BRANE ACTIONS AND OPERADS

MARCO ROBALO

1. Introduction

Let X be a topological space and consider the free loop space $\mathrm{Map}(\mathrm{S}^1, X)$. This space carries a circle action, obtained by pre-composition with the group structure on S^1. But there is something else going one: Sullivan-Chas constructed a loop product operation on the homology of the $\mathrm{Map}(\mathrm{S}^1, X)$. This operation can be modellied at the level of chains, but it boils down to the following diagram

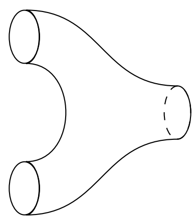
$$\mathrm{Map}(\mathrm{S}^1, X) \times \mathrm{Map}(\mathrm{S}^1, X) \leftarrow \mathrm{Map}(\mathrm{S}^1 \vee \mathrm{S}^1, X) \to \mathrm{Map}(\mathrm{S}^1, X)$$

induced by composition with

$$\mathrm{S}^1 \coprod \mathrm{S}^1 \to 8 = \mathrm{S}^1 \vee \mathrm{S}^1 \leftarrow \mathrm{S}^1 \tag{1}$$

where the first map is the canonical map and the second is map given by going around the figure 8.

Up to homotopy, one can understand the diagram 1 as the pair of pants



where the first map gives the inclusion of the two circles on the left and the second map is the inclusion of the circle on the right.

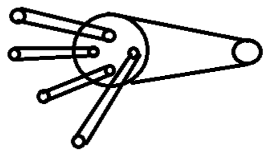
This map can also be seen as a map

$$\mathrm{S}^1 \coprod \mathrm{S}^1 \dashrightarrow \mathrm{S}^1$$

in the category of cobordisms: objects are topological spaces and morphisms are cobordisms. In this case, the algebra structure on the homology of the free loop space is a consequence of the following fact:

Proposition 1.1. *The object S^1, seen in the ∞-category of cobordisms in spaces, carries the structure of $\mathbb{E}_2$-algebra with multiplication given by the operation 1. More generally, $\sigma \in \mathbb{E}_2(n)$ seen as a configuration of n little disks, acts via the n-pants*

$$\coprod_n \mathrm{S}^1 \to \vee_n \mathrm{S}^1 \leftarrow \mathrm{S}^1$$



The goal of this talk is to illustrate the general nature of this phenomenon and explain why the operad $\mathbb{E}_2$ appears here.

Before that, let me explain how to pass from this statement to the free loop space: applying the functor $\mathrm{Map}(-, X)$ to 1 we obtain a correspondence/span in the category of spaces: objects are spaces, and morphisms from X to Y are diagrams

$$\begin{array}{ccc} & Z & \\ \swarrow & & \searrow \\ X & & Y \end{array}$$

with composition obtained by taking (homotopy) pullbacks. Finally, as the operation $\mathrm{Map}(-, X)$ sends coproducts to products, we get:

Proposition 1.2. *The free loop space* $\mathrm{Map}(\mathrm{S}^1, X)$*, seen in the* ∞*-category of spans in spaces, carries a structure of* $\mathbb{E}_2$*-algebra.*

Before unravelling the magic behind, let me give you a hint, by saying that this result has analogues versions for the higher dimensional spheres S^n, namely:

Proposition 1.3 (Ginot-Tradler-Zeinalian)**.** *The sphere* S^n*, seen as an object in the* ∞*-category of cobordisms in spaces, carries a structure of* $\mathbb{E}_{n+1}$*-algebra with multiplication given by co-spans*

$$\mathrm{S}^n \coprod \mathrm{S}^n \to \vee \mathrm{S}^n \leftarrow \mathrm{S}^n \tag{2}$$

By transport, $\mathrm{Map}(\mathrm{S}^n, X)$*, seen as an object in the* ∞*-category of spans in spaces, carries a structure of* $\mathbb{E}_{n+1}$*-algebra.*

2. Brane Actions

At this point I have to illustrate the mechanism responsable for these algebra structures. The following remark unravels some of the mistery:

Remark 2.1. The circle S^1 is weakly equivalent to the space $\mathbb{E}_2(2)$ of binary operations in the little disks operad. Similarly, S^n is weakly equivalent to $\mathbb{E}_{n+1}(2)$.

To really explain what is going on we need a definition:

Definition 2.2 (J.Lurie)**.** Let O be a monochromatic ∞-operad with $O(0) \simeq O(1) \simeq *$. Let $\sigma \in O(n)$ be a n-ary operation. The space of extensions of σ - $\mathrm{Ext}(\sigma)$ - is the homotopy fiber product

$$\{\sigma\} \times_{O(n)} O(n+1)$$

where the map $O(n+1) \to O(n)$ forgets the last entry. We say that O is coherent if for each pair of composable operations σ, τ, the natural square

$$\begin{array}{ccc} \mathrm{Ext}(Id) & \longrightarrow & \mathrm{Ext}(\sigma) \\ \downarrow & & \downarrow \\ \mathrm{Ext}(\tau) & \longrightarrow & \mathrm{Ext}(\sigma \circ \tau) \end{array}$$

is homotopy-cocartesian.

