Choices of HKR isomorphisms

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Abstract

We record the fact that the set of chain-level multiplicative HKR natural equivalences defined simultaneously for all derived schemes, functorially splitting the HKR-filtration and rendering the circle action compatible with the de Rham differential, is, via Cartier duality, in a natural bijection with the set of filtered formal exponential maps $\widehat{\mathbb{G}}_a \to \widehat{\mathbb{G}}_m$. In particular, when the base k is a field of characteristic zero, the set of choices is k^{*}.

Contents

1	Introduction	1
2	The space of functorial HKR isomorphisms: Theorem 2.6	2
3	Proof of Theorem 2.6	4

1 Introduction

Let k be a commutative ring. The Hochschild-Kostant-Rosenberg (HKR) theorem [HKR62] establishes for any smooth k-scheme $X = \operatorname{Spec}(R)$ an identification of the Hochschild homology groups $\operatorname{HH}_i(R/k) := \operatorname{Tor}^i_{R\otimes_k R}(R, R)$ with the modules of *i*-differential forms $\Omega^i_{X/k}$, given by the anti-symmetrization map

$$\Omega^{i}_{R/\mathsf{k}} \to \mathsf{HH}_{i}(R/\mathsf{k}), \quad r_{0}.dr_{1} \wedge \cdots \wedge dr_{i} \mapsto \sum_{\sigma \in \Sigma_{i}} (-1)^{\mathsf{sign}(\sigma)} [r_{0} \otimes r_{\sigma(1)} \otimes \cdots \otimes r_{\sigma(i)}].$$

where R is seen as an $R \bigotimes_{k}^{\mathbb{L}} R$ -algebra using the multiplication map $R \bigotimes_{k}^{\mathbb{L}} R \to R$. In particular, this shows that HH(R/k) carries the structure of an object in SCR_k. Also, for a general $R \in SCR_k$ we replace $\Omega_{R/k}^1$ by the *cotangent complex* $\mathbb{L}_{R/k}$ and independently of the characteristic of k, we have the HKR filtration on HH(R/k)that has $(\Lambda^i \mathbb{L}_{R/k})[i]$ as associated graded piece of weight i (see [NS18: IV. 4.1]). When k is a field with char(k) = 0, the anti-symmetrization map induces a splitting of the HKR filtration and gives a k-linear quasi-isomorphism [Lod92:Prop. 1.3.16, Remark 3.2.3, Prop. 5.4.6]

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^(*)Here SCR_k denotes the ∞ -category of simplicial commutative rings. See [Lu-SAG:§25.1.1]

$$\mathsf{HH}(R/\mathsf{k}) \xrightarrow{\sim} \bigoplus_{i=1}^{n} (\Lambda^{i} \mathbb{L}_{R/\mathsf{k}})[i] \tag{1}$$

Derived geometry [TV11; BZN12] offers another perspective: since in SCR_k , derived tensor products are pushouts, we find an equivalence in SCR_k , $HH(R/k) \simeq R \bigotimes_k^{\mathbb{L}} S^1$ that presents HH(R/k) as the derived ring of functions \mathcal{O}_{LX} on the derived loop scheme $LX := \mathbb{R}\underline{Map}(S^1, X)$ where by $\mathbb{R}\underline{Map}$ we mean the derived mapping scheme cf. [Toë14:§3.2].

Similarly, derived geometry offers a geometric incarnation for $\bigoplus_{i=1}^{n} (\Lambda^{i} \mathbb{L}_{R/k})[i]$ as the derived ring of functions of the shifted tangent bundle $\mathsf{T}[-1]X = \mathsf{Spec}(\mathsf{Sym}^{\Delta}(\mathbb{L}_{X/k}[1]))$ where Sym^{Δ} corresponds to the symmetric algebra construction in the setting of commutative simplicial rings, with the \mathbb{G}_{mk} -action corresponding to the natural grading. When k is a Q-algebra, for any affine derived scheme X, the results of [TV11; BZN12] provide an isomorphism of derived schemes functorial in X

$$\mathsf{T}[-1]X \xrightarrow{\sim} \mathsf{L}X \tag{2}$$

that recovers a quasi-isomorphism of the type (1) after passing to global functions. However, it is unclear if the equivalence obtained through derived geometry coincides with the anti-symmetrization map of (1).

