Donaldson-Thomas invariants in Aussois (Oct 2022)

Introduction (Damien)

Casson-invariants : (in differential geometry)

M compact oriented manifold dim 3

Assume M integral homology sphere.

Look at

$$\underbrace{R^{\text{irr}}}_{\text{moduli of irreducible representations}}(M) = \mathrm{Hom}(\pi_1(M), SU(2)) \Big/ \underset{\text{conjugacy}}{SU(2)}$$

definition

$$\text{stabilizer} = \underbrace{Z}_{\text{center}}(SU(2)) = \mathbb{Z}/2\mathbb{Z}$$

The only irreducible rep is the trivial one.

There is a decomposition of M:

$M = H_1 \coprod_{\Sigma} H_2$

(*)

H_1, H_2 are genus g handlebodies

Σ boundary sphere

$\underbrace{\phantom{\Sigma \text{ boundary sphere}}}$

g has to be the same for H_1, H_2 & Σ.

$\Downarrow$

van Kampen

$$\begin{tikzcd}
R^{irr}(M) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & R^{irr}(H_1) \arrow[d, dash, "i_2"] \\
R^{irr}(H_2) \arrow[r, "i_1"] & R^{irr}(\Sigma)
\end{tikzcd}$$

is a manifold

is manifold

Moreover: One can prove that i_1 & i_2 are submanifolds.

dim $6g-6$

Claim: $R^{irr}(\Sigma)$ is a symplectic manifold and both $R^{irr}(H_1)$ & $R^{irr}(H_2)$ are Lagrangians!

dim $3g-3$

Casson:

$$\underbrace{\lambda(M)}_{\text{invariant}} = \frac{1}{2}\left(\text{intersection number of } R^{irr}(H_1)\ \&\ R^{irr}(H_2) \text{ in } R^{irr}(\Sigma)\right)$$

counts the points in the intersection. $R^{irr}(M)$

↑ problem with transversality.

> **claim:** this number is independent of the splitting (*)

claim: one can prove that these are oriented manifolds ($\underbrace{\omega\wedge\dots\wedge\omega}_{3g-3}$ is a volume form)

> **Goal:** Do this for bundles / sheaves on Calabi-Yau 3-folds.

↳ **problem**:
- intersection theory
- the splitting (*) does not exist globally in general.

(not globally a Lagrangian intersection!)

Yet Another approach (via Gauge theory)

↳ identify $$\text{Rep. of } \pi_1 / \sim \;\overset{(SU(2))}{\simeq}\; \text{flat connections} / \text{gauge equivalence}$$

$$\mathrm{Conn}(M, SU(2)) = \text{connections on the trivial principle } SU(2)\text{-bundle}$$

$$= \{ d + A \mid A \in \Omega^1(M, su(2)$$

<u>Define</u> a <u>functional on this space</u>:

$$S: \mathrm{Conn}(M, SU(2)) \xrightarrow{\text{action}} \mathbb{R}$$

$$d + A \longmapsto \int_M \mathrm{tr}\left(dA \wedge A + \frac{2}{3} A \wedge A \wedge A\right)$$

$$\circlearrowleft$$

action of

$g \in C^\infty(M, SU(2))$

via $$g^{-1}(d+A)g = d + \underbrace{g^{-1}dg + g^{-1}Ag}_{A^g}$$

claim: $\exists$ c a constant such that

$$S(d+A^g) = S(d+A) + c\cdot\mathbb{Z}$$

$\Rightarrow$ dS is a well-define one-form on

$$X = \text{Conn}(M, SU(2)) \big/ C^\infty(M, SU(2))$$

we can compute the tangent of X at a given connection $\nabla = d + A$

$$T_{[\nabla]}X \cong \Omega^1(M, su(2)) \big/ \text{Im}\,\nabla$$

Computation: $dS = 0 \iff$ flat connections.

so $$\boxed{\{\pi_1\text{-rep}\} \simeq \{\text{flat-connections}\} = \text{Critical points of } S}$$

Issues when trying to count the number of critical points of S, but, this is possible

Theorem (Taubes)

$$\text{Casson invariant} = \frac{1}{2}\left(\#\{dS = 0\}\right)$$

Issues: ① X may not be a manifold

② transversality of the intersection $dS \cap \{0\}$ in T^*X.

Talk 4: Thomas paper about holomorphic analogue of this story

$$M \xrightarrow{\text{replace}} CY \text{ 3-fold}$$

$$SU(2) \xrightarrow{\text{replace}} GL_n(\mathbb{C})$$

Trivial principal $SU(2)$-bundle $\xrightarrow{\text{replace}}$ by any C^∞-bundle of rank n.

Connections $\xrightarrow{\text{replace}}$ (0,1) - connections

Action functional $\xrightarrow{\text{replace}}$ $S(\nabla + A) = \mathrm{Tr}\Big(\underbrace{\nabla A\wedge A + \frac{2}{3} A\wedge A\wedge A}_{0,3-\text{form}}\Big)$

A is (0,1)-form

$\downarrow$ M. Calabi-yau

can wedge with Ω holomorphic 3 form.

Serre duality + Calabi-yau

$\leftarrow$ this can be integrated $\leftarrow$

Counting $dS = 0$ is the first definition of DT-invariants

Rmk: a priori this does not use the symmetry of the obstruction theory to be defined. However, to show independence of the choice, we show that all symmetric obstruction theories give the same number.

- Back to problem ②: We are in the following general situation: (algebraic geometry)

$$\begin{tikzcd} E \arrow[d] \\ X \arrow[u, bend left, "s"] \end{tikzcd}$$ vector bundle

$$Z(s) \xhookrightarrow{i} X$$

Expected dim of $Z = \dim X - \mathrm{Rnk}(E)$

Intersection product to get a class in $H_{\dim X - \mathrm{rk}(E)}(Z)$

Construction:

$$\begin{tikzcd} \mathbb{L}_Z = (I_Z/I_Z^2 \arrow[r, "d"] & i^*\Omega^1_X) \\ F = (i^*E^\vee \arrow[u] \arrow[r] \arrow[u, phantom, "\phi" left] & i^*\Omega^1_X) \arrow[u, equal, "id"] \end{tikzcd}$$

take v a section of $E^\vee$ to $v \circ s$.

fixed if we want the diagram to commute.

$$v \longmapsto d(v \circ s)$$

this is an example of a perfect obst. theory in particular

- $H^0(\phi)$ is an iso
- $H^{-1}(\phi)$ is surjective.

Talk #2 and talk #11

interpretation in terms of derived geometry.

F = cotangent complex of the derived zero locus of s.

Rmk: If the perfect obstruction theory is symmetric ($F^\vee \simeq F[1]$) then the virtual fundamental class is independent of the choice of symmetric perfect obst. theory.

talk 3

(Examples: derived Lagrangian intersections).

talk 13

- Back to problem ① : locally around a flat connection, elliptic regularity tells us

that one can find "critical charts",

(the pde that computes $ds=0$) (Euler-Lagrange equations) (because it comes from a deformation of d $(d+\dots)$) ?

ie locally holomorphic

$$\{ds=0\} \simeq \{df=0\}$$

for f defined in some U where U is actually a manifold.

↗

Darboux-lemma.

↑

talks 6 & 7 $\left\{ \begin{array}{l} \cdot \text{ critical virtual manifold (Kiem-Li)} \\ \cdot \text{ d-critical locus (Joyce)} \end{array} \right.$

↓

gives rise to a symmetric semi-perfect obstruction theory.

Categorification :

(talks 6 8 7)

+ derived symplectic approach (last approach)

$U \xrightarrow{f} \mathbb{R}$ U smooth.

$\mathrm{crit}(f) = \{df = 0\}$ has a natural derived enhancement $\underline{\mathrm{dcrit}(f)}$

↙

derived intersection is (-1) shifted symplectic

↙

$$(\overbrace{i^{-1}\mathrm{Sym}(T_U[1])}^{\text{polyvectors}}, \iota_{df}) = \mathcal{O}_{\mathrm{dcrit}(f)}$$

⤳ inherits a poisson structure.

$\wr$ Hodge star operator.

assuming we are in the differential setting, U orientable

$$(i^{-1}\,\mathrm{Sym}(\Omega^1_U[1]), -\wedge df)\,[\dim X]$$

$\underbrace{+\, d_R . u}$ ← deformation

u is the deformation parameter.

twisted de Rham complex is a deformation of $\underline{\mathcal{O}_{\mathrm{dcrit}}(f)}$.

$$\underbrace{\qquad\qquad\qquad\qquad}_{\text{extend scalars to } u[[u]][u^{-1}]}$$

Theorem: (Sabbah - Saito) this coincides with the $\underbrace{\text{sheaf of vanishing}}_{\substack{\text{defined on the zero locus} \\ \text{but supported on the critical locus}}}$ cycles of f.

(conjectured by Kontsevich- Soibelman)

Problem: there is a gluing problem controlled by a $\mathbb{Z}/2\mathbb{Z}$ -gerbe orientation data. Given such an orientation data we can glue these twisted deRham <u>complexes</u>.

Rmk: in physics papers on the subject, physicists compute integrals such as $\int_X e^{f/u}\alpha$

$$d\left(e^{f/\mu}\alpha\right) = 0 \iff \frac{1}{\mu}\underbrace{df\wedge\alpha + d\alpha}=0$$

$$\underbrace{\left(d + \frac{1}{\mu}df\wedge -\right)}_{\text{twisted de Rham differential}}\alpha = 0$$

$$\int_X e^{f/\mu}\cdot\alpha$$

---

use stationary phase method

$$= \sum_{x\in \mathrm{crit}(f)} \text{something that depends only of the behavior at } x.$$

Talk 2: Virtual fundamental classes

after Behrendi – Fantechi

Kontsevich – Manin: GW-invariants should be computed as integrals over $\boxed{\overline{\mathcal{M}}_{g,n}(X,\beta)}$.

Example:

$$\underbrace{N_d}_{} = \int_{\overline{\mathcal{M}}_{0,3d-1}(\mathbb{P}^2,d)} ev_1^* pt \cdots\cdots ev_{3d-1}^* pt$$

curves of degree d of genus 0 passing through 3d-1 points

in general: $\overline{\mathcal{M}}_{g,n}(X,\beta)$ can be singular (have the wrong dimension)

solution in differential geometry: "perturb" the moduli space to make it smooth"

Solution : construct a virtual fundamental class $[M]^{vir}$

$$[M]^{vir} \in A_{\text{virtual dim}}(M)$$

$$\downarrow$$

$$H_{2.vd}(M)$$

$[M]^{vir}$ constructed by [Li-Tian, Behrend-Fantechi]
(95-96)

input : obstruction theory on M $\leftarrow$ moduli theory

$$E^{\bullet} \longrightarrow \mathbb{L}_M$$ (Damien's talk #1)

$\rightsquigarrow$

output : $[M]^{vir}$

$\rightsquigarrow$

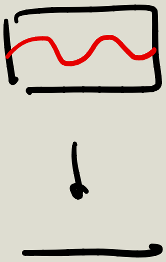
better output : enumerative invariants $\int_{[M]^{virT}}$

(this talk follows the survey of Pandharipande-thomas)

1) Local model:

$$\begin{tikzcd} E \arrow[d, shift left] & \text{Rank} = R \\ A \arrow[u, shift left, "s"] & \text{smooth } \dim = n \end{tikzcd}$$

$M := \underbrace{Z(s)}_{\text{zeros of } s.} \subseteq A$



ideal case: s is transverse to the zero section

$\Downarrow$

$Z(s)$ is smooth of dim $\underline{n-R}$

less ideal case: $\exists$ splitting $E \simeq E' \oplus E/E'$

and $s = (s', 0)$ with s' transversely intersects 0 section on E'

↙

in this case

$Z(s)$ is still smooth but the dim is $n-R'$

where $R' = \text{Rank } E'$

↙

So, not the expected dim.

idea: perturb s to $(s', \varepsilon) = s_\varepsilon$ $\quad \varepsilon$ - $\overset{\text{small}}{C^\infty}$-perturbation

In this case $Z(s_\varepsilon) \subseteq Z(s) \subseteq A$

then $$[Z(s)]^{virT} := [Z(s_\varepsilon)] \in H_{2(n-R)}(M)$$

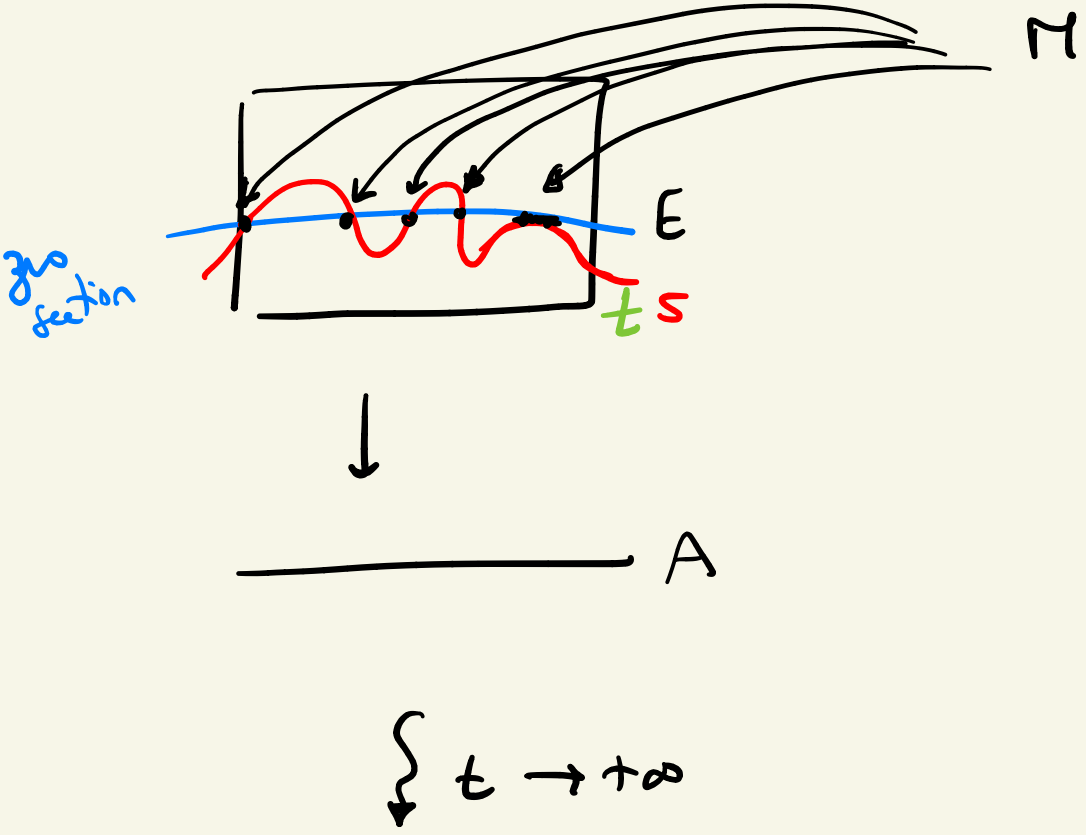
$$= c_{R-R'}(E/E') \cap [Z(s)].$$

top chern class

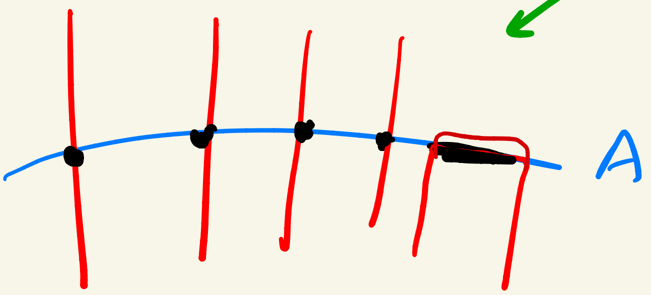
idea: defined it as the class of the perturbation.

More general:

$$\boxed{[M]^{virT} := 0_E^! [C_s] \in A_{n-R}(M)}$$



(intrinsic normal cone)



C_S is pure of n-dim.

$C_S \subseteq E$

<u>idea</u>: we want to use an intrisic version of this <u>cone</u> using "infinitesimal data"

$p \in M$

$$0 \to T_pM \to \underbrace{T_pA \xrightarrow{ds|_p} E_p}_{\text{obstruction theory}}.$$

(the cokernel of $T_pA \xrightarrow{ds|_p} E_p$) := obstruction bundle <u>obs</u>.

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② (Perfect) obstruction theory : <u>X scheme</u>, or <u>DM stack</u>

<u>Cotangent complex</u> : $\Omega_X = T_X^\vee$

<u>Functoriality</u> : $\cdots \to f^*\Omega_Y \to \Omega_X \to \Omega_f \to 0$

$\mathbb{L}_X :=$ LEFT DERIVED Functor of Ω_X

$\overset{\cap}{D(X)}$ (derived cat. of X)

- $h^0(\mathbb{L}_X) = \Omega_X$
- $h^i(\mathbb{L}_X) = 0$ for $i>0$ (X a scheme)
- functoriality in derived categories:

$$f: X \to Y \qquad f^*\mathbb{L}_X \to \mathbb{L}_Y \to \mathbb{L}_f$$

- X smooth $\Rightarrow$ $\mathbb{L}_X \simeq \Omega^1_X$
- $\underset{\text{Regular}}{X \hookrightarrow} \overset{\text{smooth}}{A}$, $\mathbb{L}_X = [I/I^2 \to \Omega_A|_X]$

Definition: An obstruction theory on X is a complex $E^\bullet \in D(X)$ together with a map $E^\bullet \to \mathbb{L}_X$ such that

- $h^0(\phi)$ is an iso.
- $h^{-1}(\phi)$ is surjective.

Rmk: Moduli spaces M usually come with obstruction theories.

since $TM = \mathrm{Hom}(\mathrm{Spec}(\mathbb{C}[\varepsilon]/_{\varepsilon^2}), M)$

Exemple : $M := \mathrm{Maps}(C, X)$

curve

smooth projective

$$\begin{tikzcd} M \times C \arrow[r, "ev"] \arrow[d, "\pi"] & X \\ M & \end{tikzcd}$$

obstruction theory

$$E^\bullet = \mathbb{R}\pi_*(ev^* T_X)^\vee \xrightarrow{\phi} \mathbb{L}_X$$

complicated to construct if we don't know that $E^\bullet$ is actually the cotangent complex to a derived enhancement of the mapping M.

If we use derived geometry, ϕ comes from functoriality for cotangent complexes.

Definition : $E^\bullet \xrightarrow{\phi} \mathbb{L}_X$ is a <u>2-term</u> perfect obs. theory (POT)

if locally we can write

$$E^\bullet = [E^{-1} \longrightarrow E^0]$$

Rmk : $\mathrm{Rank}(E^\bullet)|_p := \mathrm{Rnk}(E^0)|_p - \mathrm{Rnk}(E^{-1})|_p$

is a locally constant function.

Theorem: (Behrend-Fantechi, Li-Tian)

If M has a <u>2-term</u> poT, there is

$$\begin{tikzcd}[M]^{vir} = [M, E^{\bullet} \longrightarrow \mathbb{L}_x]^{vir} \in A_{v.d}(M) \arrow[d] \\ H_{2.vdim}(M)\end{tikzcd}$$

$vd := \mathrm{Rnk}(E^{\bullet})$

idea of proof

in this talk: "stack" = locally looks like $[X/G]$

Definition: A vector bundle/(cone stack) over M is a stack $V \longrightarrow M$ /($C \longrightarrow M$) That locally looks like a quotient of two vector bundles $[E_1/E_0]$ (resp, $[C/E]$)

$$\underbrace{\text{cone}}_{\text{in the sense of Fulton.}} \qquad \text{vect bundle}$$

Definition / Proposition

- $P^\bullet \in D(M)$ with $h^{\geq 0}(P) = 0$

then there is a cone stack such that locally

$$P_0 := (P^\bullet)^\vee = [P_0 \to P_1 \to P_2 \to \cdots]$$

$$C(P^\bullet) = \left[\frac{\ker(P_1 - P_2)}{P_0}\right] = h^1/h^0\,(P_\bullet)$$

associated cone stack :

claim: if $P^\bullet$ is a 2-term perfect obst. theory

then $C(P^\bullet)$ is a vector bundle stack.

$$\begin{tikzcd}
E^\bullet \arrow[r] & \mathbb{L}_X \arrow[r, squiggly] & C(\mathbb{L}_X) \arrow[r, "C(\phi)"] \arrow[d, equal] & C(E^\bullet) \arrow[d, equal] \\
 & & N_X & \mathcal{E}
\end{tikzcd}$$

⇝ normal sheaf.

thm (Behrend–Fantechi) : $C(\phi)$ is a closed embedding

Problem: N_X is not pure dimension $(X = M)$

$\cup$

$\mathfrak{C}_X$ an intrinsic normal cone intrinsic normal sheaf : locally modelled on

locally

$$\begin{tikzcd} U \arrow[r, phantom, "\overset{\text{open}}{\subseteq}"] \arrow[d, hook, "\text{regul}"', "\text{closed}"] & M \\ A & \\ \text{smooth} & \end{tikzcd}$$

$$\begin{tikzcd} \mathfrak{C}_M|_U \arrow[r, phantom, "\subseteq"] \arrow[d, "\text{def}"] & N_M|_U \arrow[d, phantom, "\cong" rotate=90] \\ \left[C_{U/A}/TA|_U\right] \arrow[r, phantom, "\subseteq"] & \left[N_{U/A}/TA|_U\right] \end{tikzcd}$$

FACT: $\mathfrak{C}_M$ is pure dim 0

Definition: $$[M]^{virt} := 0^!_{E}([\mathfrak{C}_M])$$

How to work with virtual fund. classes

(Basic toolkit)

- M smooth, $[M]^{virt} = e_{Top}(h^1(E^\bullet)^\vee) \cap [M]$
- (Manolache) $f: M \to N$ with relative p.o.T

$$\rightsquigarrow f^!: A_\bullet(N) \longrightarrow A_\bullet(M)$$

and in some cases, $f^!([N]^{virt}) = [M]^{virt}$

- **Siebert** $[M]^{\text{vir}} = \Big[\underbrace{s(E^\bullet)}_{\text{Segre class}} \cap \underbrace{c_F(M)}_{\text{Fulton's class}}\Big]_{\text{v.d.}}$

- **Graber - Pandharipande** :

  Torus $T \circlearrowleft M$

  then $[M]^{\text{vir}} =$ something on fixed points M^T

- (Kiem-Li)(cosection localization.)

Talk #3 : DT-type invariants after Behrend

Motivational example:

X 3 CY, smooth / projective, $\omega_X \simeq \mathcal{O}_X$

$\mathcal{M}_{ST}^{\alpha}$:= moduli space of stable sheaves

proper

$\rightsquigarrow$ can construct a POT on $\mathcal{M}_{ST}^{\alpha}$

using the CY condition, this POT is symmetric

Definition: A perfect symmetric obst. th. (S.O.T) on X is a P.O.T $E^{\bullet} \longrightarrow \mathbb{L}_X$ in $D(X)$

+ $\exists$ quasi-isomorphism

$$E \xrightarrow[\eta]{\sim} E^{\vee}[1]$$

with $\eta^{\vee}[1] \simeq \eta$

Talk #2 $\Rightarrow$ given P.O.T can construct virtual fund. class

$$[X]^{vir} \in A_{\mathrm{rk}E}(X)$$

$$\mathrm{rk}E^{0} - \mathrm{rk}E^{-1}$$

Now: If E is also symmetrical, then

$$\mathrm{rk}E \underset{\eta}{=} \mathrm{rk}E^{\vee}[1] = -\mathrm{rk}E^{\vee} = -\mathrm{rk}E$$

$$\Downarrow$$

$$\boxed{\mathrm{rk}E = 0}$$

since M^{α}_{ST} is proper, we can define

$$\text{DT-invariants} := \int_{[M^{\alpha}_{ST}]_{vir}} 1 = \deg [M^{\alpha}_{ST}]^{vir}$$

in general, can define DT-type invariants on any S.O.T

Behrend's main result is that in fact we do not need virtual fundamental classes to do the counting:

Thm (Behrend) X proper scheme with S.O.T

then $$DT(X) = \chi_e(X, \nu_X) = \sum_{i \in \mathbb{Z}} i\, \chi(\nu_X = i)$$

Sum is finite because of compact support

where $\nu_X : X \longrightarrow \mathbb{Z}$ is the Behrend function.

Main Results :

- DT invariants do not depend on the choice of the S.O.T
- This allows for a definition even when X is not proper.
- this leads to Joyce's categorification of DT-invariants.

Example: when X is smooth with S.O.T

then $$[X]^{vir} = c^{top}\left(H^1(E)^\vee\right) \cap [X]$$

is ← symmetric obsT They

$H^0(E)$

is p.o.T

$$\underbrace{H^0(\mathbb{L}_X) = \Omega^1_X}_{c^{top}(\Omega^1_X)\,\cap\,[X]}$$

Confused: how can X be smooth and have a (-1) shifted symplectic form?

In this case

$$DT = \int_{[X]^{vir}} 1 = \int_X c^{top}(\Omega^1_X) = (-1)^{\dim X}\int_X c^{top}(T_X)$$

$$= (-1)^{\dim X}\,\underbrace{\chi(X)}$$

Gauss-BoneT Euler-characteristic

Local model for Behrend's computation:

$$\underbrace{X}_{\text{crit}(f)} \xhookrightarrow{i} U \xrightarrow{f} \mathbb{C}$$

this has a natural S.O.T

$$\begin{tikzcd}
E \arrow[d] & = & {[T_U} \arrow[r, "\text{Hessian } df"] \arrow[d, "df"] & {\Omega^1_U|_X]} \arrow[d, equal, "id"] \\
\mathbb{L}_X & = & {[I/I^2} \arrow[r] & {\underbrace{i^\vee \mathbb{L}_U]}_{\Omega^1_U|_X}}
\end{tikzcd}$$

$T_U = (\Omega^1_U)^\vee$

Symmetry comes from the symmetry of the Hessian

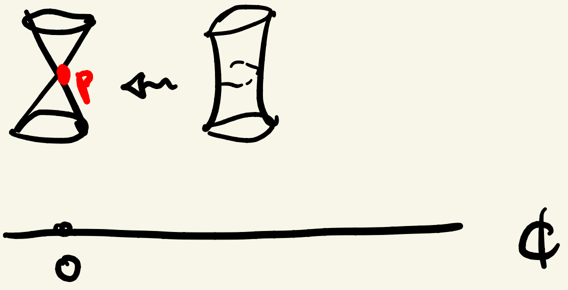
$$\left[\overset{-1}{T_U} \xrightarrow{Hf} \overset{0}{\Omega^1_X}\right]$$

$$\left(\underset{0}{T_U} \xrightarrow{Hf^\vee} \underset{1}{\Omega^1_U}\right)$$

Behrend's function:

step 1: Milnor fiber = $B_\varepsilon(p)$ — Milnor fiber at p.

