Choices of HKR isomorphisms

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Abstract

In this short note we record the fact that the set of multiplicative HKR natural equivalences defined simultaneously for all derived schemes, functorialy splitting the HKR-filtration and rendering the circle action compatible with the de Rham differential, is, via Cartier duality, in a natural bijection with the set of filtered formal exponential maps $\widehat{\mathbb{G}_a} \to \widehat{\mathbb{G}_m}$. In particular, when the base k is a field of characteristic zero, the set of choices is k^*.

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1 Introduction

Let k be a commutative ring. The Hochschild-Kostant-Rosenberg (HKR) theorem [HKR62] establishes for any smooth k-scheme $X = \mathsf{Spec}(R)$ an identification of the Hochschild homology groups $\mathsf{HH}_i(R/\mathsf{k}) := \mathrm{Tor}^i_{R\otimes_{\mathsf{k}} R}(R, R)$ with the modules of i-differential forms $\Omega^i_{X/\mathsf{k}}$, given by the anti-symmetrization map

$$\Omega^i_{R/\mathsf{k}} \to \mathsf{HH}_i(R/\mathsf{k}) \qquad r_0.dr_1 \wedge \cdots \wedge dr_i \mapsto \sum_{\sigma\in\Sigma_i} (-1)^{\mathsf{sign}(\sigma)}[r_0 \otimes r_{\sigma(1)} \otimes \cdots \otimes r_{\sigma(i)}]$$

The groups $\mathrm{HH}_i(R/\mathsf{k})$ are actually defined for every derived k-algebra $R \in \mathsf{SCR}_{\mathsf{k}}$(*) as the homology groups of the derived tensor product of k-algebras $\mathsf{HH}(R/\mathsf{k}) := R \overset{\mathbb{L}}{\underset{R\overset{\mathbb{L}}{\underset{\mathsf{k}}{\otimes}}R}{\otimes}} R$ where R is seen as an $R \overset{\mathbb{L}}{\underset{\mathsf{k}}{\otimes}} R$-algebra using the multiplication map $R \overset{\mathbb{L}}{\underset{\mathsf{k}}{\otimes}} R \to R$. In particular, this shows that $\mathsf{HH}(R/\mathsf{k})$ carries the structure of an object in $\mathsf{SCR}_{\mathsf{k}}$. Also, for a general $R \in \mathsf{SCR}_{\mathsf{k}}$ we replace $\Omega^1_{R/\mathsf{k}}$ by the *cotangent complex* $\mathbb{L}_{R/\mathsf{k}}$ and independently of the characteristic of k, we have the HKR filtration on $\mathsf{HH}(R/\mathsf{k})$

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(*)See [Lu-SAG:§25.1.1]

that has $(\Lambda^i \mathbb{L}_{R/\mathsf{k}})[i]$ as associated graded piece of weight i (see [NS18: IV. 4.1]). When k is a field with $\mathrm{char}(\mathsf{k}) = 0$, the anti-symmetrization map induces a splitting of the HKR filtration and gives a k-linear quasi-isomorphism [Lod92:Prop. 1.3.16, Remark 3.2.3, Prop. 5.4.6]

$$\mathsf{HH}(R/\mathsf{k}) \xrightarrow{\sim} \bigoplus_{i=1}^{n} (\Lambda^i \mathbb{L}_{R/\mathsf{k}})[i] \tag{1}$$