Observation 1.1. When char(k) = 0, Kapranov [Kap99] explains another way to produce HKR isomorphisms (1) by considering smooth schemes X with a torsion-free flat connection ∇ on their tangent bundle. In this case the connection provides a formal exponential $\exp^{\nabla} : \widehat{\mathsf{T}X} \simeq \widehat{\Delta}$ establishing an isomorphism between the formal completion of $X \times X$ along the diagonal and the formal completion of $\mathsf{T}X$ along the zero section. Passing to the self-intersections with X, we obtain another equivalence of derived schemes of the type (2).

2 The space of functorial HKR isomorphisms: Theorem 2.6

The goal of this note is prove Theorem 2.6 below, describing the collection of HKR isomorphisms (1). We start by noticing though, that without further assumptions, this space can be significantly large: Observation 1.1 shows that every torsion-free connection on a scheme X provides one, and the space of connections is affine. But clearly, connection-induced HKR isomorphisms are not functorial unless the maps preserve the connection. Therefore we will only consider chain level HKR equivalences enhanced with:

(i) functoriality for all derived k-rings as part of a natural equivalence of ∞ -functors on the ∞ -category of derived schemes

$$\mathsf{T}[-1](-) \xrightarrow{\sim} \mathsf{L}(-) \tag{3}$$

- (ii) functorial splittings of the HKR-filtration;
- (iii) functorial matchings of the circle action on loop spaces with the de Rham differential on forms.

Observation 2.1. In particular, chain level HKR-equivalences as in (i) are automatically multiplicative by passing to the derived rings of functions in (3).

Before formulating our main result we must first describe how derived geometry helps combining the structures in (i)-(iii), culminating with Definition 2.4 below. Using the formalism of affine stacks [Toe06], it is shown in the combined results of [MRT22; Toë20; Mou24b] that over any commutative ring k there exists a flat affine filtered abelian group stack (underived)

$$\mathsf{S}^1_{\mathsf{Fil}} o [\mathbb{A}^1_{\mathsf{k}}/\mathbb{G}_{\mathrm{m\,k}}]$$

which we call the filtered circle, and such that for any derived scheme X, the relative derived mapping stack

$$\mathbb{R}\underline{\mathrm{Map}}_{[\mathbb{A}^1_{\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}]}(\mathsf{S}^1_{\mathsf{Fil}}, X \times [\mathbb{A}^1_{\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}]) \to [\mathbb{A}^1_{\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}]$$

provides the HKR-filtration on the derived loop space LX, with associated graded stack given by $(T[-1]X)/\mathbb{G}_{mk} \to \mathbb{B}\mathbb{G}_{mk}$. More precisely, it describes the HKRfiltration with its multiplicative structure as the universal filtered algebra with an action of the filtered circle. As a consequence, asking for HKR-isomorphisms (i)-(iii), is to ask for *splittings* of the filtered circle compatible with the group structure. Let us then recall the construction of a split filtered stack associated to a graded stack:

Construction 2.2. Let $q : [\mathbb{A}_{k}^{1}/\mathbb{G}_{m\,k}] \to \mathbb{B}\mathbb{G}_{m\,k}$ be the map induced by the projection $\mathbb{A}_{k}^{1} \to \operatorname{Spec}(k)$ and let Y be a stack endowed with a \mathbb{G}_{m} -action. Take $Z = [Y/\mathbb{G}_{m}] \to \mathbb{B}\mathbb{G}_{m\,k}$. We define the associated *split* filtered stack $Z^{\operatorname{split}} \to [\mathbb{A}_{k}^{1}/\mathbb{G}_{m\,k}]$ to be the pullback

$$\begin{array}{ccc} Z^{\mathsf{split}} & \longrightarrow Z \\ \downarrow & {\color{black}{_}} & \downarrow \\ [\mathbb{A}^1_{\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}] & \longrightarrow \mathbb{B}\mathbb{G}_{\mathrm{m}\,\mathsf{k}} \end{array}$$