$f^{-1}(0) \hookrightarrow U$



$X = \text{crit}(f)$

$p \in \operatorname{crit} f$:

$$\nu_X(p) := \left[1 - \chi\left(MF(p)\right)\right](-1)^{\dim U}$$

Exemples: $\mathbb{C}^2 \xrightarrow{f} \mathbb{C}$ $\quad x^2+y^2$

$\operatorname{crit} f = 0$ $\qquad \underbrace{MF}_{\text{milnor fib}}(f,0) = S^1$

$$\nu_X(0) = 1$$

Exemple $\mathbb{C} \longrightarrow \mathbb{C}$ $\qquad \operatorname{crit} f = \{0\}$

$x \longmapsto \underbrace{x^3}_{f}$ $\qquad$ scheme theoretically

$$\shortparallel$$

$f' = 3x^2$ $\qquad \operatorname{Spec}\left(u[\varepsilon] / 3x^2\right)$

In this case

$$MF(0) = 3 \text{ points} \left(\text{third roots}\right)$$

$$\nu_X(0) = (-1)^{\dim \mathbb{C}} [1 - \underset{\text{3 connected components}}{3}] = 2$$

$$\Downarrow$$

ν_X counts the point 0 but with multiplicity 2 (which is also in this case the scheme theoretic multiplicity $\mathrm{Spec}(k[\varepsilon]/\varepsilon^2)$)

$\boxed{\text{Step 2}}$: Relation to vanishing cycles

$$\underbrace{\phi_f}_{\text{usual definition of vanishing cycles via six operations}} (\underset{\uparrow \text{ constructible sheaf}}{\underline{\mathbb{Q}}} [\dim U]) \in \mathrm{Perv}(U_0 = f^{-1}(0))$$

- is actually supported on the critical locus

- $\mathcal{H}^k(\phi_f \underline{\mathbb{Q}}[\dim U])|_p \simeq H^{\circ}(MF(p), \mathbb{Q})$

- $\nu_X(p) = \chi(\phi_f \underline{\mathbb{Q}}[\dim U]|_p)$

$$= \sum_{n\in\mathbb{Z}} (-1)^n \dim\left(\mathcal{H}^n(\phi_f \mathbb{Q}[\dim U])\right)$$

Rmk

Joyce et al : these perverse sheaves glue on X to a globally define perverse sheaf.

Back to our exemples

exemple x^2+y^2

$$\phi_f = \mathbb{Q}_0$$

exemple x^3

$$\phi_f = \mathbb{Q}_0^2$$

exemple : $U \xrightarrow{0 \text{ function}} \mathbb{C}$

$$\nu_X(p) = (-1)^{\dim M}.$$

The General construction

$$\underbrace{Z(X)}_{\substack{\text{integral}\\ \text{cycles}}} \xrightarrow[\substack{\text{Euler}\\ \text{obstruction}\\ (\mathrm{Eu})}]{\text{local}} \underbrace{\mathrm{Const}(X)}_{\substack{\text{constructible}\\ \text{functions}}}$$

$$\underset{\dim d}{\overset{\rotatebox{90}{$\in$}}{V}} \longmapsto X \to \mathbb{Z}$$

?

$$p \longmapsto \int e(\hat{T}) \cap s(\mu^{-1}(p), \hat{V})$$

$\mu^{-1}(p)$

$\mu : \hat{V} \longrightarrow V$?

Nash Blow up

local model:

$$\begin{tikzcd} X \arrow[r, hook] & M^{\text{smooth}} \\ \underbrace{C_{X/M}}_{\substack{\text{normal}\\ \text{cone}}} \arrow[u, "\pi"] & \end{tikzcd}$$

Def:

$$\underset{Z(X)}{\overset{c_X}{\rotatebox{-90}{$\in$}}} := \sum_{\substack{C' \subseteq C_{X/M}\\ \text{irreducible components}}} (-1)^{\dim \pi(C')} \operatorname{mult}(C') \cdot \pi([C'])$$

Definition: Global definition of ν_X

$$\nu_X := Eu(c_X)$$

this is very confusing because we don't seem to need the S.O.T to define any of this.

Properties

① this agrees with local model by derived critical locus.

② If $p \in X$ and X smooth at p

$$\nu_X(p) = (-1)^{\dim X}$$

③ $X \xrightarrow{f} Y$ smooth morphism

then $f^* \nu_Y = (-1)^{\dim X/Y} \nu_X$

Ideas about the proof:

① singular Gauss-Bonnet
(Mcpherson's theorem)

For $v \in Z(X)$, $\int_X \underbrace{c^M(X)}_{\text{Chern-Mather class}} = \chi(X, Eu(v))$

defined in terms of the "Nash Blow up"

$\cap$

$A(X)$

Proposition (Behrend)

$$[X]_{vir} = 0^!_{T^*M}(C)$$

where $C = C_{X/M}$ in the local model.

③ commutativity of

(**)

$$\begin{tikzcd}
Z(X) \arrow[r, "Eu"] \arrow[dr, "(c^M)_0"'] & \mathrm{Cons}(X) \arrow[r, "CC"', "\text{characteristic}"] \arrow[d, phantom, "\circlearrowright"] & L_X(M) \arrow[dl, "0^!_{T^*M}"] \\
 & A_0(X) &
\end{tikzcd}$$

cycle map

conic Lagrangian cycles inside T^*M

supported on X

to show that it is Lagrangian we need the symmetry!!

$$C_X \xmapsto{Eu} \nu_X \xmapsto{CC} C \neq C_{X/M}$$

Key fact is to show that C is Lagrangian.

<u>Finally</u>: the commutativity of (**) implies

commutativity of (**)

$$DT(X) = \int_{[X]^{vir}} 1 = \int_{0^!_{T^*M}[C]} 1 = \int_X c^M(C_X)$$

Behrends Theorem

|| Gauss Bonnet

$\chi(X, \underbrace{Eu([X])}_{\nu_X})$

in particular

<u>S.O.T</u> is needed to define $\int_{\underline{[X]}}$ viz :

but one can show that in fact the result is independent of the S.O.T

Talk #4 : Holomorphic casson invariants

Plan :

1) Deformation of sheaves, sketch of moduli problem

2) Stability for sheaves

3) sketch of DT-invariants for CY 3-folds

Part 1: X a smooth scheme$/\mathbb{C}$

$$\mathcal{F}_0 \in \mathrm{coh}(X)$$

A deformation of $\mathcal{F}_0$ is a sheaf F on $X \times \mathbb{D}$ where $\mathbb{D} = \mathrm{Spec}(\mathbb{C}[t]/t^2)$ such that

$$F|_{X \times \{0\}} \simeq \mathcal{F}_0,$$ and such that F is flat $/\mathbb{D}$.

More generally, a B-family of coherent sheaves on X is a sheaf on $X \times B$ flat over B.

Proposition: $\mathcal{F}_0 \in \mathrm{Coh}(X)$. then

$$\left\{ \text{infinitesimal deformations of } \mathcal{F}_0 \right\} \simeq \mathrm{Ext}^1_X(\mathcal{F}_0, \mathcal{F}_0)$$

sketch of proof:

suppose we have a deformation $\mathcal{F}$ on $\underline{X \times \mathbb{D}}$. with $\mathcal{F}$ flat $/\mathbb{D}$. then we have an exact sequence on $\mathrm{coh}(X \times \mathbb{D})$

$$\mathcal{F} \xrightarrow{\cdot t} \mathcal{F} \longrightarrow \pi^* \mathcal{F}_0 \longrightarrow 0$$

$\pi : \mathbb{D} \times X \longrightarrow X$

$$\Downarrow$$

$$0 \longrightarrow \pi^* \mathcal{F}_0 \longrightarrow \mathcal{F} \longrightarrow \pi^* \mathcal{F}_0 \longrightarrow 0$$

on $\mathrm{coh}(X \times \mathbb{D})$

$\underbrace{\pi_*}\, = i^*$

$\underline{i^* \pi^* = \mathrm{id}}$

$\downarrow$ $\underline{\text{restiction}}$

$$0 \longrightarrow \mathcal{F}_0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}_0 \longrightarrow 0 \quad \text{on } X$$

given an element in $\mathrm{Ext}^1_X(\mathcal{F}_0, \mathcal{F}_0)$

Conversely: Given

$$\left[0 \to F_0 \xrightarrow{i} F \xrightarrow{R} F_0 \to 0\right] \in \mathrm{Ext}^1_X(F_0, F_0)$$

then F_0 is an $\mathcal{O}_X$-module; want to upgrade to give F_0 the structure of $\mathcal{O}_{X \times D}$ $= \mathcal{O}_X \otimes \mathbb{C}[t]/t^2$ -module

Let t act by $i \circ R$

and we are done. $(i \underbrace{R i}_{0} R) = 0$.

---

obstructions: F deforms F_0 over D.

let $D' := \mathrm{Spec}(\mathbb{C}[t]/t^3)$

Answer: obstructions are classified by

$$\mathrm{Ext}^2_X(F_0, F_0)$$

↓

Suppose F is a 1st order deformation of F_0.

and F' is an extension to D'

$$\begin{tikzcd}
X \arrow[r, "F_0"] \arrow[d] & coh \\
D \arrow[ur, dashed, "F"'] \arrow[d] & \\
D' \arrow[uur, dashed, "F'"'] &
\end{tikzcd}$$

then we get a short exact sequence on $coh(X \times D)$

$$0 \to t^2F_0 \to F' \to F \to 0$$

but we also have another

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & t^2F_0 \arrow[r] \arrow[d, equal] & tF \arrow[r] \arrow[d] & tF_0 \arrow[d] & \\
0 \arrow[r] & t^2F_0 \arrow[r] & F' \arrow[r] \arrow[d] & F \arrow[r] & 0 \\
 & & F_0 \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

(

by ↓ restricting to X we get a class in

@ $\mathrm{ExT}_X^1(F, F_0)$

which restricts to the class of $F \in \mathrm{ExT}_X^1(F_0, F_0)$

↓

this is part of a long exact sequence of

ExT groups:

$$\mathrm{ExT}^1(F, F_0) \longrightarrow \mathrm{ExT}_X^1(F_0, F_0) \longrightarrow \mathrm{ExT}_X^2(F_0, F_0)$$

$$e \longmapsto e \cup e.$$

lifts to second order - - - → 1st order deformation

iff ⟶ (the image here vanishes!)

PART 2 : stability

- X projective smooth / $\mathbb{C}$
- choose ample line bundle $\mathcal{O}_X(1)$
- $H := c_1(\mathcal{O}_X(1))$

associated to any sheaf $\mathcal{F}$ we have an hilbert function

$$P(F,t) = \chi\left(F \otimes \mathcal{O}_X(t)\right)$$

thm (serre) this is a polynomial for large t,

$$a_N t^N + a_{n-1} t^{n-1} + \dots$$

Definition : slope of the sheaf F

$$\shortparallel$$

$$\mu_H(F) = \frac{a_{n-1}}{a_n}$$

Fact:

$$\mu_H(F) = \frac{c_1(F)\cdot H^{n-1}}{\text{rank } F}$$

comes from Hirzebruch-Riemann-Roch

Definition: F is slope semi-stable iff

$\forall$ short exact sequence (non-trivial)

$$0 \longrightarrow A \longrightarrow F \longrightarrow B \longrightarrow 0$$

then

$$\mu_H(A) \leq \mu_H(F)$$

(if dim F = dim B) Then this is an equality.

Properties

① Slope semi-stable $\Rightarrow$ torsion free

② semi-stability is an open condition.

③ "schur property" ("scalar endomorphisms")

suppose F, G stable with same rank for the same chern class

then $\mathrm{Hom}(F,G) = \begin{cases} 0 & F \neq G \\ \mathbb{C} \cdot \mathrm{Id}_F & F \simeq G \end{cases}$

Proof: suppose we have $\phi : F \longrightarrow G$.

we can factor as

$$0 \longrightarrow \mathrm{Kernel}\,\phi \longrightarrow F \longrightarrow \mathrm{Im}\,\phi \longrightarrow 0$$

Since F is stable

$$\mu_H(F) < \mu_H(\mathrm{Im}\,\phi)$$

but we also have

$$0 \longrightarrow \mathrm{Im}\,\phi \longrightarrow G \longrightarrow \mathrm{Coker}\,\phi \longrightarrow 0$$

Then $\mu_H(\mathrm{Im}\,\phi) < \mu_H(G)$

but we have assume F & G with

the same slope $\Rightarrow$ contradiction.

$$\Downarrow$$

Either $\underline{\phi = 0}$ or $\underline{\phi \text{ is an isomorphism}}$

Property: the moduli of stable sheaves is separated:

Proof: let F and G on $X \times \mathbb{A}^1$, F_t, G_t stable $\forall t$

with $F_t \simeq G_t$

then $F \simeq G$ for all t (in particular $t=0$)

$$\Big\} \text{ computation}$$

$\pi_* \underline{Hom}(G, F)$ is a line bundle (because of the previous result)

away from 0.

but $Hom(G,F)$ is torsion free

So $\pi_* \underline{Hom}(G,F)$ is torsion-free

hence a line bundle on $\mathbb{A}^1$, but all line bundles

on <u>$\mathbb{A}^1$ are trivial</u>.

$\downarrow$

Pick $\phi \in \Gamma(\mathbb{A}^1, Hom(G,F))$.

non-vanishing.

Part 3 Holomorphic Casson inv.

- X smooth projective CY
- Fix Rank R, if $R>0$ fix a line bundle L
- Fix $c_i \in H^{2i}(X, \mathbb{C})$

then we have a moduli space of <u>semi</u>-stable sheaves as a <u>Artin stack</u>

$\mathcal{M}_L(X, c_i)$ of rank R

$\det F = L$ (fixed determinant)

$c_i(F) = c_i$

Thm (Huybrechts)

Definition: G and F on $\overset{\text{semi-stable}}{\mathrm{coh}}(X \times S)$ are equivalent if $\exists$ P a line bundle on S such that $G \simeq F \otimes \pi^* P$

$\pi : X \times S \to S$.

claim: The tangent complex of $\mathcal{M}_L(X, c_i)$ at F is given by the ext-complex

$$\underline{\mathrm{Ext}}_X(F, F)$$

Serre duality: **correct:** $\underline{R\mathrm{Hom}}_X(F_0,F_0)$

$$\mathrm{Ext}^i_X(F,G) \simeq \mathrm{Ext}^{n-i}_X(G, F\otimes\omega_X)^\vee$$

dualizing sheaf

$+$ CY $\omega_X \simeq \mathcal{O}_X$

So:

$$\mathrm{Ext}^1_X(F_0,F_0) \simeq \mathrm{Ext}^2(F_0,F_0)^\vee$$

$\Downarrow$ by the P.O.T formalism of talk #2

Can define

$$[M_L(X,c_i)]^{vir}$$

$\boxed{\text{Talk \#5}}$ DT-invariants in CY 4-folds.

Let $\mathcal{M}$ be a moduli space of stable = semistable sheaves on a CY 4fold X

$$\begin{tikzcd} \mathcal{E} \arrow[d, bend right] & \\ X \times M \arrow[d, "\pi"] & \\ M & \end{tikzcd}$$

$\mathcal{E}$ univeral sheave

[Huybrechts–Thomas]

Ext^0 and Ext^4

cut away

⚠ why? 3 and not 1?

$$E_M := \underbrace{\pi_*}\, R\mathcal{H}om(\mathcal{E},\mathcal{E})_0\,\boxed{[3]} \longrightarrow \mathbb{L}_M$$

is an obstruction theory in the sense of Behrend-Fantechi

has non-zero h^2-term

$\searrow$ **NOT** quasi-smooth, but perfect in degree $[2,0]$

- For

$$\begin{tikzcd} {[F]} = x \in M \arrow[d] \\ X \end{tikzcd}$$

$$h^{-2}(E_M|_x) = T_{M,x} = Ext^1_X(F,F)$$

$$h^{-1}(E_M|_x) = ob^* = Ext^2_X(F,F)$$

Following Li-Tian / Behrend-Fantechi's idea,

if M is smooth, then we can define

$$[M]^{vir} = \underset{\text{(Euler)}}{e}(\mathrm{ExT}^2) \quad \text{as they are bundles.}$$

But this is wrong. why?

exp. dimension $= ext^1 - ext^2$ may not be constant in general.

<u>Serre duality</u>: $\mathrm{ExT}^1 \simeq (\mathrm{ExT}^3)^\vee$, $\mathrm{ExT}^2 \simeq (\mathrm{ExT}^2)^\vee$

(More precisely $\underbrace{E_M \xrightarrow[\theta]{\sim} E_M^\vee[2]}_{}$ $\theta = \theta^\vee[2]$)

Residue of a (-2)-shifted symplectic form

- Using this, if $\mathrm{rnk}(\mathrm{ExT}^2)$ is even, then

$$\begin{tikzcd} \mathrm{ExT}^2 \arrow[r, "\sim", "\text{locally}"'] \arrow[d, phantom, "\wr"] & \Lambda \oplus \Lambda^* \arrow[d, "s/id"] \\ (\mathrm{ExT}^2)^\vee \arrow[r, "\sim"] & \Lambda \oplus \Lambda^* \end{tikzcd}$$

locally this exists

(Darboux lemma ?)

Then, the correct one is

$$[M]^{\mathrm{vir}} = \pm \underbrace{e}_{\text{euler class.}}(\Lambda)$$

in this case

$$\text{exp. dim} = \mathrm{ext}^1 - \frac{1}{2}\mathrm{ext}^2$$

$$= \frac{1}{2}\left(\mathrm{ext}^1 - \mathrm{ext}^2 + \mathrm{ext}^3\right)$$

$$= \frac{1}{2}\left(-\chi(F,F) + 2\right) = \text{constant}$$

<u>why is this correct?</u> → because of <u>curve counting theory</u> (GW)

[Cao, Leung]

<u>Example:</u>

Let $C = \mathbb{P}^1 \hookrightarrow \underbrace{X}_{\text{CY 4 fold.}}$ with ideal $I_C =: F$

Then we claim

$$\mathrm{Ext}^1(I_C, I_C) \simeq H^0(C, N_{C/X})$$

$$\mathrm{Ext}^2(I_C, I_C) \simeq \underbrace{H^1(C, N_{C/X})}_{\Lambda} \oplus \underbrace{H^1(C, N_{C/X})}_{\Lambda^*}{}^\vee$$

Proof :

Since X is CY, $h^i(X, \mathcal{O}_X) = 0$ $\quad i=1,2,3$

and $h^0(\mathcal{O}_X) = h^4(\mathcal{O}_X) = 1$

$$h^i(X, \mathcal{O}_C) = h^i(C, \mathcal{O}_C) = 0 \quad i = 1,2,3,4$$

$$h^0(C, \mathcal{O}_C) = 1$$

using $0 \to \mathcal{I}_C \to \mathcal{O}_X \to \mathcal{O}_C \to 0$

we have

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{Ext}^i(\mathcal{I}_C, \mathcal{I}_C) \arrow[r] & \mathrm{Ext}^i(\mathcal{I}_C, \mathcal{O}_X) \arrow[r] & \mathrm{Ext}^i(\mathcal{I}_C, \mathcal{O}_C) \arrow[d] \\
 & & & \mathrm{Ext}^{i+1}(\mathcal{I}_C, \mathcal{I}_C) \arrow[d] \\
 & & & \mathrm{Ext}^{i+1}(\mathcal{I}_C, \mathcal{O}_X) \arrow[d] \\
 & & & \vdots
\end{tikzcd}$$

Fr $\underline{i=1}$

$$\begin{tikzcd}
\mathrm{Ext}^1(\mathcal{I}_C, \mathcal{O}_X) \arrow[r, phantom, "\simeq"] & \mathrm{Ext}^3(\mathcal{O}_X, \mathcal{I}_C)^\vee \arrow[d, phantom, sloped, "\cong"] & \\
 & H^3(\mathcal{I}_C)^\vee \arrow[r] \arrow[d, phantom, sloped, "="] & H^3(\mathcal{O}_X) = 0 \\
H^2(\mathcal{O}_C)_{=0} \arrow[ur] & 0 &
\end{tikzcd}$$

$$\mathrm{Ext}^2(\mathcal{I}_C, \mathcal{O}_X) \simeq H^2(\mathcal{I}_C)^\vee = 0$$

$$\mathrm{Ext}^1(\mathcal{I}_C, \mathcal{O}_C) \simeq \mathrm{Ext}^2(\mathcal{I}_C, \mathcal{I}_C)$$

<u>$i=0$</u>:

$$\begin{tikzcd} \mathrm{Ext}^0(\mathcal{I}_C, \mathcal{O}_X) \simeq H^4(\mathcal{I}_C)^\vee & H^4(\mathcal{O}_X)^\vee \simeq \mathbb{C} \arrow[l] \\ 0 = H^3(\mathcal{O}_C)^\vee \arrow[u, phantom, "\checkmark"] & \end{tikzcd}$$

$$\mathrm{Ext}^0(\mathcal{I}_C, \mathcal{I}_C) = \mathbb{C}$$

<u>Hence</u>: $\mathrm{Ext}^0(\mathcal{I}_C, \mathcal{O}_X) \longrightarrow \mathrm{Ext}^0(\mathcal{I}_C, \mathcal{O}_C)$

is <u>zero</u>

<u>So</u>: $\mathrm{Ext}^0(\mathcal{I}_C, \mathcal{O}_C) \simeq \mathrm{Ext}^1(\mathcal{I}_C, \mathcal{I}_C)$

<u>Similarly</u>: we can show that

- $\mathrm{Ext}^1(\mathcal{I}_C, \mathcal{O}_C) \simeq \mathrm{Ext}^2(\mathcal{O}_C, \mathcal{O}_C)$

So it remains to relate $\mathrm{Ext}^2(\mathcal{O}_C, \mathcal{O}_C)$

and $\mathrm{Ext}^1(\mathcal{O}_C, \mathcal{O}_C)$

to GW-numbers :

<u>Computation</u> :

$$\mathrm{Ext}^*(\mathcal{O}_C, \mathcal{O}_C) \simeq H^*(X, \underline{\mathcal{E}xt}^*(\mathcal{O}_C, \mathcal{O}_C))$$

$\cong$ adjunction formula

$\iota : C \hookrightarrow X$

$$H^*(X, \iota_* \Lambda^* N_{C/X})$$

$$\cong$$

$$H^*(C, \Lambda^* N_{C/X})$$

$\Rightarrow$ when $\underline{*=1}$

$$\mathrm{Ext}^1(\mathcal{O}_C, \mathcal{O}_C) \simeq H^1(C, N_{C/X}) \oplus \underbrace{H^1(\mathcal{O}_C)}_{\substack{\| \\ 0}}$$

when $*=2$

$$\mathrm{Ext}^2(\mathcal{O}_C, \mathcal{O}_C) \simeq H^2(C, \Lambda^2(N_{C/X}))$$

(using Koszul resolution) ⟶ $\cong$

$$H^0(\Lambda^2 N) \oplus H^1(N) \oplus \underbrace{H^2(\mathcal{O}_C)}_{\| 0}$$

using $\Lambda^2 N \otimes N \longrightarrow \Lambda^3 N \simeq \omega_X \otimes \omega_C^{-1}$

$$H^0(\Lambda^2 N) = H^0(N^\vee \otimes \omega_C)$$

IS serre duality

$$H^1(N)^\vee$$

<u>So</u>: $\mathrm{Ext}^2(\mathcal{O}_C, \mathcal{O}_C) \simeq \underbrace{H^1(N)^\vee}_{\Lambda^\vee} \oplus \underbrace{H^1(N)}_{\Lambda}$

□

<u>Rmk</u>: There is also a Gauge theoretic reason see (Joyce - Borisov). They use differential geometry to find a decomposition

$$\mathrm{Ext}^2 \simeq \underbrace{\mathrm{Ext}^2_{\mathbb{R}}}_{\Lambda} \oplus i\, \mathrm{Ext}^2_{\mathbb{R}}$$

and define $[M]^{vir} \in H^{BM}$, $[M]^{vir} = \mathrm{eul}(\mathrm{Ext}^2_{\mathbb{R}})$

Brief idea of Borisov- Joyce

using symplectic derived geometry we get a local model (Daboux lemma!)

$$\begin{tikzcd} & & (E, q) \arrow[d, shift right, "\pi"'] & q(s,s)=0 \\ M \underset{\text{open}}{\supseteq} & U \arrow[r, hook, "s^{-1}(0)"'] & A \arrow[u, shift right, bend right, "s"'] & \end{tikzcd}$$

[Rmk given such a local model we get a canonical Δ and $\Delta^\vee$ as above!]

such that

$$\left[T_A|_U \xrightarrow{ds} E|_U \xrightarrow{(ds)^*} \Omega_A|_U\right] \simeq E_M|_U$$

local model for the obstruction theory.

using decomposition

$$E \simeq E_{\mathbb{R}} \oplus i E_{\mathbb{R}}$$

we have $(A, \underbrace{E_{\mathbb{R}}, s_+})$; $s_+ : A \to E \to E_{\mathbb{R}}$

Joyce calls these μ-Kuranishi charts of M

Joyce uses this to define

$$[M]^{\text{vir}} \in H^{BM}(M)$$

---

suppose $R = 2n$. then

$$\Lambda \hookrightarrow E \longrightarrow E_{\mathbb{R}}$$ is iso of $\mathbb{R}$-bundles

we choose an orientation on E such that the induced orientations on Λ and $E_{\mathbb{R}}$ are compatible.

Algebraic Construction

(finding a lift to chow)

Need coefficients containing $\frac{1}{2}$.

↳ Need **cosection localization**:

$$\begin{tikzcd} & F \arrow[r, "\sigma"] \arrow[d, shift left] & \mathcal{O}_V \\ U \arrow[r, hook] & V \arrow[u, shift left, "t"] & \end{tikzcd}$$

vector bundle

with

$\sigma t = 0$

$U = t^{-1}(0)$

} context

$$(V, F, t) \rightsquigarrow [U]^{vir} \in A_*(U)$$

but with the extra data of σ,

$$(V, F, t, \sigma) \rightsquigarrow \exists\ [U]^{vir}_{\sigma} \in A_*(U \cap Z(\sigma^\vee))$$

such that the pushforward along the inclusion

$$Z(\sigma^\vee) \hookrightarrow U$$

Recovers $[U]^{vir}$.