Lemma 2.3. *(Lurie HA, Toen) The ∞-operads $\mathbb{E}_k$ are coherent.*

Proof. The proof boils down to the fact that given $\sigma \in \mathbb{E}_k(n)$, the space of extensions computed by the homotopy pullback gives

$$\begin{array}{ccc} \mathrm{Ext}(\sigma) = \vee_n \mathrm{S}^n & \longrightarrow & \mathbb{E}_k(n+1) \\ \downarrow & & \downarrow{\scriptstyle \text{Forget Last entry}} \\ * & \longrightarrow & \mathbb{E}_k(n) \end{array}$$

and the diagram in the coherence condition becomes

$$\begin{array}{ccc} \mathrm{Ext}(Id) = \mathrm{S}^n & \longrightarrow & \mathrm{Ext}(\sigma) = \vee_n \mathrm{S}^n \\ \downarrow & & \downarrow \\ \mathrm{Ext}(\tau) = \vee_m \mathrm{S}^n & \longrightarrow & \mathrm{Ext}(\sigma \circ \tau) = \vee_{n+m-1} \mathrm{S}^n \end{array}$$

□

Theorem 2.4 (Toen)**.** *Let O be a coherent ∞-operad. Then, $O(2) = \mathrm{Ext}(Id)$, seen as an object in the ∞-category of co-spans in spaces, carries an action of O with multiplication given by the*

$$\sigma \in O(n) \mapsto \coprod_n \mathrm{Ext}(Id) \to \mathrm{Ext}(\sigma) \leftarrow \mathrm{Ext}(Id)$$

Proof. (Sketch) Let C be a category with products. Then C^{corr} has a monoidal structure induced by $\times$. An O-algebra in C^{corr} is the same as a compatiblity family of maps $O(n) \mapsto Map_{C^{corr}}(X^n, X)$ assemblying to a map of props

$$O^{\otimes} \to C^{corr,\times}$$

This is the same as a symmetric monoidal functor

$$Env(O)^{\otimes} \to C^{corr,\times}$$

We now use the fact that the construction $C \mapsto C^{corr}$ admits a left adjoint $D \mapsto Tw(D)$ given by the category of twisted arrows: objets = morphisms in D and morphisms between $u : X \to Y$ and $v : A \to B$ are given by commutative diagrams

$$\begin{array}{ccc} X & \longrightarrow & A \\ \downarrow u & & \downarrow v \\ Y & \longleftarrow & B \end{array}$$

In this case, the data of a symmetric monoidal functor $Env(O)^{\otimes} \to C^{corr,\times}$ is equivalent to the data of a functor

$$Tw(Env(O))^{\otimes} \to C^{op}$$

Finally, in the case C=S^{op} (when we obtain co-correspondences) this is

$$Tw(Env(O))^{\otimes} \to S^{op}$$

which via the Grothendieck construction, this becomes equivalent to the data of a cartesian firation

$$B \to Tw(Env(O))^{\otimes}$$

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Finally, the construction of this boilds down to the fact that the operad is coherent. □

3. Applications in algebraic geometry

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To conclude my talk I would like to explain some applications of this result in algebraic geometry. This time, we fix X a smooth projective variety over $\mathbb{C}$. Gromov-Witten invariants are numerical invariants obtained by counting curves of degree d in X passing by n marked points. This is a story in itself which I will not have time to narrate here today. However, there is something I can say: these invariants are known to satisfy certain compatibilities and recursive relations obtained as d and n vary, obtained as a result of the dynamics of gluing curves

$$\overline{\mathscr{M}}_{g,n} \times \overline{\mathscr{M}}_{g',m} \to \overline{\mathscr{M}}_{g+g',n+m-2}$$

which endow the collection of moduli spaces of stable curves $\overline{\mathscr{M}}_{g,n}$ with a structure of operad in stacks.

Theorem 3.1. *(Manin-Kontsevich) Let X be a smooth projective variety over $\mathbb{C}$. Then the cohomology of X carries an extra multiplicative structure known as* Quantum product

$$\mathrm{H}(X)^{\otimes}_{n} \to \mathrm{H}(X)$$

This operation is part of a data that endows $\mathrm{H}(X)$ *with the structure of algebra over the operad* $\{\mathrm{H}_*(\overline{\mathscr{M}}_{g,n})\}_{n\geq 1}$.

The dynamics of brane actions allows us, as for the results of Sullivan, to show that this algebra structure exists before passing to cohomology, and in fact:

Theorem 3.2. *(Mann-R.) Let* X *be a smooth projective variety over* $\mathbb{C}$*. Then* X*, seen as an object in the* ∞*-category of spans in derived stacks, carries the structure of* $\{\overline{\mathscr{M}}_{g,n}\}$*-algebra. In particular, by applying the functor* K*-theory we get an algebra structure on* $K(X)$ *which recovers the Quantum K-theory of Givental-Lee.*

Institut Mathematiques de Jussieu - Paris Rive Gauche, Université Pierre et Marie Curie, 4 Place Jussieu, 75005, Paris, France
E-mail address: marco.robalo@imj-prg.fr