By construction, it is equivalent to the quotient stack $[Y \times \mathbb{A}^1/\mathbb{G}_m]$ where we let \mathbb{G}_m act on the product coordinate-wise. The associated graded stack $(Z^{\text{split}})^{\text{gr}}$ is canonically equivalent to Z because q is a right inverse to the inclusion $0 : \mathbb{B}\mathbb{G}_{mk} \to [\mathbb{A}^1_k/\mathbb{G}_{mk}]$. Finally, when $S \to [\mathbb{A}^1_k/\mathbb{G}_{mk}]$ is a filtered stack, we denote by $S^{\text{triv}} := (S^{\text{gr}})^{\text{split}}$ the associated split filtered stack where S^{gr} is the pullback of S along the inclusion $\mathbb{B}\mathbb{G}_{mk} \to [\mathbb{A}^1_k/\mathbb{G}_{mk}]$.

Observation 2.3. Since Construction **2.2** is monoidal with respect to cartesian products, $(S_{Fil}^1)^{triv}$ is still a group object.

We can finally formulate how to combine the enhanced structures of (i)-(iii) as part of a single object:

Definition 2.4. We define the set of chain-level HKR-isomorphisms enhanced with (i)-(iii) as the set of connected components of the mapping space of invertible maps of group (higher) stacks

$$\mathsf{Map}^{\mathrm{inv}}_{\mathrm{group},[\mathbb{A}^1_k/\mathbb{G}_{\mathrm{m\,k}}]}\big(\mathsf{S}^1_{\mathsf{Fil}}\,,\,(\mathsf{S}^1_{\mathsf{Fil}})^{\mathrm{triv}}\big)$$

ie, universal splittings of the HKR filtration compatible with the action of the filtered circle.

Observation 2.5. Given a splitting $S^1_{Fil} \simeq (S^1_{Fil})^{triv}$ as in Definition 2.4 we obtain the associated HKR-natural transformation (3) by pre-composition with the relative derived mapping spaces over $[\mathbb{A}^1_k/\mathbb{G}_{m\,k}]$

$$\mathbb{R}\underline{\mathsf{Map}}_{[\mathbb{A}^{1}_{\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}]}((\mathsf{S}^{1}_{\mathsf{Fil}})^{\mathrm{triv}}, X \times [\mathbb{A}^{1}_{\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}]) \xrightarrow{\sim} \mathbb{R}\underline{\mathsf{Map}}_{[\mathbb{A}^{1}_{\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}]}(\mathsf{S}^{1}_{\mathsf{Fil}}, X \times [\mathbb{A}^{1}_{\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}])$$

and extracting the fibers over $1 : \operatorname{Spec}(\mathsf{k}) = [\mathbb{G}_{\mathrm{m}\,\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}] \to [\mathbb{A}^1_{\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}].$

We state our main result:

Theorem 2.6. Let k be a field. Then, the set of chain-level HKR equivalences enhanced with (i)-(iii) cf. Definition 2.4 is in bijection with the set of formal exponentials, ie, group homomorphisms of formal groups,

$$\operatorname{Hom}_{\mathsf{FGr}}(\widehat{\mathbb{G}_{\mathsf{a}\,\mathsf{k}}},\widehat{\mathbb{G}_{\mathsf{m}\,\mathsf{k}}}) \simeq \begin{cases} \mathsf{k}^* & \text{if } \operatorname{char}(\mathsf{k}) = 0, \\ \emptyset & \text{otherwise.} \end{cases}$$