In the language of perfect obs. th.

$$\begin{tikzcd} E_M \arrow[r, "\sigma"] \arrow[d, "\phi"'] & O_M[-1] \\ L_M & \end{tikzcd}$$

so take $Z = O_M / \mathrm{Im}\, h^1(\sigma)$

$\rightsquigarrow$ closed substack of M.

produces $[M]^{vir}$

but adding σ, $(\phi, \sigma) \rightsquigarrow$ produces

$[M]^{vir}_{\sigma} \in A_*(Z)$

whose pushforward to M gives back

$[M]^{vir}$

$\downarrow$

Now lets return to M: in our local model

$$\begin{tikzcd} (E, q) \arrow[d, "s"] & \simeq \left(\Lambda \oplus \Lambda^*, q = \text{pairing}\right) \\ s^{-1}(0) = U \subset A & \end{tikzcd}$$

then $s = (s_1, s_2)$

Λ = maximal isotropic

then: This gives a new local model

(local model for what?)

$$\begin{tikzcd}
\Lambda^\vee \arrow[r, "s_1^*"] \arrow[d, shift right] & \mathcal{O}_A \\
A \arrow[u, bend right, "s_2"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
s_2^{-1}(0) \arrow[r, hook] & A \\
s_2^{-1}(0) \cap s_1^{-1}(0) \arrow[u, hook] &
\end{tikzcd}$$

$$s_2^{-1}(0) \cap s_1^{-1}(0) = s^{-1}(0) = U$$

→ we obtain

$$[U]^{vir}_{s_1^*} \xmapsto{\text{pushfwd}} e(\Lambda^\vee) = (-1)^n e(\Lambda)$$

$\rightsquigarrow$

We define

$$[U]^{virT}_{DT4} := (-1)^n [U]^{virT}_{s_1^*}$$

Proposition: E_M is represented by

$$T \xrightarrow{a} \underbrace{E_{,q} \xrightarrow{a^*} T^*}_{\text{This part becomes a perfect obs. theory.}}$$

C BF cone.

Now we have

$$\begin{tikzcd} p^* E = \Lambda \oplus \Lambda^* \arrow[d, shift right] \\ {} \arrow[u, shift right, bend right, "\tau = \text{tautological}"'] \end{tikzcd} \qquad \begin{tikzcd} C \arrow[d, "p"] \\ M \end{tikzcd}$$

$$\tau^{-1}(0) = M \hookrightarrow C$$

<u>τ is isotopic</u>

Talk #6

- Critical Virtual Manifolds (CVM)
- semi PoT's.

(following Kiem-Li)

Question: (Joyce - Song)

- X moduli of stable sheaves on Y

does there exists $P \in \mathrm{Perv}(X)$ on X^{an}, locally isomorphic to sheaf of vanishing cycle of f holomorphic.

Positive answer: when Y is CY-3 fold, if X^{red} is of finite type

and if admits a tautological family ?

(like pulling back the universal family along $X_{red} \hookrightarrow X$)

+ orientation data

Definition: an LG-pair (Landau-Ginzburg) is a pair (V, f)

V: complex manifold

f: holomorphic function. $V \to \mathbb{C}$ such that only critical value is 0

Definition: A CVM is a analytic space X with an open covering $\{X_\alpha\}$ and for each α, an LG pair (V_α, f_α) and embedding

$$X_\alpha \hookrightarrow V_\alpha$$

such that

$$X_\alpha \simeq \mathrm{Crit}(V_\alpha, f_\alpha)$$

biholomorphism. (as analytic spaces)

and for every intersection α, β, there should exists

$$\begin{tikzcd}
 & \varphi_\alpha(X_{\alpha\beta}) \subseteq \exists\, V_{\alpha\beta} \arrow[r, hook, "\text{open}"] \arrow[d, "\wr", "\circlearrowleft\ \exists\,\varphi_{\alpha\beta}"'] \arrow[dr] & V_\alpha \arrow[d, "f_\alpha"] \\
\underset{X_{\alpha\beta}}{\underbrace{X_\alpha \cap X_\beta}} \arrow[ur, "\varphi_\alpha"] \arrow[dr, "\varphi_\beta"'] & & \mathbb{C} \\
 & \varphi_\beta(X_{\alpha\beta}) \subseteq \exists\, V_{\beta\alpha} \arrow[r, hook, "\text{open}"'] \arrow[ur] & V_\beta \arrow[u, "f_\beta"']
\end{tikzcd}$$

with

$\varphi_{\alpha\beta} = \varphi_{\beta\alpha}^{-1}$

- $\boxed{\varphi_{\alpha\alpha} = \mathrm{id}}$

↙

⚠ **No cocycle condition!**

otherwise we could glue vanishing cycles directly without problem.

<u>Notation:</u> for CVM

$$X = \left(X_\alpha \xhookrightarrow{\varphi_\alpha} V_\alpha \xrightarrow{f_\alpha} \mathbb{C} \right)$$

Exemples :

(i) complex manifolds V $(V, f=0)$

(ii) $Z(s)$ when smooth (? how)

(iii) moduli of stable sheaves

(iv) Joyce d-critical loci

(v) analytic space associated to (−1)-shifted derived scheme.

---

§ orientability

- X a CVM

set $$K_\alpha^\vee = \left(\varphi_\alpha^* \det T_{V_\alpha}\right)\Big|_{X_\alpha^{red}}$$

the $\varphi_{\alpha\beta}$'s induce via (short exact sequences)

$$K_\alpha^\vee\Big|_{X_{\alpha\beta}^{red}} \xrightarrow[\mathcal{E}_{\alpha\beta}]{} K_\beta^\vee\Big|_{X_{\alpha\beta}^{red}}$$

Proposition: set $X_{\alpha\beta\gamma} = X_\alpha \cap X_\beta \cap X_\gamma$

the

$$\varepsilon_{\alpha\beta\gamma} := \varepsilon_{\gamma\alpha}\varepsilon_{\beta\gamma}\varepsilon_{\alpha\beta}$$

are locally constant with value $\{\pm 1\}$

$$\Downarrow$$

we have a 2-cocycle with values in $\mathbb{Z}/2\mathbb{Z}$

$$\Downarrow$$

$\varepsilon = \{\varepsilon_{\alpha\beta\gamma}\}$ defines an element in $H^2(X, \mathbb{Z}_2)$

Definition: X is orientable if $\varepsilon = 0$

In this case, $\exists$ local cochain $\mu = \{\mu_{\alpha\beta}\}$ with values in $\mathbb{Z}/2\mathbb{Z}$ such that

$$\varepsilon_{\alpha\beta\gamma} = 1$$

$\Downarrow$

these cocycles glue $\{K_\alpha^\vee\}$ into a $K_X^\vee$ line bundle on X^{red}.

($K_X^\vee \otimes K_X^\vee \simeq$ canonical bundle of the derived) .

ie: $K_X^\vee$ is a square root of the canonical bundle of $X^{derived}$. So in this case this $K_X^\vee$ squares to the determinant of the perfect obstruction theory!

ie: $$\boxed{``K_X^\vee = \sqrt{\det(\mathrm{POT})}"}$$

$\overset{\text{semi}}{\oint}(\tfrac{1}{2})$POT : X analytic space with cov $\{X_\alpha\}$ with $\underset{\text{semi}}{\underset{\sim}{\tfrac{1}{2}}}$POT's E_α on each X_α.

This is automatic if X has a derived structure! !!}

can be glued to a $\frac{1}{2}$ PoT on X if :

(i) $\forall \alpha, \beta$

$$\exists\ \psi_{\alpha\beta} : H^1(E_\alpha^\vee)|_{X_{\alpha\beta}} \xrightarrow{\sim} H^1(E_\beta^\vee)|_{X_{\alpha\beta}}$$

such that $\psi_{\alpha\alpha} = \mathrm{id}$, $\psi_{\alpha\beta}^{-1} = \psi_{\beta\alpha}$

and $\psi_{\beta\gamma}\psi_{\alpha\beta} = \psi_{\alpha\gamma}$

(ii) Via $\psi_{\alpha\beta}$, $E_\alpha|_{X_{\alpha\beta}}$ and $E_\beta|_{X_{\alpha\beta}}$ define the same obstruction assignment

Rmks

(i) $\Rightarrow$ $\exists$ $\underbrace{\text{obs}_X}_{\substack{\text{obstruction}\\ \text{sheaf}}}$ gluing $\{ \text{obs}_{X_\alpha} = H^1(E_\alpha^\vee) \}$

(ii) [BF] Definition: infinitesimal Lifting problem of X at x.

$$(*)\ 0 \to I \to B \to \bar{B} \to 0$$

of Artin local rings $(I \cdot m_B = 0)$

$$(*)\ \bar{g} : \operatorname{spec} \bar{B} \longrightarrow X$$
$$m_{\bar{B}} \longmapsto x.$$

Set $\overset{(-)}{\Delta} = \operatorname{spec}(\overset{(-)}{B})$

then $\bar{g}$ lifts to Δ iff $\omega(\bar{g}, B, \bar{B}) = 0$

where

$$\begin{tikzcd} \omega(\bar{g}, B, \bar{B}) := \Big(\bar{g}^* \mathbb{L}_X \arrow[r] & \mathbb{L}_{\bar{\Delta}} \arrow[r] & \mathbb{L}_{\bar{\Delta}/\Delta} \arrow[d, "\tau^{\geq 1}"] \\ \mathrm{Ext}^1(\bar{g}^* \mathbb{L}_X, I) \arrow[u, phantom, "\rotatebox{90}{$\in$}"] & & I[1] \Big) \end{tikzcd}$$

Then: if $\phi : E \longrightarrow \mathbb{L}_X$ is a PoT

the obstruction assignment is

$ob_X(\phi, \bar{g}, \bar{B}, B)$ is a composition

$$\left(\bar{g}^* E \xrightarrow{\omega} I[1]\right) \in H^1(E^\vee)\big|_X \otimes I$$

$\sim$

crucial result in BF paper.

Definition: A semi-PoT ($\frac{1}{2}$ PoT) is symmetric

if all ϕ_α's are (as perfect ob. assignments)

+ all $\varphi_{\alpha\beta}$ are identities on

symmetry of Thomas

$$\Omega^1_{X_\alpha} \simeq Ob_{X_\alpha}$$

seen yday in talk 4.

Proposition: $X = (X_\alpha \xleftarrow{\varphi_\alpha} V_\alpha \xrightarrow{f_\alpha} \mathbb{C})$ CVM

with $\frac{1}{2}$ POT's $= \{ E_\alpha = [T_{V_\alpha} \xrightarrow{Hf_\alpha} T^\vee_{V_\alpha}] \}$

admits a semi-perfect ob. theory in

$ob_X \simeq \Omega_X$. independent of the charts.

More precisely: given two presentations of X as a CVM, we get the same obs. assignment

Rmk: $\{E_\alpha\}$ do not glue (" $\{\varphi_{\alpha\beta}\} \neq id$) as a strict complex

⚠ but they glue in the derived category

$\downarrow$

Back to DT-invariants

because these are the tangent complex of the derived enhancement of X

[BF]

$X \hookrightarrow M$ complex manifold

(analytic space with $\frac{1}{2}$POT $\{E_\alpha \to \mathbb{L}_{X_\alpha}\}$

intrinsic normal cone $\quad \mathfrak{C}_{X_\alpha/\mathcal{U}} = \left[C_{\mathcal{U}/Y} \big/ T_{Y|U} \right]$

$\mathcal{U} \subseteq X$
open

$$\begin{tikzcd} \forall\, \mathcal{U} \arrow[r, hook] \arrow[d, hook, "\text{local embedding}"'] & X_\alpha \\ Y \text{ smooth} & \end{tikzcd}$$

$$\mathfrak{C}_{X_\alpha} \hookrightarrow \eta_{X_\alpha} := h^1/h^0\left(\mathbb{L}^\vee_{X_\alpha}\right) \to h^1/h^0\left(E^\vee_\alpha\right)$$

- Consider a local Resolution in $D(X_\alpha)$

$$F \longrightarrow E^\vee_\alpha[1]\big|_U$$

locally free

obstruction cone

$$\begin{tikzcd}
\mathcal{C}_F \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & F \arrow[d] \\
\mathcal{C}_{X_\alpha}|_U \arrow[r] & h^1/h^0(E_\alpha^\vee)|_U
\end{tikzcd}$$

Proposition: [Behrend + gluing] $\exists!\ \mathcal{C}_M \subseteq \Omega^1_M$

such that $\forall U, \forall F,$ $\forall$ lift

$$\begin{tikzcd}
 & F \arrow[d] \\
\Omega_M|_U \arrow[r] \arrow[ur, "\eta"] & \Omega_U
\end{tikzcd}$$

we have :

$$\mathcal{C}_M|_U = \underset{\substack{\text{part of}\\ \text{the}\\ \text{PoT data}}}{\eta^{-1}}(\mathcal{C}_F)$$

we have

$$\mathscr{C}_M \overset{?}{=} \underbrace{Z_{\dim M}}_{\substack{\text{integral}\\ \text{cycles}}}\Big(\underbrace{\Omega_M}_{\substack{\text{seen as its}\\ \text{total space}}}\Big)$$

<u>better</u>:

$$\mathscr{C}_M \in \underbrace{\mathscr{L}_X}_{\substack{\text{subgroup spanned}\\ \text{by conical lagrangean}\\ \text{cycles with}\\ \text{support in } X.}}(\Omega_M)$$

<u>Definition</u>:

X compact $\rightsquigarrow$

$$[X]^{vir} = 0^!_{\Omega_M}[\mathscr{C}_M] \in A_0(X)$$

$$DT := \deg [X]^{vir}$$

<u>Some facts</u>: * independent from $X \hookrightarrow M$

where

* $\mathscr{C}_M = \ell(c_X)$

$\uparrow$ when $W \subseteq M$

$(\ell(W) =$ closure of conormal of $W^{sm} \subseteq \Omega_M$

extends to

$$\begin{tikzcd}
Z_X(M) \arrow[rr, "\ell"] \arrow[dr] & & \mathcal{L}_X(M) \arrow[dl, "0^!"] \\
& A_0(X) &
\end{tikzcd}$$

Thm: X a CVM, compact, $\hookrightarrow M$ along with its $\frac{1}{2}$ PoT. Then

$$DT(X) = \chi(X, \nu_X)$$

only depends on the analytic structure of X

Talk #7 categorifying DT-invariants on C.V.M (Talk #6) of perverse sheaves of vanishing cycles.

(Kiem-Li)

Part ① complexes of vanishing cycles.

1) construction

- $D \subseteq \mathbb{A}^1$ a small (analytic) disk around 0.

- V a complex manifold.

$$f : V \longrightarrow D \subseteq \mathbb{A}^1$$

Definition let $D^\times = D \setminus \{0\}$ and $\omega : \underbrace{\widetilde{D^\times}}_{\text{universal cover}} \longrightarrow D^\times$

Then we form:

$$\begin{tikzcd}
\widetilde{V}^\times \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] \arrow[dd, bend right=60, "\overline{\omega_f}:="'] & \widetilde{D}^\times \arrow[d] \\
V^\times \arrow[r, "f|"'] \arrow[d, hook, "j"'] \arrow[dr, phantom, "\lrcorner"] & D^\times \arrow[d, hook] \\
V \arrow[r] & D \\
V_0 \arrow[r] \arrow[u, hook, "i"] \arrow[ur, phantom, "\urcorner"] & 0 \arrow[u, hook]
\end{tikzcd}$$

the functor of vanishing cycles is

$$\psi_f := i^* \overline{\omega_f}_* \overline{\omega_f}^* : D^b_c(\overbrace{\mathbb{Q}_V}^{\text{constant sheaf.}}) \longrightarrow D^b_c(\mathbb{Q}_V)$$

<u>Proposition</u> At $x \in V_0$

and $\forall \mathcal{M} \in D^b_c(\mathbb{Q}_x)$

$\forall u \in \mathbb{Z}$

we have $H^u(\psi_f \mathcal{M})_x \simeq \mathbb{R}\Gamma^u(MF_x, \mathcal{M})$

where $MF_x = \underset{\text{at } x}{\text{Milnor fiber}} = V^\times \cap B_\varepsilon(x)$

where $B_\varepsilon(x)$ is a ball centered at x

of Radius $\varepsilon << 1$

Such that Radius of $D^\times << \varepsilon$

Proposition : (Goresky - MacPherson)

$\exists$ retraction $\underbrace{sp : V}_{} \longrightarrow V_0$ and an ⚠

topologically this is easy !

(complicated as analytic space ??)

equivalence

$$\psi_f \mathcal{M} \simeq sp_*\left(\mathcal{M}|_{V_s}\right)$$

(V_s generic fiber)

Rmk : Description of sp_* and sp^* in cohomology

since D is small enough $V \underset{\substack{\text{homot} \\ \text{equivalence}}}{\sim} V_0$.

write $i : V_s \longrightarrow V \sim V_0$

then $sp : H^\circ(V_0, \mathbb{Q}) \simeq H^\bullet(V, \mathbb{Q})$

$$\begin{array}{c}\downarrow \\ H^{\bullet}(V_s, \mathbb{Q})\end{array}$$

Note that the natural transformation

$$sp^*: i^* \longrightarrow i^*\overline{\omega}_{f*}\overline{\omega}_f^*$$

induced by the unit η of $\overline{\omega}_{f*} \dashv \overline{\omega}_f^*$

induces the above after applying it to $\mathbb{Q}_V$

and taking cohomology of global sections.

__Definition__ The vanishing cycles functor ϕ_f is

the cofiber of sp.

It follows that

$$H^k(\phi_f \mathcal{M})_x \simeq \mathbb{R}^{k+1}\Gamma(\underbrace{B_\varepsilon(x), MF_x}_{\text{relative cohomology}}; \mathcal{M})$$

In particular,

with $\mathcal{M} = \mathbb{Q}_V$, we get

$$\tilde{H}^k(MF_x, \mathbb{Q})$$

since $B_\varepsilon(x) \cap V_0$ is contractible.

↳ As a consequence, the support of ϕ_f is contained on the critical locus of f.

Alternate construction (used by Kiem-Li)

Write $V_{\geq 0} := \{x \in V \mid \mathrm{Re}(f(x)) > 0\}$

$$\underbrace{V \setminus V_{\geq 0}}_{\substack{\shortparallel \\ V_{<0}}} \underset{\substack{\shortparallel \\ j_{<0}}}{\hookrightarrow} V$$

Define Γ_Z as the fibre of the sequence

$$\Gamma_Z \to \mathrm{id} \to j_{<*}\, j_<^{*}$$

So it is the derived functor of

$$\mathcal{M} \longmapsto \mathbb{R}^0 \underbrace{\Gamma_Z}_{} \mathcal{M}$$

explicitely given by

$$\left(U \longmapsto \mathrm{Ker}\left(\mathbb{R}^0\Gamma(U,\mathcal{M}) \to R^0\Gamma(U \cap V_{<0}, \mathcal{M}) \right) \right)$$

then $\boxed{\phi_f[-n] = i^{-1}\Gamma_Z}$

Notation

- $\boxed{\underbrace{{}^P\psi_f}_{\text{perverse sheaf of } \underline{\text{nearby}} \text{ cycles}} := \psi_f[-1]}$

- $\boxed{\underbrace{{}^P\phi_f}_{\text{perverse sheaf of vanishing cycles}} := \phi_f[-1]}$

- $\boxed{P_f := {}^P\phi_f\left(\underline{\mathbb{Q}}_V[\dim V]\right)}$

Corollary: At any $x \in \mathrm{Crit}(f)$

- $$\chi(P_f)_x = \sum (-1)^n \mathcal{H}^n(P_f)_x = (-1)^{\dim V}\left(1 - \chi(MF_x)\right) = \underset{\text{function}}{\nu_{\text{Behrend}}}(x)$$

- $$\chi\left(\Gamma_c\left(\mathrm{crit}(f), P_f\right)\right) = \text{DT-invariants}$$

Upshot: on a local critical chart

P_f is a categorification of the

Behrend function.

Part II Properties of ${}^p\!f$

Proposition the functors ${}^p\psi_f$ & ${}^p\phi_f$ commute with Verdier duality, i.e

$$\mathbb{D}({}^p\phi_f) = {}^p\phi(\mathbb{D}(-))$$

Rmk: the shifts of $[\dim V]$ in the definition were made to have this compatibility on the nose.

Lemma: $\pi : W \longrightarrow V$ proper analytic morphism, $g = f\circ\pi$, $\pi_0 : W_0 \longrightarrow V_0$ the restriction. Then

$$(\pi_0)_*({}^p\phi_g) \simeq ({}^p\phi_f)(\pi_*)$$

and by

By adjunction:

$$ {}^p\phi_g \, \pi^* \simeq \pi_0^* \, {}^p\phi_f $$

Proof: (Diagram chase. + proper base change.)

IN PARTiculr, when π is an homeomorphism (so $\dim W = \dim V$), we get

$$ \Sigma : P_g \simeq \pi_0^* P_f $$

Corollary: $X = (X_\alpha \xhookrightarrow{\varphi_\alpha} V_\alpha \xrightarrow{f} \mathbb{A}^1)$

a C.V.M.

$\rightsquigarrow$

$\forall \alpha$ we get a $P_\alpha = \underbrace{\varphi_\alpha^*}\, P_{f,\alpha}$

Recall that f vanishes on the critical points

so $P_{f,\alpha} \in \mathrm{Perv}(V_{\alpha,0})$

and can be pulled back

$$X_\alpha \subseteq (V_\alpha)_0$$

and $\forall \alpha, \beta$

$$P_\alpha|_{\alpha\beta} \simeq P_\beta|_{\alpha\beta}$$

Problem is that the gluing isomorphisms do NOT glue!

This morning

<u>Proof</u>: the $\varphi_{\alpha\beta} : V_{\alpha\beta} \longrightarrow V_{\beta\alpha}$ are compatible with the f_α, so $\Sigma : P_{f_\alpha} \simeq \varphi_{\alpha\beta}^* P_{f_\beta}$

Finally, apply φ_α^* to get

$$P_\alpha := \varphi_\alpha^* P_{f_\alpha} \simeq \underbrace{\varphi_\alpha^* \varphi_{\alpha\beta}^*}_{} P_{f_\beta}$$

$$\varphi_\beta^* P_{f_\beta}$$

$$\wr\wr$$

$$P_\beta$$

PART III: Gluing the perverse sheaves

Definition: $\mathcal{M} \in D^b_c(\mathbb{Q}_X)$

with X analytic space

then $\mathcal{M}$ is a perverse sheaf if

(i) support condition:

$$\dim \operatorname{supp} \mathcal{H}^i(\mathcal{M}) \leq -i \quad \forall i$$

(ii) cosupport condition

$$\dim \operatorname{supp} \mathcal{H}^i(\mathbb{D}(\mathcal{M})) \leq i \quad \forall i$$

These 2 conditions define a t-structure on $D^b_c(\mathbb{Q}_X)$ and we define

$$\boxed{\operatorname{Perv}(X) := D^b_c(\mathbb{Q}_X)^{\heartsuit}}$$

⇙

Corollary: $D^b_c(\mathbb{Q}_X)^{\heartsuit}$ is an abelian category.

Proposition: on an LG-pair (V, f) the functors ${}^p\psi_f$ and ${}^p\phi_f$ are t-exact so induce

$$ {}^p\psi_f, {}^p\phi_f : \mathrm{Perv}(V) \to \mathrm{Perv}(V_0)$$

Thm: $\mathcal{U} \longmapsto \mathrm{Perv}(\mathcal{U})$ defines a stack on X (When X is reduced .... ← why do we need this hypothesis?)

In particular, this means that if $X_\alpha \subseteq X$ is a covering, then

$$\mathrm{Perv}(X) \xrightarrow{\sim} \varprojlim \left(\mathrm{Perv}(X_\alpha) \rightrightarrows \mathrm{Perv}(X_{\alpha\beta}) \overset{\rightarrow}{\rightrightarrows} \cdots \right)$$

ie: to define a perverse sheaf on X

all we need is

- $\forall \alpha$, $P_\alpha \in \mathrm{Perv}(X_\alpha)$

- $\forall_{\alpha,\beta} \quad \sigma_{\alpha\beta} : P_\alpha|_{\alpha\beta} \simeq P_\alpha|_{\alpha\beta}$

- $\forall_{\alpha,\beta,\gamma} \quad \sigma_{\gamma\alpha}\sigma_{\beta\gamma}\sigma_{\alpha\beta} = id$

△

only this data is missing to finish / garantee

the gluing in our case

$\rightsquigarrow$

we will come back to this in future talks

this gluing is possible if we have a square Root of the canonical bundle

Proposition: Let P & P' be such gluings. Then there exists a $\mathbb{Z}/2$-local system $\rho \in H^1(X, \mathbb{Z}/2)$ and then $P' = P \otimes \rho$.

Proposition: Let $\{\varepsilon_{\alpha\beta\gamma}\}$ be the data we saw this morning $\rightarrow$ the 2-coycle obstruction of the gluing of the $K_\alpha^\vee$ (talk # 6)

Then this coycle coincides with the ones of this talk:

$$\{\varepsilon_{\alpha\beta\gamma}\} \simeq \{\sigma_{\alpha\beta\gamma}\}$$

Main LEMMA of the paper

- (V,f) LG-pair
- $\mathrm{Cut}(f) \subset U \underset{\text{open}}{\subset} V$
- $\varphi : U \longrightarrow V$ biholomorphic onto its image.

with $f \circ \varphi = f$ and $\varphi|_{\mathrm{Cut}(f)} = \mathrm{id}$

then the isomorphism $\Sigma\ f\ \varphi$

is equal to $\det\left(d\varphi|_{\mathrm{Cut}(f)} \right) \cdot \mathrm{id}$

..

$$\begin{tikzcd} P_f \arrow[d] \\ P_f \end{tikzcd}$$

Finally:

Main theorem: X orientable. Then the locally defined sheaves of vanishing cycles glue, in a unique way up to a twist by a $\mathbb{Z}/2\mathbb{Z}$ - local system.

if we fix a particular orientation, then the gluing is unique!

Talk #8 Motivic DT-invariants

(Kontsevich-Soibelman)

KS

starting point:

$$DT(X) = \int_{[X]^{vir}} \overset{\text{Behrend function}}{\tilde{\nu}_B}$$

KS: Replace all these things by their motivic analogue.

$$[X]^{mot} \quad \& \quad \nu_B^{mot}$$

KS input: start with a calabi-yau category $\mathcal{C}$ instead of a CY 3-fold.

$$\begin{tikzcd} & \text{triangulated} \\ \mathcal{C} \arrow[ur] \arrow[r] \arrow[dr] \arrow[dd] & \text{ind-constructible} \\ & A_\infty \\ 3CY & \end{tikzcd}$$

Example: (Q, ω) a Quiver with potential

$$\rightsquigarrow$$

Ginzburg algebra of $(Q,\omega) = G(Q,\omega)$

$$\rightsquigarrow$$

$$\underbrace{\mathrm{Perf}}_{\text{has a t structure}}\big(G(Q,\omega)\big)$$

$\downarrow$ can exhact is $\boxed{\text{heart}}$

$\qquad\qquad ||$

abe. Representation cat. of the Jacobi algebra

IN our case, we take

② , $\mathcal{M}$ = moduli space of objects in $\mathcal{C}$

$\mathcal{M}$ = moduli space of Quiver with potential.