Derived geometry [TV11; BZN12] offers another perspective: since in SCR_k, derived tensor products are pushouts, we find an equivalence in SCR_k, $\mathsf{HH}(R/k) \simeq R \overset{\mathbb{L}}{\underset{\mathsf{k}}{\otimes}} \mathsf{S}^1$ that presents $\mathsf{HH}(R/\mathsf{k})$ as the derived ring of functions $\mathscr{O}_{\mathsf{L}X}$ on the loop derived loop scheme $\mathsf{L}X := \mathbb{R}\mathsf{Map}(\mathsf{S}^1, X)$. Similarly, derived geometry offers a geometric incarnation for $\bigoplus_{i=1}^{n} (\Lambda^i \mathbb{L}_{R/\mathsf{k}})[i]$ as the derived ring of functions of the shifted tangent bundle $\mathsf{T}[-1]X{=}\mathsf{Spec}(\mathsf{Sym}^{\Delta}(\mathbb{L}_{X/\mathsf{k}}[1]))$ with the $\mathbb{G}_{m\,\mathsf{k}}$-action corresponding to the natural grading. When k is a $\mathbb{Q}$-algebra, for any affine derived scheme X, the results of [TV11; BZN12] provide an isomorphism of derived schemes functorial in X

$$\mathsf{T}[-1]X \xrightarrow{\sim} \mathsf{L}X \tag{2}$$

that recovers a quasi-isomorphism of the type (1) after passing to global functions. However, it is unclear if the equivalence obtained through derived geometry coincides with the anti-symmetrization map of (1).

Remark 1.1. When $\mathrm{char}(\mathsf{k}) = 0$, Kapranov [Kap99] explains another way to produce HKR isomorphisms (1) by considering smooth schemes X with a torsion-free flat connection ∇ on their tangent bundle. In this case the connection provides a formal exponential $\mathsf{exp}^{\nabla} : \widehat{\mathsf{T}X} \simeq \widehat{\Delta}$ establishing an isomorphism between the formal completion of $X \times X$ along the diagonal and the formal completion of $\mathsf{T}X$ along the zero section. Passing to the self-intersections with X, we obtain another equivalence of derived schemes of the type (2).

2 The space of functorial HKR isomorphisms

The goal of this note is prove the Corollary **3.2** below, describing the collection of HKR isomorphisms (1). We see though that without further assumptions, this space can be significantly large: the Remark **1.1** shows that every torsion-free connection on a scheme X provides one, and the space of connections is affine. But clearly, connection-induced HKR isomorphisms are not functorial unless the maps preserve the connection. Therefore we will only consider HKR isomorphisms that:

(i) are defined as part of a natural equivalence of ∞-functors on the ∞-category of derived schemes

$$\mathsf{T}[-1](-) \xrightarrow{\sim} \mathsf{L}(-) \tag{3}$$

(ii) the natural equivalence defines functorial splittings of the HKR-filtration;

(iii) functorialy match the circle action on loop spaces with the de Rham differential on forms.

Derived algebraic geometry again helps understanding these properties: using the formalism of affine stacks [Toe06], it is shown in the combined results of [MRT22; Toë20; Mou21] that over any commutative ring k there exists a flat affine filtered abelian group stack (underived)

$$\mathsf{S}^1_{\mathsf{Fil}} \to [\mathbb{A}^1_{\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}]$$

which we call the filtered circle, and such that for any derived scheme X, the relative derived mapping stack

$$\mathbb{R}\underline{\mathsf{Map}}_{[\mathbb{A}^1_{\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}]}(\mathsf{S}^1_{\mathsf{Fil}}, X \times [\mathbb{A}^1_{\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}]) \to [\mathbb{A}^1_{\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}]$$

provides the HKR-filtration on the derived loop space $\mathsf{L}X$, with associated graded stack given by $(\mathsf{T}[-1]X)/\mathbb{G}_{\mathrm{m}\,\mathsf{k}} \to \mathsf{B}\mathbb{G}_{\mathrm{m}\,\mathsf{k}}$. More precisely, it describes the HKR-filtration with its multiplicative structure as the universal filtered algebra with an action of the filtered circle. As a consequence, asking for HKR-isomorphisms respecting (i)-(iii), is to ask for *splittings* of the filtered circle compatible with the group structure:

Construction 2.1. Let $q : [\mathbb{A}^1_{\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}] \to \mathsf{B}\mathbb{G}_{\mathrm{m}\,\mathsf{k}}$ be the map induced by the projection $\mathbb{A}^1_{\mathsf{k}} \to \mathsf{Spec}(\mathsf{k})$ and let Y be a stack endowed with a $\mathbb{G}_{\mathrm{m}}$-action. Take $Z = [Y/\mathbb{G}_{\mathrm{m}}] \to \mathsf{B}\mathbb{G}_{\mathrm{m}\,\mathsf{k}}$. We define the associated *split* filtered stack $Z^{\mathsf{split}} \to [\mathbb{A}^1_{\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}]$ to be the pullback

$$\begin{array}{ccc} Z^{\mathsf{split}} & \longrightarrow & Z \\ \downarrow & & \downarrow \\ [\mathbb{A}^1_{\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}] & \longrightarrow & \mathsf{B}\mathbb{G}_{\mathrm{m}\,\mathsf{k}} \end{array}$$

By construction, it is equivalent to the quotient stack $[Y \times \mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]$ where we let $\mathbb{G}_{\mathrm{m}}$ act on the product coordinate-wise. The associated graded stack $(Z^{\mathsf{split}})^{\mathsf{gr}}$ is canonically equivalent to Z because q is a right inverse to the inclusion $0 : \mathsf{B}\mathbb{G}_{\mathrm{m}\,\mathsf{k}} \to [\mathbb{A}^1_{\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}]$. Finally, when $S \to [\mathbb{A}^1_{\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}]$ is a filtered stack, we denote by $S^{\mathrm{triv}} := (S^{\mathsf{gr}})^{\mathsf{split}}$ the associated split filtered stack where S^{gr} is the pullback of S along the inclusion $\mathsf{B}\mathbb{G}_{\mathrm{m}\,\mathsf{k}} \to [\mathbb{A}^1_{\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}]$.

Since the Construction **2.1** is monoidal with respect to cartesian products, $(\mathsf{S}^1_{\mathsf{Fil}})^{\mathrm{triv}}$ is still a group. We finally narrow down the choices of HKR-isomorphisms:

Definition 2.2. The space of HKR-isomorphisms compatible with (i)-(iii) is the space of invertible maps of group (higher) stacks

$$\mathsf{Map}^{\mathrm{inv}}_{\mathrm{group},[\mathbb{A}^1_{\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}]}\left(\mathsf{S}^1_{\mathsf{Fil}}\,,\,(\mathsf{S}^1_{\mathsf{Fil}})^{\mathrm{triv}}\right)$$

ie, universal splittings of the HKR filtration compatible with the action of the filtered circle.

Remark 2.3. Given a splitting $\mathsf{S}^1_{\mathsf{Fil}} \simeq (\mathsf{S}^1_{\mathsf{Fil}})^{\mathrm{triv}}$ as in Definition **2.2** we obtain the associated HKR-natural transformation (3) by pre-composition with the relative derived mapping spaces over $[\mathbb{A}^1_{\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}]$

$$\mathbb{R}\underline{\mathsf{Map}}_{[\mathbb{A}^1_{\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}]}((\mathsf{S}^1_{\mathsf{Fil}})^{\mathrm{triv}}, X\times[\mathbb{A}^1_{\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}]) \xrightarrow{\sim} \mathbb{R}\underline{\mathsf{Map}}_{[\mathbb{A}^1_{\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}]}(\mathsf{S}^1_{\mathsf{Fil}}, X\times[\mathbb{A}^1_{\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}])$$

and extracting the fibers over $1 : \mathsf{Spec}(\mathsf{k}) = [\mathbb{G}_{\mathrm{m}\,\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}] \to [\mathbb{A}^1_{\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}]$.

