathrm{S}^1_{\epsilon-\mathrm{gr}}$$

Construction

Consider the family of abelian groups $\mathbb{H} \to \mathbb{A}^1$ *given by*

$$\begin{array}{ccc} \mathbb{H}_\lambda := \bigcap_n \left(\mathrm{Kernel\ Frob}_n - \lambda^{n-1}\mathrm{Id}\right) & \hookrightarrow & \mathbb{H} \\ \vdots & & \downarrow \\ \{\lambda\} & \hookrightarrow & \mathbb{A}^1 \end{array}$$

Interpolates between $\mathbb{H}_{\lambda=0} = \mathrm{Ker}$ *and* $\mathbb{H}_{\lambda=1} = \mathrm{Fix}$

Remark

$\mathbb{G}_m$*-action on* $\mathbb{W}$ *restricts to* $\mathbb{G}_m$*-action on* $\mathbb{H}$ *compatible with group structure.*

Definition

$$S^1_{\mathrm{Fil}} := B_{[\mathbb{A}^1/\mathbb{G}_m]}(\mathbb{H}/\mathbb{G}_m)$$

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Functorial HKR isomorphisms

Definition

Universal HKR isomorphisms ⇔ universal splittings of the HKR-filtration ⇔ group splittings

$$\mathrm{S}^1_{\mathrm{Fil}} \simeq (\mathrm{S}^1_{\mathrm{Fil}})^{\mathrm{triv}}$$

Proposition

$$\{\textit{group splittings } \mathrm{S}^1_{\mathrm{Fil}} \simeq (\mathrm{S}^1_{\mathrm{Fil}})^{\mathrm{triv}}\} \underset{\textit{Cartier Duality}}{\simeq} \{\textit{exponentials } \widehat{\mathbb{G}_a} \to \widehat{\mathbb{G}_m}\}$$

$$\simeq \begin{cases} \emptyset & \textit{if } \mathrm{char}(\mathsf{k}) \neq 0 \\ \mathsf{k}^* \underbrace{=}_{\textit{Toën-Vezzosi}} \{\textit{Chern Characters } K_0 \to H^*_{\mathrm{d_{dR}}}\} & \textit{if } \mathrm{char}(\mathsf{k}) = 0 \end{cases}$$

Warning: The circle S^1 also admits a cogroup structure.

$$\mathrm{S}^1 \to \mathrm{S}^1 \vee \mathrm{S}^1$$

This cogroup structure extends to the filtered circle $\mathrm{S}^1_{\mathrm{Fil}}$.

Theorem (Moulinos)

There are no cogroup splittings $\mathrm{S}^1_{\mathrm{Fil}} \simeq (\mathrm{S}^1_{\mathrm{Fil}})^{\mathrm{triv}}$. The obstruction is the universal Todd class.

Corollary

None of the group splittings induced by the choice of a Chern character is a cogroup splitting. The consequence is the Grothendieck-Riemann-Roch theorem.

Back to the Schematization Problem

Remark

Toën's extension of the result $\mathrm{BFix} = \mathrm{Aff}(\mathrm{S}^1)$ *from* $\mathbb{Z}_{(p)}$ *to* $\mathbb{Z}$ *answers one of the affinization problems*

$$\mathrm{Aff}(\mathrm{S}^1) = \mathrm{Aff}(K(\mathbb{Z}, 1)) = \mathrm{BFix} = K(\mathrm{Fix}, 1)$$

Theorem (Toën, 2020)

$$\mathrm{Aff}(K(\mathbb{Z}, n)) = K(\mathrm{Fix}, n)$$