③ stability conditions

$$Z : K_0(\mathcal{C}) \longrightarrow \mathbb{C}$$

Exemple :

$$\tilde{\mathcal{C}} = \underset{\sim}{\langle} E_1, E_2 \rangle$$

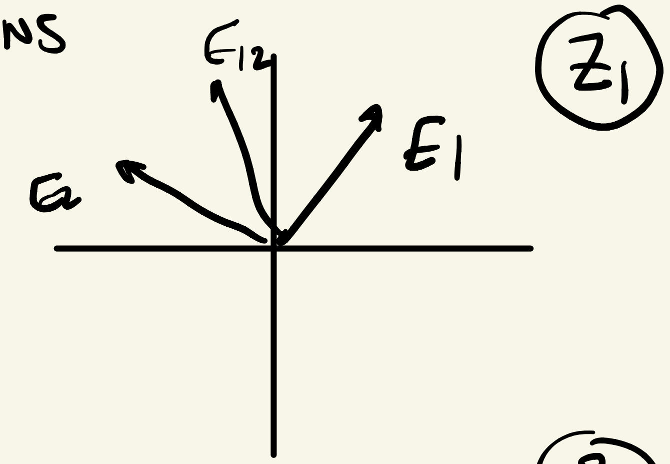
generated by 2 objects

+ extensions

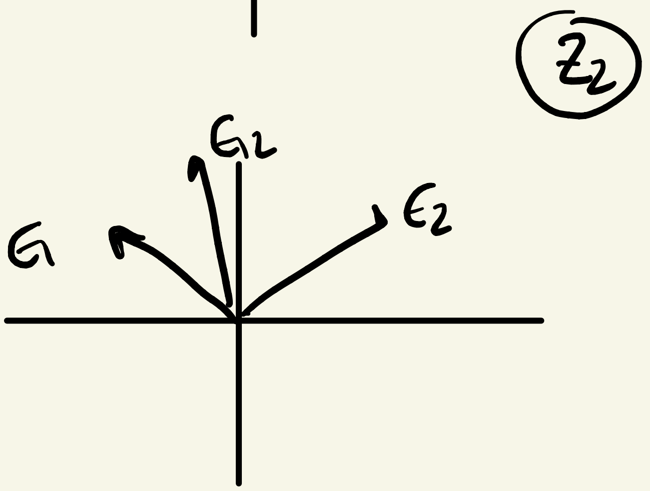
$$0 \to E_1 \to E_{12} \to E_2 \to 0$$

in this case ↓

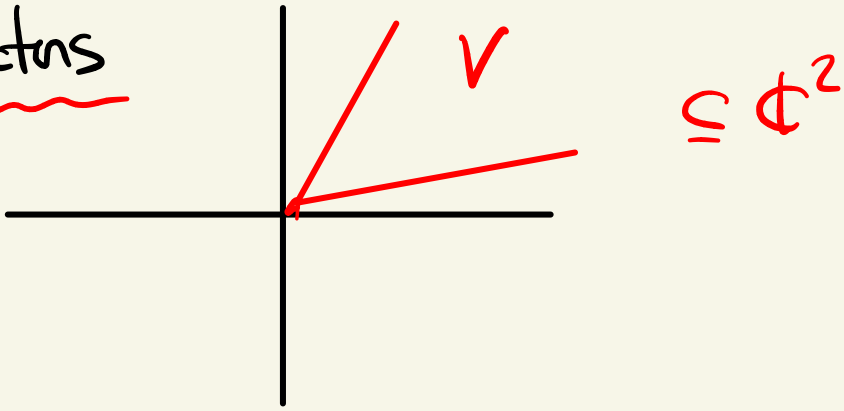
stability conditions are given by



associate to these stability conditions DT-invariants.



KS consider sectors



and to each sectr KS associate

$$\underbrace{\mathcal{C}_V}_{\text{generators given by semi-stable objects in } V.} \subseteq \mathcal{C}$$

---

④ $A_V \in H(\mathcal{C}_V)$ motivic Hall algebra.

$$K^{\hat{\mu}}\left(\text{stacks of} / \mathcal{M}_Q\right) - ?$$

characteristic cycle of $\mathcal{M}_{\mathcal{C}_V}$

---

⑤ Integration map (using motivic vanishing cycle)

$$I : H(\mathcal{C}_V) \longrightarrow \underbrace{R}_{\text{motivic Quantum torus.}}$$

$$\boxed{DT(\mathcal{C}_V) := I(A_V)}$$

Requires orientation data

given by a square root of $\sqrt{\det(\underline{\underline{Ext}}(E,E))}$

$E \in \mathcal{C}$

(tangent at the point E in the moduli of objects)

$\boxed{\text{Part } \underline{\text{II}}}$ Motives

invariants :

observation : χ_{sing} , Serre polynomial, #

1) $X \simeq Y$ then $\chi(X) \simeq \chi(Y)$

2) $\chi(X \times Y) \simeq \chi(X) \cdot \chi(Y)$

3) If $\underset{\text{closed}}{A \subseteq X}$ then

$$\chi(X) = \chi(A) + \chi(X \setminus A)$$

Definition : A generalized Euler characteristic is a Ring homomorphism

$$\bigoplus_{[X] \in \pi_0(Var)} \mathbb{Z}[X] \longrightarrow R$$

if it satisfies 3)

the Quotient

$$\boxed{\left.\bigoplus_{[x]\in\pi_0(\mathrm{Var})} \mathbb{Z}\cdot[x] \middle/ (3)\right. =: K_0(\mathrm{Var})}$$

is called the Grothendieck ring of varieties. its <u>elements</u> are called "motives"

<u>Rmk</u>: one can also do a <u>Relative</u> version of this $K_0(\mathrm{Var}/X)$

<u>Exemple</u>: of generalized euler char:

$$\begin{aligned} K_0(\mathrm{Var}/X) &\longrightarrow \mathrm{Const}(X,\mathbb{Z}) \\ [Y \xrightarrow{\pi} X] &\longmapsto \Big(x \mapsto \underbrace{\chi^c_{\mathrm{sing}}(\pi^{-1}(x))}_{\text{euler characteristic of the fiber.}}\Big) \end{aligned}$$

Exeple

$$[\mathbb{A}^1] =: \mathbb{L} \qquad [\mathbb{P}^1] = [\mathbb{L}]+1$$

$$[\mathbb{P}^n] = [\mathbb{P}^{n-1}] + [\mathbb{A}^n] = \sum_{i=0}^{n} \mathbb{L}^i$$

Proposition $f : Z \to Y \xrightarrow{g} W$

then $f^*[X \to Y] = [Z \times_Y X \to X]$

$g_*[X \to Y] = [X \to Y \to W]$.

Generalization: $\mathcal{M}$ Artin stack of f.type.

We can also define these Grothendieck rings

only those with affine stabilizers $\underline{\underline{G}}$

$$K_0\left(\text{stacks}/\mathcal{M}\right)$$

Moreover, if we assume that $\mathcal{M}$ is a commutative monoid in stacks (such as $B\mathbb{G}_m$)

then we have a new ring structure

$$\begin{bmatrix} X \\ \downarrow \\ M \end{bmatrix} \bullet \begin{bmatrix} Y \\ \downarrow \\ M \end{bmatrix} \longmapsto \begin{bmatrix} X \times Y \\ \downarrow \\ M \times M \\ \downarrow m \\ M \end{bmatrix}$$

$$\Downarrow$$

can also do symmetric powers

(make $K_0(/M)$ a λ-Ring)

can also include G-equivariance

$$K^G(-/M) = \frac{K_0^G(-/M)}{\langle [X \to Y \to M] - \coprod^R [Y \to M]}$$

X vect bndle over Y of rank R.

Proposition $K^G(\mathrm{Stck}^{aff}/M) \xrightarrow{\sim} K^G(\mathrm{Var}/M)[Gl_n, u^{-1}]$

Inertia stack?

Proof: unclear

Motivic vanishing cycles

Slogan: "Vanishing cycle $= [f^{-1}(0)] - [f^{-1}(1)]$"

Monodromy: $M : H^\bullet(MF_x, \mathbb{Q}) \longrightarrow H^\bullet(MF_x, \mathbb{Q})$

(Eigenvalues are roots of unity)

Definition the monodromy Grothendieck Ring

is $K^{\hat{\mu}}(\mathrm{Var}/X) = \operatorname{colim}_{n} K^{\mu_n}(\mathrm{Var}/X)$

Given this we can construct the motivic vanishing cycle

Construction:

$$\begin{tikzcd} X \arrow[r, "f"] & \mathbb{A}^1 \\ f^{-1}(0) \arrow[u, phantom, sloped, "\subseteq"] & \end{tikzcd}$$

smooth

$f^{-1}(0) = X_0$

$$\pi : Y \to X$$

smooth and proper

$$Y \setminus \pi^{-1}(x_0) \to X \setminus x_0 \quad , \quad \pi^{-1}(x_0) = \cup E_i$$

divisor with normal crossings.

Normal crossing

$$E_I^\circ = \bigcap_{i \in I} E_i \Big\backslash \cup E_i$$

assume multiplicity $\mathrm{mult}(E_i) = m_i$

$$V \subseteq Y \quad \text{such that} \quad f \circ \pi = u z^{m_I}$$

invertible

$$m_I = \sum_{i \in I} m_i$$

Construct a cover

$$E_I^\circ \cap V$$

$$\tilde{E}_I^\circ \cap V = \{ (z, w) \in \mathbb{A}^1 \times (E_I^\circ \cap V) \Big| \underset{u^{-1}w}{\overset{\shortparallel}{z^{m_I}}}$$

$$\mathcal{M}_f^{mot} := 1 - \sum \left(1-\mathbb{L}\right)^{|I|-1} \left[\widehat{E_I^\circ} \to X_0\right]$$

$$\in K^{\hat{\mu}}\left(\mathrm{Var}/X_0\right)\left[\mathbb{L}^{-1/2}\right]$$

Exemple $\quad \mathbb{A}^1 \xrightarrow{z^n} \mathbb{A}^1$

$$\widehat{E_0} = \left\{ (z,u) \in \mathbb{A}^1 \mid z^n = 1 \right\} = \mu_n$$

$$\boxed{\mathcal{M}F_{z^n} = \left[\{0\} \to \{0\}\right] - \left[\mu_n \to \{0\}\right]) = 1 - \left[\mu_n\right]}$$

---

back to Behrend function

$$f : X \to \mathbb{A}^1, \quad Z = [df = 0]$$

Relative virtual motive

$$[Z]_{\substack{\text{relative}\\\text{virtual}}} = \mathbb{L}^{-\frac{1}{2}\dim X} \left[\mathcal{M}_f^{mot}\right]$$

$$\in K^{\mu}\left(\mathrm{Var}/Z\right)$$

then the fibrewise Euler characteristic of $[Z]_{\mathrm{Rel\,virt}}$ is the Behrend function.

Exemple: back to the exemple of $(\mathbb{A}^1, z^n)$

we get

$$[\{0\}]_{\mathrm{R.virt}} = \mathbb{L}^{1/2}\left(1 - [\mu_n]\right)$$

Talk #9

Motivic DT-invariants for Quivers with Potential.

① Quivers, Jacobian algebra and associated moduli

② Motivic DT-partition function

$\downarrow$ "log"

$BPS_{Q,W}$ invariants

③ <u>Examples</u> : Hall algebra & Integration

$\Downarrow$

wall crossing

---

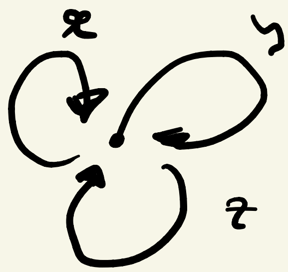
① Q = oriented graph

$\parallel$

(Q_0, Q_1)

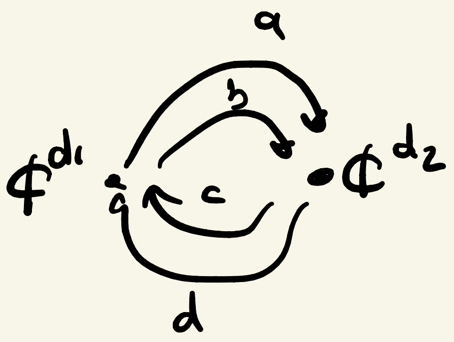
$\nearrow$ vertex $\uparrow$ edges

or



with $\mathbb{C}Q$:= path algebra.

We want to study representations of $\underline{\mathbb{C}Q}$.



Let $d=(d_i)$ be the "dimension vector".

Set $\mathcal{M}_d(Q)$ be the moduli of representations

$$\parallel$$

$$\left[\mathrm{Rep}_d(Q) \Big/ G_d(Q) \right]$$

where $\mathrm{Rep}_d(Q) = \prod \mathrm{Hom}(\mathbb{C}^{d_i}, \mathbb{C}^{d_j})$

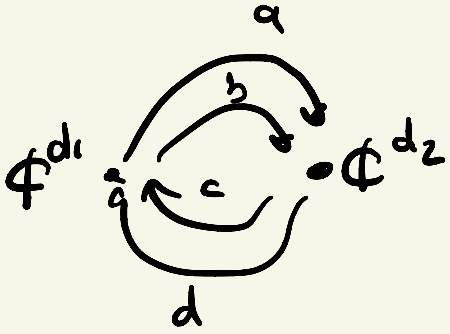
$$G_d(Q) = \prod_{i \in Q_0} Gl_{d_i}$$

$W \overset{df}{=}$ potential $:=$ linear combination of cycle elements

Exeple :

$$\boxed{W = acbd - adbc}$$

in



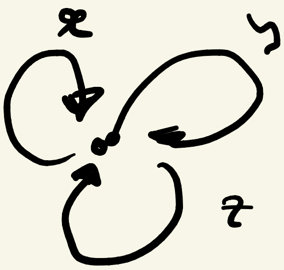
If c is a cyclic element

then $$\frac{\partial c}{\partial a} = \sum_{c = c' a c''} c''.c'$$

$a \in Q_1$

This remains cycls if a is a cycle

Exeple



$$W = xyz - xzy$$

$$\frac{\partial W}{\partial x} = yz - zy$$

$$= [y, z]$$

$$\frac{\partial W}{\partial y} = [z, x]$$

$$\frac{\partial W}{\partial z} = [x, y]$$

Then

Definition Jacobian algebra $(Q, W) = \dfrac{\mathbb{C}Q}{\langle \frac{\partial W}{\partial a} \mid a \in Q_1 \rangle}$

then:

$\mathcal{M}_d(Q, W) =$ moduli of representations of $\mathrm{Jac}(Q, W)$

Proposition

$$\mathcal{M}_d(Q, W) \simeq \left[\mathrm{Crit}(\mathrm{tr}(W_d)) \,/\, GL_d \right]$$

$$\mathrm{Tr}(W_d) : \mathrm{Rep}_d(Q) \longrightarrow \mathbb{C}$$

Stability conditions:

$$\xi : \mathbb{N}^{Q_0} \longrightarrow \mathbb{Z}$$

Notation : $\quad \xi_i = \xi(0, 0, \dots, \underbrace{1}_{i}, 0, \dots)$

Definition : If V_d is a rep. of Q of dim $d = (d_i)$ Then we can define the slope

$$\mu(V_d) := \frac{\sum \xi_i d_i}{\sum d_i}$$

We say V_d is semi-stable if $\forall\, V \subseteq V_d$

we have $\mu(V) \leq \mu(V_d)$

then we define

$$\mathcal{M}_d^{\xi\text{-semistable}}(Q) = Rep_d^{\xi\text{-ss}}(Q) \big/ G_d$$

and we have.

$$\mathcal{M}_d^{\varepsilon\text{-ss}}(Q,W) = \frac{\mathrm{Crit}(\mathrm{tr}^{\varepsilon}(W_d))}{G_d}$$

Part II : Motivic DT- partition function.

$$f : \underset{\substack{\downarrow \\ \text{smooth}}}{X} \to \mathbb{C} \quad \text{then} \quad \underbrace{[\mathrm{Crit}(f)]_{\mathrm{Relvir}} , [X]_{RV}}_{\text{virtual}}$$

Talk #8 $\quad \cap$

$$K^{\hat{\mu}}(\mathrm{Var}/k)[\mathbb{L}^{-1/2}]$$

If we define

$$\left[\frac{X}{G}\right]_{RV} := \frac{[X]_{RV}}{[G]_{RV}}$$

talk #8

and

$$\left[\frac{\mathrm{Crit}(f)}{G}\right]_{RV} = \frac{(\mathrm{Crit}(f))}{[G]} := \frac{\mathbb{L}^{-\frac{\dim X}{2}}\phi_f}{[G]} \in K^{\mu}(\mathrm{Var}/k)[\mathbb{L}^{-1/2}]$$

and $\left[\frac{X}{G}\right]_{\text{virtual}} \in K^{\hat{\mu}}(\mathrm{Sch}/k)[\mathbb{L}^{-1/2}, \mathrm{GL}_n^{-1}] =: R$

then we can define the motivic partition function

$$A(Q,W) := \sum_{d\in\mathbb{N}^{Q_0}} [\mathcal{M}_d(Q,W)]_{\mathrm{vir}}\, \underline{T}^d$$

since this is a global quotient crit/G

$$= \sum [\mathrm{crit}/G]_{\mathrm{vir}}\, \underline{T}^d$$

$$= \sum \frac{[\mathrm{crit}]}{[G]}_{\mathrm{vir}}\, \underline{T}^d.$$

$\rotatebox{90}{$\in$}$

$R[[\{t_i\}_{i\in Q}]]$ formal power series

We have

$$\left\{ P \in R[[t_i]] \,,\, P(0)=0 \right\} \xrightarrow{\mathrm{Sym}} \left\{ P \in R[[T]] : \underset{P(0)=1}{} \right\}$$

$a \in R$

$$\mathrm{Sym}(a) = \sum_{i=0}^{\infty} \mathrm{Sym}^n(a)$$

$$\mathrm{Sym}(\underline{t}^d) = \sum_i t^{id}$$

$$\mathrm{Sym}(0) = 1$$

$$\mathrm{Sym}(f+g) = \mathrm{Sym}\, f \cdot \mathrm{Sym}\, g$$

Definition

$\mathrm{BPS}_d(Q,W) \in R$ are unique elements such that:

$$-\boxed{\mathrm{Sym}\left(\frac{\sum_{d \in \mathbb{N}^{Q_0} \setminus \{0\}} \mathrm{BPS}_d(Q,W)\, t^d}{\mathbb{L}^{1/2} - \mathbb{L}^{-1/2}} \right) = A(Q,W)}$$

$\int$ for physics of BPS → see [Dimofte - Gukov]

$\downarrow$ Now with stability conditions

given ξ, slope μ

$$A^{\xi}_{\mu}(\mathbb{Q}, w) = \sum_{d} [M_d^{\xi\text{-}ss}(\mathbb{Q}, w)]_{vir} t^{d}$$

$\int$ BPS_d^{ξ} are also defined to be the terms in the logarithm of $A^{\xi}_{\mu}(\mathbb{Q}, w)$

Question: is BPS_d^{ε} represented by some moduli space?

Exemples

① $Q =$  $W = 0$

$$A(Q) = \sum_{d\in\mathbb{N}} [\mathcal{M}_d(Q)]_{vir}\, t^d$$

$$= \sum_{d\in\mathbb{N}} \frac{[Rep_d(Q)]_{vir}}{[Gl_d(Q)]}\, t^d$$

$\mathbb{L} = \mathbb{A}^{1\,d^2}$

matrices

d×d matrices

$$= \sum_{d\in\mathbb{N}} \frac{\mathbb{L}^{d^2}}{(\mathbb{L}^d - \mathbb{L}^{d-1})\cdots(\mathbb{L}^d - 1)}$$

$$= \mathrm{Sym}\left(\frac{\mathbb{L}^{1/2}\cdot t}{\mathbb{L}^{1/2} - \mathbb{L}^{-1/2}}\right)$$

So in this case:

$$\boxed{\begin{array}{l} BPS_1 = \mathbb{L}^{1/2} \\ BPS_d = 0 \quad \forall d > 1 \end{array}}$$

can have a matrix fact version?

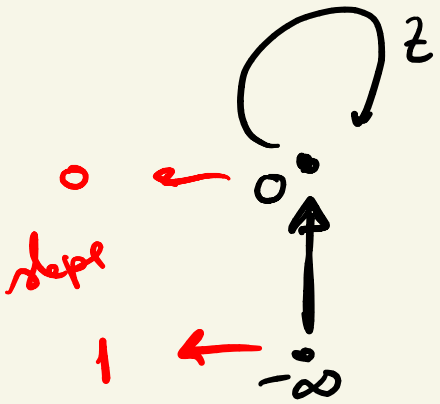
can find categorified $MF(\mathbb{A}^1, x^2)$?

Rmk: have $\mathcal{M}_d^{\text{stable}}(Q) \simeq 0$ if $d>1$

$\boxed{\mathcal{M}_d^{ss}(Q) = \mathbb{A}^1 \text{ when } d=1}$

② $Q = \bullet \circlearrowleft^{z}$, $W = z^n$.

[Davison - Meinhardt] Modify the Quiver by adding a point at ∞



Define

$$U \subseteq \mathrm{End}(\mathbb{C}^d) \times \mathbb{C}^d$$

$$\parallel$$

$$\{(A,v) : \langle v, Av, .., A^{d-1}v \rangle = \mathbb{C}^d\}$$

In this case

$$\mathcal{M}^{\text{framed}}_{(1,d)}(Q) := U/GL_d \simeq \mathrm{Hilb}^d(\mathbb{A}^1)$$

dim 1 at ∞

dim d at 0

variable z

In this case the potential comes from.

$$\begin{tikzcd}
 & & \mathrm{trace}(A^n) & \\
A \arrow[urr, bend left, mapsto] & \mathrm{Hilb}^d(\mathbb{A}^1) \arrow[rr] & & \mathbb{C} \\
(\lambda_1,\dots,\lambda_d) \arrow[u, mapsto] & \mathrm{Sym}^d(\mathbb{A}^1) \arrow[u, "\wr"] \arrow[urr] & & z_1^n + \dots + z_d^n \\
 & (z_1,\dots,z_d) \arrow[urr, mapsto] & & \mathrm{Sym}^d_+(z^n) \arrow[u, phantom, "\parallel"]
\end{tikzcd}$$

We have $z^n : \mathbb{A}^1 \longrightarrow \mathbb{A}^1$

$$\mathrm{Sym}^d(\mathbb{A}^1) \xrightarrow[\mathrm{Sym}^d(z^n)]{} \mathbb{A}^1$$

$$\mathrm{Sym}^d(z^n) = z_1^n + \dots + z_d^n$$

so $$\boxed{\phi_{\mathrm{Sym}^d(z^n)} = \mathrm{Sym}^d(\phi_{z^n})}$$

[Kontsevich- Soibelman]

take (Q,W) Quiver with potential

take $\mathfrak{M} := \prod_{d \in \mathbb{N}^{Q_0}} \mathfrak{M}_d$ moduli space of all representations

and define the Integration map

$$I^W : K\left(\mathrm{St}^{\mathrm{aff}}/\underline{\mathfrak{M}}\right) \xrightarrow{I} R[[t]]$$

no potential

$$[X \xrightarrow{f_d} \mathfrak{M}] \longmapsto \sum_{d \in \mathbb{N}^{Q_0}} p_* f_d^* [\mathrm{crit}(\mathrm{tr}\, W_d)]_{\mathrm{Rvir}} \, t^d$$

$$\downarrow p$$
$$*$$

~ no potential

the integration map depends on the potential!

This depends on the potential

Example: $\left[\begin{matrix}\mathfrak{M} = \mathfrak{M} \\ \downarrow p \\ * \end{matrix}\right]$ then $\xmapsto{I^W}$ this is $A(Q, W)$

(because $p_*[-]_{RV} = [-]_{\mathrm{virtual}}$)

can also do

$$[\mathfrak{M}_\mu^{\xi\text{-st}} \xrightarrow[\text{forget}]{} \mathfrak{M}] \xmapsto{I^W} A_\mu^\xi(Q, W)$$

Comment: In this case injection map exists because we have a global potential

---

$$[\mathcal{M}_d(Q) \times \mathbb{C}^d] = \sum_{i=0}^{d} [\mathcal{M}^{\text{framed}} \to n] \cdot [\mathcal{M}_{d-i} \to n]$$

$\Rightarrow$ replace t by $\mathbb{L}t$

$$A(Q,W)\cdot(\mathbb{L}.t) = \Big(\underbrace{\sum \mathrm{Sym}^i(\phi_Z n)}\Big) A(Q,W)$$

$$\Downarrow$$

$$\mathrm{BPS}_d(Q,W) = \begin{cases} \mathbb{L}^{-1/2}[\Gamma/\mu_n] & \text{if } d=1 \\ 0 & \text{otherwise} \end{cases}$$

Main theorem (Davison)

(BPS) = vanishing cohomology

talk #10

overview on Derived alg. Geometry to moduli Spaces

No Motivation: if you are here, you are already motivated!

① Derived schemes

Definition cdga ≤ 0

Commutative differential graded algebras

$$A = \bigoplus_{i \geq 0} A^{-i} \quad \text{(positively graded)}$$

$$A^{-i} \times A^{-j} \longrightarrow A^{-j-i} \quad \text{multiplication}$$

$$A = \left[\cdots \longrightarrow A^{-2} \xrightarrow{d} A^{-1} \xrightarrow{d} A^{0} \right]$$

with sign rule

$$ab = (-1)^{|a||b|} ba$$

where $a \in A^{-|a|}$ $b \in A^{-|b|}$

and $d(ab) = (da)b \pm a\,db$

---

idea: $\mathrm{cdga}_{\leq 0}$ are affine derived schemes

$\mathbb{R}\mathrm{Spec}(A) \leftarrow$ Notation.