3 Computation

We are interested in computing π_0 of the space in Definition **2.2**. Thanks to [Mou21:Theorem 1.7] we have an explicit formula for the filtered group circle in terms of the relative Cartier dual of the relative formal group scheme over $[\mathbb{A}^1_{\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}]$ given by the total space $\mathrm{Def} \to [\mathbb{A}^1_{\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}]$ of the deformation to the normal bundle at the unit, from the formal group $\widehat{\mathbb{G}_{\mathrm{m}\,\mathsf{k}}}$ to its lie algebra $\widehat{\mathbb{G}_{\mathsf{a}\,\mathsf{k}}}$

$$\mathsf{S}^1_{\mathsf{Fil}} \simeq \mathsf{B}_{[\mathbb{A}^1_{\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}]}(\mathrm{Def}^\vee)$$

Here, Cartier duality is given by the $[\mathbb{A}^1_{\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}]$-relative construction of [Haz12: 37.3.4]:

$$(-)^\vee := \mathrm{Hom}_{\mathsf{FGr}}(-, \widehat{\mathbb{G}_{\mathrm{m}\,\mathsf{k}}})$$

(the hom is taken inside classical formal group schemes, not as derived schemes) and $\widehat{\mathbb{G}_{\mathrm{m}\,\mathsf{k}}}$ is the multiplicative formal group. Since the construction of Cartier duality is the relative one, we can freely interchange

$$(\mathrm{Def}^\vee)^{\mathrm{triv}} \simeq (\mathrm{Def}^{\mathrm{triv}})^\vee$$

$$\mathsf{B}_{[\mathbb{A}^1_{\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}]}(\mathrm{Def}^\vee)^{\mathrm{triv}} \simeq \mathsf{B}_{[\mathbb{A}^1_{\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}]}((\mathrm{Def}^\vee)^{\mathrm{triv}})$$

As a consequence, the space of HKR-isomorphisms of Definition **2.2** is equivalent to

$$\mathsf{Map}^{\mathrm{inv}}_{\mathrm{group},[\mathbb{A}^1_{\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}]}\Big(\mathsf{B}_{[\mathbb{A}^1_{\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}]}(\mathrm{Def}^\vee)\,,\,\mathsf{B}_{[\mathbb{A}^1_{\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}]}((\mathrm{Def}^{\mathrm{triv}})^\vee)\Big)$$

Since all group stacks being used are abelian, the Eckmann–Hilton delooping at the unit provides a map

$$\begin{array}{c}\mathsf{Map}^{\mathrm{inv}}_{\mathrm{group},[\mathbb{A}^1_{\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}]}\Big(\mathsf{B}_{[\mathbb{A}^1_{\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}]}(\mathrm{Def}^\vee)\,,\,\mathsf{B}_{[\mathbb{A}^1_{\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}]}((\mathrm{Def}^{\mathrm{triv}})^\vee)\Big)\\ \big\downarrow{\scriptstyle \Omega_*}\\ \mathsf{Map}^{\mathrm{inv}}_{\mathrm{group},[\mathbb{A}^1_{\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}]}\big(\mathrm{Def}^\vee\,,\,(\mathrm{Def}^{\mathrm{triv}})^\vee\big)\end{array}$$

which induces an isomorphism of π_0 with inverse given by the B-construction.

Finally, we consider the map induced by the functor of Cartier duality

$$\begin{array}{c}\mathsf{Map}^{\mathrm{inv}}_{\mathrm{group},[\mathbb{A}^1_{\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}]}\big(\mathrm{Def}^\vee\,,\,(\mathrm{Def}^{\mathrm{triv}})^\vee\big)\\ {\scriptstyle (-)^\vee}\big\uparrow\\ \mathsf{Map}^{\mathrm{inv}}_{\mathsf{FGr},[\mathbb{A}^1_{\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}]}\big(\mathrm{Def}^{\mathrm{triv}},\mathrm{Def}\big)\end{array} \tag{4}$$

which is an equivalence, thanks to the fully faithfulness of Cartier duality [Mou21:Const 3.6, Prop 3.12, Const 3.16, Prop. 3.17]. Here, FGr denotes the category of relative smooth formal groups. Notice that, independently of $\mathrm{char}(\mathsf{k})$, both mapping spaces in the last formula are discrete. Moreover, thanks to [Hen17:1.4.2 and 1.4.5] we can either see the last mapping space as maps of prestacks or as continuous maps.

