Choices of HKR isomorphisms

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Algebra Seminar

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HKR-isomorphisms

Theorem (Hochschild-Kostant-Rosenberg)

$X = \mathrm{Spec}(R)$ *a smooth affine scheme over a commutative ring* k. *Then the anti-symmetrization map induces an isomorphism*

$$\Omega^n_{R/\mathsf{k}} \to \mathrm{HH}_n(R/k) := Tor^n_{R\otimes_{\mathsf{k}} R}(R, R)$$

$$r_0.dr_1 \wedge \cdots \wedge dr_n \mapsto \sum_{\sigma\in\Sigma_n} (-1)^{\mathrm{sign}(\sigma)}[r_0 \otimes r_{\sigma(1)} \otimes \cdots \otimes r_{\sigma(n)}]$$

Question

Are there other HKR-isomorphisms?

$$\Omega^*_R \simeq \mathrm{HH}_*(R)$$

How many?

HKR-isomorphisms

Proposition (Kapranov)

When char$(\mathsf{k}) = 0$*, any torsion-free flat connection* ∇ *on* $X = \mathrm{Spec}(R)$ *induces an exponential isomorphism*

$$\exp^{\nabla} : \widehat{\mathrm{T}X} \simeq \widehat{\Delta_X}$$

which by passing to the self-intersection of the zero section, induces an HKR-isomorphism.

Remark

The HKR-isomorphism above is not functorial in X*. Depends on* ∇*.*

Goal

Describe the collection of functorial HKR-isomorphisms. This space is still too big so we need to rigidify the problem

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The HKR-Filtration

Remark

HH *is a chain complex:*

$$\mathrm{HH}(R/\mathsf{k}) := R \overset{\mathbb{L}}{\underset{R \overset{\mathbb{L}}{\underset{\mathsf{k}}{\otimes}} R}{\otimes}} R$$

Construction (Whitehead filtration)

HH *has a descending filtration* $\mathrm{HH}_{\mathrm{Fil}} :=$

$$\cdots \to \tau_{\geq 2}\,\mathrm{HH}(R/\mathsf{k}) \longrightarrow \tau_{\geq 1}\,\mathrm{HH}(R/\mathsf{k}) \longrightarrow \tau_{\geq 0}\,\mathrm{HH}(R/\mathsf{k}) = \mathrm{HH}(R/\mathsf{k})$$

$$\cdots \quad \mathrm{gr}^2 = \Omega^2_{R/\mathsf{k}}[2] \qquad \mathrm{gr}^1 = \Omega^1_{R/\mathsf{k}}[1] \qquad \mathrm{gr}^0 = \Omega^0_{R/\mathsf{k}}[0]$$

The HKR-Filtration

Remark

(chain level) Splittings of the HKR-filtration produce HKR-isomorphisms:

$$\mathrm{HH}_{\mathrm{Fil}} \simeq \bigoplus_{i \geq} \mathrm{gr}^i$$

Proposition

If $\mathrm{char}(\mathsf{k}) = 0$*, the anti-symmetrization map induces such a splitting.*

Rigified Goal

Describe the collection of HKR-isomorphisms obtained via functorial splittings of $\mathrm{HH}_{\mathrm{Fil}}$.

Still too big!

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The naive circle action...

Remark

Algebra structure

$$\mathrm{HH}(R/\mathsf{k}) = R \underset{R \overset{\mathbb{L}}{\underset{\mathsf{k}}{\otimes}} R}{\overset{\mathbb{L}}{\otimes}} R \simeq R \overset{\mathbb{L}}{\underset{\mathsf{k}}{\otimes}} \mathrm{S}^1$$

Remark

Group structure $\mathrm{S}^1 \times \mathrm{S}^1 \to \mathrm{S}^1$

Theorem (Universal Property)

$\mathrm{S}^1 \circlearrowright \mathrm{HH}(R/\mathsf{k})$ *is the universal* R*-algebra with a* S^1*-action.*

… and the de Rham differential …

Remark

Chain Level:

$$\mathrm{H}_1(\mathrm{S}^1, \mathsf{k}) \ni B := \textit{Connes operator} : \mathrm{HH}(R/\mathsf{k}) \to \mathrm{HH}(R/\mathsf{k})[1]$$