Construction $\underbrace{A^0/dA^{-1} = H^0(A)}$ is a classical ring

$\mathrm{Spec}(H^0(A))$ is called the truncation of $\mathbb{R}\mathrm{Spec}(A)$

Definition: A derived scheme $=$

$$\Big(\underset{\text{top. space}}{X}, \underset{\text{sheaf of cdga's}}{\mathcal{O}_X^{\bullet}}\Big)$$

↳ with 2 conditions:

- $(X, H^0(\mathcal{O}_X))$ is a classical scheme

- $H^{-i}(\mathcal{O}_X^\bullet)$ is a quasi-coherent over $H^0(\mathcal{O}_X)$

Rmk we have closed immersions

$$\begin{tikzcd}
t^0(X) \arrow[r, hook, "i"] \arrow[d, equal] & X \\
(X, H^0(\mathcal{O}_X)) &
\end{tikzcd}$$

Example: Koszul complex

$$\begin{tikzcd}
E \arrow[d, shift right] \\
X \arrow[u, shift right, "s"']
\end{tikzcd}$$

E vect bndle

X smooth scheme

$$E = \operatorname{Spec}(\operatorname{Sym}_{\mathcal{O}_X}(\mathcal{E}^\vee))$$

$$s : \operatorname{Sym}_{\mathcal{O}_X}(\mathcal{E}^\vee) \longrightarrow \mathcal{O}_X$$

as $\mathcal{O}_X$-algbs

$$\Leftrightarrow \quad \mathcal{E}^\vee \xrightarrow{s^\#} \mathcal{O}_X$$

as $\mathcal{O}_X$-mod

can form the Koszul complex

$$\underbrace{\Lambda^{\mathrm{rnk}}\mathcal{E}^\vee \cdots \to \Lambda^3\mathcal{E}^\vee \to \Lambda^2\mathcal{E}^\vee \xrightarrow{i_{s^\#}} \mathcal{E}^\vee \xrightarrow{s^\#} \mathcal{O}_X}$$

this is a sheaf of cdga's $/X$ $\quad \mathrm{Kos}(\mathcal{E}, s)$

If we compute the function

$$H^0 = \mathcal{O}_X \big/ \mathrm{Im}\, s^\# = \underbrace{\mathcal{O}_{Z(s)}}$$

functions on the zero locus of s.

$$\Downarrow$$

$$\begin{tikzcd}
Z(s) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "0"] \\
X \arrow[r, "s"'] & E
\end{tikzcd}$$

So $\mathbb{R}\mathrm{Spec}_X\left(\mathrm{Kos}\left(E, s\right)\right)$ is a derived

$$\underbrace{\text{scheme with function. } Z(s)}_{\text{derived zero locus.}}$$

② How to construct derived schemes?

↳ 3 canonical ways → Ⓒ Key lemma

Ⓐ fiber products (derived) of usual schemes (Koszul resolutions)

Ⓑ derived mapping spaces of usual schemes

Ⓐ

$$\begin{tikzcd} X \times^h_Z Y \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & Z \end{tikzcd}$$

$\times^h$ = homotopical fiber product.

$t_0\left(X \times^h_Z Y\right) :=$ usual fiber product

compute $\mathcal{O}^{\text{classical}}_{X\times_Z Y} = \mathcal{O}_X \overset{\text{classical}}{\underset{\mathcal{O}_Z}{\otimes}} \mathcal{O}_Y$

$$\mathcal{O}_{X \times^h_Z Y} := \mathcal{O}_X \overset{\mathbb{L}}{\underset{\mathcal{O}_Z}{\otimes}} \mathcal{O}_Y$$

have to derive the classical functor $\otimes$

Example (of the bundle)

$$\begin{tikzcd} E \arrow[d, shift right] \\ X \arrow[u, shift right, "s"'] \end{tikzcd}$$

We want to compute

$$\begin{tikzcd} \mathbb{R}Z(s) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner h"] & X \arrow[d, "0"] \\ X \arrow[r, "s"'] & E \end{tikzcd}$$

in particular

$$t(\mathbb{R}Z(s)) = Z(s)$$

Now we can compute

$$\boxed{\mathbb{R}Z(s) = \mathbb{R}\mathrm{Spec}_X(\mathrm{Kos}(E, s))}$$

We can compute its <u>tangent space</u>

$$\mathbb{T}_{\mathbb{R}Z(s)} = \left[TX\big|_{Z(s)} \xrightarrow{ds} \mathcal{E}\big|_{Z(s)} \right]$$

<u>Subexample</u>: <u>Critical loci</u>

$f : X \to \mathbb{A}^1$, $E = T^\vee X$, $s = df$

then

$$\boxed{\mathbb{R}\mathrm{Crit}(f) := \mathbb{R}Z(df)}$$

$$\mathbb{T}_{\mathbb{R}\mathrm{Crit}(f)} = \Big[\underset{0}{T_X} \xrightarrow{H_f} \underset{1}{T^\vee X} \Big]$$

$$\mathbb{L}_{\mathbb{R}\mathrm{Crit}(f)} = \mathbb{T}^\vee = \Big[\underset{-1}{TX} \xrightarrow{H_f^\vee} \underset{0}{T^\vee X} \Big]$$

and because $\underline{H_f^\vee = H_f}$

$$\boxed{\mathbb{L}_{\mathbb{R}\mathrm{Crit}(f)}[-1] \simeq \mathbb{T}_{\mathbb{R}\mathrm{Crit}(f)}}$$

↓ (-1) shifted symplectic structure

Deeper

$$\begin{tikzcd}
\mathbb{R}\mathrm{Crit}(f) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner h"] & X \arrow[d, "df"] \\
X \arrow[r, dash, "0"'] & \underset{\sim}{T^*X}
\end{tikzcd}$$

Lagrangian

0-symplectic

Lagrangian

Result: Intersection of Lagrangians is symplectic with a shift

slogan:

$$\boxed{X \overset{h}{\underset{Z}{\times}} Y = X \underset{Z}{\times} Y + \mathrm{Tor}^{i\text{'s}}_{\mathcal{O}_Z}(\mathcal{O}_X, \mathcal{O}_Y)}$$

Ⓑ Derived Mapping Spaces

$X, Y / S$ two classical schemes

$\underline{\mathrm{Maps}}^{\mathrm{classical}}_{/S}(X, Y)$ can be made into an algebraic space.

by

$$\underbrace{\begin{array}{c} T \\ \downarrow \\ S \end{array} \longmapsto \overset{\text{set}}{\mathrm{Maps}_S}(X \times_S T, Y)}$$

$\mathrm{Maps}^{cl}_S(X,Y)$ is a $\underline{\text{scheme}}$

when X & Y are good.

$\swarrow$ $\underline{\text{proof}}$: use the graph to embedd into the $\underline{\underline{\text{Hilbert scheme}}}$.

now we can also do a derived version of this

$$\boxed{\begin{array}{l} \mathbb{R}\mathrm{Map}_S(X,Y) \\ \equiv \text{derive the functor } \mathrm{Maps}^{cl}_S(X,Y) \end{array}}$$

claim $t_0(\mathbb{R}\mathrm{Map}) = \mathrm{Map}$

slogan $\mathbb{R}\mathrm{Map} = \mathrm{Map} + \text{"}\mathrm{Ext}^i\text{"}$

↳ all examples of GW & DT will be defined as '<u>open</u>' in $\underline{\mathbb{R}\mathrm{Map}}$'s

Ⓒ (Schürg-Toën-Vezzosi)

$$\begin{tikzcd}
\underset{\text{classical}}{\mathcal{M}} \arrow[r, phantom, "\overset{\text{truncation}}{\subseteq}"] & \underset{\text{derived}}{\underbrace{\mathbb{R}\mathcal{M}}} \\
\text{open } \rotatebox{90}{$\subseteq$} & \rotatebox{90}{$\subseteq$} \text{ open} \\
U \arrow[r, phantom, "\underset{\text{truncation}}{\subseteq}"] & \exists!\ \underset{\text{derived enhancement of } U}{\underbrace{\mathbb{R}U}}
\end{tikzcd}$$

Exemples

$\boxed{\text{GW theory}}$

X smooth proj $/\mathbb{C}$

$$\overline{\mathcal{M}}_{g,n}(X,\beta) \quad \text{DM-stack}$$

$$\| $$

$$\left\{ (C, x_1, \dots, x_N) \xrightarrow{f} X \ , \ \begin{array}{l} f_*[C]=\beta \\ C \text{ nodal} \\ Aut(C_i, f_i) < \infty \end{array} \right\}$$

define

$$\mathbb{R}M := \mathbb{R}\mathrm{Map}_{\mathfrak{M}_{g,n}}\left(\underset{\sim}{\mathcal{C}_{g,n}} \ , \ X \times \mathfrak{M}_{g,n} \right)$$

classical moduli prestable curves

universal pre-stable curve (this is flat) $/\mathfrak{M}_{g,n}$

⚠

(this is only in this case derived fiber product = classical fiber product)

canonical derived enhancement

$$M = t_0(\mathbb{R}M)$$

apply the lemma

$$\begin{tikzcd}
\mathbb{R}M & \boxed{\exists\ \mathbb{R}\overline{M}_{g,n}(X,\beta)} \arrow[l, hook] \arrow[d, dashed] \\
M \arrow[u, hook, "\text{truncation}"] & \overline{M}_{g,n}(X,\beta) \arrow[l, hook, "\text{open}"']
\end{tikzcd}$$

canonical derived enhancement

We can also compute the tangent space

$\mathbb{R}Map_S(X,Y)$ (Lurie representability theorem)

derived Artin stack → PhD thesis

$$\boxed{\mathbb{T}_{\mathbb{R}Maps(X,Y)/S} = prj_{*}\, ev^{*}\, \mathbb{T}_Y}$$

where

$$\begin{tikzcd}
\mathbb{R}Maps(X,Y) \times X \arrow[r, "ev"] \arrow[d, "proj"] & Y \\
\mathbb{R}Map_S(X,Y) &
\end{tikzcd}$$

In the case of stable curves

$$\Pi_{\mathbb{R}\overline{M}_{g,n}(X,\beta)/\overline{M}_{g,n}} = \underbrace{p_{vj*}\, ev_v^{*}\, \Pi_X}_{\text{PoT in BF}}$$

Relation from DAG to PoT

$$\mathcal{M} \overset{j \ \text{function}}{\subseteq} \mathbb{R}M \quad \text{derived enhancement}$$

LEMMA: $j_* : G_0(\mathcal{M}) \xrightarrow{\sim} G_0(\mathbb{R}M)$

then by definition

$$[\mathcal{O}_M^{\text{vir, DAG}}] = (j_*)^{-1}[\mathcal{O}_{\mathbb{R}M}] = \sum (-1)^i \mathcal{H}^i(\mathcal{O}_{\mathbb{R}M})$$

this sum is only well defined if the moduli space is qusismooth.

$\mathbb{R}M$

Proposition (Key) $M \underset{j}{\hookrightarrow} \underset{\sim}{\mathbb{R}M}$

(Lurie) then

$$j^\vee \mathbb{L}_{\mathbb{R}M} \xrightarrow{Dj^*} \mathbb{L}_M \iff \mathbb{R}M$$

is a PoT quasi smooth

Behrend-Fantechi construction

with the PoT $j^\vee \mathbb{L}_{\mathbb{R}M} \longrightarrow \mathbb{L}_M$

$$\begin{tikzcd}
M \arrow[r, hook] \arrow[dr, bend right] \arrow[rr, hook, bend left] & \mathbb{R}M^{PoT} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner h"] & C_M \arrow[d] \\
& M \arrow[r, "0"] & t_0(\mathbb{V}(j^*\mathbb{L}^\vee_{\mathbb{R}M}))
\end{tikzcd}$$

$$'' \quad h^1/h^0(E)$$

another derived enhancement of M

claim

$\mathbb{R}M^{POT}$ & $\mathbb{R}M$

are two derived enhancements of M $\quad j^{POT} : M \hookrightarrow \mathbb{R}M^{POT}$

Thm: $$\mathcal{O}^{vir,POT}_{BF,loc} := (j_*^{POT})^{-1}(\mathcal{O}_{\mathbb{R}M^{POT}})$$

the existence of a retract tell us that the tangent space of $\mathbb{R}M^{POT}$ splits

Proposition (Kapranov - Fontanine, Schurg - Toën - Vezzosi)

Proposition (Kapranov - Fontanine, Schurg -lowney)

$$\boxed{\mathcal{O}^{vir,DAG} = \mathcal{O}^{vir,POT}}$$

in $G_0(M)$

(Rmk : not all PoT come from a
derived enhancent)

(schrg)

$\boxed{\text{Talk \#11}}$ Intro to DAG

Basic blocks $\quad \text{cdga}^{\le 0}_{\mathbb{C}}$

Serre intersection formula

$$\begin{tikzcd} Y \arrow[dr, hook, "\text{closed}"] & \\ & X \\ Z \arrow[ur, hook] & \end{tikzcd} \qquad W \subseteq Y \times_Z X$$

irr. component.

$$\underbrace{i(X, Y, Z, W)}_{\text{multiplicity}} = \sum_{i \ge 0} (-1)^i \, \text{length}_{O_{X,W}} \left(\text{Tor}_i^{O_{X,W}}(O_{Y,W}, O_{Z,W}) \right)$$

$$\| $$

$$\chi \left(\underbrace{O_{Y,W} \overset{\mathbb{L}}{\otimes}_{O_{X,W}} O_{Z,W}}_{\text{connective cdga}} \right)$$

$$F : \underbrace{\text{dAff}_{\mathbb{C}}^{op}}_{\text{cdga}^{\le 0}} \longrightarrow S = \left(\begin{array}{c} \text{modelled by} \\ \text{simplicial sets} \\ \text{or} \\ \text{CW-complexes} \end{array} \right)$$

- when is this functor a sheaf ?

Definition : derived stacks $:= Sh_\infty(dAff)$

$$\{U^\bullet \to X\} \quad , \quad F(X) \simeq \underleftarrow{\mathrm{holim}}\, F(U^\bullet)$$

From classical stacks to higher stacks to derived stacks

$$G : Aff^{op} \longrightarrow \text{groupoid} \xrightarrow{N} S$$

$$\begin{tikzcd} \text{Stack} \arrow[r, hook, "\text{inclusion}"', "i"' near end] & \text{derived stacks} \arrow[l, bend right, "\text{truncation to}"'] \end{tikzcd}$$

i

i does not commute with limits / neither with mapping stacks.

Quasi-coherent sheaves

$$\boxed{QCoh(\mathbb{R}Spec\, A) := dgMod_A}$$

we take as a definition on any derived stack F

$$\boxed{\mathrm{Qcoh}(F) := \underset{\overleftarrow{\ \mathrm{Spec}\,A \to F\ }}{\mathrm{holim}}\ \mathrm{dgMod}_A}$$

all categories are infinite

this is where the cotangent complex

$$\begin{tikzcd} \mathbb{R}\mathrm{Spec}(B) \arrow[d, "f"] \\ \mathbb{R}\mathrm{Spec}\,A \end{tikzcd} \rightsquigarrow \quad \exists\ \mathbb{L}_{B/A} \in \mathrm{dgMod}_B$$

where we have

$$\begin{tikzcd} B \arrow[r, phantom, "\simeq"] & Q_A B \\ A \arrow[u] \arrow[ur, "\text{cofibrant.}"'] & \end{tikzcd}$$

$$\mathbb{L}_{B/A} \simeq \Omega^1_{\underbrace{Q_A B}_{\text{resolution of } B \text{ as } A\text{-module}}} \otimes_{Q_A B} B$$

Def: $\mathbb{L}_F \in \mathrm{Qcoh}(F)$

Thm [Avramov]

$X \in \mathrm{Affsch}^{f.t}/\mathbb{C}$

$\rightsquigarrow$

$\mathbb{L}_{X/\mathbb{C}}$ either

- $[-1,0]$ & perfect $\iff$ X lci
- totally unbounded.

Rmk: • $H^0(\mathbb{L}_X) = \Omega_1^X$

$\mathbb{L}_{B/A} =$ can use Dold-Kan to B, take simplicial resolution by free algebras, apply Ω^1 levelwise and apply Dold-Kan^{-1}

Properties

① Base Change

$$\begin{tikzcd}
P \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[dr, phantom, "\lrcorner\, h" near start] & X \arrow[d, "g"] \\
Y \arrow[r, "f"'] & Z
\end{tikzcd}$$

then

$$(f')^* \mathbb{L}_g \simeq \mathbb{L}_{g'}$$

---

②

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr] & & Y \arrow[dl] \\
& Z &
\end{tikzcd}
\implies
\begin{tikzcd}
f^* \mathbb{L}_{Y/Z} \arrow[r] \arrow[d] & \mathbb{L}_{X/Z} \arrow[d] \\
0 \arrow[r] \arrow[ur, phantom, "\lrcorner" near end] & \mathbb{L}_f .
\end{tikzcd}$$

cofiber sequence

Exemples

$$\begin{tikzcd} Y = \mathrm{Spec}\left(\underbrace{\dfrac{k[x_1,\dots,x_n]}{(\underbrace{f_1,\dots,f_R}_{\text{regular sequence}})}}_{A}\right) \arrow[r, hook] & \mathbb{A}^N \arrow[d, equal] \\ & \underbrace{\mathbb{C}[x_1,\dots,x_n]}_{B} \end{tikzcd}$$

want to compute the cotangent complexes

$$\mathbb{L}_{Y/\mathbb{A}^n},\ \mathbb{L}_{Y/\mathbb{C}}$$

$$\begin{tikzcd} Y \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathbb{A}^N \arrow[d, "f_1,\dots,f_R"] \\ 0 \arrow[r, hook] & \mathbb{A}^R \end{tikzcd}$$

Compute $Q_B A$:= Take the Koszul complex

$$\begin{tikzcd} \Lambda^2 B^R \arrow[r, "e_i\wedge e_j \mapsto f_i e_i - f_j e_j"'] \arrow[d, dash] & B^{\oplus R} \arrow[r, "(e_i \mapsto f_i)"'] \arrow[d, dash] & B \arrow[r] \arrow[d] & 0 \arrow[d, dash] \\ 0 \arrow[r, equal] & 0 \arrow[r] & B/I = A \arrow[r] & 0 \end{tikzcd}$$

$\downarrow$ this vertical map is a quasi-isomorphism because the sequence is regular

$$\Omega^1_{Q_B A/B} = Q_B A\, \delta e_1 \oplus \cdots \oplus Q_B A\, \delta e_R$$

$$\Big\downarrow - \otimes_{Q_B A} A$$

$$\mathbb{L}_{Y/\mathbb{A}^n} = \left(A\,\delta e_1 \oplus \cdots \oplus A\,\delta e_R\right)[1]$$

$$\simeq I/I^2[1]$$

Now we want to compute $\mathbb{L}_{Y/\mathbb{C}}$

$$Q_B A \simeq B[\overset{-1}{e_1}, \ldots, \overset{-1}{e_R}]$$

then

$$Q_{\mathbb{C}} A = \mathbb{C}[\overset{0}{x_1}, \overset{\cdots}{\ldots}, \overset{0}{x_n}, \overset{-1}{e_1}, \overset{\cdots}{\ldots}, \overset{-1}{e_R}]$$

$$de_i = f_i$$

$$\mathbb{L}_{A/\mathbb{C}} \simeq \Omega^1_{Q_{\mathbb{C}} A/\mathbb{C}} \otimes_{QA} A =$$

$$= \left(\mathcal{O}_{\mathbb{C}}{}^{A}\,\delta x_i \oplus \dots \oplus \mathcal{O}_{\mathbb{C}}{}^{A}\,\delta e_i \right) \otimes_{\mathcal{O}_{\mathbb{C}}{}^{A}} A$$

$$= \left[A^{\oplus k} \longrightarrow A^{\oplus n} \right]$$

$$\delta c_i \longmapsto df_i$$

$$\simeq \left[I/I^2 \xrightarrow[\text{Jacobian}]{} i^* \Omega^1_{\mathbb{A}^n/\mathbb{C}} \right]$$

---

it would be easier to just compute directly from the derived fiber product

$$\begin{tikzcd} Y \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\boxed{dh}"] & \mathbb{A}^n \arrow[d, "f_1, \dots, f_k"] \\ 0 \arrow[r, hook] & \mathbb{A}^k \end{tikzcd}$$

is a derived fiber product because it is a regular sequence.

Exemple 2

$$\begin{tikzcd}
 & W \arrow[dl] \arrow[dr] \arrow[dd, "h"] & \\
Y = \mathrm{Spec}\Big(\overset{A_2}{\boxed{\frac{\mathbb{C}[x,y]}{x}}}\Big) \arrow[dr] & & Z = \mathrm{Spec}\Big(\underset{A_1}{\boxed{\frac{\mathbb{C}[x,y]}{y}}}\Big) \arrow[dl] \\
 & X = \mathrm{Spec}\Big(\underset{B}{\boxed{\frac{\mathbb{C}[x,y]}{xy}}}\Big) &
\end{tikzcd}$$

$$W = \mathbb{R}\mathrm{Spec}\Bigg(\underbrace{\overset{A_2}{\boxed{\frac{\mathbb{C}[x,y]}{x}}} \overset{\mathbb{L}}{\underset{\underset{B}{\boxed{\frac{\mathbb{C}[x,y]}{xy}}}}{\otimes}} \overset{A_1}{\boxed{\frac{\mathbb{C}[x,y]}{y}}}}_{\text{need to compute this}}\Bigg)$$

Find Resolution of A_1 over B

$\downarrow$ Koszul complex

$$\overset{-1}{}\overset{-2}{}$$

$B[\overset{-1}{t_1}, \overset{-2}{t_2}] = \cdots$

$dt_1 = y$

$dt_2 = x \cdot t_1$

$$\begin{tikzcd}
\arrow[r, "y"] & B \arrow[r, "\cdot x"] \arrow[d, dash] & B \arrow[r, "\cdot y"] \arrow[d, dash] & B \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] & 0 \arrow[r, dash] & B/(y) &
\end{tikzcd}$$

$\Downarrow$ quasi-iso

$\Downarrow$

$$C := A_1 \overset{4}{\underset{B}{\otimes}} A_2 = B[t_1, t_2] \underset{B}{\otimes} B/(x) \simeq \begin{bmatrix} B/x \\ \downarrow y \\ B/x \end{bmatrix}[t_2]$$

$$\xrightarrow{\not\simeq}\ \mathbb{C}[T_2] \qquad \underset{q.iso}{\simeq} B/(x,y)[t_2]$$

Then
$$H^p(C) = \begin{cases} \mathbb{C} & p \text{ even} \\ 0 & p \text{ odd} \end{cases}$$

Then as algebras

$$\boxed{C \overset{\cdot}{\simeq} \mathbb{C}[t^{-2}]}$$

← unbounded

and

to $(\cdot\mathbb{R}\mathrm{Spec}(C))$
$\shortparallel$
$\mathrm{Spec}(\mathbb{C})$

and

$$\mathbb{L}_{C/\mathbb{C}} \simeq \overset{-2}{\mathbb{C}[t]} \cdot \overset{\cancel{-2}}{\delta_t}$$

Example 3

$$\underset{\text{smooth}}{X = \operatorname{Spec} A} \hookleftarrow Y = \operatorname{Spec} A/I$$

regular sequence

then we can compute the self-intersection

$$Z := \mathbb{R}\operatorname{Spec}\left(A/I \overset{\mathbb{L}}{\underset{A}{\otimes}} A/I\right) = Y \times_X Y$$

So we need a resolution of A/I over A

$\downarrow$ since we are assuming regular sequence, we can use the Koszul complex $K(A,I)$ as a resolution.

We get

$$A/I \overset{\mathbb{L}}{\underset{A}{\otimes}} A/I \simeq \left(\overbrace{\mathrm{Sym}_{A/I}\left(I/I^2[1]\right)}^{S}, 0\right)$$

0 differential.

$$\mathbb{L}_{Z/Y} = i^*\mathbb{L}_{Y/X}$$

$$\simeq i^* I/I^2[1] \simeq I/I^2[1] \underset{A/I}{\otimes} S$$

$$\simeq S^{\oplus k}[1].$$

Again this could be deduced from.

$$\begin{tikzcd} Z \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner"] & Y \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner"] & 0 \arrow[d] \\ Y \arrow[r, hook] & \mathbb{A}^n \arrow[r, "f_1, \dots, f_n"] & \mathbb{A}^k \end{tikzcd}$$

So

$$\boxed{\begin{tikzcd} Z \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner"] & 0 \arrow[d, no head] \\ Y \arrow[r] & \mathbb{A}^k \end{tikzcd}}$$

Relation with POT

$X \in ST$, X^{dr} a derived enhancement

$$X \xhookrightarrow{j} X^{dr}$$

<u>Proposition</u>

(Lurie)

$$\left(j^* \mathbb{L}_{X^{dr}} \longrightarrow \mathbb{L}_X \quad \text{has a cofiber whose cohomology is in } \underline{\underline{\text{degree} \geq -1}} \right)$$

if $f: A \to B$ is n-connective then $\mathbb{L}_f$ is $(n+1)$-connective.

---

claim when X^{der} is quasi smooth

$$j^* \mathbb{L}_{X^{der}} \longrightarrow \mathbb{L}_X \text{ is a PoT}$$

Functoriality

$$\begin{tikzcd}
X \arrow[r, hook] \arrow[d, "f"] & \mathbb{R}X \arrow[d, "F"] \\
Y \arrow[r, hook] & \mathbb{R}Y
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
(t_0 F)^* j_Y^* \mathbb{L}_{\mathbb{R}Y} \arrow[r] \arrow[d] & (t_0)^* F^* \mathbb{L}_Y \arrow[d] & \\
j^* \mathbb{L}_{\mathbb{R}X} \arrow[r] & \mathbb{L}_X \arrow[r] & \mathbb{L}_{X/\mathbb{R}X}
\end{tikzcd}$$

Talk # 12

shifted symplectic structures

classical picture

symplectic manifold

$$(X,\omega) \quad , \quad \omega \in \Gamma(X, \underbrace{\Lambda^2 \Omega^1_X}_{2\text{-form}})$$

with $\underset{\text{closed}}{d\omega = 0}$ & $\underset{\text{non-deg.}}{\omega^\flat : T_X \xrightarrow{\sim} T^*X}$

Example M manifold with $q_1, \dots, q_N$ coordinates

$X = T^*M$ with q_i, p^i coordinates

$\omega = \sum dp_i \wedge dq_i$

} exact structure given by Liouville form

Darboux If (X,ω) is symplectic, then locally

$$(X,\omega) \xrightarrow{\sim} (T^*M, \omega)$$

⚠
These result is false for affine schemes ?

? I think this is true!

because this requires ω to be locally exact!