Since $\mathrm{Def}\to[\mathbb{A}^1_{\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}]$ is a smooth formal group relative to $[\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]$, we can identify the trivial filtration $\mathrm{Def}^{\mathrm{triv}}\to[\mathbb{A}^1_{\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}]$ with the affine linear formal group associated to its relative Lie algebra. In particular, following Construction **2.1**, it is given by the constant family over $[\mathbb{A}^1_{\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}]$

$$\mathrm{Def}^{\mathrm{triv}}\simeq[(\widehat{\mathbb{G}_{\mathsf{a}\,\mathsf{k}}}\times\mathbb{A}^1_{\mathsf{k}})/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}]$$

and the set of functorial HKR-isomorphisms is given the set of filtered formal exponentials

$$\mathsf{Map}^{\mathrm{inv}}_{\mathsf{FGr},[\mathbb{A}^1_{\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}]}\Big([(\widehat{\mathbb{G}_{\mathsf{a}\,\mathsf{k}}}\times\mathbb{A}^1_{\mathsf{k}})/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}]\,,\,\mathrm{Def}\Big)$$

Remark 3.1. By extracting the underlying groups of the filtration (ie, the fibers over 1 in $[\mathbb{A}^1_{\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}]$) we find a map

$$\mathsf{Map}^{\mathrm{inv}}_{\mathsf{FGr},[\mathbb{A}^1_{\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}]}\Big([(\widehat{\mathbb{G}_{\mathsf{a}\,\mathsf{k}}}\times\mathbb{A}^1_{\mathsf{k}})/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}]\,,\,\mathrm{Def}\Big)\to\mathsf{Map}^{\mathrm{inv}}_{\mathsf{FGr}}\Big(\widehat{\mathbb{G}_{\mathsf{a}\,\mathsf{k}}},\widehat{\mathbb{G}_{\mathrm{m}\,\mathsf{k}}}\Big) \tag{5}$$

By height reasons, since $\widehat{\mathbb{G}_{\mathsf{a}\,\mathsf{k}}}$ is of height ∞ and $\widehat{\mathbb{G}_{\mathrm{m}\,\mathsf{k}}}$ is of height 1, the target of (5) is empty when k is of $\mathrm{char}(p)>0$. Therefore, so is the source of (5).

Finally, when $\mathrm{char}(\mathsf{k})=0$, the relative exponential map (see for instance [Dem:Exposé VIIB - §3] or [GR17:Chapter 7, Cor. 3.2.2]) defines an isomorphism of filtered formal group schemes

$$[(\widehat{\mathbb{G}_{\mathsf{a}\,\mathsf{k}}} \times \mathbb{A}^1_{\mathsf{k}})/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}] \xrightarrow[\sim]{\mathsf{exp}_{\mathrm{rel}}} \mathrm{Def}$$

Composition with $\mathsf{exp}_{\mathrm{rel}}$ defines a bijection

$$\begin{array}{c} \mathsf{Map}^{\mathrm{inv}}_{\mathsf{FGr},[\mathbb{A}^1_{\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}]}\Big([(\widehat{\mathbb{G}_{\mathsf{a}\,\mathsf{k}}} \times \mathbb{A}^1_{\mathsf{k}})/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}]\,,\,\mathrm{Def}\Big) \\ \sim\Big\uparrow \mathsf{exp}_{\mathrm{rel}} \circ - \\ \mathsf{Map}^{\mathrm{inv}}_{\mathsf{FGr},[\mathbb{A}^1_{\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}]}\Big([(\widehat{\mathbb{G}_{\mathsf{a}\,\mathsf{k}}} \times \mathbb{A}^1_{\mathsf{k}})/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}]\,,\,[(\widehat{\mathbb{G}_{\mathsf{a}\,\mathsf{k}}} \times \mathbb{A}^1_{\mathsf{k}})/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}]\Big) \end{array} \tag{6}$$