3 Proof of Theorem 2.6

We are interested in computing π_0 of the space in Definition 2.4. Thanks to [Mou24b:Theorem 1.8] we have an explicit formula for the filtered group circle in terms of relative Cartier duality over $[\mathbb{A}^1_k/\mathbb{G}_{m\,k}]$

$$\mathsf{S}^1_{\mathsf{Fil}} \simeq \mathsf{B}_{[\mathbb{A}^1_{\mathsf{k}}/\mathbb{G}_{\mathrm{m\,k}}]}(\mathrm{Def}^\vee)$$

where $\text{Def} \to [\mathbb{A}^1_k/\mathbb{G}_{m\,k}]$ is the formal group scheme over $[\mathbb{A}^1_k/\mathbb{G}_{m\,k}]$ given by the total space of the deformation to the normal bundle at the unit from the formal group $\widehat{\mathbb{G}_{m\,k}}$ to its lie algebra $\widehat{\mathbb{G}_{a\,k}}$ (cf. [Mou24b:Construction 5.6, Proposition 5.12, Theorem

1.6].). Here, relative Cartier duality is given by the $[\mathbb{A}_{k}^{1}/\mathbb{G}_{m\,k}]$ -relative construction of [Haz12: 37.3.4]:

$$(-)^{\vee}:=\operatorname{Hom}_{\mathsf{FGr}}(-,\widehat{\mathbb{G}_{\operatorname{m}}{}_{\mathsf{k}}})$$

(the hom is taken inside the category of classical formal group schemes, not as derived schemes) and $\widehat{\mathbb{G}_{m\,k}}$ is the multiplicative formal group. Since the construction of Cartier duality is the relative one, we can freely interchange

$$(\mathrm{Def}^{\vee})^{\mathrm{triv}} \simeq (\mathrm{Def}^{\mathrm{triv}})^{\vee}, \qquad \mathsf{B}_{[\mathbb{A}^1_k/\mathbb{G}_{\mathrm{m\,k}}]}(\mathrm{Def}^{\vee})^{\mathrm{triv}} \simeq \mathsf{B}_{[\mathbb{A}^1_k/\mathbb{G}_{\mathrm{m\,k}}]}((\mathrm{Def}^{\vee})^{\mathrm{triv}})$$

As a consequence, the space of HKR-isomorphisms of Definition 2.4 is equivalent to

$$\mathsf{Map}^{\mathrm{inv}}_{\mathrm{group},[\mathbb{A}^1_k/\mathbb{G}_{\mathrm{m\,}k}]} \Big(\mathsf{B}_{[\mathbb{A}^1_k/\mathbb{G}_{\mathrm{m\,}k}]}(\mathrm{Def}^{\vee}) \,, \, \mathsf{B}_{[\mathbb{A}^1_k/\mathbb{G}_{\mathrm{m\,}k}]}((\mathrm{Def}^{\mathrm{triv}})^{\vee}) \Big)$$

Since all group stacks being used are abelian, the Eckmann–Hilton delooping at the unit provides a map

which induces an isomorphism of π_0 with inverse given by the B-construction.

Finally, we consider the map induced by the functor of Cartier duality

$$\begin{aligned} \mathsf{Map}_{\mathrm{group},[\mathbb{A}^{1}_{\mathsf{k}}/\mathbb{G}_{\mathsf{m}\,\mathsf{k}}]}^{\mathrm{inv}}\left(\mathrm{Def}^{\vee},\ (\mathrm{Def}^{\mathrm{triv}})^{\vee}\right) \\ (-)^{\vee\uparrow} \\ \mathsf{Map}_{\mathsf{FGr},[\mathbb{A}^{1}_{\mathsf{k}}/\mathbb{G}_{\mathsf{m}\,\mathsf{k}}]}^{\mathrm{inv}}\left(\mathrm{Def}^{\mathrm{triv}},\mathrm{Def}\right) \end{aligned}$$
(4)

which is an equivalence, thanks to the fully faithfulness of Cartier duality [Mou24b:Const 3.7, Prop 3.12, Const 3.17, Prop. 3.19]. Here, FGr denotes the category of relative smooth formal groups. Notice that, independently of char(k), both mapping spaces in (4) are discrete. Moreover, thanks to [Hen17:1.4.2 and 1.4.5] we can either see the last mapping space as maps of prestacks or as continuous maps.