Next talk → Darboux for (-1)-shifted derived schemes

Generalization → X singular scheme → X derived scheme

Generalization → X stack → X derived scheme

want to have

$$\mathbb{T}_X \underset{\omega}{\simeq} \mathbb{L}_X$$

or more generally

$$\mathbb{T}_X \simeq \mathbb{L}_X[n]$$

Definition An n-shifted symplectic structure on X is a closed n-shifted 2-form $\omega = (\dots, \omega_1, \omega_0)$ such that

$$\omega_0 \in \underbrace{\mathrm{Map}_{\mathcal{D}_0(X)}\left(\mathcal{O}_X, \Lambda^2 \mathbb{L}_X[n]\right)}_{\text{Space of } n\text{-shifted forms } \mathcal{A}^2(X,n)}$$

$$\simeq \mathrm{Map}\left(\mathbb{T}_X, \mathbb{L}_X[n]\right)$$

such that $\omega_0 : \mathbb{T}_X \xrightarrow{\sim} \mathbb{L}_X[n]$ quasi-iso.

$A^{2,cl}(X,n)$ = closed 2-forms

$$\downarrow$$

$A^{2}(X,n)$ = space of 2-forms

→ closure data = $(\, , \dots , \omega_1 , \omega_0)$ ⚠

is a data not a property

→ we will avoid given the definition of $A^{2,cl}(X,n)$.

Insted we will illustrate it via an exemple

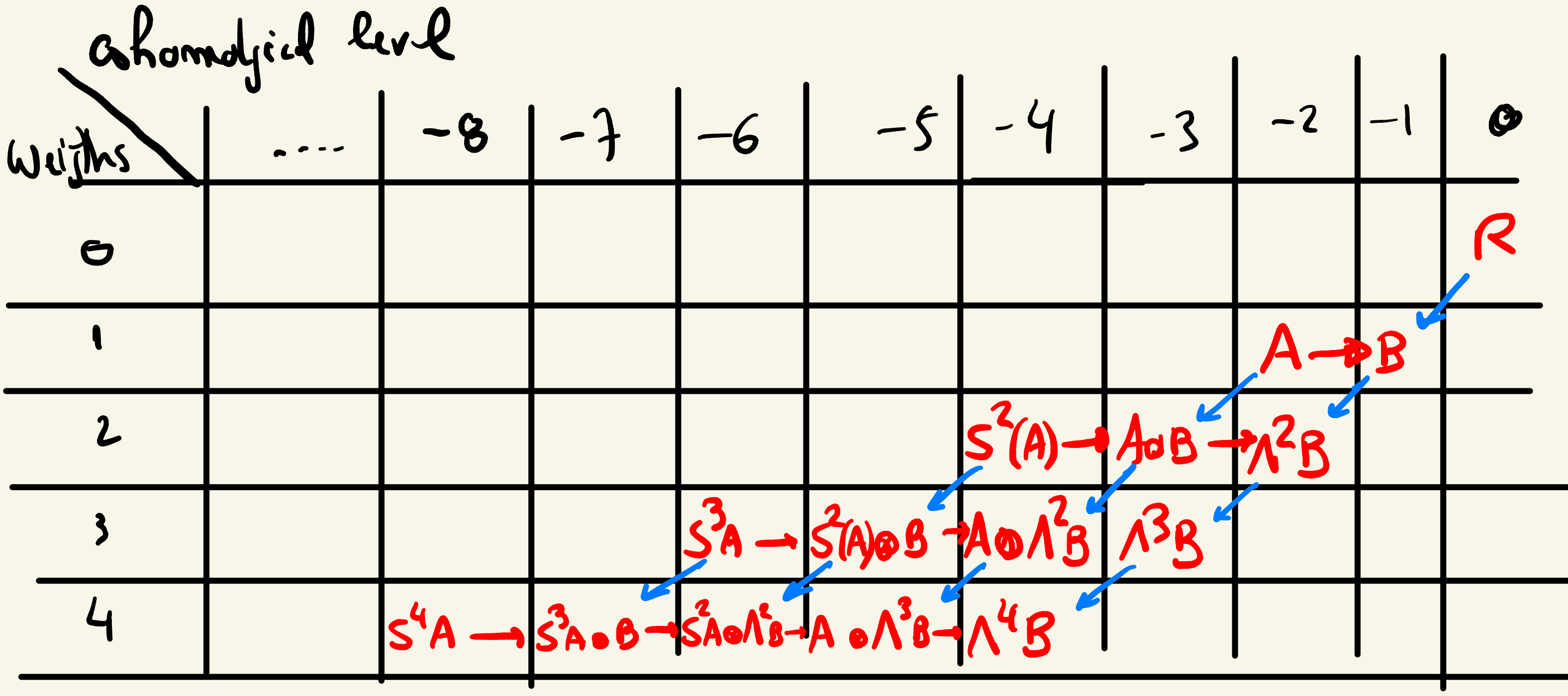
Exemple: $X = \mathrm{Spec}(R)$ affine derived scheme

$$\mathbb{L}_X = [\overset{-1}{A} \xrightarrow{d} \overset{0}{B}] \qquad A, B$$

free R-moduls

$$\Lambda^{\bullet} \mathbb{L}_X = \mathrm{Sym}_{\mathcal{O}_X}(\mathbb{L}_X[1])$$

$$= \mathrm{Sym}_{\boxed{\mathcal{O}_X} R}\left([\overset{-2}{A} \to \overset{-1}{B}] \right)$$



What is a (-1)-shifted 2-form in this case?

$$\downarrow$$

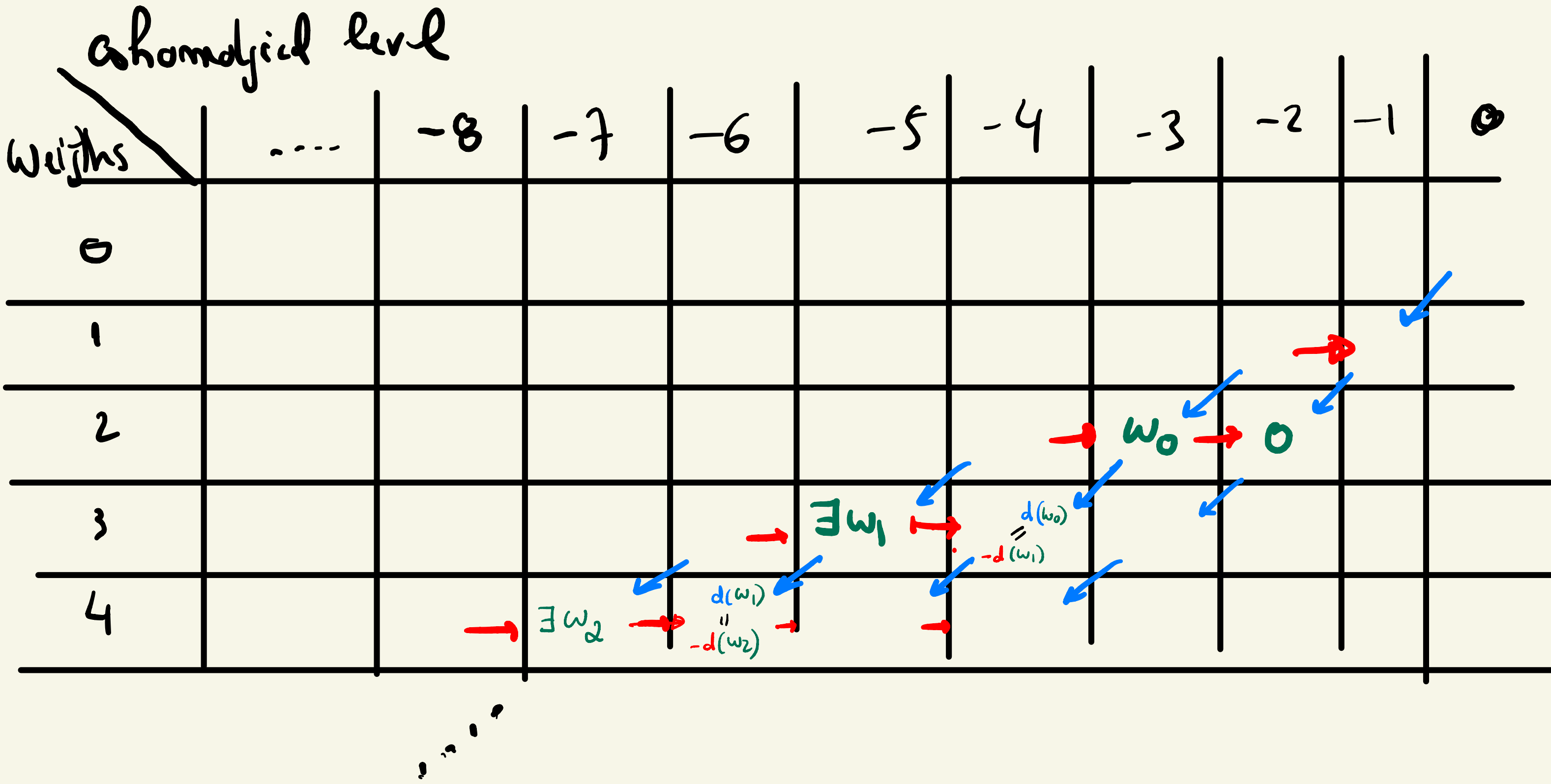
$$\mathbb{T}_X \simeq \mathbb{L}_X[-1]$$

The classical structure

$$\begin{tikzcd}
\mathbb{T}_X \simeq \Big[\overset{0}{B^\vee} \arrow[r, "d^\vee"] \arrow[d, "?"] & \overset{1}{A^\vee} \Big] \arrow[d, "?"] \\
\mathbb{L}_X[-1] = \Big[\underset{0}{A} \arrow[r, "d"] & \underset{0}{B} \Big]
\end{tikzcd}$$

$$\boxed{w_0 \in A \otimes B}$$

Now we want to say that it is closed



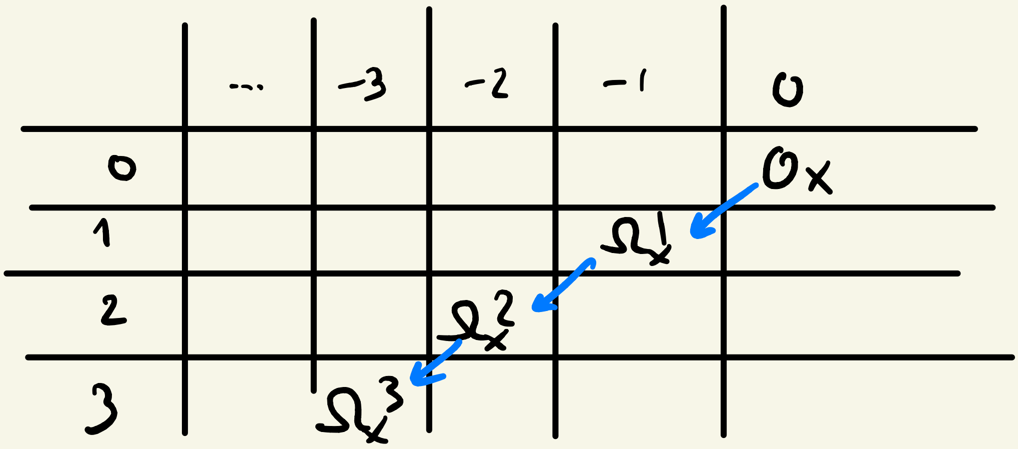
$w_0, w_1, w_2, \ldots$ closure data

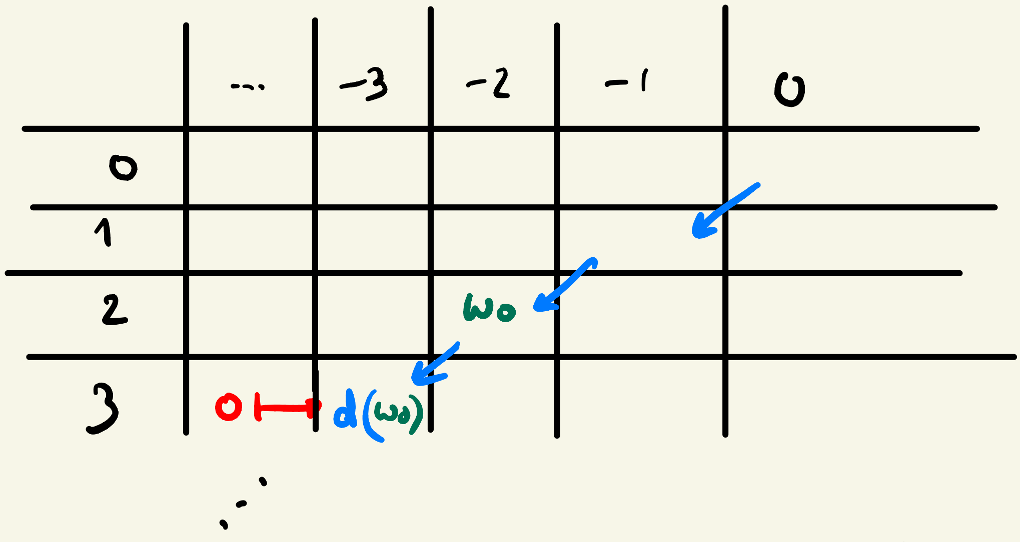
Example

What if X is a smooth affine scheme

$$\mathbb{L}_X = \underbrace{\Omega^1_X}_{\deg 0}$$

In this case we have





no closure data became all relevant

closure data is zero!

High Road to define closure

$$\begin{tikzcd}
\mathcal{L}X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d] \\
X \arrow[r] & X \times X
\end{tikzcd}$$

$$\mathcal{L}X \simeq \mathrm{RMap}(S^1, X)$$

HKR $:= \; p_* \mathcal{O}_{\mathcal{L}X} \simeq \Lambda^\bullet \mathbb{L}_X$

Definition : functions on $\mathcal{L}X$ $\longleftrightarrow$ forms

S¹-eq. functions on $\mathcal{L}X$ $\longleftrightarrow$ closed forms.

Exemples

Exemple 0 : Derived critical locus

$$Y \xrightarrow{\quad f \quad} \mathbb{A}^1$$

smooth scheme

derived critical locus

$$\begin{tikzcd}
X = \mathbb{R}\mathrm{Crit}(f) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "df"] \\
Y \arrow[r, "0"] & T^*Y
\end{tikzcd}$$

h

$$\mathbb{T}_X \simeq \Big[\underset{0}{\mathbb{T}_Y|_X} \xrightarrow{\text{Hessian of } f} \underset{1}{\mathbb{L}_Y|_X} \Big]$$

$$\mathbb{L}_X[-1] \simeq \Big[\underset{0}{\mathbb{T}_{Y}|_X} \xrightarrow{H^\vee} \underset{1}{\mathbb{L}_{Y}|_X}\Big]$$

and $\mathbb{T}_X \xrightarrow{\sim} \mathbb{L}_X[-1]$ because of the symmetry of the Hessian.

Next talk: all (−1) shifted derived schemes are Zariski locally modelled on this exemple.

Exemple 1 all $T^*[n]X = \mathbb{R}\mathrm{Spec}(\mathrm{Sym}(\mathbb{L}_X[-n]))$

are n-shifted symplectic (Damien's proof)

idea: use the Liouville form

Exemple 2: $\underbrace{BGL_n}_{} = [*/GL_n]$

moduli of GL_n-torsors $\Leftrightarrow$ Vect bndls of rank n.

$$[*/Gl_n](S) = \left\{ \begin{array}{c} P \circlearrowleft Gl_n \\ \downarrow \\ S \end{array} \right\}$$

$$= \left\{ \begin{array}{c} E \\ \downarrow \\ S \end{array} \text{ rank } n \quad E = P \times_{Gl_n} \mathbb{C}^n \right\}$$

$$\mathbb{T}_{BGl_n} = [\mathfrak{gl}_n \longrightarrow 0] \simeq \underset{\circlearrowleft}{\mathfrak{gl}_n}[1]$$

adjoint representation

$$\mathbb{L}_{BGl_n} \simeq \mathfrak{gl}_n^\vee[-1]$$

<u>claim</u> this carries a <u>2-shifted symplectic</u> form:

$$\Lambda^\bullet \mathbb{L}_{BGl_n} = \mathrm{Sym}^\bullet(\mathfrak{gl}_n^\vee)$$

2-forms $\Longleftrightarrow$ $\mathrm{Sym}^2(\mathfrak{gl}_n^\vee)^{Gl_n\text{-inv.}}$

ie, $\omega_0 \longleftrightarrow$ symmetric map which is Gl_n-eq.

There is a canonical one

$$\mathfrak{gl}_n \otimes \mathfrak{gl}_n \xrightarrow{\text{Killing form}} \mathbb{C}$$

$$x, y \longmapsto \mathrm{Tr}(xy)$$

$$\begin{tikzcd} \mathbb{T}_X \arrow[d] & = & \mathfrak{gl}_n[1] \arrow[d] \\ \mathbb{L}_X^{[2]} & = & \mathfrak{gl}_n^\vee[1] \end{tikzcd}$$

$$\mathfrak{gl}_n[1] \otimes \mathfrak{gl}_n[1] \longrightarrow \mathcal{O}_{BGl_n}[2]$$

$$S^2(\mathfrak{gl}_n)[2] \longrightarrow \mathbb{C}$$

fund'l rep.

Works for any reductive affine G ⚠

Example 3 $\mathbb{R}\mathrm{Perf}$ ⇝ moduli stack of perfect complexes.

$$\mathbb{R}\mathrm{Map}(X, \mathrm{Perf})$$

$$\shortparallel$$

derived mapping stack

turns out 2-symplectic

Thm: • X smooth & proper CY dim d

• F n-shifted symplectic

$$\boxed{\text{then} \quad \mathbb{R}\mathrm{Map}(X,F) \text{ is } (n-d)\text{-symplectic}}$$

Exemple $\mathbb{R}\mathrm{PERF}(X) := \mathbb{R}\mathrm{Map}(X, \mathbb{R}\mathrm{Perf})$

X CY 3 fold → (-1)-symp.

X CY 4 fold → (-2)-symp.

symplectic structure induced by serre duality

Lagrangian structures $L \xrightarrow{f} X$

n-shifted symplectic derived stack.

$$f^*\omega \in A^{2,cl}(L,n)$$

we can define a space of isotropic structures on f

as the space of $\underbrace{\text{homotopies } \{f^*\omega \overset{h}{\sim} 0\}}_{\boxed{\mathrm{Iso}(f,\omega)}}$ in $A^{2,cl}(L,n)$

We say h is a Lagrangian structure if the induced map

$$\mathbb{T}_f \longrightarrow \mathbb{L}_L[n-1] \qquad (**)$$

is an equivalence.

Example:

Back to classical symplectic geometry:

$L \subseteq X$ is Lagrangian when $\omega|_L = 0$

and $\dim L = \frac{\dim X}{2}$.

$$\begin{tikzcd}
\mathbb{T}_L \arrow[r] \arrow[rr, bend left, "\omega|_L"] \arrow[d, dash] & \mathbb{T}_X \arrow[r, "\omega", "\sim"'] \arrow[d, dash] & \mathbb{L}_X \arrow[r] & \mathbb{L}_L \\
0 \arrow[r] & \mathrm{Norml}_{L/X} \arrow[urr, dashed, "\sim"] & &
\end{tikzcd}$$

we h

short exact sequence

being Lagrangian (dim. counting)

Conversely: if $\mathrm{Norml}_{L/X} \xrightarrow{\sim} \mathbb{L}_L$

Then L is Lagrangian.

This is exactly the particular case of definition (∗∗) above, when $\underline{\underline{n=0}}$.

~~Finally:~~

Thm: the derived intersection of Lagrangians in a n-shifted symplectic is $(n-1)$-symplectic.

$(n-1)$-symplectic

$$\begin{tikzcd}
L_1 \overset{h}{\underset{X}{\times}} L_2 \arrow[r] \arrow[d] & L_2 \arrow[d, "\text{Lag}"] \\
L_1 \arrow[r, "\text{Lag}"] & X \arrow[ul]
\end{tikzcd}$$

n-shifted

Talk #13

Darboux theorem for (-1)-shifted symp. schemes.

all blackboards are derived IR

⓪ (-1)-shifted symplectic (X, ω)

at $x \in X$

$$\underbrace{\mathbb{T}_{X,x}}_{\substack{\text{complex} \\ \text{of vector space } /k(x)}} = \left[\mathbb{T}^{\text{Zar}}_{t(X),x} \xrightarrow{0} \mathrm{Ob}_{X,x} \to \cdots \right]$$

ω_X non-deg.

(-1)-shifted

⇓

nothing can exist here ↓

$$\underbrace{\mathbb{T}_{X,x}}_{} = \left[\mathbb{T}^{\text{Zar}}_{t(X),x} \xrightarrow{0} \underbrace{\mathrm{Ob}_{X,x}}_{\substack{\| \\ \Omega^{\text{Zar}}_{t(X),x}}} \cancel{\to \cdots} \right]$$

<u>Global analogue</u> $\mathbb{T}_X$ in tor-amplitude $[0,1]$

$\Downarrow$

$\mathbb{L}_X$ in tor amplitude $[-1,0]$

$\Downarrow$

$t(X) \xhookrightarrow{j} X$ $\qquad$ X quasi-smooth

$\Downarrow$

$$\boxed{j^*\mathbb{L}_X \longrightarrow \mathbb{L}_{t_0(X)} \text{ is a PoT}}$$

(-1)-symplectic

$\Downarrow$

$$\boxed{\mathbb{L}_X \underset{\omega^{-1}}{\simeq} \mathbb{L}_X^\vee[1]}$$

symmetric

Symmetric perfect obs. theory

---

§ 1 Exact forms

$X = \mathrm{Spec}(A)$ affine derived scheme.

$\rightsquigarrow$

$$\mathbb{L}\widehat{dR}(X) = \underbrace{\mathrm{Tot}^{\Pi}}\left[A \to \mathbb{L}_A \to \Lambda^2\mathbb{L}_A \to \cdots\right]$$

Hodge completed de Rham cohomology.

$\rightsquigarrow$ comes with the Hodge filtration

$\uparrow$ $\downarrow$ can talk about weight 2 part

$$\mathbb{L}\widehat{dR}^{\geq 2}(X) = \mathrm{Tot}^{\Pi}\left(0 \to 0 \to \Lambda^2 \mathbb{L}_A \xrightarrow{d_R} \Lambda^3 \mathbb{L}_A \to \cdots\right)$$

$$\boxed{H^{2+n}\left(\mathbb{L}\widehat{dR}^{\geq 2}(X)\right) = \left\{ \begin{array}{c} \text{closed} \\ \text{n-shifted} \\ \text{2-forms} \end{array} \right\}}$$

$$\begin{tikzcd} \mathbb{L}\widehat{dR}^{\geq 2}(X) \arrow[r, hook] & \mathbb{L}\widehat{dR}(X) \arrow[d] \\ & \mathbb{L}\widehat{dR}^{\leq 1}(X) \arrow[d, phantom, "\parallel"] \\ & {[A \xrightarrow{d_R} \mathbb{L}_A]} \end{tikzcd}$$

fibre sequence

Rotation

boundary map

$$\boxed{\begin{tikzcd} \mathbb{L}\widehat{dR}^{\leq 1}(X)[-1] \arrow[r, "\delta"] & \underbrace{\mathbb{L}\widehat{dR}^{\geq 2}(X)}_{\text{closed 2 forms}} \arrow[r, hook] & \underbrace{\mathbb{L}\widehat{dR}(X)}_{\text{de Rham cohomology}} \end{tikzcd}}$$

what does δ do?

on $(2+n)$-cycles

degree of n.

$$\Phi \in A^{1+\textcircled{n}}, \quad \Theta \in L^{2+n}$$

such that $d_A(\Phi)=0 \qquad d_{dR}\Phi = d\Theta$

$$\big\downarrow \delta$$

$$(d_{dR}\Theta, 0, \dots, 0) = \delta(\Phi, \Theta)$$

Definition: A closed n-shifted 2-form is exact if there is an homotopy in

$$\begin{tikzcd}
 & K \arrow[d, "\omega"] \arrow[dr, "\sim 0"] \arrow[dl, dashed, "\substack{\text{exact structure}\\ \exists\text{ lift}}"'] & \\
\widehat{L_{dR}}^{\leq 1}(X)[-1] \arrow[r] & \widehat{L_{dR}}^{\geq 2}(X) \arrow[r] & \underbrace{\widehat{L_{dR}}(X)}_{\text{de Rham cohomology}}
\end{tikzcd}$$

Exemple

$Y = \mathrm{Spec}(B)$ smooth

$f : Y \to \mathbb{A}^1$

$$\begin{tikzcd}
\mathrm{DCrit}(f) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "df"] \\
Y \arrow[r, "0"] & T^*Y
\end{tikzcd}$$

claim : the canonical (-1)-form on $\underline{\mathrm{DCrit}(f)}$

Ⓐ ① is $\underline{\text{exact}}$

Ⓑ ② the map $\underline{\mathrm{DCrit}(f) \longrightarrow Y}$ is a $\underline{\text{lagrangian fibration}}$ (ie $\Pi_{\mathrm{DCrit} f/Y}$ is a lagrangian subbundle of Π_{DCrit})

Proof : $\underline{\text{Exactness}}$, comes from the Liouville form on T^*Y. ⌋

•

the 1-form is precisely the homotopy α in

$$\begin{tikzcd} \mathrm{DCrit}(f) \arrow[r] \arrow[d] & Y \arrow[d] \arrow[dl, Rightarrow, "\alpha"'] \\ Y \arrow[r, dash] & T^*Y \end{tikzcd}$$

- for the Lagrangian fibration, this comes from the fact that $T^*Y \longrightarrow Y$ is a Lagrangian fibration.

---

§3 Darboux theorem (Brav-Bussi-Joyce)

Thm: X (-1)-shifted symplectic. then

Zariski locally $X \simeq \mathrm{dcrit}(f \text{ on } Y_{\text{smooth}})$

↗ symplectic

Corollary: X is quasi-compact (-1)-shifted symplectic scheme $\rightsquigarrow$ can produce virtual critical manifold on $t_0(X)^{an}$

<u>idea</u>: proof that $\overset{\text{exact}}{Ⓐ}$ & $\overset{\text{log-filtration}}{Ⓑ}$ hold

Zariski locally on $X = \mathbb{R}\mathrm{Spec}(A)$.