Let us compute the last space: since $\mathrm{char}(\mathsf{k}) = 0$, the category of formal groups relative to $[\mathbb{A}^1_{\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}]$ is equivalent to the category of Lie algebra objects in $\mathsf{QCoh}([\mathbb{A}^1_{\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}])$ [GR17:Chapter 7]. The Lie algebra associated to $[(\widehat{\mathbb{G}_{\mathsf{a}\,\mathsf{k}}} \times \mathbb{A}^1_{\mathsf{k}})/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}]$ is the structure sheaf $\mathscr{O}_{[\mathbb{A}^1_{\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}]}(1)$ with the weight-(1) action of $\mathbb{G}_{\mathrm{m}\,\mathsf{k}}$ (see [Mou21:§5]), endowed with the abelian Lie bracket. Since $\mathsf{QCoh}([\mathbb{A}^1_{\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}])$ is symmetric monoidal equivalent to filtered k-modules $\mathsf{Fil}(\mathsf{Mod}_{\mathsf{k}})$ [Mou19], $\mathscr{O}_{[\mathbb{A}^1_{\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}]}(1)$ corresponds to the abelian Lie algebra given by $\mathsf{k}(1)$. It follows that

$$\begin{array}{c} \mathsf{Map}^{\mathrm{inv}}_{\mathsf{FGr},[\mathbb{A}^1_{\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}]}\Big([(\widehat{\mathbb{G}_{\mathsf{a}\,\mathsf{k}}} \times \mathbb{A}^1_{\mathsf{k}})/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}]\,,\,[(\widehat{\mathbb{G}_{\mathsf{a}\,\mathsf{k}}} \times \mathbb{A}^1_{\mathsf{k}})/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}]\Big) \\ \sim\Big\uparrow \\ \pi_0\,\mathsf{Map}^{\mathrm{inv}}_{\mathrm{Lie},\mathsf{Fil}(\mathsf{Mod}_{\mathsf{k}})}(\mathsf{k}(1),\mathsf{k}(1)) = \mathsf{k}^* \end{array}$$

In particular, the map (6) sends $\lambda \in \mathsf{k}^*$ to $\mathsf{exp}(\lambda.(-))$. In summary:

Corollary 3.2. *Let* k *be a field. Then, the set of functorial multiplicative HKR equivalences simultaneously defined for all derived* k*-schemes, splitting the HKR-filtration and matching the circle action to the de Rham differential, is the set of exponential maps,* $\mathrm{Hom}_{\mathsf{FGr}}(\widehat{\mathbb{G}_{\mathsf{a}\,\mathsf{k}}}, \widehat{\mathbb{G}_{\mathrm{m}\,\mathsf{k}}}) \simeq \mathsf{k}^*$ *if* $\mathrm{char}(\mathsf{k}) = 0$, *and empty otherwise.*

Remark 3.3. The computations in this section describe the *group splittings* of the filtered circle as exponentials. The results of [Mou22] show that even in characteristic zero, the filtered circle does not admit splittings as a *pointed cogroup* with co-multiplication given by the pinch map. The universal obstruction is the Todd class. Recall that the splitting principle for algebraic K-theory implies that the collection of Chern characters from K-theory to de Rham cohomology coincides with the collection of exponential maps - see [TV15:Lemma 5.5]. In summary, the existence of group splittings of $\mathsf{S}^1_{\mathsf{Fil}}$ allows the Chern characters to exist, and the fact that none of those are cogroup splittings, imposes the Grothendieck-Riemann-Roch theorem.

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References

[BZN12] David Ben-Zvi and David Nadler. “Loop spaces and connections”. In: *J. Topol.* 5.2 (2012), 377–430. ISSN: 1753-8416. DOI: 10.1112/jtopol/jts007.

[Dem] *Schémas en groupes. II: Groupes de type multiplicatif, et structure des schémas en groupes généraux*. Vol. Vol. 152. Lecture Notes in Mathematics. Séminaire de Géométrie Algébrique du Bois Marie 1962/64 (SGA 3), Dirigé par M. Demazure et A. Grothendieck. Springer-Verlag, Berlin-New York, 1970, ix+654.