Proposition

$$\begin{array}{ccc} \mathrm{HH}_n(R/\mathsf{k}) & \xrightarrow{B} & \mathrm{HH}_{n+1}(R/\mathsf{k}) \\ \downarrow \wr & & \downarrow \wr \\ \Omega^n_{R/\mathsf{k}} & \xrightarrow{\mathrm{d}_{\mathrm{dR}}} & \Omega^{n+1}_{R/\mathsf{k}} \end{array}$$

Main result

Theorem (Slogan version)

The collection of HKR-isomorphisms

- *Chain level;*
- *Arising as functorial splittings of the HKR-filtration;*
- *matching the circle action with the de Rham differential;*
- *compatible with multiplicative structures on both sides;*

are in bijection with

$$\{\textit{exponentials } \widehat{\mathbb{G}_a} \to \widehat{\mathbb{G}_m}\} \simeq \begin{cases} \emptyset & \textit{if } \mathrm{char}(\mathsf{k}) \neq 0 \\ \mathsf{k}^* & \textit{if } \mathrm{char}(\mathsf{k}) = 0 \end{cases}$$

Problem (With the slogan)

"$B = \mathrm{d}_{\mathrm{dR}}$" is not immediately compatible with the HKR-filtration:

$$\cdots \to \underset{\mathrm{S}^1 \circlearrowright}{\tau_{\geq 2} \mathrm{HH}(R/\mathsf{k})} \longrightarrow \underset{\mathrm{S}^1 \circlearrowright}{\tau_{\geq 1} \mathrm{HH}(R/\mathsf{k})} \longrightarrow \underset{\mathrm{S}^1 \circlearrowright}{\tau_{\geq 0} \mathrm{HH}(R/\mathsf{k})}$$

$$\mathrm{S}^1 \circlearrowright \mathrm{gr}^n = \Omega^n_{R/\mathsf{k}}[n], \qquad \Omega^n_{R/\mathsf{k}}[n] \underset{0}{\to} \Omega^n_{R/\mathsf{k}}[n+1]$$

Levelwise circle action on the filtration is too naive. Does not capture graded weights.

What exactly is the de Rham differential?

First step: Consider $\bigoplus_{n\geq 0} \Omega^n_{R/\mathsf{k}}[n]$ as a graded module and d_{dR} as an extra operator that increases the weight.

Construction (Graded modules)

$$\mathrm{Ch}^{\mathrm{gr}}_{\mathsf{k}} := \prod_{n\in\mathbb{Z}} \mathrm{Ch}_{\mathsf{k}} \quad \ni E = (E_n)_{n\in\mathbb{Z}}, \qquad \bigoplus : \mathrm{Ch}^{\mathrm{gr}}_{\mathsf{k}} \to \mathrm{Ch}_{\mathsf{k}}$$

$$E \overset{\mathbb{L}}{\underset{\mathsf{k}}{\otimes}} F = \Big(\bigoplus_{n+m=\ell} E_n \overset{\mathbb{L}}{\underset{\mathsf{k}}{\otimes}} F_m \Big)_{\ell\in\mathbb{Z}}$$

Construction

$\mathsf{k}[\epsilon]_{\mathrm{gr}}$ = *graded strictly associative dg-algebra freely generated by an element* ϵ *in homological degree* 1 *and weight* 1, *and strictly verifying* $\epsilon^2 = 0$.