<u>step 1</u>: locally any (-1)-shifted form is <u>exact</u>

Ⓐ

↓

this uses a theorem of (Bloom - Herrera, Deligne (Hodge III) + Goodwillie)

<u>Thm</u>: For any finite type $\mathrm{cdga}^{\leq 0}$ A, $X = \mathrm{Spec}\, A$,

the canonical map

$$\underbrace{\widehat{LdR}(X)}_{\text{Hodge completed de Rham}} \longrightarrow H^0\big[H^0(A) \to \underbrace{\Omega^1_{H^0(A)} \to \Lambda^2\Omega^1_{H^0A} \cdots}_{\text{underived de Rham cohomology.}}\big]$$

has a <u>retract</u>

<u>Proof</u>: uses <u>resolution of singularities.</u>

Consequence: the map

$$H^{*}\left(\mathbb{L}dR^{\geq 2}(X)\right) \longrightarrow H^{*}\left(\widehat{\mathbb{L}dR}(X)\right)$$

is zero for $* \leq 1$

Consequence: any $(-n)$-shifted closed 2-form on $\mathbb{R}\mathrm{Spec}\, A$ for $n \geq 1$, is exact

because of Deligne theorem.

$$\begin{tikzcd}
 & K \arrow[ld, dashed, "\exists\, \text{lift}"', "\text{exact structure}"] \arrow[d, "\omega"] \arrow[rd, "\sim 0"] & \\
\widehat{\mathbb{L}dR}^{\leq 1}(X)[-1] \arrow[r] & \widehat{\mathbb{L}dR}^{\geq 2}(X) \arrow[r] & \underbrace{\widehat{\mathbb{L}dR}(X)}_{\text{de Rham cohomology}}
\end{tikzcd}$$

Proof of the Corollary

$$\begin{tikzcd}
H^*(\mathbb{L}\widehat{dR}(X)) \arrow[r, hook, "\text{injective for } *\leq 1"] \arrow[d, "\substack{\text{injective}\\ \text{because of Deligne}}"'] & H^\bullet(A \to \mathbb{L}_A) \arrow[d] \\
H^\bullet(\cancel{H^0(A) \to \Omega^1_{H^0A} \to \cdots}) \arrow[r, "\substack{\text{canonically}\\ \text{injective for } *\leq 1}"'] & H^\bullet(H^0 \to \Omega^1_{H^0A})
\end{tikzcd}$$

Local properties of $\mathbb{R}\mathrm{Spec}(A), \omega$

We can now assume ω to be exact

$$\omega \sim d_R(\Phi, \theta)$$

$\rightsquigarrow$ almost a critical locus.

Step 2 Find the smooth scheme on which the function is defined.

idea: take $x \in X$

then we can present $\mathbb{T}_{X,x}$ as a 2-term complex

$$\begin{tikzcd}
\mathbb{T}_{X,x} = \Big[\mathbb{T}^{\mathrm{Zar}}_{t(X),x} \arrow[r, "0"] \arrow[r, leftrightarrow, bend right, "\omega_X \text{ canonical pairing.}"'] & \Omega^1_{t(X),x} \Big]
\end{tikzcd}$$

$$\Downarrow$$

$\mathbb{T}_{X,x}$ has a lagrangian subspace

$$\begin{tikzcd}
\text{lagrangian subspace} \arrow[d, phantom, "\shortparallel"] \\
\Omega^1_{t(X),x}[1]
\end{tikzcd}$$

$\Downarrow$ <u>Nakayama trick</u>

locally around $x \in X$

$$\mathbb{T}_X := [E^0 \xrightarrow{d} E^1]$$

such that $d = 0$ at x

this has a canonical <u>lagrangian subbundle</u>

$$\underbrace{\mathcal{L} := E^1[-1]}_{\text{is lagrangian distribution}}$$

→ what is integrability condition ?

Morally $\mathcal{L}$ is an integrable distribution

we want to take "$Y = X/\mathcal{L}$ leaf space" → smooth formal scheme

↓

they do integrability condition

by saying that since π_X is in $[0,1]$, there the integrability conditions vanishes.

$$\underbrace{\hphantom{\text{they do integrability}}}_{\text{need nice cdga model.}}$$

More Hands-on-approach $\qquad$ $x \in X$

$$t^0(X) \hookrightarrow U$$

choose embedding $\qquad$ smooth affine.

can make factorize

$$\begin{tikzcd} t^0(X) \arrow[rr, hook] \arrow[dr, hook] & & \mathbb{A}^N \\ & Y \arrow[ur] & \end{tikzcd}$$

induces iso on Zariski tangent spaces

smooth subscheme.

$= \operatorname{Spec}(B)$.

$$T^{Zar}_{X,x} \simeq T_x Y$$

$(\mathbb{R}\mathrm{spec}(A), \omega)$ locally

$$\omega \sim d_R(\Phi, \theta)$$

$$\mathbb{R}\mathrm{spec}\, A \hookrightarrow \underset{\uparrow \text{smooth}}{\mathrm{Spec}\, B}$$

$$+ \quad H^0(T_x X) \simeq H^0(T_x Y)$$

$\Downarrow$ conseguence

use Lurie-Quillen [connectivity estimates]

$$\begin{tikzcd} H^{-1}(\mathbb{L}_{A/B}) \arrow[d] & H^0(\mathrm{fib}(B\to A)) \otimes H^0 A \arrow[l, "\sim"'] & \\ & & H^0(B) \arrow[ul, "\text{syect}"'] \\ \text{locally free on generators} & H^0(\mathrm{fib}(B\to A)) \arrow[uu] & \\ e_1, \dots, e_N & g_1, \dots, g_N \arrow[l, dashed, "\exists \text{ lift}"'] & \end{tikzcd}$$

closed imm.

$\cup$

$$\begin{tikzcd} X \arrow[r, "\exists !", "\text{étale}"'] \arrow[dr, bend right] & Z(g) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner h"] & Y \arrow[d, "g_1, \dots, g_N"] \\ & 0 \arrow[r] & \mathbb{A}^n \end{tikzcd}$$

and is a closed immersion + étale $\Longrightarrow$

$X \hookrightarrow Z(g)$ is a connected component.

$\Downarrow$

locally $A \simeq$ Koszul complex$(B, g_1, \ldots, g_N)$

$$\omega = d_R(\underset{\substack{\rotatebox{90}{$\in$} \\ B}}{\Phi}, \underset{\substack{\rotatebox{90}{$\in$} \\ L_A^{-1}}}{\theta})$$

$$A = [B \leftarrow E^\vee \leftarrow \Lambda^2_B E^\vee \leftarrow \cdots\cdots]$$

$$L_A^{-1} = A^{-1} \underset{B}{\otimes} \Omega^1_B \oplus d_{dR}(E^\vee)$$

$$\rotatebox{90}{$\in$}$$

$$\theta = \theta_1 + d_R(\eta)$$

Now: $(\Phi, \theta) \sim (\underbrace{\Phi - d\eta}_{\boxed{f \in B}}, \underbrace{\theta_1}_{\substack{\rotatebox{90}{$\in$} \\ A \underset{B}{\otimes} \Omega^1_B}})$

such that

$$d\theta_1 = d_R(f)$$

$$\Downarrow$$

$$\begin{tikzcd}
X \arrow[rr, "\boxed{\theta_1}"] \arrow[dr] & & \mathrm{Def}(f : Y \to \mathbb{A}^1) \arrow[dl] \\
& Y &
\end{tikzcd}$$

- θ_1 is étale
- and is closed immersion

$\Downarrow$

is $\boxed{\text{iso}}$

$$\boxed{X \xrightarrow[\sim]{\theta_1} \mathrm{Def}(f : Y \to \mathbb{A}^1)}$$

f = exact structure — isotopic structure

$\boxed{\text{talk } \# 14}$ hyperbolic localizations of DT-perverse sheaf

X <u>(-1)-shifted symplectic</u> <u>scheme</u>

$\circlearrowright$

$\mathbb{C}^*$ action $= \mathbb{G}_m$.

$\mathbb{C}^*$-invariant $= \mathbb{G}_m$.

$X^0 =$

$$\hookrightarrow X^0 = \coprod_{\pi \in \Pi} X^0_\pi$$

Π: fixed components.

?

Behrend: $$e^{vir}(X) = \sum_{\pi} \pm e^{vir}(X^0_\pi)$$

we will compute the formula for X smooth and extend to X (-1)-symplectic

<u>X smooth</u>:

$$\begin{tikzcd} & X^+ \arrow[dl, "\eta"'] \arrow[dr, "p"] & \\ X & & X^0 \end{tikzcd}$$

$$X^+ : y \longmapsto \mathrm{Hom}^{\mathbb{C}^*}(\mathbb{A}^1 y, X)$$

$$X^0 : Y \longmapsto \mathrm{Hom}^{\mathbb{C}^*}(Y, X)$$

$d^{\#} :=$ ~~$\#\{$ contracting~~ repelling weights in $T_X|_{X^0_\pi}$

$$T_X|_{X^0} = T_{X^0_\pi} \oplus \underbrace{\overbrace{T_{X^0_\pi}^{+}}^{d^+ \dim}}_{d^+} \oplus \underbrace{T_{X^0_\pi}^{-}}_{d^-}$$

$\sim$ eigenvectors for the $\mathbb{C}^\times$-action.

Now

$X \curvearrowleft \mathbb{C}^*$ ← (-1)-shifted symplectic with orientation $K_X^{1/2}$

$$K^{2}_{X^0_\pi} = \det(\mathbb{L}_{X^0_\pi})$$

$$\boxed{P_X = \text{perverse sheaf of vanishing cycles}}$$

$p_! \eta^*$ ← six operations for derived stacks on X^+ & X^0

and

<u>Drinfeld</u>: X^0 & X^+ are schemes

$$p_! \eta^* P_X = \bigoplus_{\pi\in\Pi} P_{X^0_\pi}[-\mathrm{ind}_\pi]$$

where $\left(\mathrm{ind}_\pi = \dim \mathfrak{g}\ T_X|_{X^0_\pi}\right)$

$$\boxed{\begin{aligned}\dim H^0_c(X, P_X) = \sum_{\pi\in\Pi} &\dim H_c^{*-\mathrm{in}_\pi}(X^0_\pi, P_{X^0_\pi}) \\ &+ \dim H_c(X_+, \eta(X'), P_X)\end{aligned}}$$

---

① hyperbolic localizations formulas

$\mathbb{C}^* \curvearrowright X$ - quasi-sep
- loc. f.t.

$\Rightarrow$ $\mathbb{C}^*$-action étale locally linearizable.

$$\begin{tikzcd}
 & X^+ \arrow[dl, "\eta = \eta^+ \text{ of before}"'] \arrow[dr, "p^+ = p \text{ of before}"] & \\
X & & X^0 \\
 & X^- \arrow[ul, "\eta^-"] \arrow[ur, "p^-"'] &
\end{tikzcd}$$

Drinfeld shows these are schemes

Drinfeld $\Rightarrow$

$p^+_!(\eta^+)^*(A)$ is $(p^-)_*(\eta^-)^*(A)$

changes ! with $(\,)_*$

uses ~~Serre~~ Verdier duality

(hyperbolic localization fcts)

$\mathbb{G}_m = \mathbb{C}^*$ action on X smooth, X^0 smooth

$$IC_X := \mathbb{Q}_X[\dim X]$$

$$IC_{X^0_\pi} := \mathbb{Q}_{X^0_\pi}[\dim X^0_\pi]$$

$$p_! \eta^* IC_X \simeq \bigoplus_{\pi \in \Pi} IC_{X^0_\pi}[-d^+_\pi + d^-_\pi]$$

Hyperbolic localization & vanishing cycles

$$\begin{tikzcd} \mathbb{C}^\times \arrow[r, phantom, "\circlearrowright"] & U \arrow[d, "f"] \\ & \mathbb{A}^1 \end{tikzcd}$$

$\mathbb{C}^\times$ preserves f $\Rightarrow$ passes to U_0

and we have

$$\begin{tikzcd} \mathbb{C}^\times \arrow[r, phantom, "\circlearrowright"] & U_0 \arrow[r, hook] \arrow[d, hook] & U_0^+ \arrow[r] \arrow[d] & U^{0,0} \arrow[d] \\ \mathbb{C}^\times \arrow[r, phantom, "\circlearrowright"] & U \arrow[d] & U^+ \arrow[l, "\eta_U^+"'] \arrow[r, "p_U^+"] \arrow[dl, "f^+"] & U^0 \arrow[dll, bend left, "0"] \\ & \mathbb{A}^1 & & \end{tikzcd}$$

Then, we can apply

$$\begin{tikzcd} \varphi_{f_0} p^+_{U!} (\eta^+_U)^* (A) \arrow[r] \arrow[d, phantom, "\wr"] & (p^+_{U_0})_! (\eta^+_{U_0})^* \varphi_f(A) \arrow[d, "\wr"] \arrow[dl, phantom, "\text{Verdier duality}"] \\ \varphi_{f_0} p^-_{U*} (\eta^-_U)^! (A) & (p^-_{U_0})_* (\eta^-_{U_0})^! \varphi_f(A) \arrow[l] \end{tikzcd}$$

critical loci

$$\begin{tikzcd}
 & R \arrow[d, hook, "i"] & R^+ \arrow[l, "(p_R)"'] \arrow[r, "(\eta_R)"] \arrow[d] & R^- \arrow[d, "i^+"] \\
\mathbb{C} \curvearrowright & U_0 \arrow[d, hook] & U_0^+ \arrow[l, hook'] \arrow[r] \arrow[d] & U^{0,0} \arrow[d] \\
\mathbb{C} \curvearrowright & U \arrow[d] & U^+ \arrow[l, "\eta_U^+"'] \arrow[r, "p_U^+"] \arrow[dl, "f^+"] & U^0 \arrow[dll, bend left, "0"] \\
 & \mathbb{A}^1 & &
\end{tikzcd}$$

$$(p_R)_! \eta_R^* i^* \simeq (i^0)^* (p_{U_0})_! (\eta_{U_0})^*$$

$$\boxed{\mathcal{P}_R := i^* \varphi_f IC_U}$$

Thm $(p_R)_! (\eta_R)^* \mathcal{P}_R = \bigoplus_{\pi \in \Pi} \mathcal{P}_{R^0_\pi} [-\operatorname{ind} \pi]$

Rmk: U^0 is smooth because the fixed points of a $\mathbb{C}^*$-action on a smooth scheme will be smooth.

Gluing (like in Joyce):

$\mathbb{C}^*$ $\mathbb{R}X$ (-1)-symplectic ⟹ d-critical charts.
equivariant. on $t(\mathbb{R}X)=X$

$$\begin{tikzcd} R \arrow[r, hook, "\text{étale}"] \arrow[d, hook, "i"] & X \\ U \end{tikzcd}$$

U smooth, $I_{R,U}$.

$$0 \to \underbrace{S_{X|R}}_{\text{Joyce sheaf}} \to \frac{i^{-1}(\mathcal{O}_U)}{I_{R,U}^2} \xrightarrow{d} \frac{i^{-1}(T^*U)}{I_{R,U}T^*U}$$

$s = f + I_{R,U}^2$ such that $df|_R = 0$.

we have a sheaf S_X on X

$$s \in \Gamma(S_X) \iff s_R = f + I_{R,u}^2 \text{ with } df|_R = 0$$

s is a <u>d-critical structure</u> if for each $x \in X$ we can find an étale neighborhood such that

$$\begin{cases} f + I_{R,U}^2 = s_R \\ R = \mathrm{crit}(f). \end{cases}$$

Restriction of charts:

$$\begin{tikzcd} R' \arrow[r, hook] \arrow[d] & U' \arrow[r, "f'"] \arrow[d] & \mathbb{C} \arrow[d, phantom, "\wr\!\!\mid"] \\ R \arrow[r, hook, "i"] & U \arrow[r] & \mathbb{C} \end{tikzcd}$$

not enough to compare critical charts

need stabilization to compare charts

$$\begin{tikzcd} (R, U, f, i) \arrow[dr] & & (S, V, g, j) \arrow[dl] \\ & {} \arrow[r, bend right] & W \end{tikzcd}$$

up to étale restriction

$$(R, U \times \mathbb{C}^n, f \boxplus q, i \times \{0\}) \simeq (S, V \times \mathbb{C}^n, g \boxplus q, j \times \{0\})$$

standard quadratic form

d-critical scheme + finiteness conditions

$\Longrightarrow$ critical virtual manifold.

② How to obtain C.V.M with equivariant critical charts.

X (-1)-shifted symplectic • with $\mathbb{C}^*$ action.

$$\Downarrow$$

$\mathbb{C}^*$-invariant d-critical chart

claim: Given $x \in X \Rightarrow \exists$ affine $\mathbb{C}^*$-invariant étale neighborhood.

Halpern?

$$\begin{tikzcd} R \arrow[r, hook, "i"] \arrow[loop below, "\mathbb{C}^*"'] & U \arrow[loop below] \end{tikzcd}$$

$\rightsquigarrow$ $\boxed{\exists \ \mathbb{C}^*\text{-action can be lifted}}$

$\Downarrow$ can be chosen such that

$$x \in R \xhookrightarrow{i} U \xrightarrow{f} \mathbb{C}$$

$$\begin{cases} S_R = f + I_{R,U}^2 \\ df = 0 \end{cases}$$

$i(R) = \mathrm{crit}(f)$

$\Downarrow$

$\boxed{\text{thm: } \exists \text{ equivariant critical chart}}$

claim $\exists$ $\mathbb{C}^*$- equivariant upper bound

$$\begin{tikzcd}
\mathbb{C}^{2*} \cup \arrow[dr] & & V \curvearrowleft \mathbb{C}^* \arrow[dl] \\
& W &
\end{tikzcd}$$

$\exists\ \mathbb{C}^* \circlearrowright$

upper bound

need to use an equivariant quadratic form (not the standard one!)

Donaldson-Thomas in Aussois

My talk: ongoing project with
Benjamin & Julian Holstein.

so far

Thm (Joyce et al) X (-1) shifted symplectic derived scheme. locally $X \simeq \mathrm{dcrit}(f \text{ on } U)$ symp.

If $\exists$ L + $L \otimes L \simeq \det(\mathbb{T}_X)$ then the line bundle locally defined $P_{U,f} \in \mathrm{Perv}(\mathrm{crit}(f))$ glue to $P_f \in \mathrm{Perv}(X)$

Problems: Joyce strategy only works because:

① Perv is a discrete category

② no use of the derived structure on X

③ gluing by hand

What we want: general gluing mechanism

that

① explains Joyce

② works for other types of local invariants (Matrix of singularities. $\underbrace{\text{factorizations}}$)

this would be another week

Step 1: Vanishing cycles depends on local model

$S \underset{\text{open}}{\subseteq} X$

$s \underset{\text{symp}}{\simeq} \mathrm{dcrit}(U,f) \hookrightarrow \overset{\text{smooth}}{U}$

Idea: look at the moduli space of choices of local models.

Naive idea: Consider the assignment

$$S \underset{\text{open}}{\subseteq} X \longmapsto \overset{\text{groupoid}}{\left\{ S \underset{\text{symp}}{\simeq} \operatorname{dcrit}(U,f) \hookrightarrow U \right\}} \quad f : U \to \mathbb{A}^1$$

Problem: this is not even $\underline{\text{functorial}}$ because

$$\begin{tikzcd}
\text{if} \quad S' & \left\{ S' \simeq \operatorname{dcrit}(f) \hookrightarrow \boxed{?} \right\} \\
S \arrow[u, phantom, "\underset{\text{open}}{\cap\!\!|}"] & \left\{ S \xrightarrow{\sim} \operatorname{dcrit}(U,f) \hookrightarrow U \right\} \arrow[u]
\end{tikzcd}$$

choice.

Solution: Re-defined what we mean by LG-pairs:

$$\underset{\substack{\nearrow \\ \text{smooth} \\ \text{scheme}}}{(U,f)} \rightsquigarrow \Big(\underset{\substack{\text{formal} \\ \text{smooth} \\ \text{scheme}}}{\hat{\mathcal{U}}} := \widehat{\operatorname{crit}(f)}^{U}, \underset{\substack{\uparrow \\ \text{taylor} \\ \text{development}}}{\hat{f}} \Big)$$

$$\boxed{\underline{\text{claim}} \quad \operatorname{dcrit}(f,U) \underset{\uparrow}{\xrightarrow{\sim}} \operatorname{dcrit}(\hat{\mathcal{U}}, \hat{f})}$$

Proof since $\hat{\mathcal{U}} \xrightarrow[\text{étale}]{} \mathcal{U}$ $\Rightarrow$ is étale with the same truncation

$\Downarrow$

iso of derived schemes. ☐

Exemple : Replace $(\mathbb{A}^1, x^2)$ by $(\widehat{\mathbb{A}^1}, x^2)$.

---

claim : the assignment

$$S \subseteq X \longmapsto \overset{\text{groupoid}}{\left\{ S \underset{\text{dcrit}}{\simeq} \mathrm{dcrit} \hookrightarrow \underset{\substack{\text{smooth} \\ \text{formal scheme} \\ \text{top. of} \\ \text{finite presentation}}}{\mathcal{U}} \xrightarrow{f} \mathbb{A}^1 \right\}}$$

is **Functorial** .

Proof:

$$\begin{tikzcd} S' \arrow[rr] & & \exists!\, V \text{ formal} \arrow[d, dashed] \\ S \arrow[u, "\text{open}"] \arrow[r, phantom, "\simeq"] & \mathrm{dcrit} \arrow[r, hook] & \mathcal{U} \end{tikzcd}$$

uniquness of extension along étale maps

☐

---

conclusion : what this teach us is that not only the assignment is functorial on $S \subseteq X$ opens, it also works for étale maps $S \xrightarrow{ét} X$.

claim the assignment

$$\underbrace{\left\{ S \xrightarrow{eT} X \right\}}_{\text{small ét. site}} \longmapsto (\infty\text{-})\text{groupoids}$$

$$X_{eT} \qquad S \longmapsto \left\{ S \underset{\text{symp}}{\simeq} \mathrm{dcrit} \hookrightarrow \mathcal{U} \xrightarrow{f} \mathbb{A}^1 \right\}$$

is a $\overset{(\infty)}{\text{stack}}$.

we call it the Darboux stack of X

$$\underbrace{}_{\mathrm{Darb}_X}$$

OUR main strategy: Vanishing cycles const. ~~Joyce's~~ construction is defined without ambiguity as a morphism of stacks on X_{eT}:

$$\underline{\mathrm{Darb}}_X \xrightarrow{\ P\ } \underline{\mathrm{Perv}}_X$$

$\rotatebox{90}{$\sim$}$ stack of perverse sheaves on X

on each $S \xrightarrow{eT} X$, given by

$$\begin{tikzcd} \underline{\mathrm{Darb}}_X(S) \arrow[r] & \underline{\mathrm{Perv}}_X(S) \arrow[d, equal] \\ & \mathrm{Perv}(S) \\ {[S \xrightarrow[\text{symp}]{\sim} \mathrm{dcrit} \hookrightarrow U \xrightarrow{f} \mathbb{A}^1]} \arrow[uu, phantom, sloped, "\in"] \arrow[r, mapsto] & P_{U,f} \arrow[u, phantom, sloped, "\in"] \end{tikzcd}$$

only depends on formal completion

The Question is why does it descend to $* =$ final object in $\mathrm{sh}(X_{eT})$

$$\begin{tikzcd} \underline{\mathrm{Darb}}_X \arrow[r, "P"] \arrow[d] & \underline{\mathrm{Perv}}_X \\ * \arrow[ur, dashed, "?"'] & \end{tikzcd} \qquad (*)$$

My talk : explain yet another description of $D_{\underline{crit}} X$ and how it also appears in 4 folds ((-2) shifted case)

Benjamin's talk : Explain (*)

Rmk: easy to see why (*) is appealing: can replace P by any other type of invariant and glue.

Part II Derived lagrangian foliations

Back to local models and to the algebraic (non-formal case!)

$$\begin{tikzcd} \underset{(-1)\text{ symplectic}}{S} \arrow[r, "\sim"] & \operatorname{dcrit}(f) \arrow[r, hook] & U \\ & & \underset{\text{smooth scheme}}{\underbrace{U}} \arrow[r, "f"] & \mathbb{A}^1 \end{tikzcd}$$

__claim__: the derived fibers of the inclusion

$$\mathrm{dcrit}(f) \xhookrightarrow{i} \mathcal{U}$$

are lagrangians.

__Proof__:

Relative tangent complex

$$\begin{tikzcd}
\mathbb{T}_i \arrow[r] \arrow[d, "h"] \arrow[rd, phantom, "(*)"] & 0 \arrow[d, dash] \\
\mathbb{T}_{\mathrm{dcrit}(f)} = \left[\begin{array}{c} i^*\mathbb{T}_U \\ \downarrow Hf \\ i^*\Omega^1_U \end{array}\right] \arrow[r, "Di"] & i^*\mathbb{T}_{\mathcal{U}}
\end{tikzcd}$$

What we need to show

$$\begin{tikzcd}
\mathbb{T}_i \arrow[r] \arrow[rrr, bend right, "\sim 0"'] & \mathbb{T}_{\mathrm{dcrit}} \underset{\omega}{\simeq} \mathbb{L}_{\mathrm{dcrit}}[-1] \arrow[r] & \mathbb{L}_i[-1]
\end{tikzcd}$$

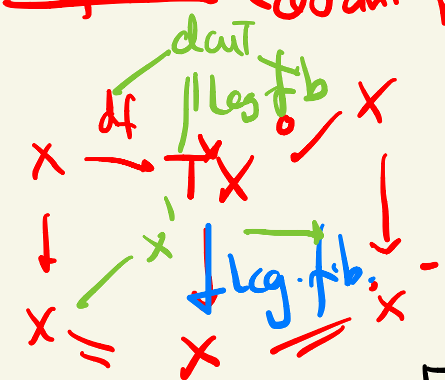
but $\mathbb{T}_i \simeq \Omega_U[-1]$

$\mathbb{L}_i \simeq \mathbb{T}_U[1]$

so the composition is just $\underline{\underline{*}}$.

honest proof →

Safranov (Geometric Quant Prop 1.14)



$\square$

Idea: think of the fibers of i as a foliation of $\mathrm{dCrit}(f)$. because the fibers are Lagrangian, this is a Lagrangian foliation.

Recall in differential geometry

$$\begin{tikzcd} X \arrow[d, "\pi"] \\ Y \end{tikzcd}$$

surjective.

Then we can consider

$$\mathcal{F} := \{ \pi^{-1}(y) \}_{y \in Y}$$

a foliation of X (not necc. smooth)

but we can actually recover π by setting

$$X/_{\mathcal{F}} := \quad x \sim x' \text{ if they are in the same leaf.}$$

claim: this construction makes sense in dgeometry but what we get

$$\mathrm{dCrit}/_{\mathcal{F}_i}$$

is not U

but the <u>formal completion</u> $\widehat{\mathrm{dcrit}(f)}^{U} = \mathcal{U}$

exactly what we needed to consider for the functoriality of the $\underline{\mathrm{Darb}}_X$.