Since $\operatorname{Def} \to [\mathbb{A}_k^1/\mathbb{G}_{m\,k}]$ is a smooth formal group relative to $[\mathbb{A}^1/\mathbb{G}_m]$, we can identify the trivial filtration $\operatorname{Def}^{\operatorname{triv}} \to [\mathbb{A}_k^1/\mathbb{G}_{m\,k}]$ with the affine linear formal group associated to its relative Lie algebra. In particular, following Construction 2.2, it is given by the constant family over $[\mathbb{A}_k^1/\mathbb{G}_{m\,k}]$

$$\mathrm{Def}^{\mathrm{triv}} \simeq [(\widehat{\mathbb{G}_{\mathsf{a}\,\mathsf{k}}} \times \mathbb{A}^1_{\mathsf{k}})/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}]$$

In conclusion, we have shown that the set of functorial HKR-isomorphisms as in Definition 2.4 is in bijection with the set of filtered formal exponentials

$$\mathsf{Map}_{\mathsf{FGr},[\mathbb{A}^1_k/\mathbb{G}_{m\,k}]}^{\mathrm{inv}} \big([(\widehat{\mathbb{G}_{\mathsf{a}\,\mathsf{k}}} \times \mathbb{A}^1_{\mathsf{k}})/\mathbb{G}_{m\,\mathsf{k}}] \,, \, \mathrm{Def} \big)$$

Observation 3.1. By extracting the underlying groups of the filtration (ie, the fibers over 1 in $[\mathbb{A}^1_k/\mathbb{G}_{m\,k}]$) we find a map

$$\mathsf{Map}_{\mathsf{FGr},[\mathbb{A}^1_k/\mathbb{G}_{\mathrm{m}\,k}]}^{\mathrm{inv}}\Big([(\widehat{\mathbb{G}_{\mathsf{a}\,k}}\times\mathbb{A}^1_k)/\mathbb{G}_{\mathrm{m}\,k}]\,,\,\mathrm{Def}\Big)\to\mathsf{Map}_{\mathsf{FGr}}^{\mathrm{inv}}\Big(\widehat{\mathbb{G}_{\mathsf{a}\,k}},\widehat{\mathbb{G}_{\mathrm{m}\,k}}\Big) \tag{5}$$

By height reasons, since $\widehat{\mathbb{G}_{ak}}$ is of height ∞ and $\widehat{\mathbb{G}_{mk}}$ is of height 1, the target of (5) is empty when k is of char(p) > 0. Therefore, so is the source of (5).

Finally, assume char(\mathbf{k}) = 0. The relative exponential map (see for instance [Dem:Exposé VIIB - §3] or [GR17:Chapter 7, Cor. 3.2.2]) defines an isomorphism of filtered formal group schemes

$$[(\widehat{\mathbb{G}_{\mathsf{a}\,\mathsf{k}}}\times\mathbb{A}^1_{\mathsf{k}})/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}] \mathop{\longrightarrow}\limits^{\mathsf{exp}_{\mathrm{rel}}} \mathrm{Def}$$