Remark

A left-$\mathsf{k}[\epsilon]_{\mathrm{gr}}$*-module in* $\mathrm{Ch}_{\mathsf{k}}^{\mathrm{gr}}$ $\Leftrightarrow$ $E = (E_n)_{n\in\mathbb{Z}}$ + *operator*

$$\epsilon : E(1)[1] \to E$$

with $\epsilon \circ \epsilon = 0$ *(strict).*

Construction

$\mathsf{k}[\epsilon]_{\mathrm{gr}}$ *carries a strictly commutative graded Hopf structure*

$$\mathsf{k}[\epsilon]_{\mathrm{gr}} \to \mathsf{k}[\epsilon]_{\mathrm{gr}} \otimes_{\mathrm{gr}} \mathsf{k}[\epsilon]_{\mathrm{gr}}$$

determined by

$$\epsilon \mapsto \epsilon \otimes 1 + 1 \otimes \epsilon$$

$\Rightarrow$ *tensor product of* $\mathsf{k}[\epsilon]_{\mathrm{gr}}$*-modules makes sense.*

Definition

- *Mixed graded modules* = left-modules over $\mathsf{k}[\epsilon]_{\mathrm{gr}}$.
- *Mixed graded algebras* = commutative algebra objects in the symmetric monoidal ∞-category of mixed graded modules.

Proposition

Over any ring k, *setting* $\epsilon := \mathrm{d}_{\mathrm{dR}}$ *endows* $\bigoplus_{n\in\mathbb{Z}} \Omega^n_{R/\mathsf{k}}[n]$ *with a structure of mixed graded algebrax.*

Problem

Identify the connection between three types of structures:

- *circle actions;*
- *mixed graded structures*
- *HKR-filtration*

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Proposition (Toën and Vezzosi, Ben-Zvi and Nadler)

Let $X = \mathrm{Spec}(R)$*:*

- $\mathrm{HH}(R/\mathsf{k}) = \mathcal{O}_{\mathbb{R}\mathrm{Map}(\mathrm{S}^1, X)}$.
- $\bigoplus_{n\geq 0} \Omega^n_{X/\mathsf{k}}[n] \simeq \mathrm{Sym}^{\Delta}(\Omega^1_{X/\mathsf{k}}[1]) \simeq \mathcal{O}_{\mathrm{T}[-1]X}$

Remark

$\mathbb{R}\mathrm{Map}(\mathrm{S}^1, X)$ *has an* S^1*- action.*

$$\mathcal{O}_{\mathbb{R}\mathrm{Map}(\mathrm{S}^1,X)/\mathrm{S}^1} = \mathrm{HH}(R/\mathsf{k}) \;+\; \mathrm{S}^1 - \textit{action}.$$

Remark

$\mathrm{T}[-1]X$ *has a* $\mathbb{G}_m$*-scaling along the fibers. This is responsible for the grading.*

Lemma

$$\{\textit{Chain complexes with additional } \mathbb{Z}\textit{-grading}\}^{\otimes} \simeq \underbrace{\mathrm{Qcoh}_{\infty}(\mathrm{B}\mathbb{G}_m)^{\otimes}}_{\mathbb{G}_m-\textit{representations}}$$

Corollary

$$\pi : (\mathrm{T}[-1]X)/\mathbb{G}_m \to \mathrm{Spec}(\mathsf{k})/\mathbb{G}_m = \mathrm{B}\mathbb{G}_m$$

$$\mathcal{O}_{(\mathrm{T}[-1]X)/\mathbb{G}_m} = \pi_* \mathcal{O} \simeq \bigoplus_{n \geq 0} \Omega^n_{X/\mathsf{k}}[n] \textit{ with grading}$$

What about d_{dR}?

Construction

$$\mathsf{k}[\eta] := \mathsf{k} \oplus \underbrace{\mathsf{k}[-1]}_{\textit{weight } -1}$$

with the trivial square zero multiplication as a cosimplicial graded commutative algebra.

$$\underbrace{\mathrm{coSpec}(\mathsf{k}[\eta])}_{\textit{Affine stack}} \circlearrowleft \mathbb{G}_m$$

.