Proposition

$$\mathrm{dcrit}(f)/\mathcal{F}_i \simeq \widehat{\mathrm{dcrit}(f)}^{U}$$

Proof: Bagher-Bhatt / Carlsson.

Another important fact:

Proposition the symplectic form on dcrit f has a canonical exact structure

Proof: the symplectic form on $\mathrm{dcrit}(f)$ comes from the one of T^*U, which is exact because of the Liouville form.

Finally

Thm (Toën-Pantev) $S = \mathrm{Spec}(A)$ (-1)-symplectic

$$\underbrace{\left\{ S \underset{\text{symp.}}{\overset{\phi}{\simeq}} \mathrm{dcrit} \xhookrightarrow{\text{formal}} \mathcal{U} \quad \mathcal{U} \to \mathbb{A}^1 \right\}}_{\mathrm{Darb}(S)} \simeq \left\{ \begin{array}{l} \text{Exact} \\ \text{structure} \\ \text{on } S \end{array} \right\} \times \left\{ \begin{array}{l} \text{Lagrangian} \\ \text{foliations} \\ \text{with smooth quotient} \end{array} \right\}$$

$$S \overset{\phi}{\simeq} \mathrm{dcrit} \overset{i}{\hookrightarrow} \mathcal{U} \longmapsto \left(\phi^*(\text{Liouville}) , \begin{array}{c} \text{Fibers} \\ \text{of} \\ i \circ \phi \end{array} \right)$$

$f : \mathcal{U} \to \mathbb{A}^1$

$\mathcal{U} := S/_F$ ⟵ (Θ, F)

smooth formal scheme

$$\boxed{f = \Theta - \text{isotopic structure}}$$

Definition S an n-shifted derived stack

$$\mathrm{Dab}(S) := \mathrm{Exact}(S) \times \mathrm{Lagfol}^{sm}(S)$$

Darboux lemma ⇒ locally these are non-empty.

Exemple

- $n = -1$ → $\mathrm{Dab}(S) \simeq$ local models

- $n = 0$ $\mathrm{Dab}(S) := \left\{ \begin{array}{c} \alpha \text{ exact struct} \\ + \\ \mathcal{F} \text{ lag. foliation} \end{array} \right\}$

$$\wr\wr$$

$$\left\{ \begin{array}{c} \text{identifications of} \\ S \simeq T^*\mathcal{U} \\ \text{with } \mathcal{U} \simeq S/\mathcal{F} \end{array} \right\} \longrightarrow \text{classical Darboux models for 0-shifted symplectic manifolds}$$

Claim $n = -2$

$$\mathrm{Dab}(S) \simeq \left\{ \text{Joyce - Borisov local models} \right\}$$

sketch of proof:

then locally we have

$$\mathbb{T}_X = \begin{array}{c|c} \left[\begin{array}{c} E_0 \\ \downarrow \\ E_1 \\ \downarrow \\ E_2 \end{array}\right] & \begin{array}{c} 0 \\ \\ -1 \\ \\ -2 \end{array} \end{array} \qquad \mathbb{L}_X = \begin{array}{c|c} \left[\begin{array}{c} E_2^\vee \\ \downarrow \\ E_1^\vee \\ \downarrow \\ E_0^\vee \end{array}\right] & \begin{array}{c} 2 \\ \\ 1 \\ \\ 0 \end{array} \end{array}$$

$$\Big\downarrow \text{shift}$$

$$\Downarrow$$

$$E_0 \simeq E_2^\vee$$

$$E_0 \simeq E_1^\vee$$

$$\boxed{E_1 \underset{q}{\simeq} E_1^\vee}$$

non-degenerated quadratic form on E_1

$$\Rightarrow \quad \mathbb{T}_X \simeq \begin{array}{c|c} \left[\begin{array}{c} V \\ \downarrow \\ E \\ \downarrow \\ V^\vee \end{array}\right] & \begin{array}{c} 0 \\ \\ 1 \\ \\ 2 \end{array} \end{array}$$

However:
not all lagrangians of $\mathbb{T}_X$ are necessary in tor-amplitude $[0,1]$, but we can restrict to those that are: for instance we can consider lagrangians of the form

$$\begin{tikzcd}
0 & 0 \arrow[d, dash] \arrow[r, "0"] & V \arrow[d, dash] \arrow[r, equal] & V \arrow[d, dash] \\
1 & P \arrow[d, dash] \arrow[r] & E \arrow[d, dash] \arrow[r] & \boxed{E/P \simeq P^\vee} \arrow[d, dash] \\
2 & V^\vee \arrow[r, equal] & V^\vee \arrow[r] & 0
\end{tikzcd}$$

condition p being lagragin

In this case

$P \to E$ is a lagrangian

in the bundle (E,q) $\Rightarrow$ $E/p \simeq P^\vee$

---

slogan: different local models $\Longleftrightarrow$ Lag. fibration + exact structure

Lagrangian distributions in (−2)-shifted stacks

(X, ω) (−2)-shifted symplectic

then locally we have

$$\mathbb{L}_X = \begin{array}{c|c} \left[\begin{array}{c} E_2^\vee \\ \downarrow \\ E_1^\vee \\ \downarrow \\ E_0^\vee \end{array}\right] & \begin{array}{c} 2 \\ \\ 1 \\ \\ 0 \end{array} \end{array}$$

$$\mathbb{T}_X = \begin{array}{c|c} \left[\begin{array}{c} E_0 \\ \downarrow \\ E_1 \\ \downarrow \\ E_2 \end{array}\right] & \begin{array}{c} 0 \\ \\ -1 \\ \\ -2 \end{array} \end{array}$$

$$\Big\downarrow \text{shift}$$

$$\Downarrow$$

$$E_0 \simeq E_2^\vee$$

$$E_0 \simeq E_1^\vee$$

$$\boxed{E_1 \underset{q}{\simeq} E_1^\vee}$$

non-degenerated quadratic form on E_1

$$\Rightarrow \quad \mathbb{T}_X \simeq \left[\begin{array}{c} V \\ \downarrow \\ E \\ \downarrow \\ V^\vee \end{array}\right] \begin{array}{l} 0 \\ \\ 1 \\ \\ 2 \end{array}$$

However:

not all lagrangians of $\mathbb{T}_X$ are necessary in tor-amplitude $[0,1]$, but we can restrict to those that are: for instance we can consider lagrangians of the form

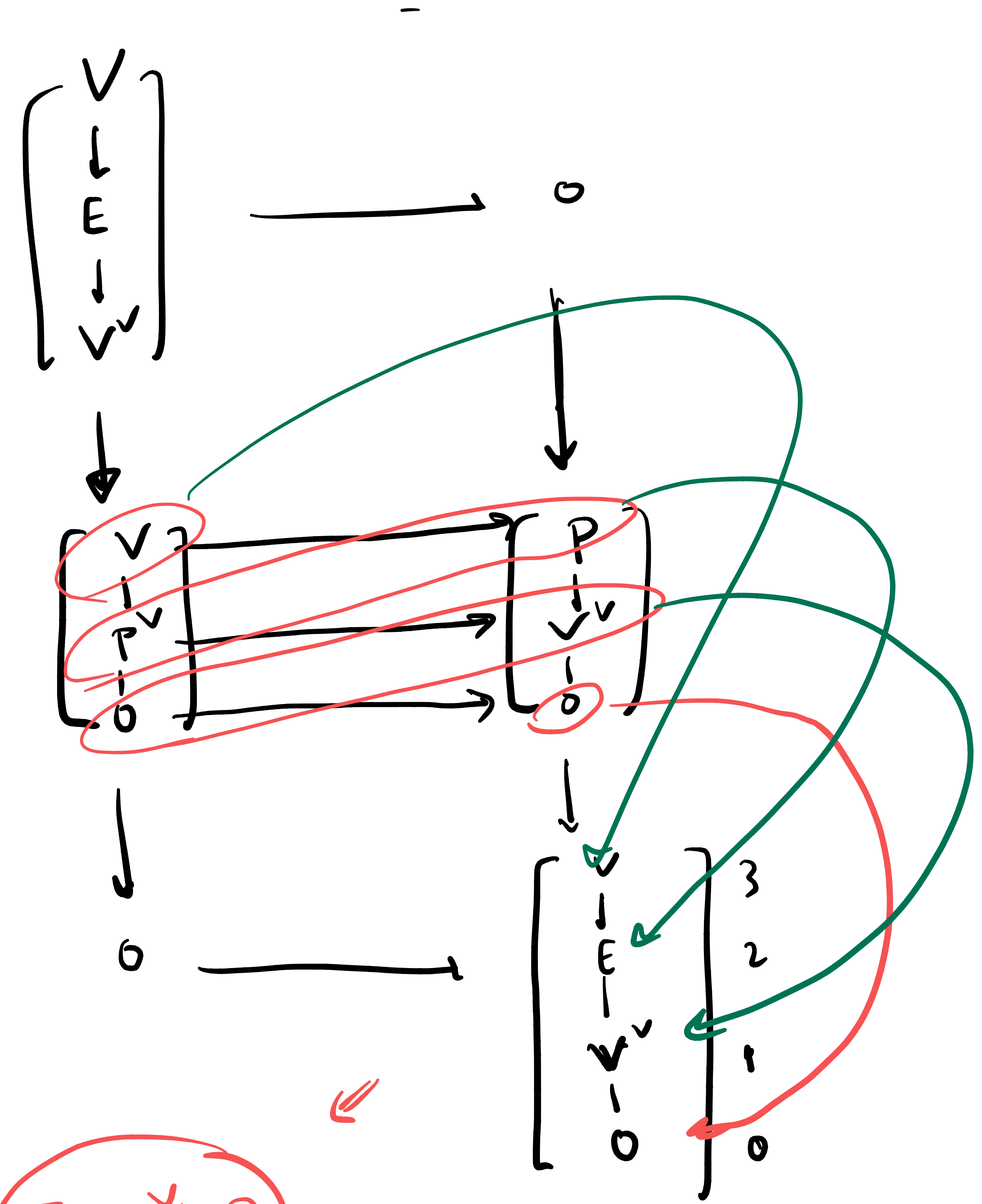
$$\begin{tikzcd}
0 & 0 \arrow[r, "0"] \arrow[d] & V \arrow[r, equal] \arrow[d] & V \arrow[d] \\
1 & P \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & E/P \arrow[d] & \simeq P^\vee \\
2 & V^\vee \arrow[r, equal] & V^\vee \arrow[r] & 0
\end{tikzcd}$$

condition of being lagrangian

in this case

$$P \longrightarrow E \text{ is a lagrangian}$$

in the bundle $(E, q) \Rightarrow E/P \simeq P^\vee$



$$\boxed{E \simeq P^\vee \oplus P}$$

So that the data of a lagrangian foliation gives

all [0,1]-foliations give such a presentation

$$\Pi_X \simeq \left[V \to \underbrace{P \oplus P^\vee}_{E} \to V^\vee \right]$$

~~So we can ask what is a morphism of lagrangian distributors:~~

~~**Remark:**~~

$$\begin{tikzcd}
L_1 \arrow[r, "\ell"] \arrow[d] & L_2 \arrow[r] \arrow[d, "\ell^\vee"] & \Pi_X \arrow[r] \arrow[d] \arrow[ddr] & L_1^\vee[-1] \\
0 \arrow[r] & N[-1] \arrow[r] \arrow[d] & Cof_1 \arrow[ur, dashed] \arrow[d, "\varepsilon"] & \\
 & 0 \arrow[r] & Cof_2 \arrow[r, dashed] & L_2^\vee[-1] \arrow[uu, "\ell^\vee[-1]"']
\end{tikzcd}$$

$$\Downarrow$$

$$\begin{tikzcd}
L_1 \arrow[r] \arrow[r, bend left, -] & L_2
\end{tikzcd}$$ ~~has a [illegible]~~

~~$\Rightarrow$ N carries a form~~

$$\begin{tikzcd}
N^\vee[-1] \arrow[r] \arrow[d, -] & 0 \arrow[d, -] \\
0 \arrow[r] & N[1][-2]
\end{tikzcd}
\quad \Rightarrow \quad
\begin{array}{l}
N^\vee[-1][1] \simeq N[-1] \\
N^\vee[-1][2] \simeq N
\end{array}$$

$$\Downarrow \qquad \overset{\wr\wr}{N[-1]}$$

$\boxed{\text{Talk \#16}}$ (Benjamin)

X (-1)-symplectic scheme

Recall:

$$\underline{\mathrm{Dab}}_X : X^{\mathrm{aff}}_{\text{ét}} \longrightarrow \infty\text{-groupoids}$$

$$S \longmapsto \left\{ S \underset{\text{symp.}}{\simeq} \mathrm{dcrit}(f) \subset \mathcal{U} \xrightarrow{f} \mathbb{A}^1 \right\}$$

$\uparrow$ affine smooth formal scheme.

$+ \quad \mathcal{U}_{\mathrm{red}} \simeq X_{\mathrm{red}}$

isomorphism?:

$$\begin{tikzcd} & U \arrow[dr, "f"] \arrow[dd, "s"', "\text{iso}"] & \\ S \arrow[ur, hook] \arrow[dr, hook] & \circlearrowleft & \mathbb{A}^1 \\ & V \arrow[ur, "g"'] & \end{tikzcd}$$

This commutativity requires an homotopy because S is a derived scheme.

Today how to glue?

$$\underline{\mathrm{Dab}}_X \xrightarrow{P} \underline{\mathrm{Perv}}_X$$

$$U, f \longmapsto P_{U,f}.$$

gluing Mechanism

$$\begin{tikzcd} \underline{D^b_c}_X \arrow[r, "P"] \arrow[d] & \underline{Perv}_X \\ *_X \arrow[ur, dashed] & \end{tikzcd}$$

this corresponds to a global section of $\underline{Perv}_X$

ie $P_X \in \underline{Perv}_X(X) = Perv(X)$

Exemple : Milnor number

$$\begin{tikzcd} \underline{D^b_c}_X \arrow[r] \arrow[d] & \underline{\mathbb{Z}}_X \\ *_X \arrow[ur, dashed, "\nu_X"] & \end{tikzcd}$$

global section $\iff$ Behrends function.

Main theorem (we want to explain)

(KL - Brav - Bussi - Dupont - Joyce - Szendroi)

Given a square root of $K_X := \det(\mathbb{L}_X)$

then there exists a canonical factorization

$$\begin{tikzcd}
\underline{\mathrm{Dcrb}}_X \arrow[r, "P"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \underline{\mathrm{Perv}}_X \\
*_X \arrow[ur, dashed] &
\end{tikzcd}$$

I] comparing local models

$U, f \in \mathrm{Dcrb}_X(S) \rightsquigarrow (U \times \widehat{\mathbb{A}^n}, f + x_1^2 + \dots + x_n^2)$

can
add
variables and
add quadratic fcn

- Same Milnor number
- only isomorphic perverse sheaves.

$$P_{U,f} \underset{\uparrow}{\simeq} P_{U \times \widehat{\mathbb{A}^N}, f + \underbrace{x_1^2 + \dots + x_n^2}_{q}}$$

Non-canonical.

- canonical isomorphism

$$\det T(U, f) \underset{\text{Symplectic}}{\simeq} \det T(U \times \mathbb{A}^N_U, f + q)$$

$\Downarrow$

cannot glue $P_{v,f}$ without some additional data.

IN General

Quad_X^{∇} := stack on $X_{\acute{e}t}^{aff}$ of non-deg. quadratic bundles with compatible flat connection

does not depend on derived structure

(Q, q, ∇) on S

U, f on S $\Rightarrow$ $S_{red} = U_{red}$

<u>then</u>: can form $\begin{array}{c} Q_U \\ \downarrow \\ U \end{array}$ a non-dg. quad. bundle on U.

using the connection.

there is an action

$$\underline{\mathrm{Darb}}_X \times \underline{\mathrm{Quad}}_X \longrightarrow \underline{\mathrm{Darb}}_X$$

$$\boxed{(U,f)\,,\ Q^{\triangledown} \longmapsto \left(\widehat{Q_U}^{\text{zero section}}, f\circ\pi+q\right)}$$

$$\underbrace{\qquad\qquad\qquad\qquad}_{\text{action of the monoid } \mathrm{Quad}_X^{\triangledown}}$$

(sum of quadratic bundles)

Ambiguity :

$$P_{\widehat{Q_U},\, f\pi+q} \underset{\substack{\uparrow \\ \text{thom-sebastiani}}}{\simeq} P_{U,f} \otimes P_{Q_0,q}$$

Example :

$$\left(P_{\mathbb{A}^n,\, x_1^2+\dots+x_n^2}\right)_{x_0\in\mathrm{crit}} \simeq H^{n-1}(S^{n-1},\mathbb{C})$$

$$\begin{array}{c} H^{n-1}(S^{n-1},\mathbb{C}) \\ \wr\| \ \ \text{non-canonical} \\ \mathbb{C} \end{array}$$

$Q = \mathbb{A}^n,\ x_1^2+\dots+x_n^2$

Fix an orientation for the circle.

$P_{\widehat{Q_U},q}$ is a line bundle over U with

$$\boxed{\text{transition functions in } \mu_2 \subseteq \mathbb{G}_m.}$$

$\downarrow$

is a μ_2- $\underline{\text{bundle}}$.

action

$$\begin{tikzcd}
\vdots \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] & \vdots \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] & \vdots \\
\underline{\mathrm{Darb}}_X \times \underline{\mathrm{Quad}}^{\nabla}_X \arrow[r] \arrow[d, shift right, "\text{projection}"'] \arrow[d, shift left, "\text{action}"] & \underline{\mathrm{Perv}}_X \times B\mu_2 \arrow[r] \arrow[d, shift right, "\text{Thom-sebastiani}"'] \arrow[d, shift left, "\text{action}"] & * \times B\mu_2 \arrow[d, shift right] \arrow[d, shift left] \\
\underline{\mathrm{Darb}}_X \arrow[r, "P"] \arrow[d] & \underline{\mathrm{Perv}}_X \arrow[r] \arrow[d] & * \arrow[d] \\
\underbrace{\underline{\mathrm{Darb}}_X / \underline{\mathrm{Quad}}^{\nabla}_X}_{\text{stacks.}} \arrow[r, dashed, "P"] & \underline{\mathrm{Perv}}_X / B\mu_2 \arrow[r] & \underbrace{*/B\mu_2}_{\text{stack classifying } \mu_2\text{-gerbes.}} = K(\mu_2, 2)
\end{tikzcd}$$

(principal bundles with fiber $B\mu_2$)

$$H^0(S, K(\mu_2,2)) \simeq H^2_{ét}(S, \mu_2)$$

Examples of a μ_2- Gerbe;

fix L a line bundle.

look at the <u>stack of square roots of L</u>

if we know that we have a factorization

$$\begin{tikzcd}
\underline{Dab}_X \arrow[r, "P"] \arrow[d] & \underline{Perv}_X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \ast \arrow[d] \\
\underbrace{\underline{Dab}_X / \underline{Quad}^{\nabla}_X}_{\text{stacks.}} \arrow[r, dashed, "P"] \arrow[d] & \underline{Perv}_X / B\mu_2 \arrow[r] & \ast / B\mu_2 = K(\mu_2, 2) \\
\ast \arrow[ur, dashed, "?\Theta"] \arrow[urr, bend right, "\text{is a } \mu_2\text{-gerbe}"'] & &
\end{tikzcd}$$

then the composition

a trivialization of this gerbe gives

$$\begin{tikzcd}
& \underline{\mathrm{Perv}}_X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & * \arrow[d] \\
* \arrow[ur, dashed, "\tilde{\theta}"] \arrow[urr, bend left=40, dashed, "\text{orientation data}"] \arrow[r, "\theta"'] \arrow[rr, bend right=40, "\mu_2\text{-gerbe of square roots of } K_X"'] & \underline{\mathrm{Perv}}_X/B\mu_2 \arrow[r] & K(\mu_2,2)
\end{tikzcd}$$

$\tilde{\theta}$ is our perverse sheaf.

→ Rmk: $\underline{\mathrm{Perv}}_X/B\mu_2$ classifies twisted perverse sheaf.

the orientation data is necessary to lift the twist. Now we want to produce the factorization:

$$\begin{tikzcd}
\underbrace{\underline{\mathrm{Darb}}_X/\underline{\mathrm{Quad}}^{\nabla}_X}_{\text{stacks.}} \arrow[rr, dashed, "P"] \arrow[dr] & & \underline{\mathrm{Perv}}_X/B\mu_2 \\
& * \arrow[ur, dashed, "\theta"] &
\end{tikzcd}$$

Naive idea: prove that

$$\underline{\mathrm{Darb}}_X / \mathrm{Qued}_X^{\nabla} \longrightarrow X \quad \text{is an}$$

equivalence. But this is not true.

thm (BBDJS)

the action of Qued_X^{∇} on $\underline{\mathrm{Darb}}_X$ is locally transitive.

(ie, the stalks of $\underline{\mathrm{Darb}}_X / \mathrm{Qued}_X^{\nabla}$ are locally connected).

"**Proofoid**":

$$\begin{tikzcd} & \mathrm{crit}(f) \arrow[r, hook] & U \arrow[r, "f"] & \mathbb{A}^1 \\ S \arrow[ur, "\sim"] \arrow[dr, "\wr"'] & & & \\ & \mathrm{crit}(g) \arrow[r, hook] & V \arrow[r, "g"] & \mathbb{A}^1 \end{tikzcd}$$

Everything is affine and $U_{red} = S_{red} = V_{red}$

$U_{red} = S_{red}$, V formally smooth

$\Downarrow$

∃ lifting

$$\begin{tikzcd} & U \arrow[d, "\theta"] \\ S \arrow[ur] \arrow[r] & V \end{tikzcd}$$

but it does not necessarily commute with functions.

⇓

can localize to assume that $\mathbb{L}_U$ comes from $\mathbb{L}_V$

i.e., ∃ $\tilde{T}_U$ such that $\theta^* \tilde{T}_U = T_U$

then form a <u>quadratic bundle</u>

$$Q_V = \tilde{T}_U \oplus \tilde{T}_U^\vee$$ with

canonical pairing.

here

$$\begin{tikzcd} U \arrow[r, bend left, "{(0,df)}"] \arrow[d] & Q_V \arrow[dl] \\ V \arrow[ur, bend left, "0"] & \end{tikzcd}$$

this does not preserve the functions

$$\begin{tikzcd}
U \arrow[r] \arrow[d] \arrow[rr, bend left, "f"] & Q_V \arrow[r, "\cancel{\circlearrowleft}", "g+q"'] & A_1' \\
V \arrow[ru, "0"'] \arrow[rru, bend right, "g"'] & &
\end{tikzcd}$$

does not commute.

<u>tecnical part: using Lagrangian foliations:</u>

idea, modify the function and the homotopies

$$\begin{tikzcd}
& U \arrow[d] \\
S \arrow[ru, hook] \arrow[r] & V
\end{tikzcd}$$

Lemma We can modify the Lagrangian foliation structure corresponding to $S \hookrightarrow Q_V$ with function $g+q$ so that everything commutes.

$$\begin{tikzcd}
& U \arrow[rd] \arrow[rrd, bend left, "f"] & & \\
S \arrow[ru, hook] \arrow[rd, bend right, no head] & & W \arrow[r, "h"] & A_1' \\
& V \arrow[ru, "\sim"'] \arrow[rru, bend right, "g"'] & &
\end{tikzcd}$$

W no longer a priori have a Quadratic bundle

but $\mathrm{dcrit}(h) \simeq S$.

+ formal Morse lemma :

locally $h = f + q_f$

$h = g + q_g$

$W \cong N_{V/W}$

$W \cong N_{U/W}$

$\mathrm{Darb}_X / \mathrm{Quad}_X^{\nabla}$ locally connected

but it is not contractible.

Issue with π_1 : there can be automorphisms

$$X \simeq \mathrm{dcrit}\, f \hookrightarrow U \xrightarrow[f]{2\theta} \mathbb{A}^1$$

that do not come from automorphisms of

the Quadratic bundle.

Example:

$$\begin{tikzcd}
 & & \widehat{\mathbb{A}}^{2\circ} \arrow[r, "x^3+y^2"] \arrow[dd, "\varphi"] & \mathbb{A}^1 \\
\mathrm{dcut}(\widehat{\mathbb{A}}^1, x^3) \arrow[r, hook] & \widehat{\mathbb{A}}^0 \arrow[ur, hook] \arrow[dr, hook] \arrow[r, phantom, "2"] & & \\
 & & \widehat{\mathbb{A}}^2 \arrow[r, "x^3+y^2"] & \mathbb{A}^1
\end{tikzcd}$$

$$\varphi(x) = x + y^2$$

$$\varphi(y) = y\,h \qquad \text{where} \qquad h = \underbrace{\sqrt{1-3x^2-3xy^2-y^4}}$$

exists because everything is formal.

is an iso

does not come from an automorphism of quadratic form.

idea: add a t:

$$h_t := \sqrt{1-t3x^2-t^2 3xy^2-t^3y^4}$$

φ is $\mathbb{A}^1$-homotopic to the identity

claim Need to mod out $\mathbb{A}^1$-homotopies and Quadratic bundles.

thm (BBDJS) any automorphism

$$\begin{tikzcd} S \arrow[r, hook] & \mathrm{dcrit} \arrow[loop below, "\varphi"] & U \arrow[l, hook'] \arrow[r, "f"] & \mathbb{A}^1 \end{tikzcd}$$

any φ is $\mathbb{A}^1$-homotopic to an automorphism of the form

$$U \simeq U_0 + \text{Quadratic bundle}$$

$$\circlearrowleft \varphi \qquad \circlearrowright \begin{pmatrix} \mathrm{id} & 0 \\ 0 & \mu \end{pmatrix} \qquad \underline{\mu \in \mathbb{O}(n)}.$$

Corollary $\left(\underline{\mathrm{Darb}}_X \big/ \mathrm{Quad}_X^T \right)_{\mathbb{A}^1}$ has no local automorphisms.

this + locally connected implies that

$$\tau_{\leq 1}\left(\left(\mathcal{D}ab/\mathcal{Q}ucd\right)_{A^1}\right) = *_X$$

Back to our main result:

$$\begin{tikzcd}
\mathcal{D}ab/\mathcal{Q}ucd \arrow[rr] \arrow[d] & & \mathcal{P}erv/B\mu_2 \\
\left(\mathcal{D}ab/\mathcal{Q}ucd\right)_{A^1} \arrow[urr, dashed, "\exists\ \text{fact.}"] \arrow[dr] & & \\
& \tau_{\leq 1}(\ ) = *_X \arrow[uur, dash dot] &
\end{tikzcd}$$

$\rightsquigarrow$ are A^1 inv.

and 1-truncated

and like this we construct P_X. □