[GR17] Dennis Gaitsgory and Nick Rozenblyum. *A study in derived algebraic geometry. Vol. II. Deformations, Lie theory and formal geometry*. Vol. 221. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2017, xxxv+436. ISBN: 978-1-4704-3570-7.

[Haz12] Michiel Hazewinkel. *Formal groups and applications*. Corrected reprint of the 1978 original. AMS Chelsea Publishing, Providence, RI, 2012, xxvi+573. ISBN: 978-0-8218-5349-8. DOI: 10.1090/chel/375.

[Hen17] Benjamin Hennion. “Higher dimensional formal loop spaces”. In: *Ann. Sci. Éc. Norm. Supér. (4)* 50.3 (2017), 609–663. ISSN: 0012-9593,1873-2151. DOI: 10.24033/asens.2329.

[HKR62] G. Hochschild, Bertram Kostant and Alex Rosenberg. “Differential forms on regular affine algebras”. In: *Trans. Amer. Math. Soc.* 102 (1962), 383–408. ISSN: 0002-9947. DOI: 10.2307/1993614.

[Kap99] M. Kapranov. “Rozansky-Witten invariants via Atiyah classes
”. In: *Compositio Mathematica* 115.1 (1999), 71–113. ISSN: 0010-437X. DOI: 10.1023/A:1000664527238.

[Lod92] Jean-Louis Loday. *Cyclic homology*. Vol. 301. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Appendix E by María O. Ronco. Springer-Verlag, Berlin, 1992, xviii+454. ISBN: 3-540-53339-7. DOI: 10.1007/978-3-662-21739-9.

[Lu-SAG] Jacob Lurie. *Spectral Algebraic Geometry*. 2018. URL: http://www.math.harvard.edu/~lurie/papers/SAG-rootfile.pdf.

[Mou19] Tasos Moulinos. *The geometry of filtrations, arXiv:1907.13562*. 2019. eprint: arXiv:1907.13562.

[Mou21] Tasos Moulinos. “Filtered formal groups, Cartier duality, and derived algebraic geometry”. In: (25th Jan. 2021). arXiv: 2101.10262 [math.AG].

[Mou22] Tasos Moulinos. “Cogroupoid structures on the circle and the Hodge degeneration”. In: (22nd Mar. 2022). arXiv: 2203.11806 [math.AT].

[MRT22] Tasos Moulinos, Marco Robalo and Bertrand Toën. “A universal Hochschild-Kostant-Rosenberg theorem”. In: *Geom. Topol.* 26.2 (2022), 777–874. ISSN: 1465-3060. DOI: 10.2140/gt.2022.26.777.

[NS18] Thomas Nikolaus and Peter Scholze. “On topological cyclic homology”. English. In: *Acta Mathematica* 221.2 (2018), 203–409. ISSN: 0001-5962. DOI: 10.4310/ACTA.2018.v221.n2.a1.

[Toe06] Bertrand Toen. “Champs affines”. In: *Selecta Math. (N.S.)* 12.1 (2006), 39–135. ISSN: 1022-1824. DOI: 10.1007/s00029-006-0019-z.

[Toë20] Bertrand Toën. “Grothendieck’s schematization problem revisited”. English. In: *Épijournal de Géométrie Algébrique. EPIGA* 4 (2020). Id/No 14, 21. ISSN: 2491-6765.

[TV11] Bertrand Toën and Gabriele Vezzosi. “Algèbres simpliciales S^1-équivariantes, théorie de de Rham et théorèmes HKR multiplicatifs”. In: *Compos. Math.* 147.6 (2011), 1979–2000. ISSN: 0010-437X. DOI: 10.1112/S0010437X11005501.

[TV15] Bertrand Toën and Gabriele Vezzosi. “Caractères de Chern, traces équivariantes et géométrie algébrique dérivée”. In: *Selecta Math. (N.S.)* 21.2 (2015), 449–554. ISSN: 1022-1824. DOI: 10.1007/s00029-014-0158-6.