Proposition

$\mathrm{coSpec}(\mathsf{k}[\eta])$ *admits a unique abelian group structure, compatible with the grading.*

Definition

$$\underbrace{\mathrm{S}^1_{\epsilon-\mathrm{gr}}}_{\text{Mixed graded circle}} \quad := [\mathrm{coSpec}(\mathsf{k}[\eta])/\mathbb{G}_m] \to \mathrm{B}\mathbb{G}_m$$

Lemma

$$\mathrm{Qcoh}(\mathrm{B}_{\mathrm{B}\mathbb{G}_m}(\mathrm{S}^1_{\epsilon-\mathrm{gr}}))^{\otimes} \simeq \{\textit{Mixed graded modules}\}^{\otimes}$$

Remark (BenZvi-Nadler)

When $\mathrm{char}(\mathsf{k}) = 0$

$$\mathrm{coSpec}(\mathsf{k}[\eta]) \underbrace{\simeq}_{\mathbb{G}_m-eq.} \mathrm{B}\mathbb{G}_a$$

Proposition

Equivalence of $\mathbb{G}_m$-stacks (independent of $\mathrm{char}(\mathsf{k})$)

$$\mathbb{R}\mathrm{Map}(\mathrm{coSpec}(\mathsf{k}[\eta]), X) \simeq \mathrm{T}[-1]X$$

Construction

$$\mathrm{S}^1_{\epsilon-\mathrm{gr}} \circlearrowright (\mathrm{T}[-1]X)/\mathbb{G}_m$$

$$[(\mathrm{T}[-1]X)/\mathbb{G}_m]/\mathrm{S}^1_{\epsilon-\mathrm{gr}} \to \mathrm{B}_{\mathrm{B}\mathbb{G}_m}(\mathrm{S}^1_{\epsilon-\mathrm{gr}})$$

Proposition

$$\mathcal{O}_{[(\mathrm{T}[-1]X)/\mathbb{G}_m]/\mathrm{S}^1_{\epsilon-\mathrm{gr}}} = \bigoplus_{n\geq 0} \Omega^n_{X/\mathsf{k}}[n] \textit{ with grading } + \mathrm{d}_{\mathrm{dR}}$$

Geometrization of the filtration

Need to explain

$$\mathrm{S}^1 \circlearrowright \mathrm{L}X \Leftarrow \underbrace{\text{geometrization of } \mathrm{HH}_{\mathrm{Fil}}}_{\overset{\circlearrowright}{?}} \Rightarrow \mathrm{S}^1_{\epsilon-\mathrm{gr}} \circlearrowright \mathrm{T}[-1]X$$

Bridge between the underlying object and the associated graded.

Lemma (Simpson)

The quotient stack $[\mathbb{A}^1/\mathbb{G}_m]$ *encodes filtrations, ie,*

$$\{\textit{chain complexes with additional } \mathbb{Z}\textit{-filtration}\}^{\otimes} \underbrace{\simeq}_{\textit{Rees}} \mathrm{Qcoh}_\infty([\mathbb{A}^1/\mathbb{G}_m])^{\otimes}$$

Remark

$$\begin{array}{ccccc}
\mathrm{Qcoh}_\infty(\mathrm{Spec}(k)) & \xleftarrow{1^*} & \mathrm{Qcoh}_\infty([\mathbb{A}^1/\mathbb{G}_m]) & \xrightarrow{0^*} & \mathrm{Qcoh}_\infty(B\mathbb{G}_m) \\
\downarrow \sim & & \downarrow \sim & & \downarrow \sim \\
\{\textit{complexes}\} & \xleftarrow{\text{underlying}} & \{\textit{Filtered complexes}\} & \xrightarrow{\text{ass}-\text{gr}} & \{\textit{Graded complexes}\}
\end{array}$$

Definition

A *filtered (derived) stack* is a (derived) stack Z together with a map $Z \to [\mathbb{A}^1/\mathbb{G}_m]$.

$$\begin{array}{ccccc}
\text{underlying} & \longrightarrow & Z & \longleftarrow & \text{associated graded} \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Spec}(\mathsf{k}) & \xrightarrow{1} & [\mathbb{A}^1/\mathbb{G}_m] & \xleftarrow{0} & \mathrm{B}\mathbb{G}_m
\end{array}$$

Remark

Given $p : Z \to [\mathbb{A}^1/\mathbb{G}_m]$, $p_*\mathcal{O}_Z \in \mathrm{Qcoh}_\infty([\mathbb{A}^1/\mathbb{G}_m])$ *has is a filtered chain complex.*

Theorem (Moulinos-R-Toën)

- *There exists a filtered abelian group stack*

$$\mathrm{S}^1_{\mathrm{Fil}} \to [\mathbb{A}^1/\mathbb{G}_m]$$

that implements a filtration on the topological circle with associated graded $\mathrm{S}^1_{\epsilon-\mathrm{gr}}$.