Composition with exp_{rel} defines a bijection

$$\begin{aligned} \mathsf{Map}_{\mathsf{FGr},[\mathbb{A}^{1}_{\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}]}^{\mathrm{inv}} \left(\left[(\widehat{\mathbb{G}_{\mathsf{a}\,\mathsf{k}}} \times \mathbb{A}^{1}_{\mathsf{k}})/\mathbb{G}_{\mathrm{m}\,\mathsf{k}} \right], \, \mathrm{Def} \right) \\ & \stackrel{?\uparrow \mathsf{exp}_{\mathrm{rel}}\,\circ\,-}{\cong} \end{aligned} \tag{6} \\ \mathsf{Map}_{\mathsf{FGr},[\mathbb{A}^{1}_{\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}]}^{\mathrm{inv}} \left(\left[(\widehat{\mathbb{G}_{\mathsf{a}\,\mathsf{k}}} \times \mathbb{A}^{1}_{\mathsf{k}})/\mathbb{G}_{\mathrm{m}\,\mathsf{k}} \right], \, \left[(\widehat{\mathbb{G}_{\mathsf{a}\,\mathsf{k}}} \times \mathbb{A}^{1}_{\mathsf{k}})/\mathbb{G}_{\mathrm{m}\,\mathsf{k}} \right] \right) \end{aligned}$$

Let us compute the last space: since char(k) = 0, the category of formal groups relative to $[\mathbb{A}_{k}^{1}/\mathbb{G}_{m\,k}]$ is equivalent to the category of Lie algebra objects in $\mathsf{QCoh}([\mathbb{A}_{k}^{1}/\mathbb{G}_{m\,k}])$ [GR17:Chapter 7]. The Lie algebra associated to $[(\widehat{\mathbb{G}}_{a\,k} \times \mathbb{A}_{k}^{1})/\mathbb{G}_{m\,k}]$ is the structure sheaf $\mathcal{O}_{[\mathbb{A}_{k}^{1}/\mathbb{G}_{m\,k}]}(1)$ with the weight-(1) action of $\mathbb{G}_{m\,k}$, endowed with the abelian Lie bracket (see [Mou24b:§5]). Since $\mathsf{QCoh}([\mathbb{A}_{k}^{1}/\mathbb{G}_{m\,k}])$ is symmetric monoidal equivalent to filtered k-modules Fil(Mod_k) [Mou21], $\mathcal{O}_{[\mathbb{A}_{k}^{1}/\mathbb{G}_{m\,k}]}(1)$ corresponds to the abelian Lie algebra given by k(1). It follows that

$$\begin{split} \mathsf{Map}_{\mathsf{FGr},[\mathbb{A}^1_{\mathsf{k}}/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}]}^{\mathrm{inv}} \Big([(\widehat{\mathbb{G}_{\mathsf{a}\,\mathsf{k}}} \times \mathbb{A}^1_{\mathsf{k}})/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}] \,, \, [(\widehat{\mathbb{G}_{\mathsf{a}\,\mathsf{k}}} \times \mathbb{A}^1_{\mathsf{k}})/\mathbb{G}_{\mathrm{m}\,\mathsf{k}}] \Big) \\ & \stackrel{\wedge}{\longrightarrow} \\ \pi_0 \, \mathsf{Map}_{\mathrm{Lie},\mathsf{Fil}(\mathsf{Mod}_{\mathsf{k}})}^{\mathrm{inv}}(\mathsf{k}(1),\mathsf{k}(1)) = \mathsf{k}^* \end{split}$$

In particular, the map (6) sends $\lambda \in k^*$ to $\exp(\lambda(-))$, thus proving Theorem 2.6.

Remark 3.2. Theorem **2.6** describes the space of group splittings of the filtered circle as exponentials (cf. Definition **2.4**). The results of [Mou24a] show that even in characteristic zero, the filtered circle does not admit splittings as a *pointed cogroup* with co-multiplication given by the pinch map. The universal obstruction is the Todd class. Recall that the splitting principle for algebraic K-theory implies that the collection of Chern characters from K-theory to de Rham cohomology coincides with the collection of exponential maps - see [TV15:Lemma 5.5]. In summary, the existence of group splittings of S_{Fil}^1 allows the Chern characters to exist, and the fact that none of those are cogroup splittings, imposes the Grothendieck-Riemann-Roch theorem.

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