- *Universal property of the HKR-filtration:*

$$\mathrm{HH}_{\mathrm{Fil}}(R/\mathsf{k}) \simeq \mathcal{O}_{\mathbb{R}\mathrm{Map}(\mathrm{S}^1_{\mathrm{Fil}}, X)}$$

is the universal filtered R-algebra with an action of the filtered circle $\mathrm{S}^1_{\mathrm{Fil}}$.

Remark

A similar universal property has been obtained by Raksit.

Example

X quasi-smooth

$$\mathrm{HH}_{\mathrm{Fil}}(X)^{h\mathrm{S}^1_{\mathrm{Fil}}} = \mathrm{HC}^-(X/\mathsf{k}) \text{ with Antieau filtration}$$

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Functorial HKR isomorphisms

Definition

$$\{\text{Universal HKR isomorphisms}\} := \{\text{group splittings } \mathrm{S}^1_{\mathrm{Fil}} \simeq (\mathrm{S}^1_{\mathrm{Fil}})^{\mathrm{triv}}\}$$

Theorem (Precise formulation)

$$\{\textit{Universal HKR isomorphisms}\} \simeq \{\textit{exponentials } \widehat{\mathbb{G}_a} \to \widehat{\mathbb{G}_m}\} \simeq$$

$$\simeq \begin{cases} \emptyset & \textit{if } \mathrm{char}(\mathsf{k}) \neq 0 \\ \mathsf{k}^* & \textit{if } \mathrm{char}(\mathsf{k}) = 0 \end{cases}$$

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Witt vectors

Definition

$$\mathbb{W}(R) := \underbrace{1 + tR[[t]]}_{\text{invertible formal power series}}$$

$$(\mathbb{W}, +) : \{\text{Commutative rings}\} \to \{\text{Abelian groups}\}$$

Remark

Underlying scheme

$$\mathbb{W} \simeq \prod_{i \geq 0}^{\infty} \mathbb{A}^1$$

As a group, built from successive extensions by $\mathbb{G}_a$*.*

Construction

- *Ghost coordinates*

$$-\frac{d}{dt}\log : (1 + t.R[[t]])^{\times} \to R[[t]], \qquad f \mapsto -\frac{f'}{f}$$

transforms products of formal power series into sums.

$$Ghost : (\mathbb{W}, +) \to (\prod_{i=1}^{\infty} \mathbb{G}_a, +)$$

- *Frobenius endomorphisms*

$$\mathrm{Frob}_n : \mathbb{W} \to \mathbb{W}, \quad \forall n \in \mathbb{N}$$

determined by the shift maps on Ghost coordinates

$$\mathrm{Shift}_n : \prod_{i=1}^{\infty} \mathbb{G}_a \to \prod_{i=1}^{\infty} \mathbb{G}_a \quad (\omega_i) \mapsto (\omega_{ni})$$

- $\mathbb{G}_m \circlearrowright \mathbb{W}$ *given by multiplication by Teichmuller representative.*

Definition

$$\mathrm{Fix} := \bigcap_n (\mathrm{Frob}_n - \text{fixed points}) \subseteq \mathbb{W}$$

$$\mathrm{Ker} := \bigcap_n (\mathrm{Kernel}\ \mathrm{Frob}_n) \subseteq \mathbb{W}$$

Remark

When char(k) = 0, $\mathrm{Fix} = \mathrm{Ker} = \mathbb{G}_a$.

Remark

Ker *is closed under* $\mathbb{G}_m$*-action.*

Proposition (Moulinos-R-Toën over $\mathbb{Z}_{(p)}$, J. Tapia, J.Sauloy and Toën over $\mathbb{Z}$)

$$\mathrm{BFix} \simeq \mathrm{coSpec}(\mathrm{C}^\bullet(\mathrm{S}^1, \mathbb{Z})) \qquad (\mathrm{BKer})/\mathbb{G}_m \simeq \mathrm{S}^1_{\epsilon-\mathrm{gr}}$$

Construction

Consider the family of abelian groups $\mathbb{H} \to \mathbb{A}^1$ *given by*

$$\begin{array}{ccc} \mathbb{H}_\lambda := \bigcap_n \left(\mathrm{Kernel}\ \mathrm{Frob}_n - \lambda^{n-1}\mathrm{Id}\right) & \hookrightarrow & \mathbb{H} \\ \vdots & & \downarrow \\ \{\lambda\} & \hookrightarrow & \mathbb{A}^1 \end{array}$$

Interpolates between $\mathbb{H}_{\lambda=0} = \mathrm{Ker}$ *and* $\mathbb{H}_{\lambda=1} = \mathrm{Fix}$

Remark

$\mathbb{G}_m$*-action on* $\mathbb{W}$ *restricts to* $\mathbb{G}_m$*-action on* $\mathbb{H}$ *compatible with group structure.*

Definition

$$\mathrm{S}^1_{\mathrm{Fil}} := \mathrm{B}_{[\mathbb{A}^1/\mathbb{G}_m]}(\mathbb{H}/\mathbb{G}_m)$$

Cartier duality

Reminder (Cartier Duality)

Over a commutative ring k, construction sending a smooth commutative formal group $\mathcal{G}$

$$\mathcal{G} \mapsto \mathcal{G}^\vee := \mathrm{HOM}_{\mathrm{CommFGr}}(\mathcal{G}, \widehat{\mathbb{G}_m})$$

defines an equivalence of categories

$$\mathrm{CommFGr}_k \simeq \{ \textit{commutative affine algebraic groups over } k \}$$

Proposition (Moulinos, Sekiguchi-Suwa)

- $\mathrm{Ker} \simeq \widehat{\mathbb{G}_a}^\vee$;
- $\mathrm{Fix} \simeq \widehat{\mathbb{G}_m}^\vee$
- *The filtered group $\mathbb{H}/\mathbb{G}_m$ is the (filtered) Cartier dual of the family Def over $[\mathbb{A}^1/\mathbb{G}_m]$ implementing the deformation to the normal bundle of $1 \in \widehat{\mathbb{G}_m}^\vee$.*

Functorial HKR isomorphisms

Definition

Universal HKR isomorphisms $\Leftrightarrow$ universal splittings of the HKR-filtration $\Leftrightarrow$ group splittings

$$S^1_{\mathrm{Fil}} \simeq (S^1_{\mathrm{Fil}})^{\mathrm{triv}}$$

Sketch of proof of the main result.

$$\{\text{group splittings } S^1_{\mathrm{Fil}} \simeq (S^1_{\mathrm{Fil}})^{\mathrm{triv}}\} \underset{\text{Cartier Duality}}{\simeq} \{\text{exponentials } \widehat{\mathbb{G}_a} \to \widehat{\mathbb{G}_m}\}$$

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Relation with GRR

Remark (Toën-Vezzosi)

When $char(k) = 0$*, because of the splitting principle*

$$\{exponentials\ \widehat{\mathbb{G}_a} \to \widehat{\mathbb{G}_m}\} \simeq \mathsf{k}^* \simeq \{Chern\ Characters\ K_0 \to H^*_{\mathrm{d_{dR}}}\}$$

in particular, universal HKR-isomorphisms are in bijection with Chern characters.

Relation with GRR

Remark

The circle S^1 also admits a cogroup structure.

$$\mathrm{S}^1 \to \mathrm{S}^1 \vee \mathrm{S}^1$$

This cogroup structure extends to the filtered circle $\mathrm{S}^1_{\mathrm{Fil}}$.

Theorem (Moulinos)

There are no cogroup splittings $\mathrm{S}^1_{\mathrm{Fil}} \simeq (\mathrm{S}^1_{\mathrm{Fil}})^{\mathrm{triv}}$. The obstruction is the universal Todd class.

Corollary

None of the group splittings induced by the choice of a Chern character is a cogroup splitting. The consequence is the Grothendieck-Riemann-Roch theorem.

Thank you for your attention.